An analytic local sector subtraction at NNLO

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  - * We have well established methods at NLO:
    - Frixione-Kunst-Signer (FKS) subtraction — Frixione, Kunszt, Signer 9512328; Frixione 9706545
    - Catani-Seymour (CS) Dipole subtraction — Catani, Seymour 9605323; Catani et al. 0201036
    - Nagy-Soper subtraction — Nagy, Soper, 0308127

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- Search for a "minimal" subtraction procedure at NNLO:
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    - Nagy-Soper subtraction — Nagy, Soper, 0308127
  - * Understand their basic features
  - * Try to find a simpler subtraction at NLO, by merging them
  - * Then generalize to NNLO

Structure of subtraction at NLO

$$\frac{d\sigma_{\rm NLO} - d\sigma_{\rm LO}}{dX} = \int d\Phi_n V \, \delta_{X_n} + \int d\Phi_{n+1} R \, \delta_{X_{n+1}} = \text{finite.}$$

X = IRC safe observable $\qquad \delta_{X_m} = \delta(X - X_m)$

X_m = observable computed with m-body kinematics

V has explicit poles in ϵ, R diverges in phase space integration

* Introduce counterterms K and their integral I

$$\int d\Phi_{n+1} \, K \, \delta_{X_n} = \int d\Phi_n \, I \, \delta_{X_n}$$

$$\frac{d\sigma_{\rm NLO} - d\sigma_{\rm LO}}{dX} = \int d\Phi_n (V + I) \, \delta_{X_n} + \int d\Phi_{n+1} \Big(R \, \delta_{X_{n+1}} - K \, \delta_{X_n} \Big)$$

V+I is finite in ϵ, R-K converges in phase space integration

Some notations

Center of mass (CM) momentum: $\mathrm{q} = (\sqrt{s}, \vec{0})$

$$s_{\mathrm{q}i} = 2\,\mathrm{q}\cdot k_i \qquad s_{ij} = (k_i + k_j)^2 = 2k_i\cdot k_j$$

$$s_{ijk} = (k_i + k_j + k_k)^2$$

$$s_{ijkl} = (k_i + k_j + k_k + k_l)^2$$

$\mathcal{E}_i = \frac{s_{\mathrm{q}i}}{s}$ = rescaled energy of particle i in CM frame

$$w_{ij} = \frac{s\, s_{ij}}{s_{\mathrm{q}i}\, s_{\mathrm{q}j}} = \frac{1 - \cos\theta_{ij}}{2}$$

θ_{ij} = angle between i and j in CM frame

* In the following we consider massless QCD just in final state

Primary IRC limits at NLO

* Soft limit:

$$\mathbf{S}_i \quad \Leftrightarrow \quad k_i^\mu \to 0 \quad \Rightarrow \quad \mathcal{E}_i \to 0 \quad \Leftrightarrow \quad \begin{cases} \dfrac{s_{ih}}{s_{kl}} \to 0 & (k,l \neq i) \\ \dfrac{s_{ik}}{s_{il}} \to \text{finite} & (k,l \neq i) \end{cases}$$

Limit on the real matrix element:

$$\mathbf{S}_i\, R(\{k\}) = -\,\mathcal{G}_1 \sum_{k\neq i,\, l\neq i} \frac{s_{kl}}{s_{ik}s_{il}} B_{kl}(\{k\}_{\not i})$$

$$\mathcal{G}_1 = 8\pi\alpha_{\rm s} \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon}$$

Primary IRC limits at NLO

* Collinear limit: $k^\mu = k_i^\mu + k_j^\mu$ $\qquad a = i, j$

Sudakov parametrization

$$\bar{k}^\mu = k^\mu - \frac{k^2}{2k\cdot r} r^\mu \qquad z_a = \frac{2k_a\cdot r}{2k\cdot r} \qquad \tilde{k}_a^\mu = k_a^\mu - z_a k^\mu - \left(\frac{2k\cdot k_a}{k^2} - 2z_a\right)\frac{k^2}{2k\cdot r} r^\mu$$

$$\bar{k}^2 = 0 \qquad z_i + z_j = 1 \qquad \tilde{k}\cdot\bar{k} = \tilde{k}\cdot r = 0 \qquad \tilde{k}_i^\mu + \tilde{k}_j^\mu = 0$$

$$k_a^\mu = z_a \bar{k}^\mu + \tilde{k}_a^\mu - \frac{1}{z_a}\frac{\tilde{k}_a^2}{2k\cdot r} r^\mu$$

$$\mathbf{C}_{ij} \;\Leftrightarrow\; \tilde{k}_i^\mu \to 0 \;\Rightarrow\; w_{ij} \to 0 \;\Leftrightarrow\; \begin{cases} \dfrac{s_{ij}}{s_{kl}} \to 0 & (kl \neq ij,\; k \neq l) \\ \dfrac{s_{ik}}{s_{jk}} \to \text{independent on } k & (k \neq i, j) \end{cases}$$

Limit on the real matrix element:

$$\mathbf{C}_{ij}\, R(\{k\}) = \frac{\mathcal{G}_1}{s_{ij}} \left[P_{ij}\, B(\{k\}_{\not i \not j}, k) - Q_{ij}^{\mu\nu}\, B_{\mu\nu}(\{k\}_{\not i \not j}, k)\right]$$

$$Q_{ij}^{\mu\nu} = Q_{ij}\left[g^{\mu\nu} - (d-2)\frac{\tilde{k}_i^\mu \tilde{k}_i^\nu}{\tilde{k}_i^2}\right]$$

Derived IRC limits at NLO

* Soft-collinear limit:

$$z_i = \frac{s_{ir}}{s_{ir} + s_{jr}} \xrightarrow[\mathbf{S}_i]{} 0 \qquad z_j = \frac{s_{jr}}{s_{ir} + s_{jr}} \xrightarrow[\mathbf{S}_i]{} 1$$

$$\frac{z_j}{z_i} = \frac{s_{jr}}{s_{ir}} \xrightarrow[\mathbf{S}_i]{} \frac{s_{jr}}{s_{ir}} = \frac{z_j}{z_i} \qquad \frac{z_i}{z_j} \xrightarrow[\mathbf{S}_i]{} 0$$

$$P_{ij} \xrightarrow[\mathbf{S}_i]{} \frac{2\, C_{f_j}}{s_{ij}} \frac{z_j}{z_i} \delta_{f_i g}$$

$$\frac{s_{kl}}{s_{ik} s_{il}} \xrightarrow[\mathbf{C}_{ij}]{} \mathcal{O}(1) \quad \text{if } k, l \neq j$$

$$\frac{s_{jl}}{s_{ij} s_{il}} \xrightarrow[\mathbf{C}_{ij}]{} \frac{1}{s_{ij}} \frac{z_j}{z_i} \qquad \frac{s_{kj}}{s_{ik} s_{ij}} \xrightarrow[\mathbf{C}_{ij}]{} \frac{1}{s_{ij}} \frac{z_j}{z_i}$$

$$\mathbf{S}_i\, \mathbf{C}_{ij}\, R(\{k\}) = \mathbf{C}_{ij}\, \mathbf{S}_i\, R(\{k\}) = \mathcal{G}_1 \frac{2\, C_{f_j}}{s_{ij}} \frac{z_j}{z_i} B(\{k\}_{\not i})\, \delta_{f_i g}$$

FKS subtraction procedure

- Divide the phase space through sector functions

$$\mathcal{W}_{ij} = \frac{\sigma_{ij}}{\sigma} \qquad \sigma_{ij} = \frac{1}{\mathcal{E}_i \, w_{ij}} = \frac{s_{qj}}{s_{ij}} \qquad \sigma = \sum_{i,j\neq i} \sigma_{ij}$$

* Basic properties:

$$\boxed{\sum_{i,j\neq i} \mathcal{W}_{ij} = 1}$$

$$\mathbf{S}_i \, \mathcal{W}_{ij} = \frac{\frac{1}{w_{ij}}}{\sum_{j'\neq i} \frac{1}{w_{ij'}}} \qquad \boxed{\sum_{j\neq i} \mathbf{S}_i \, \mathcal{W}_{ij} = 1}$$

$$\mathbf{C}_{ij} \, \mathcal{W}_{ij} = \frac{\mathcal{E}_j}{\mathcal{E}_i + \mathcal{E}_j} \qquad \boxed{\mathbf{C}_{ij} \, \mathcal{W}_{ij} + \mathbf{C}_{ij} \, \mathcal{W}_{ji} = 1}$$

$$\boxed{\mathbf{S}_i \, \mathbf{C}_{ij} \, \mathcal{W}_{ij} = \mathbf{C}_{ij} \, \mathbf{S}_i \, \mathcal{W}_{ij} = 1}$$

FKS subtraction procedure

- Divide the phase space through sector functions
- Each sector reparametrized differently

Sector $\mathcal{W}_{ij}$

$$d\Phi_{n+1}(\{k\}) = d\Phi_n(\{\bar{k}\}_{\not{i}\not{j}}, \bar{k}) d\Phi_1(s, \zeta; \mathcal{E}_i, w_{ij}, \phi)$$

$$\int d\Phi_1(s, \zeta; \mathcal{E}_i, w_{ij}, \phi) = G\, s^{1-\epsilon} \int_0^{\pi} d\phi \sin^{-2\epsilon}\phi \int_0^{\zeta} d\mathcal{E}_i \int_0^1 dw_{ij} \left[\frac{\mathcal{E}_i^2(\zeta - \mathcal{E}_i)^2 w_{ij}(1 - w_{ij})}{\zeta^2(1 - \mathcal{E}_i w_{ij})^2}\right]^{-\epsilon} \frac{\mathcal{E}_i(\zeta - \mathcal{E}_i)}{\zeta(1 - \mathcal{E}_i w_{ij})^2}$$

$$\zeta = \frac{2\bar{k} \cdot q}{s} \qquad G = \frac{(4\pi)^{\epsilon-2}}{\pi^{1/2}\Gamma(1/2 - \epsilon)}.$$

FKS subtraction procedure

- Divide the phase space through sector functions
- Each sector reparametrized differently
- Identify counterterms through parametrization

Sector $\mathcal{W}_{ij}$

$$\mathcal{E}_i^{1-2\epsilon}\, w_{ij}^{-\epsilon}\, R \;=\; \mathcal{E}_i^{-1-2\epsilon}\, w_{ij}^{-1-\epsilon}\, [\mathcal{E}_i^2\, w_{ij} R]$$

$$= \left[-\frac{1}{2\epsilon}\delta(\mathcal{E}_i) + \left(\frac{1}{\mathcal{E}_i} - 2\epsilon \frac{\ln \mathcal{E}_i}{\mathcal{E}_i} \right)_+ \right] \left[-\frac{1}{\epsilon}\delta(w_{ij}) + \left(\frac{1}{w_{ij}} \right)_+ \right] [\mathcal{E}_i^2\, w_{ij} R]$$

- Terms containing **δ**'s $\longrightarrow$ $I\,\delta_{X_n}$
- Remaining term $\longrightarrow$ $R\,\delta_{X_{n+1}} - K\,\delta_{X_n}$

FKS subtraction procedure

- Divide the phase space through sector functions
- Each sector reparametrized differently
- Identify counterterms through parametrization
- Integrate analytically after getting rid of sector functions

$$\sum_{i,j\neq i} \delta(\mathcal{E}_i)\, \mathcal{W}_{ij} = \sum_{i} \delta(\mathcal{E}_i)\, \underbrace{\sum_{j\neq i} \mathbf{S}_i\, \mathcal{W}_{ij}}_{1}$$

$$\begin{aligned}
\sum_{i,j\neq i} \delta(w_{ij})\, \mathcal{W}_{ij} &= \sum_{i,j>i} \delta(w_{ij})\, (\mathcal{W}_{ij} + \mathcal{W}_{ji}) \\
&= \sum_{i,j>i} \delta(w_{ij})\, \underbrace{\mathbf{C}_{ij}\, (\mathcal{W}_{ij} + \mathcal{W}_{ji})}_{1}
\end{aligned}$$

FKS subtraction procedure

- Divide the phase space through sector functions
- Each sector reparametrized differently
- Identify counterterms through parametrization
- Integrate analytically after getting rid of sector functions

* The integration of some counterterms can be non trivial:

$$\int d\Phi_1 \sum_j K_{ij}^{(\mathrm{soft})} \sim \sum_{kl} \int d\bar{\Omega}_i \frac{1-\cos\bar{\theta}_{kl}}{(1-\cos\bar{\theta}_{ki})(1-cos\bar{\theta}_{il})}$$

* Sector parametrization not always optimal

* Can one do something simpler?

CS subtraction procedure

- Counterterms mimic the IRC behaviour in **all** phase space

CS subtraction procedure

- Counterterms mimic the IRC behaviour in **all** phase space
- Counterterms written as sum of terms

$$K = \sum_{\text{pairs } ij} \sum_{k \neq i,j} K_{ijk}$$

$$K_{ijk}(\{k\}) = \frac{\mathcal{G}_1}{s_{ij}} \left[V^{[ij]k} B_{[ij]k}(\{k\}_{\not{i}\not{j}\not{k}}, \bar{k}, \bar{r}) + V_{\mu\nu}^{[ij]k} B^{\mu\nu}_{[ij]k}(\{k\}_{\not{i}\not{j}\not{k}}, \bar{k}, \bar{r}) \right]$$

$$\bar{k}^\mu = k_i^\mu + k_j^\mu - \frac{s_{ij}}{s_{ik} + s_{jk}} k_k^\mu \qquad \bar{r}^\mu = \frac{s_{ijk}}{s_{ik} + s_{jk}} k_k^\mu$$

* $V^{[ij]k}$ and $V_{\mu\nu}^{[ij]k}$ need to reproduce **both** soft and collinear limits:

$$\mathbf{S}_i \, V^{[ij]k} = \frac{s_{jk}}{s_{ij} + s_{ik}} \qquad \mathbf{S}_i \, V_{\mu\nu}^{[ij]k} = 0$$

$$\mathbf{C}_{ij} \, V^{[ij]k} B_{[ij]k} = -P_{ij} \, B \qquad \mathbf{C}_{ij} \, V_{\mu\nu}^{[ij]k} B^{\mu\nu}_{[ij]k} = -Q_{ij}^{\mu\nu} B_{\mu\nu}$$

CS subtraction procedure

- Counterterms mimic the IRC behaviour in **all** phase space
- Counterterms written as sum of terms
- Phase space reparametrized differently for each term of the sum

$$d\Phi_{n+1}(\{k\}) = d\Phi_n(\{k\}_{\not{i}\not{j}\not{k}}, \bar{k}, \bar{r}) d\Phi_1(p^2; y, z, \phi)$$

$$p^2 = (k_i + k_j + k_k)^2 = (\bar{k} + \bar{r})^2$$

$$y = \frac{s_{ij}}{p^2} \qquad z = \frac{s_{ik}}{s_{ik} + s_{jk}}$$

$$\int d\Phi_1(p^2; y, z, \phi) = G\,(p^2)^{1-\epsilon} \int_0^\pi d\phi \sin^{-2\epsilon}\phi \int_0^1 dy \int_0^1 dz\ \left[y\,z(1-y)^2(1-z)\right]^{-\epsilon}(1-y)$$

CS subtraction procedure

- Counterterms mimic the IRC behaviour in **all** phase space
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- Integrate analytically each term

* Integration can be non trivial if counterterms are complicated

* Can one introduce simpler counterterms?

FKS subtraction procedure

- Divide the phase space through sector functions
- Phase space reparametrized differently for each sector
- Identify counterterms through reparametrization
- Integrate analytically after getting rid of sector functions

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FKS subtraction procedure

- Divide the phase space through sector functions ✓
- Phase space reparametrized differently for each sector ✗
- Identify counterterms through reparametrization ✗
- Integrate analytically after getting rid of sector functions ✓

CS subtraction procedure

- Counterterms mimic the IRC behaviour in **all** phase space ✗
- Counterterms written as sum of terms ✓
- Phase space reparametrized differently for each term of the sum ✓
- Integrate analytically each term ✓

A “minimal” subtraction procedure at NLO

- Divide the phase space through sector functions

$$\mathcal{W}_{ij} = \frac{\sigma_{ij}}{\sigma} \qquad \sigma_{ij} = \frac{1}{\mathcal{E}_i \, w_{ij}} = \frac{s_{\mathrm{q}j}}{s_{ij}} \qquad \sigma = \sum_{i,j\neq i} \sigma_{ij}$$

* Basic properties:

$$\sum_{i,j\neq i} \mathcal{W}_{ij} = 1$$

$$\mathbf{S}_i \, \mathcal{W}_{ij} = \frac{\frac{1}{w_{ij}}}{\sum_{j'\neq i} \frac{1}{w_{ij'}}} \qquad \sum_{j\neq i} \mathbf{S}_i \, \mathcal{W}_{ij} = 1$$

$$\mathbf{C}_{ij} \, \mathcal{W}_{ij} = \frac{\mathcal{E}_j}{\mathcal{E}_i + \mathcal{E}_j} \qquad \mathbf{C}_{ij} \, \mathcal{W}_{ij} + \mathbf{C}_{ij} \, \mathcal{W}_{ji} = 1$$

$$\mathbf{S}_i \, \mathbf{C}_{ij} \, \mathcal{W}_{ij} = \mathbf{C}_{ij} \, \mathbf{S}_i \, \mathcal{W}_{ij} = 1$$

A “minimal” subtraction procedure at NLO

- Divide the phase space through sector functions
- Identify counterterms through IRC kernels

Sector $\mathcal{W}_{ij}$

$\mathbf{S}_i$ and $\mathbf{C}_{ij}$ commute

$(1-\mathbf{S}_i)(1-\mathbf{C}_{ij})R$ ⟶ finite ⟶ Candidate for $R - K_{ij}$

Can we define the counterterms as $K_{ij} = \Big[1-(1-\mathbf{S}_i)(1-\mathbf{C}_{ij})\Big]R$?

Momenta in $\mathbf{S}_i R$ and $\mathbf{C}_{ij} R$ do not satisfy mass-shell condition and momenta conservation

Counterterm

$$K_{ij} = \Big[1-(1-\bar{\mathbf{S}}_i)(1-\bar{\mathbf{C}}_{ij})\Big]R = \Big[\bar{\mathbf{S}}_i + \bar{\mathbf{C}}_{ij}(1-\bar{\mathbf{S}}_i)\Big]R$$

$(R - K_{ij})\,\mathcal{W}_{ij}$ ⟶ finite

$\bar{\mathbf{C}}_{ij} R$ and $\bar{\mathbf{S}}_i R$ are the same as $\mathbf{S}_i R$ and $\mathbf{C}_{ij} R$ with physical momenta

A “minimal” subtraction procedure at NLO

- Divide the phase space through sector functions
- Identify counterterms through IRC kernels
- Counterterms written as sum of terms

$$\bar{\mathbf{S}}_i \, R(\{k\}) = -\,\mathcal{G}_1 \sum_{k\neq i,\, l\neq i} \frac{s_{kl}}{s_{ik}s_{il}} B_{kl}(\{k\}_{\not i \not k \not l}, \bar{k}_{ikl}, \bar{r}_{ikl})$$

$$\bar{\mathbf{C}}_{ij} \, R(\{k\}) = \frac{\mathcal{G}_1}{s_{ij}} \Big[P_{ij} B(\{k\}_{\not i \not j \not r}, \bar{k}_{ijr}, \bar{r}_{ijr}) - Q^{\mu\nu}_{ij} B_{\mu\nu}(\{k\}_{\not i \not j \not r}, \bar{k}_{ijr}, \bar{r}_{ijr}) \Big]$$

$$\bar{k}^\mu_{abc} = k^\mu_a + k^\mu_b - \frac{s_{ab}}{s_{ac}+s_{bc}} k^\mu_c \qquad \bar{r}^\mu_{abc} = \frac{s_{abc}}{s_{ac}+s_{bc}} k^\mu_c$$

$\bar{\mathbf{C}}_{ij} R$ and $\bar{\mathbf{S}}_i R$ are the same as $\mathbf{S}_i R$ and $\mathbf{C}_{ij} R$,

with momenta satisfying on-shell condition and momenta conservation

A “minimal” subtraction procedure at NLO

- Divide the phase space through sector functions
- Identify counterterms through IRC kernels
- Counterterms written as sum of terms
- Phase space reparametrized differently for each term of the sum

$\bar{\mathbf{S}}_i R$, B_{kl} term	$\bar{\mathbf{C}}_{ij}(1-\bar{\mathbf{S}}_i)R$
$a,b,c=i,k,l$	$a,b,c=i,j,r$

$$d\Phi_{n+1}(\{k\}) = d\Phi_n(\{k\}_{\not a \not b \not c}, \bar{k}_{abc}, \bar{r}_{abc}) d\Phi_1(p^2; y, z, \phi)$$

$$p^2 = (k_a + k_b + k_c)^2 = (\bar{k}_{abc} + \bar{r}_{abc})^2 \qquad y = \frac{s_{ab}}{p^2} \qquad z = \frac{s_{ac}}{s_{ac} + s_{bc}}$$

$$\int d\Phi_1(p^2; y, z, \phi) = G\,(p^2)^{1-\epsilon} \int_0^\pi d\phi \sin^{-2\epsilon}\phi \int_0^1 dy \int_0^1 dz\, \left[y\, z(1-y)^2(1-z)\right]^{-\epsilon} (1-y)$$

A "minimal" subtraction procedure at NLO

- Divide the phase space through sector functions
- Identify counterterms through IRC kernels
- Counterterms written as sum of terms
- Phase space reparametrized differently for each term of the sum
- Integrate analytically each term after getting rid of the sector functions

$$I = \sum_{i,j\neq i} \int d\Phi_1 \left[\bar{\mathbf{S}}_i + \bar{\mathbf{C}}_{ij}(1-\bar{\mathbf{S}}_i) \right] R\, \mathcal{W}_{ij}$$

$$\sum_{i,j\neq i} \bar{\mathbf{S}}_i\, R\, \mathcal{W}_{ij} = \sum_i \bar{\mathbf{S}}_i\, R \underbrace{\sum_{j\neq i} \mathbf{S}_i\, \mathcal{W}_{ij}}_{1}$$

$$\begin{aligned}
\sum_{i,j\neq i} \bar{\mathbf{C}}_{ij}(1-\bar{\mathbf{S}}_i) R\, \mathcal{W}_{ij} &= \sum_{i,j\neq i} \bar{\mathbf{C}}_{ij}\, R\, \mathbf{C}_{ij} \mathcal{W}_{ij} - \sum_{i,j\neq i} \bar{\mathbf{C}}_{ij} \bar{\mathbf{S}}_i\, R \underbrace{\mathbf{C}_{ij} \mathbf{S}_i \mathcal{W}_{ij}}_{1} \\
&= \sum_{i,j>i} \bar{\mathbf{C}}_{ij}\, R \underbrace{\mathbf{C}_{ij}\left(\mathcal{W}_{ij} + \mathcal{W}_{ji}\right)}_{1} - \sum_{i,j\neq i} \bar{\mathbf{C}}_{ij} \bar{\mathbf{S}}_i\, R \\
&= \sum_{i,j>i} \bar{\mathbf{C}}_{ij}(1-\bar{\mathbf{S}}_i - \bar{\mathbf{S}}_j)\, R
\end{aligned}$$

A "minimal" subtraction procedure at NLO

- Divide the phase space through sector functions
- Identify counterterms through IRC kernels
- Counterterms written as sum of terms
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- Integrate analytically each term after getting rid of the sector functions

$$I = \sum_i \int d\Phi_1 \, \bar{\mathbf{S}}_i \, R + \sum_{i,j>i} \int d\Phi_1 \, \bar{\mathbf{C}}_{ij} (1 - \bar{\mathbf{S}}_i - \bar{\mathbf{S}}_j) \Big] R$$

$$\int d\Phi_1 \bar{\mathbf{S}}_i R = -\mathcal{G}_1 \sum_{k\neq i,\, l\neq i} B_{kl} \int d\Phi_1 \frac{s_{kl}}{s_{ik} s_{il}}$$

$$\int d\Phi_1 \frac{s_{kl}}{s_{ik} s_{il}} = G \, (p^2)^{1-\epsilon} \int_0^\pi d\phi \sin^{-2\epsilon}\phi \int_0^1 dy \int_0^1 dz \left[y \, z (1-y)^2 (1-z) \right]^{-\epsilon} (1-y) \frac{1-z}{y \, z}$$

$$= G \, (p^2)^{1-\epsilon} B(\frac{1}{2}, \frac{1}{2} - \epsilon) B(-\epsilon, 2 - 2\epsilon) B(-\epsilon, -\epsilon)$$

A "minimal" subtraction procedure at NLO

- Divide the phase space through sector functions
- Identify counterterms through IRC kernels
- Counterterms written as sum of terms
- Phase space reparametrized differently for each term of the sum
- Integrate analytically each term after getting rid of the sector functions

$$I = \sum_i \int d\Phi_1 \, \bar{\mathbf{S}}_i \, R + \sum_{i,j>i} \int d\Phi_1 \, \bar{\mathbf{C}}_{ij} (1 - \bar{\mathbf{S}}_i - \bar{\mathbf{S}}_j) \Big] R$$

$$\int d\Phi_1 \bar{\mathbf{C}}_{ij} R = \mathcal{G}_1 \left[B \int d\Phi_1 \frac{P_{ij}}{s_{ij}} - B_{\mu\nu} \underbrace{\int d\Phi_1 \frac{Q^{\mu\nu}_{ij}}{s_{ij}}}_{0} \right]$$

$$\int d\Phi_1 \bar{\mathbf{C}}_{ij} (1 - \bar{\mathbf{S}}_i - \bar{\mathbf{S}}_j) R = \mathcal{G}_1 B \int d\Phi_1 \frac{1}{s_{ij}} \left[P_{ij} - 2C_j \frac{z_j}{z_i} - 2C_i \frac{z_i}{z_j} \right]$$

Structure of subtraction at NNLO

$$\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = \int d\Phi_n VV\, \delta_{X_n} + \int d\Phi_{n+1} RV\, \delta_{X_{n+1}} + \int d\Phi_{n+2}\, RR\, \delta_{X_{n+2}}$$

VV and VR have poles in ϵ, VR and RR diverge in phase space

* Counterterms $K^{(1)}$, $K^{(12)}$, $K^{(2)}$, $K^{(RV)}$ and their integrals $I^{(1)}$, $I^{(12)}$, $I^{(2)}$, $I^{(RV)}$

$$\int d\Phi_{n+2} \left[K^{(\mathbf{1})}\, \delta_{X_{n+1}} + \left(K^{(\mathbf{12})} + K^{(\mathbf{2})}\right) \delta_{X_n}\right] = \int d\Phi_{n+1}\, I^{(\mathbf{1})}\, \delta_{X_{n+1}} + \int d\Phi_n \left(I^{(\mathbf{12})} + I^{(\mathbf{2})}\right) \delta_{X_n}$$

$$\int d\Phi_{n+2}\, K^{(\mathbf{RV})}\, \delta_{X_n} = \int d\Phi_{n+1}\, I^{(\mathbf{RV})}\, \delta_{X_n}$$

$$\begin{aligned}\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = & \int d\Phi_n \left(VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})}\right) \delta_{X_n} \\ & + \int d\Phi_{n+1} \left[\left(RV + I^{(\mathbf{1})}\right) \delta_{X_{n+1}} - \left(K^{(\mathbf{RV})} - I^{(\mathbf{12})}\right) \delta_{X_n}\right] \\ & + \int d\Phi_{n+2} \left[RR\, \delta_{X_{n+2}} - K^{(\mathbf{1})} \delta_{X_{n+1}} - \left(K^{(\mathbf{2})} + K^{(\mathbf{12})}\right) \delta_{X_n}\right]\end{aligned}$$

$(V + I^{(2)} + I^{(RV)})$, $(RV + I^{(1)})$ and $(K^{(RV)} - I^{(12)})$ are finite in ϵ

$(RR - K^{(1)} - K^{(12)} - K^{(2)})$, $(RV - K^{(RV)})$ and $(I^{(1)} + I^{(12)})$ converge in phase space

Primary IRC limits at NNLO

- Single soft limit
- Single collinear limit
- Double soft limit:

$$\mathbf{S}_{ik} \Leftrightarrow \begin{cases} k_i^\mu = \lambda k_i'^\mu \\ k_k^\mu = \lambda k_k'^\mu \\ \lambda \to 0 \end{cases} \Rightarrow \begin{cases} \mathcal{E}_i, \mathcal{E}_k \to 0 \\ \dfrac{\mathcal{E}_i}{\mathcal{E}_k} \to \text{finite} \end{cases} \Leftrightarrow \begin{cases} \dfrac{s_{ik}}{s_{il}}, \dfrac{s_{ik}}{s_{kl}}, \dfrac{s_{ih}}{s_{lm}}, \dfrac{s_{kh}}{s_{lm}} \to 0 \quad (h, l, m \neq i, k) \\ \dfrac{s_{il}}{s_{im}}, \dfrac{s_{kl}}{s_{km}}, \dfrac{s_{il}}{s_{km}} \to \text{finite} \quad (l, m \neq i, k) \end{cases}$$

Limit on the double real matrix element: Catani, Grazzini 9908523

$$\mathbf{S}_{ik}\, RR(\{k\}) = \mathcal{G}_2 \Bigg[\sum_{\substack{l \neq i,k \\ m \neq i,k}} \mathcal{I}_{lm} B_{lm}(\{k\}_{\not i \not k}) + \frac{1}{2} \sum_{\substack{l \neq i,k \\ m \neq i,k}} \mathcal{I}_{lm}^{(i)} \sum_{\substack{l' \neq i,k \\ m' \neq i,k}} \mathcal{I}_{l'm'}^{(k)} B_{lm,l'm'}(\{k\}_{\not i \not k}) \Bigg]$$

$$\mathcal{G}_2 = \mathcal{G}_1^2 = (8\pi\alpha_s)^2 \left(\frac{\mu^2 e^{\gamma_E}}{4\pi} \right)^{2\epsilon} \qquad \mathcal{I}_{ab}^{(i)} = \begin{cases} 0 & \text{if } i, k \text{ are quarks} \\ \dfrac{s_{ab}}{s_{ia} s_{ib}} & \text{if } i, k \text{ are gluons} \end{cases}$$

Primary IRC limits at NNLO

* Double collinear limit: $k^\mu = k_i^\mu + k_j^\mu + k_k^\mu$ $\qquad a = i, j, k$

$$\bar{k}^\mu = k^\mu - \frac{k^2}{2k\cdot r} r^\mu \qquad z_a = \frac{2k_a\cdot r}{2k\cdot r} \qquad \tilde{k}_a^\mu = k_a^\mu - z_a k^\mu - \left(\frac{2k\cdot k_a}{k^2} - 2z_a\right)\frac{k^2}{2k\cdot r} r^\mu$$

$$\bar{k}^2 = 0 \qquad z_i + z_j + z_k = 1 \qquad \tilde{k}\cdot\bar{k} = \tilde{k}\cdot r = 0 \qquad \tilde{k}_i^\mu + \tilde{k}_j^\mu + \tilde{k}_k^\mu = 0$$

$$k_a^\mu = z_a \bar{k}^\mu + \tilde{k}_a^\mu - \frac{1}{z_a}\frac{\tilde{k}_a^2}{2k\cdot r} r^\mu$$

$$\mathbf{C}_{ijk} \Leftrightarrow \begin{cases} \tilde{k}_i^\mu = \lambda \tilde{k}_i'^\mu \\ \tilde{k}_j^\mu = \lambda \tilde{k}_j'^\mu \\ \tilde{k}_k^\mu = \lambda \tilde{k}_k'^\mu \\ \lambda \to 0 \end{cases} \Rightarrow \begin{cases} w_{ij}, w_{jk}, w_{ik} \to 0 \\ \frac{w_{ij}}{w_{jk}}, \frac{w_{jk}}{w_{ik}}, \frac{w_{ik}}{w_{ij}} \to \text{fin.} \end{cases} \Leftrightarrow \begin{cases} \frac{s_{ij}}{s_{lm}}, \frac{s_{jk}}{s_{lm}}, \frac{s_{ik}}{s_{lm}} \to 0 & (lm \neq ij, jk, ik \;\; l \neq m) \\ \frac{s_{ij}}{s_{jk}}, \frac{s_{jk}}{s_{ik}}, \frac{s_{ik}}{s_{ij}} \to \text{finite} & \\ \frac{s_{il}}{s_{jl}}, \frac{s_{jl}}{s_{kl}}, \frac{s_{il}}{s_{kl}} \to \text{indep. on } l & (l \neq i, j, k) \end{cases}$$

Limit on the double real matrix element: Catani, Grazzini 9908523

$$\mathbf{C}_{ijk}\, RR(\{k\}) = \frac{\mathcal{G}_2}{s_{ijk}^2}\left[P_{ijk}\, B(\{k\}_{\not{i}\not{j}\not{k}}, k) - Q_{ijk}^{\mu\nu}\, B_{\mu\nu}(\{k\}_{\not{i}\not{j}\not{k}}, k)\right]$$

$$Q_{ijk}^{\mu\nu} = \sum_{a=i,j,k} Q_{ijk}^{(a)}\left[g^{\mu\nu} - (d-2)\frac{\tilde{k}_a^\mu \tilde{k}_a^\nu}{\tilde{k}_a^2}\right]$$

A “minimal” subtraction procedure at NNLO

- Divide the phase space through sector functions

$$\mathcal{W}_{ijkl} = \frac{\sigma_{ijkl}}{\sigma} \qquad \sigma = \sum_{\substack{i,\,j\neq i\\ k\neq i,\,l\neq i,k}} \sigma_{ijkl}$$

$$\sum_{\substack{i,\,j\neq i\\ k\neq i,\,l\neq i,k}} \mathcal{W}_{ijkl} = 1$$

$$\sigma_{ijkl} = \frac{1}{(\mathcal{E}_i)^\alpha (w_{ij})^\beta} \frac{1}{(\mathcal{E}_k + \delta_{kj}\mathcal{E}_i) w_{kl}} \qquad \alpha > \beta > 1$$

* Single soft and single collinear limits

$$\mathbf{S}_i\, \mathcal{W}_{ijkl} = \left(\mathbf{S}_i \mathcal{W}_{ij}^{(\beta)}\right) \mathcal{W}_{kl}$$

$$\mathbf{C}_{ij}\, \mathcal{W}_{ijkj} = \left(\mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha)}\right) \mathcal{W}_{k[ij]}$$

$$\mathbf{C}_{ij}\, \mathcal{W}_{ijjk} = \left(\mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha)}\right) \mathcal{W}_{[ij]k}$$

$$\mathbf{C}_{ij}\, \mathcal{W}_{ijkl} = \left(\mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha)}\right) \mathcal{W}_{kl}$$

$$\mathcal{W}_{ij}^{(a)} = \frac{\frac{1}{(\mathcal{E}_i w_{ij})^a}}{\sum_{i,j\neq i} \frac{1}{(\mathcal{E}_i w_{ij})^a}}$$

A "minimal" subtraction procedure at NNLO

- Divide the phase space through sector functions

$$\mathcal{W}_{ijkl} = \frac{\sigma_{ijkl}}{\sigma} \qquad \sigma = \sum_{\substack{i,\, j\neq i \\ k\neq i,\, l\neq i,k}} \sigma_{ijkl}$$

$$\sum_{\substack{i,\, j\neq i \\ k\neq i,\, l\neq i,k}} \mathcal{W}_{ijkl} = 1$$

$$\sigma_{ijkl} = \frac{1}{(\mathcal{E}_i)^{\alpha}(w_{ij})^{\beta}} \frac{1}{(\mathcal{E}_k + \delta_{kj}\mathcal{E}_i)w_{kl}} \qquad \alpha > \beta > 1$$

* Double soft and double collinear limits

$$\sum_{j\neq i, l\neq i,k} \mathbf{S}_{ik}\, \mathcal{W}_{ijkl} + \sum_{j\neq k, l\neq i,k} \mathbf{S}_{ik}\, \mathcal{W}_{kjil} = 1$$

$$\mathbf{C}_{ijk}\, \left(\mathcal{W}_{ikkj} + \mathcal{W}_{ijkj}\right) + (\text{perm. of } i, j, k) = 1$$

$$\mathbf{S}_{ik}\mathbf{C}_{ijk}(\mathcal{W}_{ikkj}+\mathcal{W}_{ijkj}+\mathcal{W}_{kiij}+\mathcal{W}_{kjij}) = \mathbf{C}_{ijk}\mathbf{S}_{ik}(\mathcal{W}_{ikkj}+\mathcal{W}_{ijkj}+\mathcal{W}_{kiij}+\mathcal{W}_{kjij}) = 1$$

A “minimal” subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IRC kernels

Sector $\mathcal{W}_{ijkj}$

$\mathbf{S}_i$, $\mathbf{C}_{ij}$, $\mathbf{S}_{ik}$ and $\mathbf{C}_{ijk}$ commute

$$(1-\mathbf{S}_i)(1-\mathbf{C}_{ij})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})RR \longrightarrow \text{finite}$$

Counterterms

$$K^{(\mathbf{2})}_{ijkj} = \left[\bar{\mathbf{S}}_{ik} + \bar{\mathbf{C}}_{ijk}(1-\bar{\mathbf{S}}_{ik})\right]RR$$

$$K^{(\mathbf{12})}_{ijkj} = -\left[\bar{\mathbf{S}}_{i} + \bar{\mathbf{C}}_{ij}(1-\bar{\mathbf{S}}_{i})\right]\left[\bar{\mathbf{S}}_{ik} + \bar{\mathbf{C}}_{ijk}(1-\bar{\mathbf{S}}_{ik})\right]RR$$

$$K^{(\mathbf{1})}_{ijkj} = \left[\bar{\mathbf{S}}_{i} + \bar{\mathbf{C}}_{ij}(1-\bar{\mathbf{S}}_{i})\right]RR \qquad K^{(\mathbf{RV})}_{ij} = \left[\bar{\mathbf{S}}_{i} + \bar{\mathbf{C}}_{ij}(1-\bar{\mathbf{S}}_{i})\right]RV$$

$$\left[RR - K^{(\mathbf{2})}_{ijkj} - K^{(\mathbf{12})}_{ijkj} - K^{(\mathbf{1})}_{ijkj}\right]\mathcal{W}_{ijkj} \longrightarrow \text{finite}$$

A “minimal” subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IRC kernels

Sector $\mathcal{W}_{ijkl}$

$\mathbf{S}_i$, $\mathbf{C}_{ij}$ and $\mathbf{S}_{ik}$ commute

$$(1-\mathbf{S}_i)(1-\mathbf{C}_{ij})(1-\mathbf{S}_{ik})RR \longrightarrow \text{finite}$$

Counterterms

$$K^{(\mathbf{2})}_{ijkl} = \bar{\mathbf{S}}_{ik}\, RR$$

$$K^{(\mathbf{12})}_{ijkl} = -\left[\bar{\mathbf{S}}_i + \bar{\mathbf{C}}_{ij}(1-\bar{\mathbf{S}}_i)\right]\bar{\mathbf{S}}_{ik}\, RR$$

$$K^{(\mathbf{1})}_{ijkl} = \left[\bar{\mathbf{S}}_i + \bar{\mathbf{C}}_{ij}(1-\bar{\mathbf{S}}_i)\right]RR \qquad K^{(\mathbf{RV})}_{ij} = \left[\bar{\mathbf{S}}_i + \bar{\mathbf{C}}_{ij}(1-\bar{\mathbf{S}}_i)\right]RV$$

$$\left[RR - K^{(\mathbf{2})}_{ijkl} - K^{(\mathbf{12})}_{ijkl} - K^{(\mathbf{1})}_{ijkl}\right]\mathcal{W}_{ijkl} \longrightarrow \text{finite}$$

A “minimal” subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IRC kernels
- Counterterms written as sum of terms

$$\bar{\mathbf{S}}_{ik}\, RR(\{k\}) = \mathcal{G}_2 \Bigg[\sum_{\substack{l\neq i,k \\ m\neq i,k}} \mathcal{I}_{lm} B_{lm}(\{k\}_{\not{i}\not{k}\not{l}\not{m}}, \bar{k}_{iklm}, \bar{r}_{iklm}) + \frac{1}{2} \sum_{\substack{l_1\neq i,k \\ m_1\neq i,k}} \mathcal{I}^{(i)}_{lm} \sum_{\substack{l_2\neq i,k \\ m_2\neq i,k}} \mathcal{I}^{(k)}_{l'm'} B_{lm,l'm'}(\{k\}_{\not{i}\not{l}\not{m}\not{k}\not{l'}\not{m'}}, \bar{k}_{ilm}, \bar{r}_{ilm}, \bar{k}_{kl'm'}, \bar{r}_{kl'm'}) \Bigg]$$

$$\bar{\mathbf{C}}_{ijk}\, RR(\{k\}) = \frac{\mathcal{G}_2}{s^2_{ijk}} \left[P_{ijk}\, B(\{k\}_{\not{i}\not{j}\not{k}\not{r}}, \bar{k}_{ijkr}, \bar{r}_{ijkr}) - Q^{\mu\nu}_{ijk}\, B_{\mu\nu}(\{k\}_{\not{i}\not{j}\not{k}\not{r}}, \bar{k}_{ijkr}, \bar{r}_{ijkr}) \right]$$

$$\bar{k}^{\mu}_{abcd} = k^{\mu}_a + k^{\mu}_b + k^{\mu}_c - \frac{s_{abc}}{s_{ad}+s_{bd}+s_{cd}} k^{\mu}_d \qquad \bar{r}^{\mu}_{abcd} = \frac{s_{abcd}}{s_{ad}+s_{bd}+s_{cd}} k^{\mu}_d$$

A “minimal” subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IRC kernels
- Counterterms written as sum of terms
- Phase space reparametrized differently for each term of the sum

$\bar{\mathbf{S}}_{ik} RR$ $\quad$ $B_{lm,l'm'}$ term

$$p_i^2 = (k_i + k_m + k_l)^2 = (\bar{k}_{iml} + \bar{r}_{kml})^2$$

$$p_k^2 = (k_k + k_{m'} + k_{l'})^2 = (\bar{k}_{im'l'} + \bar{r}_{km'l'})^2$$

$$d\Phi_{n+2}(\{k\}) = d\Phi_n(\{k\}_{\not{i}\not{l}\not{m}\not{k}\not{l}'\not{m}'}, \bar{k}_{ilm}, \bar{r}_{ilm}, \bar{k}_{kl'm'}, \bar{r}_{kl'm'}) d\Phi_1(p_i^2; y, z, \phi) d\Phi_1(p_k^2; y', z', \phi')$$

A “minimal” subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IRC kernels
- Counterterms written as sum of terms
- Phase space reparametrized differently for each term of the sum

$$\boxed{\bar{\mathbf{S}}_{ik} RR} \quad \boxed{B_{lm,l'm'} \ \text{term}}$$

$$p_i^2 = (k_i + k_m + k_l)^2 = (\bar{k}_{iml} + \bar{r}_{kml})^2$$

$$p_k^2 = (k_k + k_{m'} + k_{l'})^2 = (\bar{k}_{im'l'} + \bar{r}_{km'l'})^2$$

$$d\Phi_{n+2}(\{k\}) = d\Phi_n(\{k\}_{\not i \not l \not m \not k \not l' \not m'}, \bar{k}_{ilm}, \bar{r}_{ilm}, \bar{k}_{kl'm'}, \bar{r}_{kl'm'}) d\Phi_1(p_i^2; y, z, \phi) d\Phi_1(p_k^2; y', z', \phi')$$

$$\boxed{\bar{\mathbf{S}}_{ik} RR} \quad \boxed{B_{lm} \ \text{term}}$$

$$a, b, c, d = i, k, l, m$$

$$\boxed{\bar{\mathbf{C}}_{ijk}(1 - \bar{\mathbf{S}}_{ik}) RR}$$

$$a, b, c, d = i, j, k, r$$

$$p^2 = (k_a + k_b + k_c + k_d)^2 = (\bar{k}_{abcd} + \bar{r}_{abcd})^2$$

$$d\Phi_{n+2}(\{k\}) = d\Phi_n(\{k\}_{\not a \not b \not c \not d}, \bar{k}_{abcd}, \bar{r}_{abcd}) d\Phi_2(p^2; y, z, \phi, y', z', x')$$

A “minimal” subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IRC kernels
- Counterterms written as sum of terms
- Phase space reparametrized differently for each term of the sum

$$
\begin{aligned}
2k_a\cdot k_b &= y'\, y\, p^2 \\
2k_a\cdot k_c &= z'(1-y')y\, p^2, \\
2k_b\cdot k_c &= (1-y')(1-z')y\, p^2, \\
2k_c\cdot k_d &= (1-y')(1-y)(1-z)p^2, \\
2k_a\cdot k_d &= (1-y)\left[y'(1-z')(1-z) + z'z - 2\,(1-2x')\sqrt{y'z'(1-z')z(1-z)}\right] p^2 \\
2k_b\cdot k_d &= (1-y)\left[y'z'(1-z) + (1-z')z + 2\,(1-2x')\sqrt{y'z'(1-z')z(1-z)}\right] p^2
\end{aligned}
$$

$$
\int d\Phi_2(p^2; y, z, \phi, y', z', x') = G_2\,(p^2)^{2-2\epsilon}\int_0^1 dx' \int_0^1 dy' \int_0^1 dz' \int_0^1 dy \int_0^1 dz\, [x'(1-x')]^{-\epsilon-1/2}
$$
$$
\left[y'z'(1-y')^2(1-z')y^2 z(1-y)^2(1-z)\right]^{-\epsilon} y(1-y)(1-y')
$$

A “minimal” subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IRC kernels
- Counterterms written as sum of terms
- Phase space reparametrized differently for each term of the sum
- Integrate analytically each term after getting rid of the sector functions

$$I^{(\mathbf{2})} = \sum_{\substack{i,j\neq i \\ k\neq i, l\neq i,k}} \int d\Phi_2 \left[\bar{\mathbf{S}}_{ik} + \bar{\mathbf{C}}_{ijk}(1-\bar{\mathbf{S}}_{ik})\right] RR\, \mathcal{W}_{ijkl}$$

$$\sum_{\substack{i,j\neq i \\ k\neq i, l\neq i,k}} \bar{\mathbf{S}}_{ik}\, RR\, \mathcal{W}_{ijkl} = \sum_{i,k>i} \bar{\mathbf{S}}_{ik}\, RR$$

$$\sum_{\substack{i,j\neq i \\ k\neq i, l\neq i,k}} \bar{\mathbf{C}}_{ijk}\,(1-\bar{\mathbf{S}}_{ik}) RR\, \mathcal{W}_{ijkl} = \sum_{\substack{i,j>i \\ k>j}} \bar{\mathbf{C}}_{ijk}\,(1-\bar{\mathbf{S}}_{ij}-\bar{\mathbf{S}}_{ik}-\bar{\mathbf{S}}_{jk})\, RR$$

A "minimal" subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IRC kernels
- Counterterms written as sum of terms
- Phase space reparametrized differently for each term of the sum
- Integrate analytically each term after getting rid of the sector functions

$$I^{(\mathbf{2})} = \sum_{i,k\neq i} \int d\Phi_2\, \bar{\mathbf{S}}_{ik} RR + \sum_{\substack{i,j>i\\k>j}} \int d\Phi_2\, \bar{\mathbf{C}}_{ijk}(1-\bar{\mathbf{S}}_{ij}-\bar{\mathbf{S}}_{ik}-\bar{\mathbf{S}}_{jk})RR$$

$$\int d\Phi_2\, \bar{\mathbf{S}}_{ik} RR = \mathcal{G}_2 \Big[\sum_{\substack{l\neq i,k\\m\neq i,k}} B_{lm} \int d\Phi_2\, \mathcal{I}_{lm} + \frac{1}{2} \sum_{\substack{l\neq i,k\\m\neq i,k}} \sum_{\substack{l'\neq i,k\\m'\neq i,k}} B_{lm,l'm'} \int d\Phi_1^{(i)}\, \mathcal{I}_{lm}^{(i)} \int d\Phi_1^{(k)}\, \mathcal{I}_{l'm'}^{(k)} \Big]$$

A “minimal” subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IRC kernels
- Counterterms written as sum of terms
- Phase space reparametrized differently for each term of the sum
- Integrate analytically each term after getting rid of the sector functions

$$I^{(\mathbf{2})} = \sum_{i,k\neq i} \int d\Phi_2\, \bar{\mathbf{S}}_{ik} RR + \sum_{\substack{i,j>i\\k>j}} \int d\Phi_2\, \bar{\mathbf{C}}_{ijk}(1-\bar{\mathbf{S}}_{ij}-\bar{\mathbf{S}}_{ik}-\bar{\mathbf{S}}_{jk})RR$$

$$\int d\Phi_2 \bar{\mathbf{C}}_{ijk}\, RR = \mathcal{G}_2 \left[B \int d\Phi_2 \frac{P_{ijk}}{s_{ijk}^2} - B_{\mu\nu} \underbrace{\int d\Phi_2 \frac{Q_{ijk}^{\mu\nu}}{s_{ijk}^2}}_{0} \right]$$

$$\int d\Phi_2\, \bar{\mathbf{C}}_{ijk}(1-\mathbf{S}_{ij}-\mathbf{S}_{ik}-\mathbf{S}_{jk})RR = \mathcal{G}_2\, B \int d\Phi_2\, (1-\mathbf{S}_{ij}-\mathbf{S}_{ik}-\mathbf{S}_{jk}) \frac{P_{ijk}}{s_{ijk}^2}$$

A "minimal" subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IRC kernels
- Counterterms written as sum of terms
- Phase space reparametrized differently for each term of the sum
- Integrate analytically each term after getting rid of the sector functions

$$I^{(\mathbf{1})} = \sum_{\substack{i,j\neq i\\ k\neq i,l\neq i,k}} \int d\Phi_1 \left[\bar{\mathbf{S}}_i + \bar{\mathbf{C}}_{ij}(1-\bar{\mathbf{S}}_i)\right] RR\, \mathcal{W}_{ijkl}$$

$$\sum_{\substack{i,j\neq i\\ k\neq i,l\neq i,k}} \bar{\mathbf{S}}_i\, RR\, \mathcal{W}_{ijkl} = \sum_i \bar{\mathbf{S}}_i RR \sum_{\substack{k\neq i\\ l\neq i,k}} \mathcal{W}_{kl}$$

W_{kl} sectors match with NLO sectors for RV

$$\sum_{\substack{i,j\neq i\\ k\neq i,l\neq i,k}} \bar{\mathbf{C}}_{ij}(1-\bar{\mathbf{S}}_i)\, RR\, \mathcal{W}_{ijkl} = \sum_{i,j>i} \bar{\mathbf{C}}_{ij}(1-\bar{\mathbf{S}}_i-\bar{\mathbf{S}}_j) RR \sum_{k\neq i,j} \left[\mathcal{W}_{k[ij]} + \mathcal{W}_{[ij]k} + \sum_{l\neq i,j,k} \mathcal{W}_{kl}\right]$$

Similar for $I^{(\mathbf{12})}$

Proof of concept

- $T_R C_F$ NNLO contribution to the total cross section for $e^+e^- \to q\bar{q}$

  Just contributions from the radiation of a $q'\bar{q}'$ pair

- Known exact NNLO results:

  Hamberg, van Neerven, Matsuura 1991
  Gehrmann De Ridder, Gehrmann, Glover 0403057
  Ellis, Ross, Terrano 1980

$$VV = B\left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F \left\{ \left(\frac{\mu^2}{s}\right)^{2\epsilon}\left[\frac{1}{3\epsilon^3}+\frac{14}{9\epsilon^2}+\frac{1}{\epsilon}\left(-\frac{11}{18}\pi^2+\frac{353}{54}\right)+\left(-\frac{26}{9}\zeta_3-\frac{77}{27}\pi^2+\frac{7541}{324}\right)\right] + \left(\frac{\mu^2}{s}\right)^{\epsilon}\left[-\frac{4}{3\epsilon^3}-\frac{2}{\epsilon^2}+\frac{1}{\epsilon}\left(\frac{7}{9}\pi^2-\frac{16}{3}\right)+\left(\frac{28}{9}\zeta_3+\frac{7}{6}\pi^2-\frac{32}{3}\right)\right]\right\}$$

$$\int d\Phi_1\, RV = B\left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F \left(\frac{\mu^2}{s}\right)^{\epsilon}\left[\frac{4}{3\epsilon^3}+\frac{2}{\epsilon^2}+\frac{1}{\epsilon}\left(-\frac{7}{9}\pi^2+\frac{19}{3}\right)+\left(-\frac{100}{9}\zeta_3-\frac{7}{6}\pi^2+\frac{109}{6}\right)\right]$$

$$\int d\Phi_2\, RR = B\left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F \left(\frac{\mu^2}{s}\right)^{2\epsilon}\left[-\frac{1}{3\epsilon^3}-\frac{14}{9\epsilon^2}+\frac{1}{\epsilon}\left(\frac{11}{18}\pi^2-\frac{407}{54}\right)+\left(\frac{134}{9}\zeta_3+\frac{77}{27}\pi^2-\frac{11753}{324}\right)\right]$$

Proof of concept

- We integrate the known limits $\bar{\mathbf{S}}_{ik}RR$ and $\bar{\mathbf{C}}_{ijk}RR$

$$\int d\Phi_2\,\bar{\mathbf{S}}_{ik}RR = (4\pi\alpha_s^u\mu_0^{2\epsilon})^2 T_R \sum_{l,m=1}^{2} B_{lm}\int d\Phi_2\,\frac{4(s_{il}s_{km}+s_{im}s_{kl}-s_{ik}s_{lm})}{s_{ik}^2(s_{il}+s_{kl})(s_{im}+s_{km})}$$

$$= B\left(\frac{\alpha_s}{2\pi}\right)^2 T_R C_F\left(\frac{\mu^2}{s}\right)^{2\epsilon}\left[-\frac{1}{3\epsilon^3}-\frac{17}{9\epsilon^2}+\frac{1}{\epsilon}\left(\frac{7}{18}\pi^2-\frac{232}{27}\right)+\left(\frac{38}{9}\zeta_3+\frac{131}{54}\pi^2-\frac{2948}{81}\right)\right]$$

$$\int d\Phi_2\,\bar{\mathbf{C}}_{ijk}RR = (8\pi\alpha_s^u\mu_0^{2\epsilon})^2 B\int d\Phi_2\,\frac{2T_R C_F}{s_{ijk}s_{ik}}\left[-\frac{t_{ik,j}^2}{s_{ik}s_{ikj}}+\frac{4z_j+(z_i-z_k)^2}{z_i+z_k}+(1-2\epsilon)\left(z_i+z_k-\frac{s_{ik}}{s_{ikj}}\right)\right]$$

$$= B\left(\frac{\alpha_s}{2\pi}\right)^2 T_R C_F\left(\frac{\mu^2}{s}\right)^{2\epsilon}\left[-\frac{1}{3\epsilon^3}-\frac{31}{18\epsilon^2}+\frac{1}{\epsilon}\left(\frac{1}{2}\pi^2-\frac{889}{108}\right)+\left(\frac{80}{9}\zeta_3+\frac{31}{12}\pi^2-\frac{23941}{648}\right)\right]$$

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- And we get the 2-unresolved integrated counterterm:

$$I^{(2)} = \int d\Phi_2\left[\bar{\mathbf{S}}_{34}+\bar{\mathbf{C}}_{134}(1-\bar{\mathbf{S}}_{34})+\bar{\mathbf{C}}_{234}(1-\bar{\mathbf{S}}_{34})\right]RR$$

$$= B\left(\frac{\alpha_s}{2\pi}\right)^2 T_R C_F\left(\frac{\mu^2}{s}\right)^{2\epsilon}\left[-\frac{1}{3\epsilon^3}-\frac{14}{9\epsilon^2}+\frac{1}{\epsilon}\left(\frac{11}{18}\pi^2-\frac{425}{54}\right)+\left(\frac{122}{9}\zeta_3+\frac{74}{27}\pi^2-\frac{12149}{324}\right)\right]$$

Proof of concept

- From the explicit expression of RV se get for $I^{(\mathbf{RV})}$:

$$I^{(\mathbf{RV})} = \frac{\alpha_S}{2\pi}\frac{2}{3}\frac{T_R}{\epsilon}\left[\int d\Phi_1\, \mathbf{S}_{[34]}\, R + \int d\Phi_1\, \mathbf{C}_{1[34]}(1-\mathbf{S}_{[34]})R + \int d\Phi_1\, \mathbf{C}_{2[34]}(1-\mathbf{S}_{[34]})R\right]$$

$$= \left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F \left(\frac{\mu^2}{s}\right)^{\epsilon}\left[\frac{4}{3\epsilon^3}+\frac{2}{\epsilon^2}+\frac{1}{\epsilon}\left(-\frac{7}{9}\pi^2+\frac{20}{3}\right)+\left(-\frac{100}{9}\zeta_3-\frac{7}{6}\pi^2+20\right)\right]$$

- Analytical cancellation of poles in the subtracted VV:

$$VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})} = B\left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F\left(\frac{8}{3}\zeta_3-\frac{1}{9}\pi^2-\frac{44}{9}-\frac{4}{3}\ln\frac{\mu^2}{s}\right)$$

- NNLO corrections with the subtraction …

$$\frac{\mu}{\sqrt{s}} = 0.35$$

$$\frac{\sigma_{\rm NNLO}-\sigma_{\rm NLO}}{\sigma_{\rm LO}} = \left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F\left(1.40806 \pm 0.00040\right)$$

- … compared with the analytical result

$$\frac{\sigma_{\rm NNLO}-\sigma_{\rm NLO}}{\sigma_{\rm LO}} = \left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F\left[-\frac{11}{2}+4\zeta_3-\ln\frac{\mu^2}{s}\right] = \left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F\left(1.40787186\right)$$

Leading Outlook

- Complete the distributions
- Complete the computation of final state counterterms

Next-to-Leading Outlook

- Compute counterterms with initial state hadrons

Next-to-Next-to-Leading Outlook

- Consider the massive case

Still work in progress …

Back-up slides

$$(1-\mathbf{S}_i)[\mathbf{S}_i\mathbf{C}_{jk}]=0$$

$$(1-\mathbf{C}_{ij})\,[\mathbf{C}_{ij}\mathbf{C}_{kl}]=0$$

Sector $\mathcal{W}_{ijkj}$

$$(1-\mathbf{S}_i)(1-\mathbf{C}_{ij})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})(1-[\mathbf{S}_i\mathbf{C}_{jk}])RR=$$
$$(1-\mathbf{S}_i)(1-\mathbf{C}_{ij})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})RR$$

Sector $\mathcal{W}_{ijkl}$

$$(1-\mathbf{S}_i)(1-\mathbf{C}_{ij})(1-\mathbf{S}_{ik})(1-[\mathbf{C}_{ij}\mathbf{C}_{kl}])(1-[\mathbf{S}_i\mathbf{C}_{kl}])RR=$$
$$(1-\mathbf{S}_i)(1-\mathbf{C}_{ij})(1-\mathbf{S}_{ik})RR$$

- $[\mathbf{S}_i\mathbf{C}_{jk}]$ and $[\mathbf{C}_{ij}\mathbf{C}_{kl}]$ disappear from the subtraction procedure

$$\sum_{\substack{i,j\neq i\\ k\neq i,l\neq i,k}} \bar{\mathbf{S}}_{ik}\, RR\, \mathcal{W}_{ijkl} = \sum_{i,k>i} \bar{\mathbf{S}}_{ik}\, RR \Bigg[\underbrace{\sum_{j\neq i,l\neq i,k} \mathbf{S}_{ik}\, \mathcal{W}_{ijkl} + \sum_{j\neq k,l\neq i,k} \mathbf{S}_{ik}\, \mathcal{W}_{kjil}}_{1} \Bigg]$$

$$\begin{aligned}
\sum_{\substack{i,j\neq i\\ k\neq i,l\neq i,k}} \bar{\mathbf{C}}_{ijk}(1-\bar{\mathbf{S}}_{ik}) RR\, \mathcal{W}_{ijkl} &= \sum_{\substack{i,j\neq i\\ k\neq i,j}} \bar{\mathbf{C}}_{ijk}(1-\bar{\mathbf{S}}_{ik})\, RR\, (\mathcal{W}_{ijkj} + \mathcal{W}_{ikkj}) \\
&= \sum_{\substack{i,j>i\\ k>j}} \bar{\mathbf{C}}_{ijk}\, RR \underbrace{\sum_{\substack{\text{perm.}\\ \text{of } i,j,k}} \mathbf{C}_{ijk}(\mathcal{W}_{ijkj} + \mathcal{W}_{ikkj})}_{1} \\
&\quad - \sum_{\substack{i,j>i\\ k>j}} \bar{\mathbf{C}}_{ijk}\bar{\mathbf{S}}_{ik}\, RR \underbrace{\mathbf{C}_{ijk}\mathbf{S}_{ik}(\mathcal{W}_{ikkj} + \mathcal{W}_{ijkj} + \mathcal{W}_{kiij} + \mathcal{W}_{kjij})}_{1} \\
&\quad - \sum_{\substack{i,j>i\\ k>j}} \bar{\mathbf{C}}_{ijk}\bar{\mathbf{S}}_{ij}\, RR \underbrace{\mathbf{C}_{ijk}\mathbf{S}_{ij}(\mathcal{W}_{ijjk} + \mathcal{W}_{ikjk} + \mathcal{W}_{jiik} + \mathcal{W}_{jkik})}_{1} \\
&\quad - \sum_{\substack{i,j>i\\ k>j}} \bar{\mathbf{C}}_{ijk}\bar{\mathbf{S}}_{jk}\, RR \underbrace{\mathbf{C}_{ijk}\mathbf{S}_{jk}(\mathcal{W}_{jkki} + \mathcal{W}_{jiki} + \mathcal{W}_{kjji} + \mathcal{W}_{kiji})}_{1} \\
&= \sum_{\substack{i,j>i\\ k>j}} \bar{\mathbf{C}}_{ijk}\, (1-\bar{\mathbf{S}}_{ij}-\bar{\mathbf{S}}_{ik}-\bar{\mathbf{S}}_{jk}) RR
\end{aligned}$$

$$\sum_{\substack{i,j\neq i\\k\neq i,l\neq i,k}} \bar{\mathbf{S}}_i\, RR\, \mathcal{W}_{ijkl} = \sum_i \bar{\mathbf{S}}_i RR \underbrace{\sum_{j\neq i} (\mathbf{S}_i \mathcal{W}_{ij}^{(\beta)})}_{1} \sum_{\substack{k\neq i\\l\neq i,k}} \mathcal{W}_{kl}$$

$$\begin{aligned}
\sum_{\substack{i,j\neq i\\k\neq i,l\neq i,k}} \bar{\mathbf{C}}_{ij}(1-\bar{\mathbf{S}}_i)\, RR\, \mathcal{W}_{ijkl} &= \sum_{i,j\neq i} \left[\bar{\mathbf{C}}_{ij} RR\, \mathbf{C}_{ij}\mathcal{W}_{ij}^{(\alpha)} - \bar{\mathbf{C}}_{ij}\bar{\mathbf{S}}_i RR \underbrace{\mathbf{C}_{ij}\mathbf{S}_i\mathcal{W}_{ij}^{(\alpha)}}_{1}\right] \\
&\quad \sum_{k\neq i,j}\left[\mathcal{W}_{k[ij]} + \mathcal{W}_{[ij]k} + \sum_{l\neq i,j,k}\mathcal{W}_{kl}\right] \\
&= \sum_{i,j>i}\left[\bar{\mathbf{C}}_{ij}\, RR \underbrace{\mathbf{C}_{ij}(\mathcal{W}_{ij}^{(\alpha)} + \mathcal{W}_{ji}^{(\alpha)})}_{1} - \bar{\mathbf{C}}_{ij}(\bar{\mathbf{S}}_i + \bar{\mathbf{S}}_j) RR\right] \\
&\quad \sum_{k\neq i,j}\left[\mathcal{W}_{k[ij]} + \mathcal{W}_{[ij]k} + \sum_{l\neq i,j,k}\mathcal{W}_{kl}\right] \\
&= \sum_{i,j>i} \bar{\mathbf{C}}_{ij}(1-\bar{\mathbf{S}}_i-\bar{\mathbf{S}}_j) RR \sum_{k\neq i,j}\left[\mathcal{W}_{k[ij]} + \mathcal{W}_{[ij]k} + \sum_{l\neq i,j,k}\mathcal{W}_{kl}\right]
\end{aligned}$$

Many subtraction methods available at NNLO

- Antenna subtraction — Gehrmann De Ridder, Gehrmann, Glover, Heinrich, et al.
- STRIPPER (Sector-improved residue subtraction) — Czakon, Mitov, et al.; Boughezal, Caola, Melnikov, Petriello, et al.
- Colourful subtraction — Del Duca, Duhr, Kardos, Somogyi, Troscanyi, et al.
- qT-slicing — Catani, Grazzini, et al.
- N-jettiness slicing — Boughezal, Petriello, et al.
- Projection to Born — Cacciari, Salam, Zanderighi, et al.
- Sector decomposition — Anastasiou, Binoth, et al.
- ε-prescription — Frixione, Grazzini