Geometric IR Subtraction for Real Radiation

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DEVIL
IS IN
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DETAIL

Conventions

$$p^\mu_{ij..kl} = p^\mu_i + p^\mu_j + .. + p^\mu_k + p^\mu_l$$

$$s_{ij} = 2p_i.p_j$$

$$s_{ijk} = 2(p_i.p_j + p_i.p_k + p_j.p_k)$$

Phase Space Measures and Volumes

The familiar Lorentz invariant on-shell phase space measure:

$$\mathrm{d}\Phi_{1..n}(Q; m_1^2, .., m_n^2) \equiv (2\pi)^{D(1-n)-n} \, \delta^{(D)}\left(Q - \sum_{k=1}^{n} p_k\right) \prod_{k=1}^{n} \mathrm{d}^D p_i \, \delta^+(p_i^2 - m_i^2)$$

Shorthand for massless particles:

$$\mathrm{d}\Phi_{1..n}(Q) = \mathrm{d}\Phi_{1..n}(Q; 0, ..., 0)$$

Shorthand for massive sums of momenta:

$$\mathrm{d}\Phi_{(12)34..n}(Q; s_{12}, 0, .., 0) = \mathrm{d}\Phi_{(12)34..n}(Q; s_{12}) = \mathrm{d}\Phi_{(12)34..n}(Q)$$

The integrated volume:

$$\Phi_n(Q; m_1^2, .., m_n^2) = \int \mathrm{d}\Phi_{1..n}(Q; m_1^2, .., m_n^2)$$

Phase Space Factorisation

$$\mathrm{d}\Phi_{1..n}(Q) = \frac{\mathrm{d}s_{12..k}}{2\pi}\;\mathrm{d}\Phi_{(12..k)k+1..n}(Q; s_{12..k})\;\mathrm{d}\Phi_{12..k}(p_{12..k})$$

A simple Example

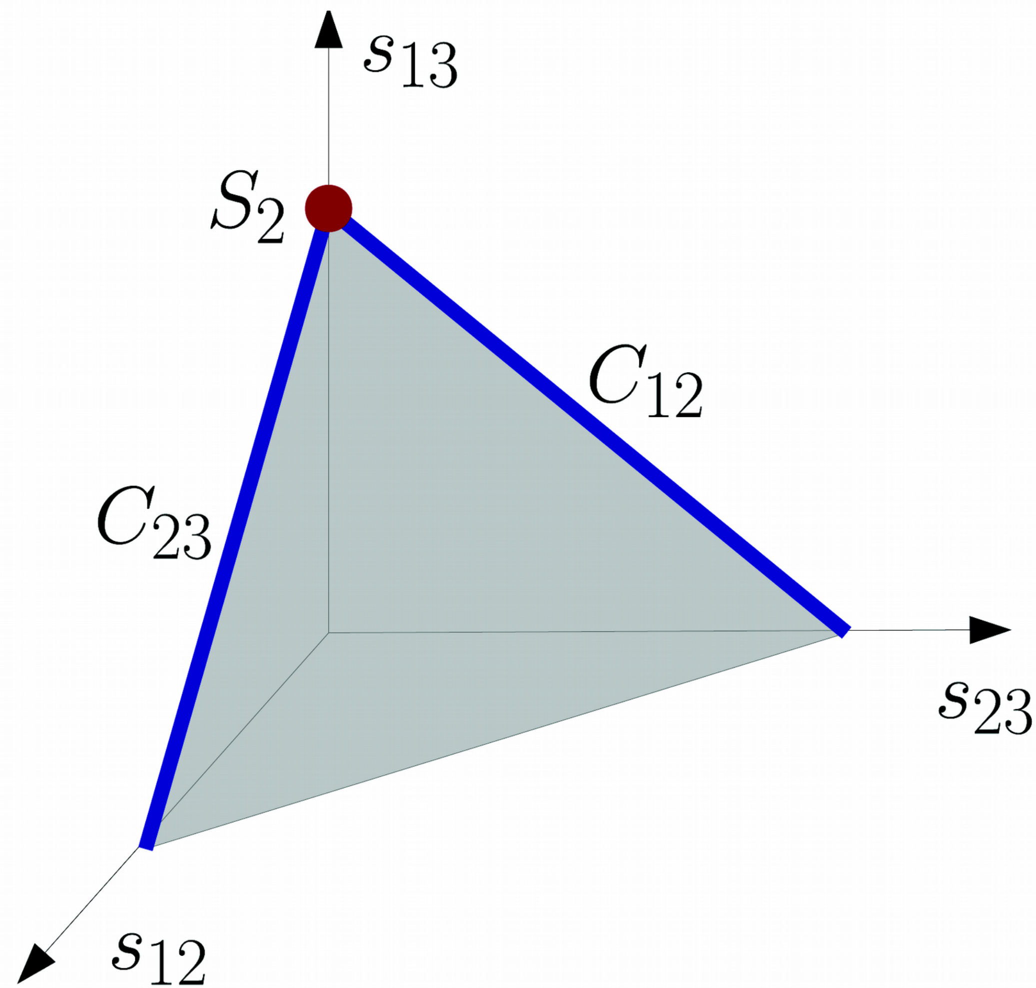
$$I(Q;D) = \int \mathrm{d}\Phi_{123}(Q)\ \frac{s_{13}}{s_{12}s_{23}}$$

Collinear singularities: 1||2 and 2||3

Soft singularity: 2→0

Singularities in Invariant Space

$$\int \mathrm{d}\Phi_{123}(Q) = (Q^2)^{-1+\epsilon}\mathcal{N}_3 \int_0^{Q^2} \mathrm{d}s_{12}\,\mathrm{d}s_{13}\,\mathrm{d}s_{23}\,\delta(Q^2 - s_{12} - s_{13} - s_{23})\ (s_{12}s_{13}s_{23})^{-\epsilon}$$



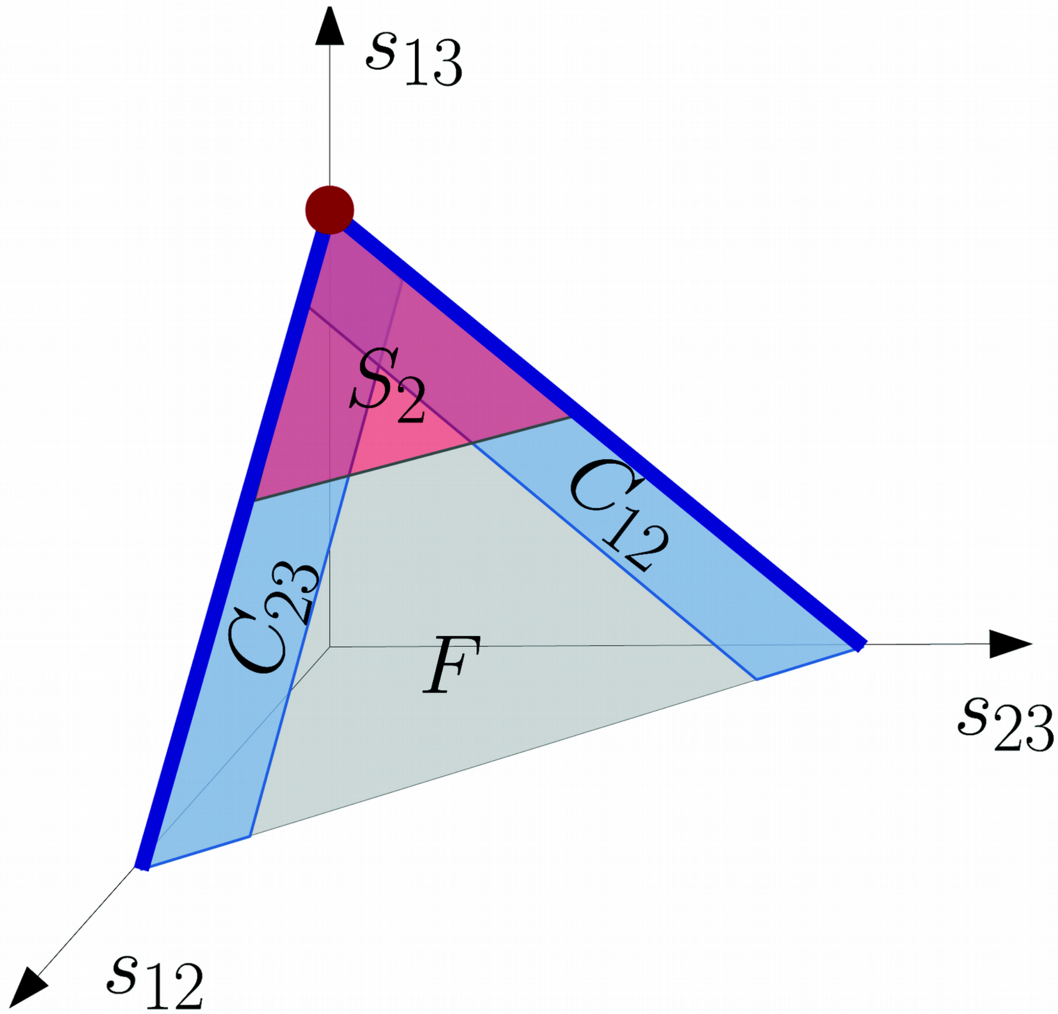
Singularities evaluate to Poles
Dimensional Regularisation

$$I(Q;D) = (Q^2)^{-2\epsilon}\mathcal{N}_3 \frac{\Gamma(-\epsilon)^2\Gamma(2-\epsilon)}{\Gamma(2-3\epsilon)} = \frac{\Phi_3(Q^2)}{(Q^2)}\left(\frac{2}{\epsilon^2} - \frac{5}{\epsilon} + 3 + \mathcal{O}(\epsilon)\right)$$

It is impractical to have to evaluate phase space integrals in D-dimensions!

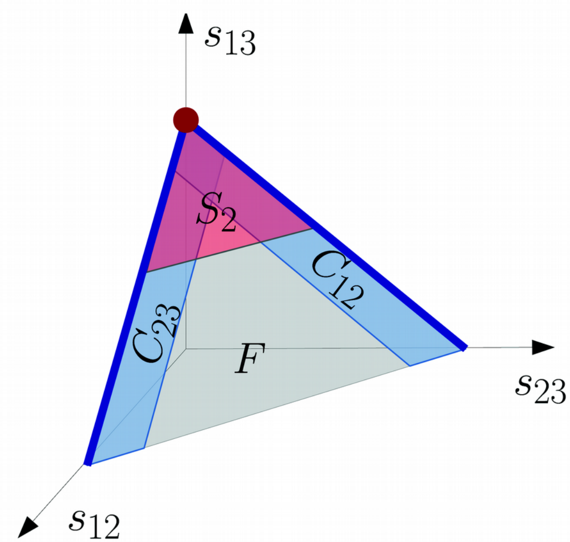
How can we subtract singularities before integration in a minimal way?

A simple Slicing Scheme



A simple Slicing Scheme

$$\begin{aligned}
\Theta(S_2) &= \Theta(s_{2(13)} < a_2 s_{13}) \\
\Theta(C_{12}) &= \Theta(s_{23} < b_{23} Q^2) \\
\Theta(C_{23}) &= \Theta(s_{12} < b_{12} Q^2) \\
\Theta(C_{23} \cap S_2) &= \Theta(s_{2(13)} < a_2 s_{13})\Theta(s_{23} < b_{23} Q^2) \\
\Theta(C_{12} \cap S_2) &= \Theta(s_{2(13)} < a_2 s_{13})\Theta(s_{12} < b_{12} Q^2) \\
\Theta(F) &= \Theta(s_{2(13)} > a_2 s_{13})\Theta(s_{23} > b_{23} Q^2)\Theta(s_{12} > b_{12} Q^2)
\end{aligned}$$



Partition of unity:

$$1 = \Theta(F) + \Theta(S_2) + \Theta(C_{12}) + \Theta(C_{23}) - \Theta(C_{12} \cap S_2) - \Theta(C_{23} \cap S_2)$$

Collinear Region

The collinear limit can be parameterised choosing s_{12} as a normal coordinate:

$$p_1 = z_1 p_{\widetilde{12}} + \frac{s_{12} z_2}{2 p_{\widetilde{12}}.n} n + \sqrt{s_{12} z_1 z_2} e^{\perp}$$

$$p_2 = z_2 p_{\widetilde{12}} + \frac{s_{12} z_1}{2 p_{\widetilde{12}}.n} n - \sqrt{s_{12} z_1 z_2} e^{\perp}$$

$$p_{12} = p_{\widetilde{12}} + \frac{s_{12}}{2 p_{\widetilde{12}}.n} n \, , \qquad p_{\widetilde{12}}^2 = 0 = n^2 \qquad z_1 + z_2 = 1$$

$$\lim_{s_{12} \to 0} p_{12} = p_{\widetilde{12}} + \mathcal{O}(s_{12})$$

Collinear Phase Space

$$\lim_{s_{12}\to 0} \mathrm{d}\Phi_{123}(Q) = \frac{\mathrm{d}s_{12}}{2\pi}\, \mathrm{d}\Phi_{12}(s_{12}) \lim_{s_{12}\to 0} \mathrm{d}\Phi_{(12)3}(Q; s_{12})$$

$$\Rightarrow \quad \lim_{s_{12}\to 0} \mathrm{d}\Phi_{123}(Q) = \mathrm{d}\Phi_{C_{12}}\, \mathrm{d}\Phi_{\widetilde{123}}(Q)$$

$$\mathrm{d}\Phi_{C_{12}} = \frac{\mathrm{d}s_{12}}{2\pi}\, \mathrm{d}\Phi_{12}(s_{12})$$

Soft Phase Space

$p_2 \to 0$ is parameterised by the normal coordinate

$$s_{2(13)} = 2p_2.p_{13}$$

since $E_2 = \frac{s_{2(13)}}{2\sqrt{s_{13}}}$ and $p_2 = E_2(1, \vec{n})$

Soft Phase Space

$$\lim_{s_{13}\to Q^2} \mathrm{d}\Phi_{123}(Q) = \lim_{s_{13}\to Q^2} \frac{\mathrm{d}s_{13}}{2\pi} \, \mathrm{d}\Phi_{13}(s_{13}) \, \mathrm{d}\Phi_{(13)2}(Q; s_{13})$$

$$\Rightarrow \lim_{s_{13}\to Q^2} \mathrm{d}\Phi_{123}(Q) = \mathrm{d}\Phi_{13}(Q^2) \, \mathrm{d}\Phi_{S_2}^{(1,3)}$$

$$\mathrm{d}\Phi_{S_2}^{(1,3)} = \frac{\mathrm{d}s_{2(13)}}{2\pi} \, \mathrm{d}\Phi_{(13)2}(Q^2; Q^2 - s_{2(13)})$$

Soft Collinear Phase Space

Order limits such that $b_{12} \ll a_2$

$$\lim_{a_2 \to 0} \lim_{b_{12} \to 0} \Theta(s_{12} < b_{12} Q^2)\, \Theta(s_{2(13)} < a_2 s_{13})$$

$s_{12} \to 0$

$$= \lim_{a_2 \to 0} \Theta(s_{12} < b_{12} Q^2)\, \Theta(z_2 s_{2\widetilde{13}} < a_2 z_1 s_{2\widetilde{13}})$$

$z_2 \to 0$

$$= \Theta(s_{12} < b_{12} Q^2)\, \Theta(z_2 < a_2)$$

Singular Phase Spaces and Integrals

C_{12}

$$\int \mathrm{d}\Phi_{C_{12}} \Theta(C_{12}) = \frac{(4\pi)^{-2+\epsilon}}{\Gamma(1-\epsilon)} \int_0^{b_{12}Q^2} \mathrm{d}s_{12} s_{12}^{-\epsilon} \int_0^1 \mathrm{d}z_1 \, \mathrm{d}z_2 \, \delta(1 - z_1 - z_2) \, (z_1 z_2)^{-\epsilon}$$

$$\int \mathrm{d}\Phi_{C_{12}} \frac{\Theta(C_{12})}{s_{12}} \frac{z_1}{z_2} = (4\pi)^{-2+\epsilon} \frac{\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)} \frac{(b_{12}Q^2)^{-\epsilon}}{\epsilon^2}$$

S_2

$$\int \mathrm{d}\Phi_{S_2}^{(1,3)} \Theta(S_2) = \frac{(4\pi)^{-2+\epsilon}}{\Gamma(1-\epsilon)} s_{13}^{-1-\epsilon} \int_0^\infty \mathrm{d}s_{12} \, \mathrm{d}s_{23} \, (s_{12}s_{23})^{-\epsilon} \, \Theta(s_{12} + s_{23} < a_2 s_{13})$$

$$\int \mathrm{d}\Phi_{S_2}^{(1,3)} \frac{\Theta(S_2) s_{13}}{s_{12}s_{23}} = (4\pi)^{-2+\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \frac{s_{13}^{-\epsilon} a_2^{-2\epsilon}}{\epsilon^2}$$

$S_2 \cap C_{12}$

$$\int \mathrm{d}\Phi_{C_{12}S_2} \Theta(C_{12} \cap S_2) = \frac{(4\pi)^{-2+\epsilon}}{\Gamma(1-\epsilon)} \int_0^{b_{12}Q^2} \mathrm{d}s_{12} s_{12}^{-\epsilon} \int_0^{a_2} \mathrm{d}z_2 \, z_2^{-\epsilon}$$

$$\int \mathrm{d}\Phi_{C_{12}S_2} \frac{\Theta(C_{12} \cap S_2)}{s_{12}z_2} = \frac{(4\pi)^{-2+\epsilon}}{\Gamma(1-\epsilon)} \frac{(a_2 b_{12} Q^2)^{-\epsilon}}{\epsilon^2}$$

Sum of Singular Regions

$$I_{\text{Singular}}(Q; a_1, b_{12}, b_{23}) = \tag{3.25}$$
$$\frac{\Phi_2}{Q^2}\left[+ I_{S1}(a_2, Q^2) + I_{C_{12}}(b_{12}Q^2) + I_{C_{12}}(b_{23}Q^2) - I_{C_{12}S_1}(b_{23}Q^2, a_2) - I_{C_{12}S_1}(b_{12}Q^2, a_2)\right]$$

$$\begin{aligned}
= \frac{\Phi_3}{(Q^2)^2}\Bigg[&+ \left(\frac{2}{\epsilon^2} + \frac{-9 - 4\ln a_2}{\epsilon} + (9 + 4\zeta_2 + 18\ln a_2 + 4\ln^2 a_2) + \mathcal{O}(\epsilon)\right) \\
&+ \left(\frac{2}{\epsilon^2} + \frac{-7 - 2\ln b_{12}}{\epsilon} + (4 + 4\zeta_2 + 7\ln b_{12} + \ln^2 b_{12}) + \mathcal{O}(\epsilon)\right) \\
&+ \left(\frac{2}{\epsilon^2} + \frac{-7 - 2\ln b_{23}}{\epsilon} + (4 + 4\zeta_2 + 7\ln b_{23} + \ln^2 b_{23}) + \mathcal{O}(\epsilon)\right) \\
&- \Bigg(\frac{2}{\epsilon^2} + \frac{-9 - 2\ln a_2 - 2\ln b_{12}}{\epsilon} + (9 + 6\zeta_2 + 9\ln a_2 + 9\ln b_{12} \\
&\qquad + 2\ln a_2 \ln b_{12} + \ln^2 a_2 + \ln^2 b_{12}) + \mathcal{O}(\epsilon)\Bigg) \\
&- \Bigg(\frac{2}{\epsilon^2} + \frac{-9 - 2\ln a_2 - 2\ln b_{23}}{\epsilon} + (9 + 6\zeta_2 + 9\ln a_2 + 9\ln b_{23} \\
&\qquad + 2\ln a_2 \ln b_{23} + \ln^2 a_2 + \ln^2 b_{23}) + \mathcal{O}(\epsilon)\Bigg)\Bigg]
\end{aligned} \tag{3.26}$$

$$= \frac{\Phi_3}{(Q^2)^2}\left[\frac{2}{\epsilon^2} + \frac{-5}{\epsilon} + (-1 - 2\ln b_{12} - 2\ln b_{23} - 2\ln a_2 \ln b_{12} - 2\ln a_2 \ln b_{23} + 2\ln^2 a_2) + \mathcal{O}(\epsilon)\right].$$

Counter terms reproduce correct poles and simple finite parts

Evaluation of finite part

- Use two different approaches:

  i) Slicing

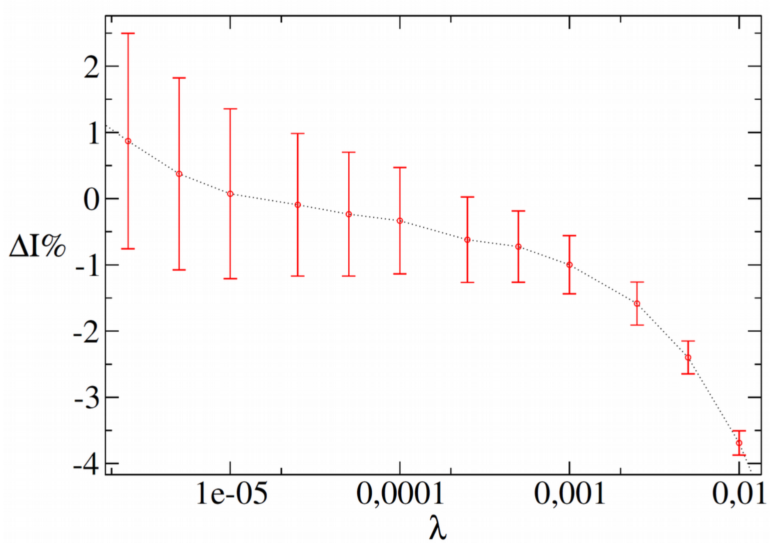
$$I_F(Q; a_1, b_{12}, b_{23}) = \int d\Phi_{123}\, \Theta(F)\, \frac{s_{13}}{s_{12}\, s_{23}}$$

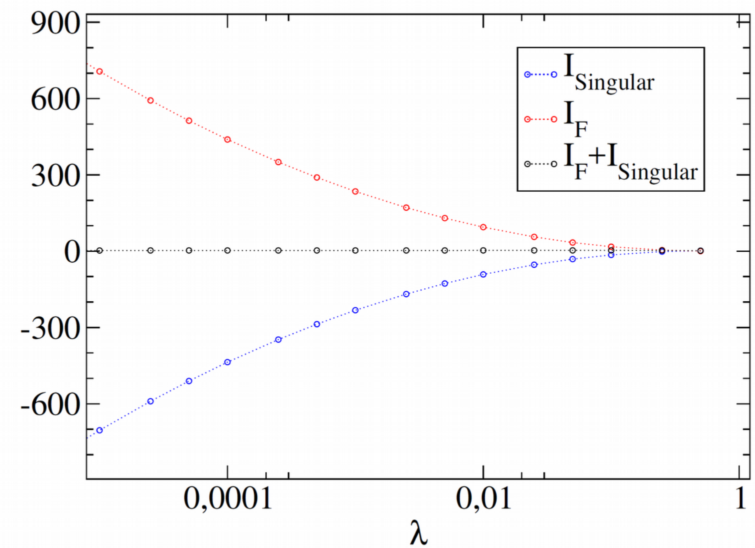
$$\Theta(F) = \Theta(s_{12} > b_{12}Q^2)\Theta(s_{23} > b_{23}Q^2)\Theta(s_{2(13)} > a_2 s_{13})$$

  ii) Subtraction

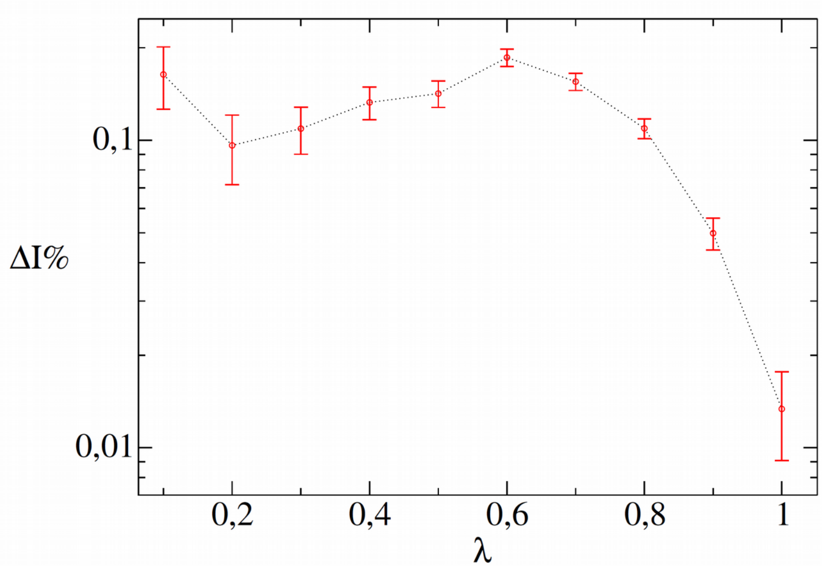
$$I_F(Q; a_1, b_{12}, b_{23}) = \int d\Phi_{123}\Big[\frac{s_{13}}{s_{12}\, s_{23}} - \frac{Q^2}{s_{12}\, s_{23}}\Theta(s_{2(13)} < a_2 Q^2) - \frac{\big(z_{12} - \Theta(z_{21} < a_2)\big)}{s_{12}\, z_{21}(1 - s_{12}/Q^2)}\Theta(s_{12} < b_{12}Q^2) - \frac{\big(z_{32} - \Theta(z_{23} < a_2)\big)}{s_{23}\, z_{23}(1 - s_{23}/Q^2)}\Theta(s_{23} < b_{23}Q^2)\Big]$$

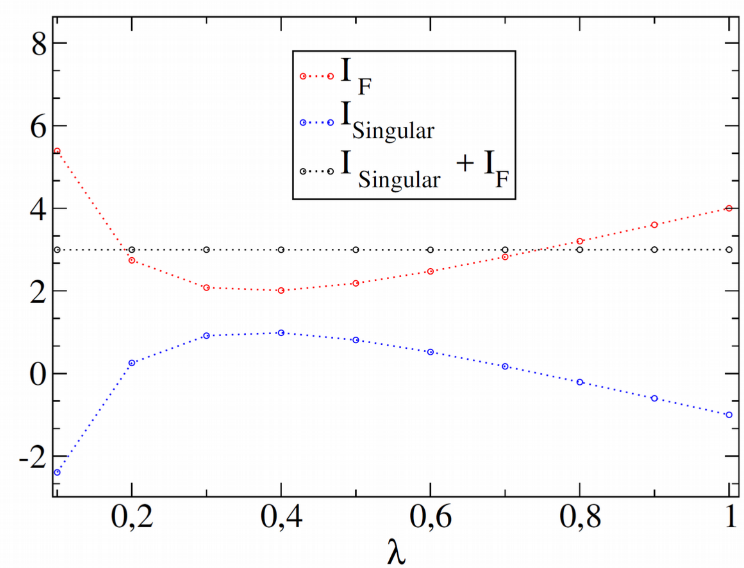
Slicing $\lambda = a_i, \quad \lambda^2 = b_{ij}$





Subtraction $\lambda = a_i = b_{ij}$



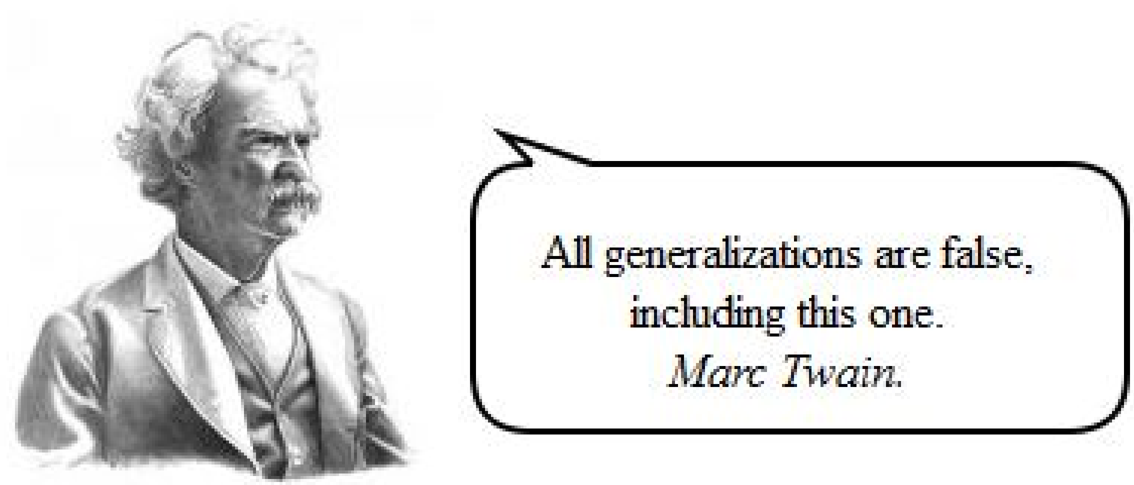


What we learned from this simple example?

- A slicing scheme can be defined based on the phase space factorisation property.
- The Slicing scheme allows to define simple (to integrate) counter terms.
- The Slicing scheme can be promoted to a fully local subtraction scheme.
- Subtraction method easily outperforms its parent slicing method numerically.

The BIG Questions:

Can we generalise to multi particle amplitudes? To NNLO? beyond?



General Formalism

Overlap contributions

Using normal coordinates to define regions we partition the phase space into a *singular* and a *finite* region

$$\Theta(\mathrm{Singular}) + \Theta(F) = 1$$

The finite region can expressed as

$$\Theta(F) = \prod_{r \in R} (1 - \Theta(r))$$

Where R is the set of all singular regions.
Such that for our simple example: $R = \{C_{12}, C_{23}, S_2\}$

Overlap contributions II

Combining and multiplying out we obtain:

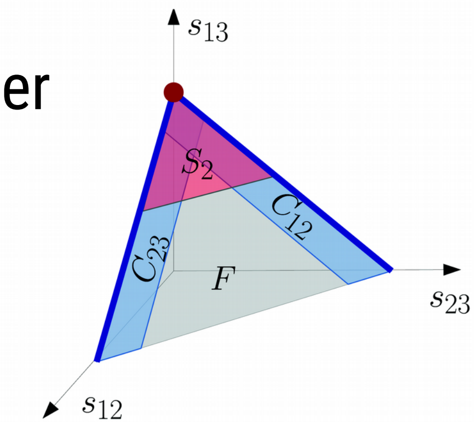
$$\Theta(\text{Singular}) = -\sum_{U \subset R} (-1)^{|U|} \prod_{r \in U} \Theta(r)$$

where the sum goes over all non empty subsets U of R .
So for our simple example we just get:

$$\begin{aligned}\Theta(\text{Singular}) = \;& \Theta(C_{12}) + \Theta(C_{23}) + \Theta(S_2) - \Theta(C_{12} \cap S_2) - \Theta(C_{23} \cap S_2) \\ & -\Theta(C_{12} \cap C_{23}) + \Theta(C_{12} \cap C_{23} \cap S_2)\,,\end{aligned}$$

Which agrees with our previous expression if we further demand the *geometric cancellation identity*:

$$\Theta(C_{12} \cap C_{23}) = \Theta(C_{12} \cap C_{23} \cap S_2)$$



Overlap contributions III

Introduce the measurement-function $J^{(l)}_{1..n}$ which allows for no more than l unresolved partons. We then obtain:

$$J^{(l)}\,\Theta(\text{Singular}) = -J^{(l)} \sum_{U \in \mathcal{U}^{(l)}} (-1)^{|U|} \prod_{r \in U} \Theta(r)$$

$\mathcal{U}^{(l)}$ is the set of soft and/or collinear singularities which:

i) pass the criteria of the the measurement function and
ii) survive the region cancellations

We will refer to the set $\mathcal{U}^{(l)}$ as the *IR forest*.

Normal coordinates and ordering of regions

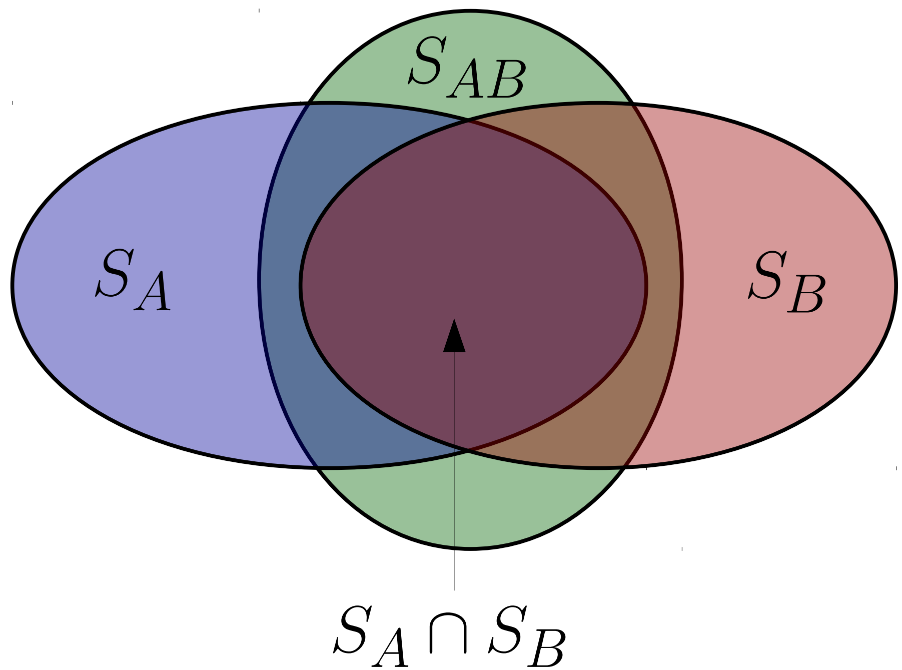
Regions are defined by:

$$\Theta(S_{i_1..i_m}) = \Theta(a_{i_1..i_m} s_{kl} \geq s_{(i_1..i_m)(kl)})$$
$$\Theta(C_{i_1..i_m}) = \Theta(b_{i_1..i_m} \geq s_{i_1..i_m})$$

We then impose the following **strict** order:

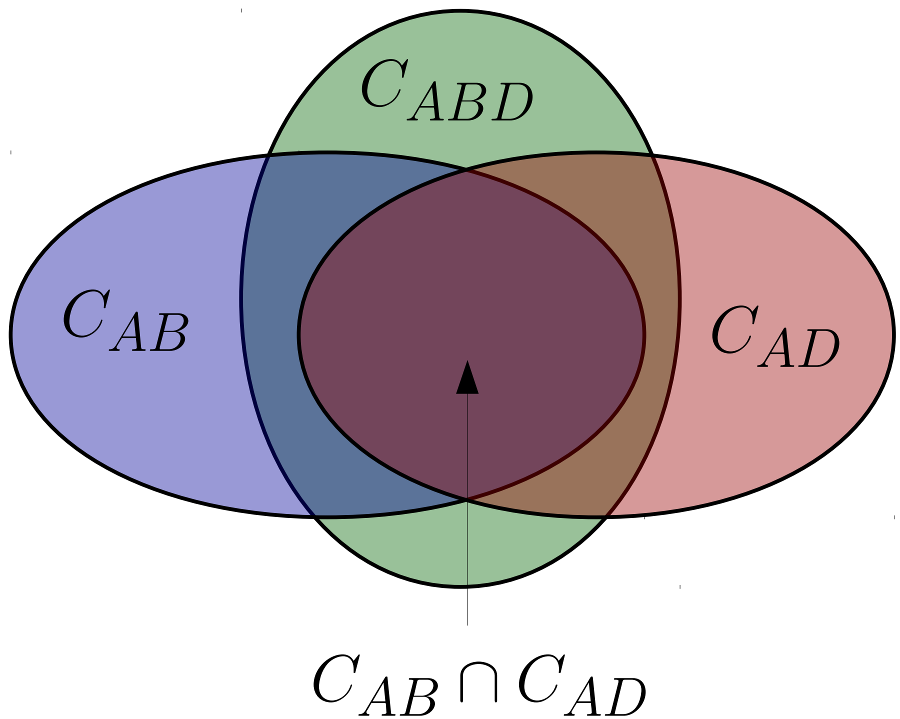
$$a_{i_1 i_2 \ldots i_l} \gg .. \gg a_{i_1} \gg b_{i_1 i_2 \ldots i_{l+1}} \gg .. \gg b_{i_1 i_2}$$

Region Cancellations I



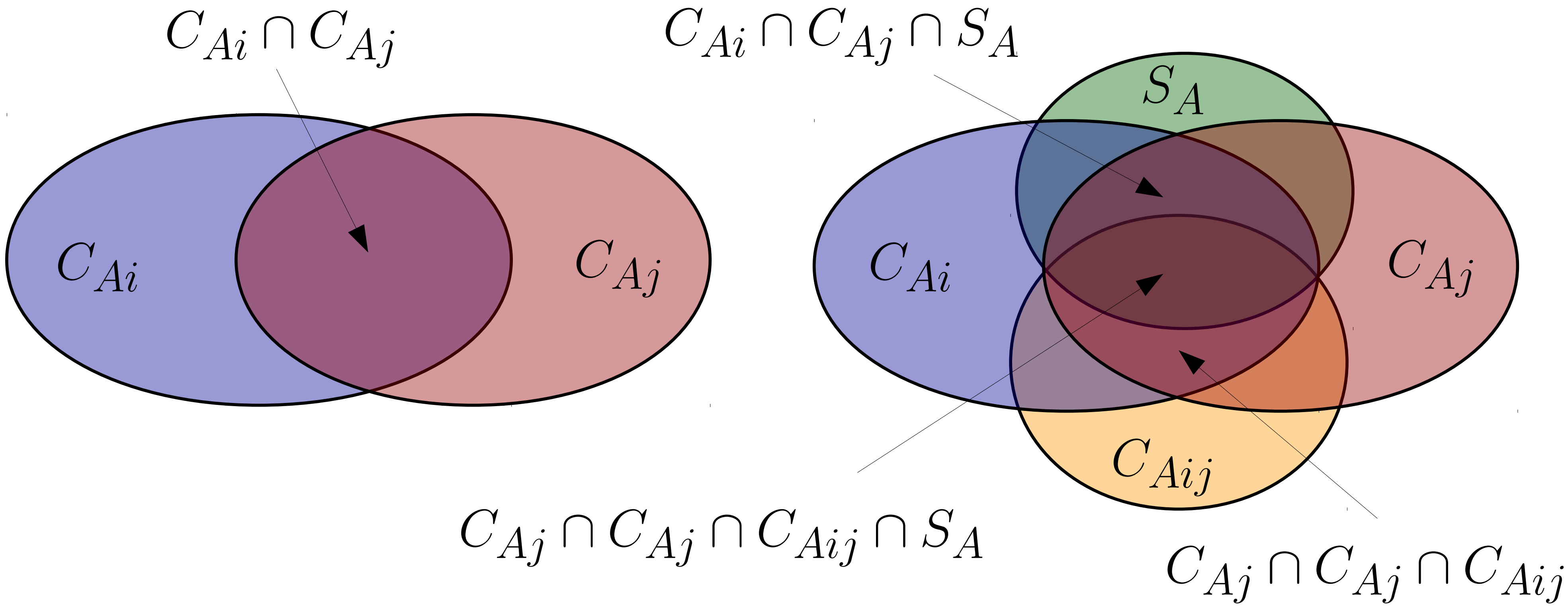
$$\Theta(S_A \cap S_B) = \Theta(S_A \cap S_B \cap S_{AB})$$

Region Cancellations II



$$\Theta(C_{AB} \cap C_{AD}) = \Theta(C_{AB} \cap C_{AD} \cap C_{ABD})$$

Region Cancellations III



$$\Theta(C_{Ai} \cap C_{Aj}) = \Theta(S_A \cap C_{Ai} \cap C_{Aj}) + \Theta(C_{Aij} \cap C_{Ai} \cap C_{Aj})$$
$$-\Theta(S_A \cap C_{Aij} \cap C_{Ai} \cap C_{Aj})\,.$$

The IR forest factorises

as a consequence of region cancellations/ordering

- Conjecture:

$$\mathcal{U}^{(l)} = \mathcal{U}_S^{(l)} \times \mathcal{U}_C^{(l)} \mod \mathcal{J}^{(l)}$$

  - $\mathcal{U}_S^{(l)}$ is a soft forest
  - $\mathcal{U}_C^{(l)}$ is a collinear forest

Counter terms for final states in Yang Mills

Define an observable

$$\mathcal{O}_{l;1\ldots n+l} = \int \mathrm{d}\Phi_{1\ldots n+l}\, \mathcal{J}^{(l)}_{1..n+l}\, |\mathcal{M}_{1..n+l}|^2$$

In the following wish to compute for l=1,2 ; the integrated counterterm:

$$\mathcal{O}^{\mathrm{Singular}}_{l;1..n+l} = \int \mathrm{d}\Phi_{1..n+l}\, \mathcal{J}^{(l)}_{1..n+l}\, \mathbf{\Theta}(\mathrm{Singular}) * |\mathcal{M}_{1..n+l}|^2$$

Key idea

Insert different volumes in different sets of Feynman diagrams

$$\boldsymbol{\Theta}(\text{Singular}) * |\mathcal{M}_{1..n+l}|^2 = \sum_{k,m} (\mathcal{M}_k^*)_{1..n+l}\,(\mathcal{M}_m)_{1..n+l}\,\Theta(\text{Singular}(k,m))$$

Independent sums/classes of Feynman Diagrams

N-particle final state at NLO

Poles of single real are isolated by singular volume contribution:

$$\mathcal{O}^{\text{Singular}}_{1;1..n+1} = -\lim_{a_i \to 0} \lim_{b_{ij} \to 0}$$

$$\cdot \sum_{U \in \mathcal{U}^{(1)}} (-1)^{|U|} \int \mathrm{d}\Phi_{1..n+1}\, \mathcal{J}^{(1)}_{1..n+1} \prod_{r \in U} \boldsymbol{\Theta}(r) * |\mathcal{M}_{1..n+1}|^2$$

$$\mathcal{U}^{(1)} = \{\{C_{ij}\}, \{S_i\}, \{C_{ij}, S_i\}\}$$

It is sufficient to define insertion in the limit

(almost any decomposition, which satisfies these will do)

Soft Region:

$$\lim_{a_k \to 0} \mathbf{\Theta}(S_k) * |\mathcal{M}_{1..n+1}|^2 = \sum_{ij} |\mathcal{M}^{(i,j)}_{1..\not{k}..n+1}|^2 \, \mathcal{S}_k^{(i,j)} \, \Theta(a_k s_{ij} - s_{k(ij)})$$

Collinear Region:

$$\lim_{b_{ij} \to 0} \mathbf{\Theta}(C_{ij}) * |\mathcal{M}_{..i..j..}|^2 = \frac{2}{s_{ij}} (P_{ij})_{\mu_1 \mu_2} |\mathcal{M}^{\boldsymbol{\mu_1 \mu_2}}{}_{..\widehat{ij}..}|^2 \, \Theta(b_{ij} Q^2 - s_{ij})$$

Integrated counter-terms are simple!

$$\mathcal{I}_g^S(s_{kl}, a_i) = \int \mathrm{d}\Phi_{S_i}^{(k,l)}(s_{kl}, a_i)\, \mathcal{S}_i^{(k,l)}$$
$$= 2c_\Gamma \frac{(a_i^2 s_{kl})^{-\epsilon}}{\epsilon^2} \frac{\Gamma(1-\epsilon)^2}{\Gamma(2-2\epsilon)}$$

$$\mathcal{I}_{gg}^C(Q^2, b_{ij}) = \int \mathrm{d}\Phi_{C_{ij}}(Q^2 b_{ij})\, \frac{2}{s_{ij}}\, \langle P_{gg}(z_i)\rangle$$
$$= 6C_A c_\Gamma \frac{(Q^2 b_{ij})^{-\epsilon}}{\epsilon^2} \frac{(1-\epsilon)(4-3\epsilon)}{(3-2\epsilon)} \frac{\Gamma(1-\epsilon)^2}{\Gamma(2-2\epsilon)}$$

$$\mathcal{I}_{gg}^{SC}(Q^2, b_{ij}, a_i) = \int \mathrm{d}\Phi_{C_{ij}S_i}(Q^2 b_{ij}, a_i)\, \frac{2}{s_{ij}}\, \langle P_{gg}(z_i)\rangle\Big|_{z_i\to 0}$$
$$= 4C_A c_\Gamma \frac{(Q^2 b_{ij} a_i)^{-\epsilon}}{\epsilon^2}$$

Convenient to define a soft subtracted collinear counterterm:

$$\mathcal{I}_{ab}^{\widehat{C}}(Q^2, b_{ij}, a_i, a_j) = \mathcal{I}_{ab}^{C}(Q^2, b_{ij}) - \mathcal{I}_{ab}^{SC}(Q^2, b_{ij}, a_i) - \mathcal{I}_{ab}^{SC}(Q^2, b_{ij}, a_j)$$

The integrated NLO counterterm for n emissions:

$$\mathcal{O}^{\text{Singular}}_{1;1..n+1} = \sum_{i>j} \mathcal{I}^{\widehat{C}}_{ij}(Q^2 b_{ij}, a_i, a_j)\mathcal{O}_{0;1..\widehat{ij}..n+1} + \sum_i \sum_{k,l\neq i} \int \mathrm{d}\mathcal{O}^{(k,l)}_{0;1..\not{i}..n+1} \mathcal{I}^S_{g_i}(s_{kl}, a_i)$$

$$\mathrm{d}\mathcal{O}^{(i,j)}_{l;1..n+l} = \mathrm{d}\Phi_{1..n+l}\, |\mathcal{M}^{(i,j)}_{1..n+l}| \mathcal{J}^{(l)}_{1..n+l}\,.$$

agrees with usual 1-loop Catani operator

N-particles final state at NNLO

Normal Coordinates and Measures at NNLO

Limits	Normal coordinate bound	Phase Space Measure
$i\|\|j\|\|k$	$s_{ijk} < b_{ijk} Q^2$	$\mathrm{d}\Phi_{C_{ijk}} = \frac{\mathrm{d}s_{ijk}}{2\pi}\,\mathrm{d}\Phi_{ijk}$
$ij \to 0$	$s_{(ij)(kl)} < a_{ij} s_{kl}$	$\mathrm{d}\Phi^{(k,l)}_{S_{ij}} = \frac{\mathrm{d}s_{(ij)(kl)}}{2\pi} \lim_{ij\to 0} \mathrm{d}\Phi_{ij(kl)}$

The Double Soft Measure

Unlike the single the double soft measure has further support:

$$\mathrm{d}\Phi_{S_{ij}}^{(k,l)} = \frac{\mathrm{d}s_{(ij)(kl)}}{2\pi} \lim_{ij\to 0} \mathrm{d}\Phi_{ij(kl)}$$

$$= \mathrm{d}s_{(ij)(kl)} \frac{d^D p_i}{(2\pi)^{D-1}} \delta^+(p_i^2) \frac{d^D p_j}{(2\pi)^{D-1}} \delta^+(p_j^2)\, \delta(s_{(ij)(kl)} - 2p_{ij}.p_{kl})$$

- Double soft integrals are not (completely) trivial.
- Evaluation can be simplified by IBPs.
- The corresponding 2 double soft Master integrals known [1208.3130]
- In fact even tripple soft masters (hard!, which enter at N3LO) are already known from Higgs soft expansion at N3LO

Tripple Collinear Masters

- Slightly harder than double soft but **same** as N-jettiness beam function
- 4 Master Integrals

$$\mathcal{M}^{(1)}_{C^{(2)}}(Q^2; b_{123}) = \int \mathrm{d}\Phi_{C_{123}}(b_{123}Q^2)$$

$$\mathcal{M}^{(2)}_{C^{(2)}}(Q^2; b_{123}) = \int \mathrm{d}\Phi_{C_{123}}(b_{123}Q^2)\, \frac{1}{s_{123}s_{12}z_{23}}$$

$$\mathcal{M}^{(3)}_{C^{(2)}}(Q^2; b_{123}) = \int \mathrm{d}\Phi_{C_{123}}(b_{123}Q^2)\, \frac{1}{s_{12}s_{13}z_2 z_3}$$

$$\mathcal{M}^{(4)}_{C^{(2)}}(Q^2; b_{123}) = \int \mathrm{d}\Phi_{C_{123}}(b_{123}Q^2)\, \frac{1}{s_{12}s_{13}z_{13}z_{12}}$$

- Evaluated by Ritzmann and Waalewijn for initial and final states (to all orders in eps in terms of 4F3 and 3F2) [1407.3272]

Double Soft - Triple Collinear Overlap

$$\Theta(s_{ijk} < b_{ijk} Q^2)\,\Theta(s_{(ij)(kl)} < a_{ij} s_{kl})$$

$$\downarrow \; b_{ijk} \to 0$$

$$\Theta(s_{ijk} < b_{ijk} Q^2)\,\Theta(z_{ij} s_{\widetilde{ij}kl} < a_{ij} z_k s_{\widetilde{ij}kl})$$

$$\downarrow \; a_{ij} \to 0 \qquad\qquad \downarrow \; a_{ij} \to 0$$

$$\Theta(s_{(ij)k} < b_{ijk} Q^2)\,\Theta(z_{ij} < a_{ij})$$

Asymptotic measure:

$$\mathrm{d}\Phi_{C_{ijk} S_{ij}} = \mathrm{d}s_{(ij)k}\;\mathrm{d}z_{ij}\,\frac{d^D p_i}{(2\pi)^{D-1}}\delta^+(p_i^2)\,\frac{d^D p_j}{(2\pi)^{D-1}}\delta^+(p_j^2)\,\delta(s_{(ij)(kl)} - 2p_{ij}.p_k)\delta(z_{ij} - \frac{p_{ij}.n}{p_k.n})$$

Double soft triple collinear Master integrals can be extracted from the double soft Masters!

Singular double real contribution

$$\mathcal{O}^{\text{Singular}}_{2;1..n+2} = -\lim_{a_{ij}\to 0}\lim_{a_i\to 0}\lim_{b_{ijk}\to 0}\lim_{b_{ij}\to 0}$$

$$\cdot \sum_{U\in\mathcal{U}^{(2)}} (-1)^{|U|} \int d\Phi_{1..n+2}\, \mathcal{J}^{(2)}_{1..n+2} \prod_{r\in U} \boldsymbol{\Theta}(r) * |\mathcal{M}_{1..n+2}|^2$$

Task is to find a suitable insertion of volumes:

-NLO limits are inserted as before!

-NNLO limits require a prescription

Collinear phase spaces factorise (in limit)

$$\lim_{b_{ijk}\to 0} |\mathcal{M}_{..i..j..k..}|^2 * \mathbf{\Theta}(C_{ijk}) = \frac{4}{(s_{ijk})^2} (P_{ijk})_{\mu_1\mu_2} |\mathcal{M}^{\boldsymbol{\mu_1\mu_2}}{}_{..\widehat{ijk}..}|^2 \, \Theta(Q^2 b_{ijk} - s_{ijk})$$

What to do with the double soft?

Soft momenta factorised but color kinematic correlations with up to 4 Wilson lines

$$\lim_{k,l\to 0} |\mathcal{M}_{1..n+2}|^2 = \frac{1}{2} \sum_{i,j,r,t=0}^{n} |\mathcal{M}_{1..\not{k}..\not{l}..n}^{(i,j)(r,t)}|^2 \, \mathcal{S}_k^{(i,j)} \, \mathcal{S}_l^{(r,t)}$$

$$- \frac{1}{2} C_A \sum_{i>j=1}^{n} |\mathcal{M}_{1..\not{k}..\not{l}..n}^{(i,j)}|^2 \left(2\, \mathcal{S}_{kl}^{(i,j)} - \mathcal{S}_{kl}^{(i,i)} - \mathcal{S}_{kl}^{(j,j)} \right)$$

Double soft momenta correlated, but only 2 Wilson lines

Let the kinematics follow the color!

$$\lim_{a_{kl}\to 0} \mathbf{\Theta}(S_{kl}) * |\mathcal{M}_{1..n+2}|^2 =$$

$$-\frac{1}{2} C_A \sum_{i,j=1\neq k,l}^{n+2} |\mathcal{M}^{(i,j)}_{1..\not{k}..\not{l}..n+2}|^2 \left(2\mathcal{S}^{(i,j)}_{kl} - \mathcal{S}^{(i,i)}_{kl} - \mathcal{S}^{(j,j)}_{kl}\right) \Theta(a_{kl}s_{ij} - s_{(kl)(ij)})$$

$$\lim_{a_{kl}\to 0} \lim_{(a_k,a_l)\to 0} (1 - \mathbf{\Theta}(S_{kl}))\mathbf{\Theta}(S_k)\mathbf{\Theta}(S_l) * |\mathcal{M}_{1..n+2}|^2 =$$

$$+\frac{1}{2} \sum_{i,j,r,t\neq k,l} |\mathcal{M}^{(i,j)(r,t)}_{1..\not{k}..\not{l}..n+2}|^2 \, \mathcal{S}^{(i,j)}_{k} \, \mathcal{S}^{(r,t)}_{l} \; \Theta(a_k s_{rt} - s_{k(rt)}) \, \Theta(a_l s_{ij} - s_{l(ij)})$$

This fixes all the overlaps at NNLO!

Iterated double soft limits: $\{S_{ij}, S_i\}$

$$\lim_{a_{kl}\to 0}\lim_{a_k\to 0}\mathbf{\Theta}(S_k)\mathbf{\Theta}(S_{kl}) * |\mathcal{M}_{1..n+2}|^2 = -C_A \sum_{i,j\neq k,l}^{n+2} |\mathcal{M}^{(i,j)}_{1..\not{k}..\not{l}..n+2}|^2\, \mathcal{S}^{(i,j)}_l\, \Theta(a_{kl}s_{ij} - s_{l(ij)})$$
$$\cdot\Big(\mathcal{S}^{(l,j)}_k \Theta(a_k s_{lj} - s_{k(lj)}) + \mathcal{S}^{(l,i)}_k \Theta(a_k s_{li} - s_{k(li)}) - \mathcal{S}^{(i,j)}_k \Theta(a_k s_{ij} - s_{k(ij)})\Big)$$

3 different eikonals in iterated limit contribute to each non-abelian double soft factor

Caveat: although $\{S_{ij}, C_{ik}\}$ vanishes.

$\{S_{ij}, C_{ik}, S_j\}$ survives, due to single soft Phase space

$$\lim_{a_{jk}\to 0}\lim_{a_k\to 0}\lim_{b_{il}\to 0}\mathbf{\Theta}(C_{il})\mathbf{\Theta}(S_k)\mathbf{\Theta}(S_{kl}) * |\mathcal{M}_{1..n+2}|^2 =$$

$$-C_A\sum_{j\neq k,l}^{n+2}|\mathcal{M}^{(i,j)}_{1..\not{k}..\not{l}..n+2}|^2\,\frac{2}{s_{il}}\langle P_{il}(z_l)\rangle\Big|_{z_l\to 0}\Theta(a_{kl}-z_l)\,\Theta(b_{kl}Q^2-s_{il})$$

$$\cdot\,\mathcal{S}_k^{(\widehat{il},j)}\Big(\Theta(a_k z_l s_{\widehat{il}j}-z_l s_{k\widehat{il}}-s_{kj})-\Theta(a_k s_{\widehat{il}j}-s_{k\widehat{il}}-s_{kj})\Big)$$

$$\mathcal{S}_k^{(\widehat{il},j)}=\mathcal{S}_k^{(z_l\widehat{il},j)}$$

Rescale invariance of eikonal factor is not satisfied by the soft volume bound

The IR forest at NNLO

$$\mathcal{U}^{(2)} = \Big\{\{S_i\}, \{S_{ij}\}, \{C_{ij}\}, \{C_{ijk}\}, \{C_{ijk}, C_{ij}\}, \{C_{ijk}, S_{ij}\}, \{C_{ijk}, S_i\}, \{C_{ij}, C_{kl}\},$$
$$\{C_{ij}, S_{ij}\}, \{C_{ij}, S_i\}, \{C_{ij}, S_k\}, \{S_{ij}, S_i\}, \{S_i, S_j\}, \{S_i, S_j, S_{ij}\}, \{C_{ijk}, C_{ij}, S_{ij}\},$$
$$\{C_{ijk}, C_{ij}, S_i\}, \{C_{ijk}, C_{ij}, S_k\}, \{C_{ijk}, S_{ij}, S_i\}, \{C_{ijk}, S_i, S_j\}, \{C_{ijk}, S_i, S_j, S_{ij}\},$$
$$\{C_{ij}, C_{kl}, S_i\}, \{C_{ij}, S_{ij}, S_i\}, \{C_{ij}, S_i, S_k\}, \{C_{ij}, S_i, S_k, S_{ik}\}, \{C_{jk}, S_{ij}, S_i\},$$
$$\{C_{ijk}, C_{ij}, S_{ij}, S_i\}, \{C_{ijk}, C_{ij}, S_{ik}, S_k\}, \{C_{ijk}, C_{ij}, S_i, S_k\}, \{C_{ijk}, C_{ij}, S_i, S_k, S_{ik}\},$$
$$\{C_{ij}, C_{kl}, S_i, S_k\}, \{C_{ij}, C_{kl}, S_i, S_k, S_{ik}\}\Big\}$$

Reality is slightly better since some terms can be combined into one term..

Primitive Measures

All limits of phase space measures at NNLO are expressable using

$$
\begin{aligned}
\mathrm{d}\Phi_{S_i}^{(j,k)}(a_i, s_{jk}) &= \mathrm{d}\Phi_{S_i}^{(j,k)}\,\Theta(s_{i(jk)} < a_i s_{jk}) \\
\mathrm{d}\Phi_{S_{ij}}^{(l,k)}(a_{ij}, s_{kl}) &= \mathrm{d}\Phi_{S_{ij}}^{(k,l)}\,\Theta(s_{(ij)(kl)} < a_{ij} s_{kl}) \\
\mathrm{d}\Phi_{C_{ij}}(b_{ij} Q^2) &= \mathrm{d}\Phi_{C_{ij}}\,\Theta(s_{ij} < b_{ij} Q^2) \\
\mathrm{d}\Phi_{C_{ijk}}(b_{ijk} Q^2) &= \mathrm{d}\Phi_{C_{ijk}}\,\Theta(s_{ijk} < b_{ijk} Q^2) \\
\mathrm{d}\Phi_{C_{ij} S_i}(b_{ij} Q^2, a_i) &= \mathrm{d}\Phi_{C_{ij} S_i}\,\Theta(s_{ij} < b_{ij} Q^2)\Theta(z_i < a_i) \\
\mathrm{d}\Phi_{C_{ijk} S_{ij}}(b_{ijk} Q^2, a_{ij}) &= \mathrm{d}\Phi_{C_{ijk} S_{ij}}\,\Theta(s_{(ij)k} < b_{ijk} Q^2)\Theta(z_{ij} < a_{ij})
\end{aligned}
$$

Other overlapping regions are all iterated or factorising integrals of the NLO ones and evaluate to Gamma-functions.

Convenient to combine sets regions:

$$
\begin{aligned}
\boldsymbol{\Theta}(\bar{C}_{12}) &= \boldsymbol{\Theta}(C_{12})\Big(1-\boldsymbol{\Theta}(S_1)-\boldsymbol{\Theta}(S_2)\Big) \\
\boldsymbol{\Theta}\big(\hat{S}_{12}\big) &= \boldsymbol{\Theta}(S_{12})\Big[\big(1-\boldsymbol{\Theta}(S_1)-\boldsymbol{\Theta}(S_2)\big)\big(1-\boldsymbol{\Theta}(C_{12})\big) \\
&\qquad +\boldsymbol{\Theta}(S_1)\sum_{k\neq 1,2}\boldsymbol{\Theta}(C_{2k})+\boldsymbol{\Theta}(S_2)\sum_{k\neq 1,2}\boldsymbol{\Theta}(C_{1k})\Big] \\
&\quad -\boldsymbol{\Theta}(S_1)\boldsymbol{\Theta}(S_2)(1-\boldsymbol{\Theta}(S_{12}))
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\Theta}(\bar{C}_{123}) = \boldsymbol{\Theta}(C_{123})\Big[&\big(1-\sum_{k=1}^{3}\boldsymbol{\Theta}(S_k)\big)\big(1-\sum_{i>j=1}^{3}\boldsymbol{\Theta}(C_{ij})\big) \\
&+\sum_{i>j=1}^{3}\sum_{k=1\neq i,j}^{3}\big(1-\boldsymbol{\Theta}(S_{ij})\big)\boldsymbol{\Theta}(S_i)\boldsymbol{\Theta}(S_j)\big(1-\boldsymbol{\Theta}(C_{ik})-\boldsymbol{\Theta}(C_{jk})\big) \\
&+\sum_{i>j=1}^{3}\sum_{k=1\neq i,j}^{3}\boldsymbol{\Theta}(S_{ij})\Big(\big(1-\boldsymbol{\Theta}(S_i)-\boldsymbol{\Theta}(S_j)\big)\big(1-\boldsymbol{\Theta}(C_{ij})\big) \\
&\qquad +\boldsymbol{\Theta}(S_j)\boldsymbol{\Theta}(C_{ik})+\boldsymbol{\Theta}(S_i)\boldsymbol{\Theta}(C_{jk})\Big)\Big]
\end{aligned}
$$

Leads to following basic integrated counterterm building blocks:

$$t_{ij..} = Q^2 b_{ij..}$$

$$\lim_{a_{ij}\to 0} \lim_{a_i\to 0} \lim_{b_{ijk}\to 0} \lim_{b_{ij}\to 0} \int \mathrm{d}\mathcal{O}_{2:123..n+2} \boldsymbol{\Theta}(\bar{C}_{123}) =$$

$$\mathcal{I}^{\bar{C}}_{g_1 g_2 g_3}(t_{123}, t_{12}, t_{13}, t_{23}, a_{12}, a_{13}, a_{23}, a_1, a_2, a_3) \int \mathrm{d}\mathcal{O}_{0;\widehat{123}..n+2}$$

$$\lim_{a_{ij}\to 0} \lim_{a_i\to 0} \lim_{b_{ij}\to 0} \int \mathrm{d}\mathcal{O}_{2:123..n+2} \boldsymbol{\Theta}(\hat{S}_{12}) =$$

$$-\frac{C_A}{2} \sum_{i,j\neq 1,2} \int \mathrm{d}\mathcal{O}^{(i,j)}_{0;\widehat{123}..n+2} \mathcal{I}^{\hat{S}}_{g_1 g_2}(s_{ij}, a_{12}, a_1, a_2, t_{12}, t_{1i}, t_{1j}, t_{2i}, t_{2j})$$

$$+ \sum_{i,j,k,l\neq 1,2} \int \mathrm{d}\mathcal{O}^{(i,j)(k,l)}_{0;\widehat{123}..n+2} \mathcal{I}^{S}_{g_1}(s_{ij}, a_1) \, \mathcal{I}^{S}_{g_2}(s_{kl}, a_2)$$

The integrated NNLO counterterm

$$
\begin{aligned}
\mathcal{O}^{\text{Singular}}_{2;1..n+2} = & \sum_{i>j} \mathcal{I}^{\bar{C}}_{g_i g_j}(t_{ij}, a_i, a_j)\, \mathcal{O}_{1;1..\widehat{ij}..n+2} \\
& - \sum_{k} \sum_{i,j\neq k} \int \mathrm{d}\mathcal{O}^{(i,j)}_{1;1..\not{k}..n+2}\, \mathcal{I}^{S}_{g_k}(s_{ij}, a_k) \\
& - \sum_{i>j>k>l} \mathcal{I}^{\bar{C}}_{g_i g_j}(t_{ij}, a_i, a_j)\, \mathcal{I}^{\bar{C}}_{g_k g_l}(t_{kl}, a_k, a_l)\, \mathcal{O}_{0;1..\widehat{ij}..\widehat{kl}..n+2} \\
& + \sum_{i>j>k} \mathcal{I}^{\bar{C}}_{g_i g_j g_k}(t_{ijk}, t_{ij}, t_{ik}, t_{jk}, a_{ij}, a_{ik}, a_{jk}, a_i, a_j, a_k)\, \mathcal{O}_{0;1..\widehat{ijk}..n+2} \\
& + \sum_{i>j} \sum_{k\neq i,j} \sum_{l,m\in\{1,..,\widehat{ij},..,\not{k},..n+2\}} \mathcal{I}^{\bar{C}}_{g_i g_j}(t_{ij}, a_i, a_j) \int \mathrm{d}\mathcal{O}^{(l,m)}_{0;1..\widehat{ij}..\not{k}..n+2}\, \mathcal{I}^{S}_{g_k}(s_{lm}, a_k) \\
& + \sum_{k,l} \sum_{i,j,m,n\neq k,l} \int \mathrm{d}\mathcal{O}^{(i,j)(m,n)}_{0;1..\not{k}..\not{l}..n+2}\, \mathcal{I}^{S}_{g_k}(s_{ij}, a_k)\, \mathcal{I}^{S}_{g_l}(s_{mn}, a_l) \\
& - \frac{C_A}{2} \sum_{k,l} \sum_{i,j\neq k,l} \int \mathrm{d}\mathcal{O}^{(i,j)}_{0;1..\not{k}..\not{l}..n+2}\, \mathcal{I}^{\hat{S}}_{g_k g_l}(s_{ij}, a_{kl}, a_k, a_l, t_{kl}, t_{ik}, t_{jk}, t_{il}, t_{jl})
\end{aligned}
$$

Check for H→gg double real emission

Analytic result is easy to obtain:

$$\mathcal{O}_{H\to g_1g_2g_3g_4} = 120(c_\Gamma)^2(C_A)^2\mathcal{O}_{H\to g_1g_2} \cdot\Big\{ -\frac{1}{\epsilon^4} - \frac{1}{\epsilon^3}\frac{121}{30} + \frac{1}{\epsilon^2}\Big[\frac{39}{5}\zeta_2 - \frac{872}{45}\Big] + \frac{1}{\epsilon}\Big[\frac{123}{5}\zeta_3 + \frac{473}{15}\zeta_2 - \frac{4691}{54}\Big] + \Big[-\frac{37}{10}\zeta_4 - \frac{304951}{810} + 99\zeta_3 + \frac{2303}{15}\zeta_2\Big] + \mathcal{O}(\epsilon)\Big\} \qquad (5.48)$$

$$Q^2 b_{ijk} = \beta_2, \quad Q^2 b_{ij} = \beta_1, \quad a_{ij} = \alpha_2, \quad a_i = \alpha_1$$

$$
\begin{aligned}
\mathcal{O}^{\text{Singular}}_{H\to g_1g_2g_3g_4} = {}& 120(c_\Gamma)^2(C_A)^2\mathcal{O}_{H\to g_1g_2} \\
&\cdot\Bigg\{ -\frac{1}{\epsilon^4} - \frac{1}{\epsilon^3}\frac{121}{30} + \frac{1}{\epsilon^2}\left[\frac{39}{5}\zeta_2 - \frac{872}{45}\right] + \frac{1}{\epsilon}\left[\frac{123}{5}\zeta_3 + \frac{473}{15}\zeta_2 - \frac{4691}{54}\right] \\
&+\Bigg[-\frac{586351}{1620} + \frac{6788}{45}\zeta_2 + \frac{1496}{15}\zeta_3 - \frac{8}{5}\zeta_4 - \frac{1}{5}L^4_{\alpha_2} - \frac{17}{3}L^2_{\alpha_1} - \frac{89}{135}L_{\beta_2} \\
&-\frac{6}{5}L^2_{\beta_2} - \frac{22}{15}L_{\beta_2}L^2_{\alpha_2} - \frac{22}{15}L^2_{\beta_2}L_{\alpha_2} - \frac{2}{5}L^2_{\beta_2}L^2_{\alpha_2} - \frac{8}{5}L^2_{\alpha_1}L^2_{\beta_2} + \frac{4}{5}L^4_{\alpha_1} \\
&-\frac{44}{15}L^2_{\alpha_1}L_{\beta_1} - \frac{22}{15}L^2_{\alpha_2}L_{\beta_1} - \frac{16}{5}L_{\beta_1}L^3_{\alpha_1} - \frac{22}{15}L_{\beta_2}L^2_{\alpha_1} - \frac{22}{5}L^2_{\beta_2}L_{\alpha_1} \\
&-\frac{4}{5}L^2_{\beta_2}\zeta_2 - \frac{16}{5}L_{\alpha_1}\zeta_3 - \frac{8}{5}L_{\alpha_2}\zeta_3 - \frac{44}{15}L_{\alpha_2}\zeta_2 + \frac{22}{15}L^3_{\alpha_2} + \frac{503}{27}L_{\alpha_1} \\
&+\frac{187}{18}L_{\beta_1} + \frac{121}{90}L^2_{\beta_1} - \frac{44}{15}L_{\alpha_1}\zeta_2 + 4\zeta_3 L_{\beta_2} + \frac{8}{5}L_{\beta_2}L_{\beta_1}\zeta_2 \\
&+\frac{16}{5}L_{\beta_1}L^2_{\alpha_1}L_{\beta_2} + \frac{44}{15}L_{\beta_1}L_{\beta_2}L_{\alpha_2} + \frac{4}{5}L^2_{\alpha_2}L_{\beta_1}L_{\beta_2} + \frac{44}{5}L_{\beta_1}L_{\alpha_1}L_{\beta_2} \\
&-\frac{8}{5}L_{\alpha_1}L_{\beta_1}\zeta_2 + \frac{8}{5}L^2_{\alpha_1}\zeta_2 - \frac{16}{5}L_{\alpha_2}L_{\alpha_1}\zeta_2 - \frac{8}{5}L_{\beta_2}L_{\alpha_1}\zeta_2 + \frac{8}{5}L^2_{\alpha_2}\zeta_2 \\
&+\frac{4}{5}L^2_{\alpha_2}L^2_{\alpha_1} + \frac{134}{45}L_{\beta_2}L_{\alpha_1} + \frac{12}{5}L_{\beta_2}L_{\beta_1} + \frac{8}{5}L_{\alpha_1}L^3_{\alpha_2} + \frac{644}{45}L_{\alpha_1}L_{\beta_1} \\
&+\frac{44}{15}L^2_{\beta_1}L_{\alpha_1} + \frac{8}{5}L^2_{\beta_1}L^2_{\alpha_1} - \frac{12}{5}L_{\beta_1}L_{\alpha_1}L^2_{\alpha_2} - \frac{8}{5}L^2_{\alpha_1}L_{\alpha_2}L_{\beta_2} \\
&-\frac{8}{5}L_{\alpha_1}L^2_{\beta_2}L_{\alpha_2} - \frac{4}{5}L_{\alpha_1}L^2_{\alpha_2}L_{\beta_2} + \frac{16}{5}L_{\beta_1}L_{\alpha_1}L_{\beta_2}L_{\alpha_2}\Bigg] + \mathcal{O}(\epsilon)\Bigg\}
\end{aligned}
$$

Poles check out!

Finite terms remain to be checked.

Outlook & Conclusion

Outlook

- Application of the differential cross section calculations still requires adequate mappings
  - They should exist, but not completely trivial

- Generalisation to initial states and real-virtual is not much work
  - Required tripple collinear integrals already known

- Generalisation to massive colored states (tops)
  - possible, but requires eikonal factors with massive Wilson lines (more challenging; integrals may not be known?)

- N3LO should be possible
  - tripple soft known; double real-virtual: double soft known also; ….

Conclusions

- Presented a new subtraction scheme based on different slicing observables for different sets of Feynman diagrams

- Integrated counterterms are simple and can be recycled from higgs soft expansion and n-jettiness jet function

- Scheme is useless as a slicing scheme!
  - Numerically unstable

- Proposition: Scheme can be promoted to a fully local subtraction scheme, after including proper mappings.. (remains to be shown!)

- $\{C_{ij}\}$:

$$\lim_{b_{ij}\to 0}\int \mathbf{\Theta}(C_{ij}) * \mathrm{d}\mathcal{O}_{2;1..i..j..n+2} = \mathcal{I}^{C}_{gg}(Q^2 b_{ij}) \int \mathrm{d}\mathcal{O}_{1;1..\widehat{ij}..n+2}$$

- $\{C_{ijk}\}$:

$$\lim_{b_{ijk}\to 0}\int \mathbf{\Theta}(C_{ijk}) * \mathrm{d}\mathcal{O}_{2;1..i..j..k..n+2} = \mathcal{I}^{C}_{ggg}(Q^2 b_{ijk}) \int \mathrm{d}\mathcal{O}_{0;1..\widehat{ijk}..n+2}$$

- $\{S_k\}$:

$$\lim_{a_k\to 0}\int \mathbf{\Theta}(S_k) * \mathrm{d}\mathcal{O}_{2;1..n+2} = -\sum_{i,j=1\neq k}^{n+2} \int \mathrm{d}\mathcal{O}^{(i,j)}_{1;1..\not{k}..n+2}\,\mathcal{I}^{S}_{gg}(s_{ij}, a_k)$$

- $\{S_{kl}\}$:

$$\lim_{a_{kl}\to 0}\int \mathbf{\Theta}(S_{kl}) * \mathrm{d}\mathcal{O}_{2;1..n+2} = -\frac{1}{2}C_A \sum_{i,j=1\neq k,l}^{n+2} \int \mathrm{d}\mathcal{O}^{(i,j)}_{0;1..\not{k}..\not{l}..n+2}\,\mathcal{I}^{S}_{gg}(s_{ij}, a_{kl})$$

- $\{C_{ijk}, C_{ij}\}$:

$$\lim_{b_{ijk}\to 0}\lim_{b_{ij}\to 0}\int \mathbf{\Theta}(C_{ijk})\mathbf{\Theta}(C_{ij}) * \mathrm{d}\mathcal{O}_{2;1..i..j..k..n+2} = \mathcal{I}^{C}_{gg}(Q^2, b_{ijk})\,\mathcal{I}^{C}_{gg}(Q^2, b_{ij}) \int \mathrm{d}\mathcal{O}_{0;1..\widehat{ijk}..n+2}$$

- $\{C_{ijk}, S_{ij}\}$:

$$\lim_{a_{ij}\to 0}\lim_{b_{ijk}\to 0}\int \mathbf{\Theta}(C_{ijk})\mathbf{\Theta}(S_{ij}) * \mathrm{d}\mathcal{O}_{2;1..i..j..k..n+2} = \mathcal{I}^{SC}_{ggg}(Q^2, a_{ij}, b_{ijk}) \int \mathrm{d}\mathcal{O}_{0;1..\widehat{ijk}..n+2}$$

- $\{C_{ijk}, S_k\}$:

$$\lim_{a_k\to 0}\lim_{b_{ijk}\to 0}\int \mathbf{\Theta}(C_{ijk})\mathbf{\Theta}(S_k) * \mathrm{d}\mathcal{O}_{2;1..i..j..k..n+2} = \int \mathrm{d}\mathcal{O}_{0;1..\widehat{ijk}..n+2}\int \mathrm{d}\mathcal{I}^{C}_{g_ig_j}(Q^2, b_{ijk}) \cdot \frac{1}{2}\Big[\mathcal{I}^{S}_{g}(s_{ij}, a_k) + \mathcal{I}^{SC}_{gg}(Q^2 b_{ijk} - s_{ij}, z_i a_k) + \mathcal{I}^{SC}_{gg}(Q^2 b_{ijk} - s_{ij}, z_j$$

- $\{C_{ij}, C_{kl}\}$

$$\lim_{b_{ij}\to 0}\lim_{b_{kl}\to 0}\int \mathbf{\Theta}(C_{ij})\mathbf{\Theta}(C_{kl}) * \mathrm{d}\mathcal{O}_{2;1..i..j..k..l..n+2} = \mathcal{I}^{C}_{gg}(Q^2, b_{ij})\,\mathcal{I}^{C}_{gg}(Q^2, b_{kl}) \int \mathrm{d}\mathcal{O}_{0;1..\widehat{ijk}..n+2}$$

- $\{C_{kl}, S_{kl}\}$

$$\lim_{a_{kl}\to 0}\lim_{a_{kl}\to 0}\int \mathbf{\Theta}(S_{kl})\mathbf{\Theta}(C_{kl}) * \mathrm{d}\mathcal{O}_{2;1..k..l..n+2} = -\mathcal{I}^{C}_{gg}(Q^2 b_{kl}) \sum_{i,j=1\neq k,l}^{n+2} \int \mathrm{d}\mathcal{O}^{(i,j)}_{0;1..\not{k}..\not{l}..n+2}\,\mathcal{I}^{S}_{g}(s_{ij}, a_{kl})$$

- $\{C_{ij}, S_i\}$:

$$\lim_{a_i\to 0}\lim_{b_{ij}\to 0}\int \mathbf{\Theta}(S_i)\mathbf{\Theta}(C_{ij}) * \mathrm{d}\mathcal{O}_{2;1..i..j..n+2} = \int \mathrm{d}\mathcal{O}_{1;1..\widehat{ij}..n+2}\,\mathcal{I}^{SC}_{gg}(Q^2 b_{ij}, a_i)$$

- $\{C_{ij}, S_k\}$:

$$\lim_{a_k\to 0}\lim_{b_{ij}\to 0}\int \mathbf{\Theta}(S_k)\mathbf{\Theta}(C_{ij}) * \mathrm{d}\mathcal{O}_{2;1..i..j..k..n+2} = -\sum_{l,m\in\{1,..,\widehat{ij},..,\not{k},..n+2\}} \int \mathrm{d}\mathcal{O}^{(l,m)}_{0;1..\widehat{ij}..\not{k}..n+2}\,\mathcal{I}^{S}_{g}(s_{lm}, a_k)\,\mathcal{I}^{C}_{gg}(Q^2 b_{ij})$$

- $\{S_{ij}, S_i\}$

$$\lim_{a_{kl}\to 0}\lim_{a_k\to 0}\int \mathbf{\Theta}(S_k)\mathbf{\Theta}(S_{kl}) * \mathrm{d}\mathcal{O}_{2;1..n+2} = -C_A \sum_{i,j\neq k,l} \mathrm{d}\mathcal{O}^{(i,j)}_{0;1..\not{k}..\not{l}..n+2} \cdot\Big[\int \mathrm{d}\mathcal{I}^{S}_{g_l}(s_{ij}, a_{kl})\big(\mathcal{I}^{S}_{g}(s_{il}, a_k) + \mathcal{I}^{S}_{g}(s_{jl}, a_k)\big) - \mathcal{I}^{S}_{g}(s_{ij}, a_{kl})\mathcal{I}^{S}_{g}(s_{ij}, a_k)\Big]$$

- $\{\{S_k, S_l\}, \{S_{kl}, S_k, S_l\}\}$

$$\lim_{a_{kl}\to 0}\lim_{(a_k,a_l)\to 0}\int \big(1 - \mathbf{\Theta}(S_{kl})\big)\mathbf{\Theta}(S_k)\mathbf{\Theta}(S_l)\big) * \mathrm{d}\mathcal{O}_{2;1..n+2} = +\frac{1}{2}\sum_{i,j,r,t\neq k,l}\int \mathrm{d}\mathcal{O}^{(i,j)(r,t)}_{0;1..\not{k}..\not{l}..n+2}\,\mathcal{I}^{S}_{g}(s_{ij}, a_k)\mathcal{I}^{S}_{g}(s_{rt}, a_l) \tag{B.1}$$

- $\{C_{ijk}, C_{ij}, S_{ij}\}$:

$$\lim_{a_{ij}\to 0}\lim_{b_{ijk}\to 0}\lim_{b_{ij}\to 0}\int \mathbf{\Theta}(S_{ij})\mathbf{\Theta}(C_{ijk})\mathbf{\Theta}(C_{ij}) * \mathrm{d}\mathcal{O}_{2;1..i..j..k..n+2} = \int \mathrm{d}\mathcal{O}_{0;1..\widehat{ijk}..n+2}\,\mathcal{I}^{SC}_{gg}(Q^2 b_{ijk}, a_{ij})\,\mathcal{I}^{C}_{gg}(Q^2 b_{ij})$$

- $\{C_{ijk}, C_{ij}, S_i\}$:

$$\lim_{a_i\to 0}\lim_{b_{ijk}\to 0}\lim_{b_{ij}\to 0}\int \mathbf{\Theta}(S_i)\mathbf{\Theta}(C_{ijk})\mathbf{\Theta}(C_{ij}) * \mathrm{d}\mathcal{O}_{2;1..i..j..k..n+2} = \int \mathrm{d}\mathcal{O}_{0;1..\widehat{ijk}..n+2}\,\mathcal{I}^{C}_{gg}(Q^2 b_{ijk})\,\mathcal{I}^{SC}_{gg}(Q^2 b_{ij}, a_i)$$

- $\{C_{ijk}, C_{ij}, S_k\}$:

$$\lim_{a_k\to 0}\lim_{b_{ijk}\to 0}\lim_{b_{ij}\to 0}\int \mathbf{\Theta}(C_{ijk})\mathbf{\Theta}(C_{ij})\mathbf{\Theta}(S_k) * \mathrm{d}\mathcal{O}_{2;1..i..j..k..n+2} = \int \mathrm{d}\mathcal{O}_{0;1..\widehat{ijk}..n+2}\int \mathrm{d}\mathcal{I}^{C}_{g_ig_j}(Q^2 b_{ijk}) \cdot \frac{1}{2}\Big[\mathcal{I}^{SC}_{gg}(Q^2 b_{ijk}, z_i a_k) + \mathcal{I}^{SC}_{gg}(Q^2 b_{ijk}, z_j a_k)\Big]$$

- $\{C_{ijk}, S_{ik}, S_k\}$:

$$\lim_{a_{ik}\to 0}\lim_{a_k\to 0}\lim_{b_{ijk}\to 0}\int \mathbf{\Theta}(C_{ijk})\mathbf{\Theta}(S_k)\mathbf{\Theta}(S_{ik}) * \mathrm{d}\mathcal{O}_{2;1..i..j..k..n+2} = \int \mathrm{d}\mathcal{O}_{0;1..\widehat{ijk}..n+2}\int \mathrm{d}\mathcal{I}^{SC}_{g_ig_j}(Q^2 b_{ijk}, a_{ik}) \cdot \frac{1}{2}\Big[\mathcal{I}^{S}_{g}(s_{ij}, a_k) + \mathcal{I}^{SC}_{gg}(Q^2 b_{ijk} - s_{ij}, z_i a_k) - \mathcal{I}^{SC}_{gg}(Q^2 b_{ijk} - s_{ij}, a_k)\Big]$$

- $\{\{C_{ijk},S_i,S_j\},\{C_{ijk},S_{ij},S_i,S_j\}\}$:

$$\lim_{a_i\to 0}\lim_{a_j\to 0}\lim_{b_{ijk}\to 0}\int(1-\mathbf{\Theta}(S_{ij}))\mathbf{\Theta}(C_{ijk})\mathbf{\Theta}(S_i)\mathbf{\Theta}(S_j)*\mathrm{d}\mathcal{O}_{2;1..i..j..k..n+2}= \int \mathrm{d}\mathcal{I}^{SC}_{g_ig_k}(Q^2b_{ijk},a_i)\,\mathcal{I}^{SC}_{gg}(Q^2b_{ijk}-s_{ik},a_j)\int \mathrm{d}\mathcal{O}_{0;1..\widehat{ijk}..n+2}$$

- $\{C_{ij},C_{kl},S_i\}$:

$$\lim_{a_i\to 0}\lim_{b_{ij}\to 0}\lim_{b_{kl}\to 0}\int\mathbf{\Theta}(S_i)\mathbf{\Theta}(C_{ij})\mathbf{\Theta}(C_{kl})*\mathrm{d}\mathcal{O}_{2;1..i..j..k..l..n+2}= \mathcal{I}^{SC}_{gg}(Q^2b_{ij},a_i)\,\mathcal{I}^{C}_{gg}(Q^2,b_{kl})\int \mathrm{d}\mathcal{O}_{0;1..\widehat{ij}..\widehat{kl}..n+2}$$

- $\{C_{kl},S_{kl},S_k\}$:

$$\lim_{a_{kl}\to 0}\lim_{a_k\to 0}\lim_{b_{kl}\to 0}\int\mathbf{\Theta}(S_{kl})\mathbf{\Theta}(C_{kl})\mathbf{\Theta}(S_k)*\mathrm{d}\mathcal{O}_{2;1..k..l..n+2}= -\mathcal{I}^{SC}_{gg}(Q^2b_{kl},a_k)\sum_{i,j=1\neq k,l}^{n+2}\int \mathrm{d}\mathcal{O}^{(i,j)}_{0;1..\not{k}..\not{l}..n+2}\,\mathcal{I}^{S}_{g}(s_{ij},a_{kl}) \tag{B.2}$$

- $\{\{C_{ij},S_i,S_k\},\{C_{ij},S_{ik},S_i,S_k\}\}$

$$\lim_{a_k\to 0}\lim_{a_i\to 0}\lim_{b_{ij}\to 0}\int(1-\mathbf{\Theta}(S_{ik}))\mathbf{\Theta}(S_k)\mathbf{\Theta}(S_i)\mathbf{\Theta}(C_{ij})*\mathrm{d}\mathcal{O}_{2;1..i..j..k..n+2}= -\mathcal{I}^{SC}_{gg}(Q^2b_{ij},a_i)\sum_{l,m\in\{1,..,\widehat{ij},..,\not{k},..n+2\}}\int \mathrm{d}\mathcal{O}^{(l,m)}_{0;1..\widehat{ij}..\not{k}..n+2}\,\mathcal{I}^{S}_{gg}(s_{lm},a_k)$$

- $\{C_{il},S_{kl},S_k\}$

$$\lim_{a_{kl}\to 0}\lim_{a_k\to 0}\lim_{b_{il}\to 0}\int\mathbf{\Theta}(C_{il})\mathbf{\Theta}(S_k)\mathbf{\Theta}(S_{kl})*\mathrm{d}\mathcal{O}_{2;1..n+2}= -C_A\sum_{j\in\{1,..,\widehat{il},..,\not{k},..n+2\}}\int \mathrm{d}\mathcal{O}^{(\widehat{il},j)}_{0;1..\not{k}..\widehat{il}..n+2}\int \mathrm{d}\mathcal{I}^{SC}_{g_lg_i}(Q^2b_{il},a_{kl}) \cdot\left(\mathcal{I}^{S}_{g}(z_l s_{\widehat{il}j},a_k)-\mathcal{I}^{S}_{g}(s_{\widehat{il}j},a_k)\right)$$

- $\{C_{ijk},C_{ij},S_{ij},S_i\}$:

$$\lim_{a_{ij}\to 0}\lim_{a_i\to 0}\lim_{b_{ijk}\to 0}\lim_{b_{ij}\to 0}\int\mathbf{\Theta}(S_{ij})\mathbf{\Theta}(S_i)\mathbf{\Theta}(C_{ijk})\mathbf{\Theta}(C_{ij})*\mathrm{d}\mathcal{O}_{2;1..i..j..k..n+2}= \mathcal{I}^{SC}_{gg}(Q^2b_{ijk},a_{ij})\,\mathcal{I}^{SC}_{gg}(Q^2b_{ij},a_i)\int \mathrm{d}\mathcal{O}_{0;1..\widehat{ijk}..n+2}$$

- $\{C_{ijk},C_{ij},S_{ik},S_k\}$

$$\lim_{a_{ik}\to 0}\lim_{a_i\to 0}\lim_{b_{ijk}\to 0}\lim_{b_{ij}\to 0}\int\mathbf{\Theta}(S_{ik})\mathbf{\Theta}(S_i)\mathbf{\Theta}(C_{ijk})\mathbf{\Theta}(C_{ij})*\mathrm{d}\mathcal{O}_{2;1..i..j..k..n+2}= \frac{1}{2}\int \mathrm{d}\mathcal{I}^{SC}_{g_ig_j}(Q^2b_{ij},a_{ik})\left(\mathcal{I}^{SC}_{gg}(Q^2b_{ijk},z_ia_k)-\mathcal{I}^{SC}_{gg}(Q^2b_{ijk},a_k)\right)\int \mathrm{d}\mathcal{O}_{0;1..\widehat{ijk}..n+2}$$

- $\{\{C_{ijk},C_{ij},S_i,S_k\},\{C_{ijk},C_{ij},S_i,S_k,S_{ik}\}\}$

$$\lim_{a_k\to 0}\lim_{a_i\to 0}\lim_{b_{ijk}\to 0}\lim_{b_{ij}\to 0}\int(1-\mathbf{\Theta}(S_{ik}))\mathbf{\Theta}(S_k)\mathbf{\Theta}(S_i)\mathbf{\Theta}(C_{ijk})\mathbf{\Theta}(C_{ij})*\mathrm{d}\mathcal{O}_{2;1..i..j..k..n+2}= \mathcal{I}^{SC}_{gg}(Q^2b_{ijk},a_k)\,\mathcal{I}^{SC}_{gg}(Q^2b_{ij},a_i)\int \mathrm{d}\mathcal{O}_{0;1..\widehat{ijk}..n+2}$$

- $\{\{C_{ij},C_{kl},S_i,S_k\},\{C_{ij},C_{kl},S_i,S_k,S_{ik}\}\}$:

$$\lim_{a_k\to 0}\lim_{a_i\to 0}\lim_{b_{ij}\to 0}\lim_{b_{kl}\to 0}\int(1-\mathbf{\Theta}(S_{ik}))\mathbf{\Theta}(S_i)\mathbf{\Theta}(S_k)\mathbf{\Theta}(C_{ij})\mathbf{\Theta}(C_{kl})*\mathrm{d}\mathcal{O}_{2;1..i..j..k..l..n+2}= \mathcal{I}^{SC}_{gg}(Q^2b_{ij},a_i)\,\mathcal{I}^{SC}_{gg}(Q^2,b_{kl},a_k)\int \mathrm{d}\mathcal{O}_{0;1..\widehat{ij}..\widehat{kl}..n+2}$$

Scalar integral Checks

Checked that sum of integrated counterterms reproduces poles of the following to integrals:

$$\int \frac{\mathrm{d}\Phi_{1234}}{s_{34}s_{134}s_{234}} = \Phi_4(Q)\left[-\frac{1}{4\epsilon^3} - \frac{1}{2\epsilon^2} + \frac{\frac{5}{12}\pi^2 - 1}{2\epsilon} + \mathcal{O}(\epsilon^0)\right]$$

$$R = \{\{C_{34}\}, \{S_{34}\}, \{C_{134}\}, \{C_{234}\}\}$$

$$\int \frac{\mathrm{d}\Phi_{1234}}{s_{13}s_{24}s_{34}} = \Phi_4(Q)\left[\frac{3}{4\epsilon^4} + \frac{-\pi^2}{\epsilon^2} + \frac{-39\zeta(3)}{2\epsilon} + \mathcal{O}(\epsilon^0)\right]$$

$$R = \{\{S_3\}, \{S_4\}, \{C_{13}\}, \{C_{24}\}, \{C_{34}\}, \{S_{34}\}, \{C_{134}\}, \{C_{234}\}\}$$

- $\{C_{34}\}$:

$$\int \frac{\mathrm{d}\Phi_{12\widehat{34}}}{s_{1\widehat{34}}s_{2\widehat{34}}} \int \frac{\mathrm{d}\Phi_{b_{34}}(b_{34})}{s_{34}} = -S_\Gamma \frac{(b_{34})^{-\epsilon}}{\epsilon^3} \frac{(1-3\epsilon)(2-3\epsilon)\Gamma^5(1-\epsilon)}{\Gamma(3-3\epsilon)\Gamma(2-2\epsilon)}$$

- $\{S_{34}\}$:

$$\int \mathrm{d}\Phi_{12} \int \frac{\mathrm{d}\Phi^{(1,2)}_{S_{34}}(s_{12}, a_{34})}{s_{34}s_{1(34)}s_{2(34)}} = -S_\Gamma \frac{(a_{34}^4)^{-\epsilon}}{2\epsilon^3} \frac{(1-4\epsilon)(3-4\epsilon)\Gamma^4(1-\epsilon)}{\Gamma(4-4\epsilon)}$$

- $\{C_{134}\}$:

$$\int \frac{\mathrm{d}\Phi_{\widehat{134}2}}{s_{\widehat{134}2}} \int \frac{\mathrm{d}\Phi_{C_{134}}(b_{134})}{s_{34}s_{134}z_{34}} = -S_\Gamma \frac{(b_{134}^2)^{-\epsilon}}{4\epsilon^3} \frac{(1-3\epsilon)(2-3\epsilon)\Gamma^5(1-\epsilon)}{\Gamma(3-3\epsilon)\Gamma(2-2\epsilon)}$$

- $\{C_{134}, S_{34}\}$:

$$\int \frac{\mathrm{d}\Phi_{\widehat{1342}}}{s_{\widehat{1342}}} \int \frac{\mathrm{d}\Phi_{C_{134}S_{34}}(b_{134}, a_{34})}{s_{34}s_{1(34)}z_{34}} = -S_\Gamma \frac{(a_{34}^2 b_{134}^2)^{-\epsilon}}{4\epsilon^3} \frac{(1-2\epsilon)\Gamma^4(1-\epsilon)}{\Gamma^2(2-2\epsilon)} \tag{4.40}$$

- $\{C_{34}, S_{34}\}$:

$$\int \mathrm{d}\Phi_{12} \int \frac{\mathrm{d}\Phi^{(1,2)}_{S_{\widehat{34}}}(s_{12}, a_{34})}{s_{1\widehat{34}}s_{2\widehat{34}}} \int \frac{\mathrm{d}\Phi_{C_{34}}(b_{34})}{s_{34}} = -S_\Gamma \frac{(a_{34}^2 b_{34})^{-\epsilon}}{\epsilon^3} \frac{(1-2\epsilon)\Gamma^4(1-\epsilon)}{\Gamma^2(2-2\epsilon)} \tag{4.41}$$

- $\{C_{34}, C_{134}\}$:

$$\int \frac{\mathrm{d}\Phi_{\widehat{1342}}}{s_{\widehat{1342}}} \int \frac{\mathrm{d}\Phi_{C_{1\widehat{34}}}(b_{134})}{s_{1\widehat{34}}z_{\widehat{34}}} \int \frac{\mathrm{d}\Phi_{C_{34}}(b_{34})}{s_{34}} = -S_\Gamma \frac{(b_{34}b_{134})^{-\epsilon}}{\epsilon^3} \frac{(1-2\epsilon)\Gamma^4(1-\epsilon)}{\Gamma^2(2-2\epsilon)} \tag{4.42}$$

- $\{S_{34}, C_{234}, C_{34}\}$:

$$\int \frac{\mathrm{d}\Phi_{\widehat{1342}}}{s_{\widehat{1342}}} \int \frac{\mathrm{d}\Phi_{C_{1\widehat{34}}S_{\widehat{34}}}(b_{134}, a_{34})}{s_{1\widehat{34}}z_{\widehat{34}}} \int \frac{\mathrm{d}\Phi_{C_{34}}(b_{34})}{s_{34}} = -S_\Gamma \frac{(a_{34}b_{34}b_{134})^{-\epsilon}}{\epsilon^3} \frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)} \tag{4.43}$$