Anatomy of CoLoRFulNNLO

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Subtracting Infrared Singularities Beyond NLO
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Outline

1. The overall picture
2. The nuts and bolts
3. Towards processes with hadronic initial states

The overall picture

The NNLO cross section

Aim: compute cross sections at NNLO with arbitrary acceptance cuts (J) in $d = 4$

$$\sigma^{\mathrm{NNLO}}[J] = \int_{m+2} \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} + \int_{m+1} \mathrm{d}\sigma^{\mathrm{RV}}_{m+1} J_{m+1} + \int_{m} \mathrm{d}\sigma^{\mathrm{VV}}_{m} J_{m}$$

- Phase space integrals must be performed numerically
- All three terms are separately IR divergent in $d = 4$ dimensions
- Infrared singularities cancel between real and virtual quantum corrections at the same order in perturbation theory, for sufficiently inclusive (i.e. IR safe) observables (KLN theorem)

How to make this cancellation explicit, so that the various contributions can be computed numerically?

CoLoRFulNNLO is built around the idea that the solution should

- Give the exact perturbative result $\Rightarrow$ subtraction
  (no slicing parameter)
- Be well-defined $\Rightarrow$ completely local counterterms with all spin and color correlations
  (no integrals that are finite but undefined in $d = 4$)
- Lead to general and explicit expressions
  (automation, we use color space notation)

It is also advantageous if in addition

- The cancellation of explicit ϵ-poles in virtual contributions is analytic
  ("mathematical rigor")
- The option exists to constrain the subtractions to near the singular regions ($\alpha_{\max}$)
  (efficiency, important check)
- The construction is algorithmic
  (valid at any order in perturbation theory, in principle)

Structure

Use the same framework that was successful at NLO: local subtraction scheme

The NLO correction to some m-jet observable J

$$\sigma^{\mathrm{NLO}}[J] = \int_{m+1} \left[\mathrm{d}\sigma^{\mathrm{R}}_{m+1} J_{m+1} - \mathrm{d}\sigma^{\mathrm{R,A_1}}_{m+1} J_m \right]_{d=4} + \int_m \left[\mathrm{d}\sigma^{\mathrm{V}}_m + \int_1 \mathrm{d}\sigma^{\mathrm{R,A_1}}_{m+1} \right]_{d=4} J_m$$

The NNLO correction is the sum of three pieces

$$\sigma^{\mathrm{NNLO}}[J] = \int_{m+2} \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} + \int_{m+1} \mathrm{d}\sigma^{\mathrm{RV}}_{m+1} J_{m+1} + \int_m \mathrm{d}\sigma^{\mathrm{VV}}_m J_m$$

The three contributions are separately IR divergent in $d = 4$

- RR: double and single unresolved real emission
- RV: single unresolved real emission $\oplus$ ϵ-poles from $m+1$ parton one-loop
- VV: ϵ poles from m parton two-loop

For the RR contribution subtractions are needed to regularize one- and two-parton emissions

$$\sigma_{m+2}^{\mathrm{NNLO}} = \int_{m+2} \left\{ \mathrm{d}\sigma_{m+2}^{\mathrm{RR}} J_{m+2} - \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2} J_m - \left[\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1} J_{m+1} - \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}} J_m \right] \right\}_{d=4}$$

- A_1 and A_2 have overlapping singularities $\Rightarrow$ A_{12} is needed to cancel

For the RV contribution emissions are like at NLO but for one-loop $\otimes$ tree interference

$$\sigma_{m+1}^{\mathrm{NNLO}} = \int_{m+1} \left\{ \left[\mathrm{d}\sigma_{m+1}^{\mathrm{RV}} + \int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1} \right] J_{m+1} - \left[\mathrm{d}\sigma_{m+1}^{\mathrm{RV},\mathrm{A}_1} + \left(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1} \right)^{\mathrm{A}_1} \right] J_m \right\}_{d=4}$$

- Notice the integrated A_1 from RR which is still singular $\Rightarrow$ subtraction is needed (last term)

The m-parton contribution contains the double virtual and integrated subtractions

$$\sigma_{m}^{\mathrm{NNLO}} = \int_{m} \left\{ \mathrm{d}\sigma_{m}^{\mathrm{VV}} + \int_2 \left[\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2} - \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}} \right] + \int_1 \left[\mathrm{d}\sigma_{m+1}^{\mathrm{RV},\mathrm{A}_1} + \left(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1} \right)^{\mathrm{A}_1} \right] \right\}_{d=4} J_m$$

The non-trivial role of $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}}$

The sum of subtractions, symbolically (r, s can become unresolved)

$$\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2} + \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1} - \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}} = \sum_{r,s} \left[\mathcal{D}_{rs} + (\mathcal{D}_r + \mathcal{D}_s) - (\mathcal{D}_{\hat{s}}\mathcal{D}_r + \mathcal{D}_{\hat{r}}\mathcal{D}_s)\right]$$

The dual role of A_{12}

- In the double unresolved limits (r, s unresolved), it cancels A_1

$$\mathrm{d}\sigma_{m+2}^{\mathrm{RR}} - \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2} = \mathrm{d}\sigma_{m+2}^{\mathrm{RR}} - \mathcal{D}_{rs} = \text{“finite”}$$

$$\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1} - \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}} = (\mathcal{D}_r + \mathcal{D}_s) - (\mathcal{D}_{\hat{s}}\mathcal{D}_r + \mathcal{D}_{\hat{r}}\mathcal{D}_s) = \text{“finite”}$$

- In the single unresolved limits (say, r unresolved), it cancels A_2 and part of A_1

$$\mathrm{d}\sigma_{m+2}^{\mathrm{RR}} - \left(\text{part of } \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1}\right) = \mathrm{d}\sigma_{m+2}^{\mathrm{RR}} - \mathcal{D}_r = \text{“finite”}$$

$$\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2} - \left(\text{part of } \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}}\right) = \mathcal{D}_{rs} - \mathcal{D}_{\hat{s}}\mathcal{D}_r = \text{“finite”}$$

$$\left(\text{part of } \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1}\right) - \left(\text{part of } \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}}\right) = \mathcal{D}_s - \mathcal{D}_{\hat{r}}\mathcal{D}_s = \text{“finite”}$$

Repeat what already worked at NLO!

1. Compute relevant IR factorization formulae for squared matrix elements
2. Use those to construct general, explicit, local subtractions
3. Integrate the subtractions once and for all, check cancellation of ϵ-poles
4. Apply to specific processs

Use known ingredients

Collinear and soft factorization of QCD matrix elements at NNLO known

- Tree level 3-parton splitting functions and double soft gg and $q\bar{q}$ currents

[Campbell, Glover 1997; Catani, Grazzini 1998; Del Duca, Frizzo, Maltoni 1999; Kosower 2002]

- One-loop 2-parton splitting functions and soft gluon current

[Bern, Dixon, Dunbar, Kosower 1994; Bern, Del Duca, Kilgore, Schmidt 1998-9; Kosower, Uwer 1999; Catani, Grazzini 2000; Kosower 2003]

Use known ingredients

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But note

- Unresolved regions in phase space overlap
- Quantities in factorization formulae are only well-defined in the strict limit

Defining the subtraction terms – issues

1. Matching of limits to avoid multiple subtraction in overlapping singular regions of phase space. General structure dictated by "sieve principle". E.g., at NLO simply: collinear limit + soft limit − collinear–soft limit.

$$\mathbf{A}_1 = \sum \Big(\mathbf{C} + \mathbf{S} - \mathbf{C} \cap \mathbf{S}\Big)$$

At NNLO for double radiation we have

$$\mathbf{A}_2 = \sum \Big[\mathbf{C}_3 + \mathbf{C}_{2;2} + \mathbf{CS} + \mathbf{S} - (\mathbf{C}_3 \cap \mathbf{CS} + \mathbf{C}_3 \cap \mathbf{S} + \mathbf{C}_{2;2} \cap \mathbf{CS} + \mathbf{C}_{2;2} \cap \mathbf{S} + \mathbf{CS} \cap \mathbf{S}) + (\mathbf{C}_3 \cap \mathbf{CS} \cap \mathbf{S} + \mathbf{C}_{2;2} \cap \mathbf{CS} \cap \mathbf{S})\Big]$$

2. Extension of IR factorization formulae over full phase space. E.g., must define the momenta entering factorized matrix elements. Requires momentum mappings that respect factorization and delicate structure of cancellations in all limits.

$$\{p\}_{m+1} \xrightarrow{r} \{\tilde{p}\}_m : \quad \mathrm{d}\phi_{m+1}(\{p\}_{m+1}; Q) = \mathrm{d}\phi_m(\{\tilde{p}\}_m; Q)[\mathrm{d}p_{1,m}]$$

$$\{p\}_{m+2} \xrightarrow{r,s} \{\tilde{p}\}_m : \quad \mathrm{d}\phi_{m+2}(\{p\}_{m+2}; Q) = \mathrm{d}\phi_m(\{\tilde{p}\}_m; Q)[\mathrm{d}p_{2,m}]$$

3. Integration of the counterterms over the phase space of unresolved emission.

Defining the subtraction terms – issues

Issues specific to NNLO

1. Matching: since limits do not commute in general, care must be taken to specify the proper ordering.

2. Extension: the A_1 counterterms for single unresolved real emission (unintegrated and integrated) must have universal IR limits, so that A_{12} can be constructed in general. This is (obviously) not guaranteed by QCD factorization.

3. Choosing the counterterms such that integration over the unresolved phase space becomes more straightforward may conflict with the delicate internal cancellations between subtractions. Integrating the counterterms is tedious.

General features of CoLoRFulNNLO

CoLoRFulNNLO: Completely Local subtRactions for Fully differential NNLO

Subtractions built using universal IR limit formulae and exact PS factorization

- Altarelli-Parisi splitting functions, soft currents
- PS factorizations based on momentum mappings that can be generalized to any number of unresolved partons

Completely local in color $\otimes$ spin space, fully differential in phase space

- No need to consider the color decomposition of real emission ME's
- Azimuthal correlations correctly taken into account in gluon splitting
- Can check explicitly that the ratio of the sum of counterterms to the real emission cross section tends to unity in any IR limit

Poles of integrated subtraction terms computed analytically

- Can check pole cancellation in (double) virtual contribution explicitly

Explicit formulae for processes with colorless initial state

- Automation is possible (`MCCSM`)

The nuts and bolts

Single unresolved A_1 subtractions – IR factorization

The symbolic operators $\mathbf{C}_{ir}$ and $\mathbf{S}_r$ denote taking the single collinear and single soft limits

- Collinear: $p_i || p_r$ $(p_i \to z_i p_{ir} + k_\perp + \mathcal{O}(k_\perp^2),\ p_r \to z_r p_{ir} - k_\perp + \mathcal{O}(k_\perp^2))$

$$\mathbf{C}_{ir} |\mathcal{M}^{(0)}_{m+2}(p_i, p_r, \ldots)|^2 = 8\pi\alpha_\mathrm{s}\mu^{2\epsilon} \frac{1}{s_{ir}} \hat{P}_{f_i f_r}(z_i, z_r, k_\perp; \epsilon) \otimes |\mathcal{M}^{(0)}_{m+1}(p_{ir}, \ldots)|^2$$

- Soft: $p_r \to 0$

$$\mathbf{S}_r |\mathcal{M}^{(0)}_{m+2}(p_r, \ldots)|^2 = -8\pi\alpha_\mathrm{s}\mu^{2\epsilon} \sum_{j,k} \frac{s_{jk}}{s_{jr} s_{kr}} |\mathcal{M}^{(0)}_{m+1,(i,k)}(\cancel{p_r}, \ldots)|^2$$

In order to avoid double subtraction when p_r is both soft and collinear to another momentum p_i, we need to remove the "collinear-soft" contribution.

However, the soft and collinear limits do not commute at the level of factorization formulae.

Non-commuting limits

Consider the soft limit of the collinear formula: $\mathbf{S}_r \mathbf{C}_{ir}$

- Momentum fractions:

$$\mathbf{S}_r z_i \to 1, \qquad \mathbf{S}_r z_r \to 0$$

- Altarelli-Parisi splitting kernels: e.g., for $q \to qg$ splitting $\left(z_i + z_r = 1\right)$

$$P_{qg}(z_i, z_r; \epsilon) = C_{\rm F} \left[\frac{1+z_i^2}{1-z_i} - \epsilon(1-z_i)\right] \quad \Rightarrow \quad \mathbf{S}_r P_{qg}(z_i, z_r; \epsilon) \to \frac{2}{z_r} C_{\rm F}$$

  and in general

$$\mathbf{S}_r P_{f_i f_r}(z_i, z_r, k_\perp; \epsilon) \to \frac{2}{z_r} T_{ir}^2$$

- Soft-collinear limit

$$\mathbf{S}_r \mathbf{C}_{ir} |\mathcal{M}_{m+2}^{(0)}(p_i, p_r, \ldots)|^2 = 8\pi\alpha_{\rm s}\mu^{2\epsilon} \frac{1}{s_{ir}} \frac{2}{z_r} \boldsymbol{T}_{ir}^2 |\mathcal{M}_{m+1}^{(0)}(p_i, \ldots)|^2$$

Non-commuting limits

Consider the collinear limit of the soft formula: $\mathbf{C}_{ir}\mathbf{S}_r$

- Two-particle invariants

$$\mathbf{C}_{ir} s_{il} \to z_i s_{(ir)l}, \qquad \mathbf{C}_{ir} s_{lr} \to z_r s_{(ir)l}, \qquad l = j, k$$

- Eikonal factor

$$\mathbf{C}_{ir} \sum_{j,k} \frac{s_{jk}}{s_{jr} s_{kr}} \boldsymbol{T}_j \boldsymbol{T}_k = \mathbf{C}_{ir} \sum_{k} \frac{2 s_{ik}}{s_{ir} s_{kr}} \boldsymbol{T}_i \boldsymbol{T}_k \to \sum_{k} \frac{2}{s_{ir}} \frac{z_i}{z_r} \boldsymbol{T}_i \boldsymbol{T}_k = -\frac{2}{s_{ir}} \frac{z_i}{z_r} \boldsymbol{T}_i^2$$

- Collinear-soft limit

$$\mathbf{C}_{ir}\mathbf{S}_r |\mathcal{M}^{(0)}_{m+2}(p_i, p_r, \ldots)|^2 = 8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon} \frac{1}{s_{ir}} \frac{2 z_i}{z_r} \boldsymbol{T}_i^2 |\mathcal{M}^{(0)}_{m+1}(p_i, \ldots)|^2$$

Non-commuting limits

Hence limits do not commute: $\mathbf{S}_r\mathbf{C}_{ir} \neq \mathbf{C}_{ir}\mathbf{S}_r$

$$\mathbf{S}_r\mathbf{C}_{ir}|\mathcal{M}^{(0)}_{m+2}|^2 \propto \frac{1}{s_{ir}}\frac{2}{z_r}\boldsymbol{T}^2_{ir}|\mathcal{M}^{(0)}_{m+1}|^2 \qquad \text{but} \qquad \mathbf{C}_{ir}\mathbf{S}_r|\mathcal{M}^{(0)}_{m+2}|^2 \propto \frac{1}{s_{ir}}\frac{2z_i}{z_r}\boldsymbol{T}^2_{i}|\mathcal{M}^{(0)}_{m+1}|^2$$

- Reason: soft operators send some momentum fractions to one: $\mathbf{S}_r z_i \to 1$
- Note: no explicit phasespace parametrization, so no specific parameter controls the approach to limits

Which ordering to use?

- $\mathbf{S}_r\mathbf{C}_{ir}$ will not work in the collinear limit

$$\mathbf{S}_r\left(\mathbf{C}_{ir} - \mathbf{S}_r\mathbf{C}_{ir}\right)|\mathcal{M}^{(0)}_{m+2}|^2 = 0 \qquad \text{but} \qquad \mathbf{C}_{ir}\left(\mathbf{S}_r - \mathbf{S}_r\mathbf{C}_{ir}\right)|\mathcal{M}^{(0)}_{m+2}|^2 \neq 0$$

- $\mathbf{C}_{ir}\mathbf{S}_r$ will work in both limits

$$\mathbf{S}_r\left(\mathbf{C}_{ir} - \mathbf{C}_{ir}\mathbf{S}_r\right)|\mathcal{M}^{(0)}_{m+2}|^2 = 0 \qquad \text{but} \qquad \mathbf{C}_{ir}\left(\mathbf{S}_r - \mathbf{C}_{ir}\mathbf{S}_r\right)|\mathcal{M}^{(0)}_{m+2}|^2 = 0$$

This phenomenon arises also in double unresolved limits. In general, limits must be ordered form "more soft" to "less soft".

Single unresolved A_1 subtractions – matching

Hence the complete single unresolved subtraction term has the structure

$$\mathbf{A}_1|\mathcal{M}^{(0)}_{m+2}|^2 = \sum_r \left[\sum_{i\neq r}\frac{1}{2}\mathbf{C}_{ir} + \mathbf{S}_r - \sum_{i\neq r}\mathbf{C}_{ir}\mathbf{S}_r\right]|\mathcal{M}^{(0)}_{m+2}|^2$$

We must still give the precise definition of each term away from the respective limit

$$\mathbf{C}_{ir}|\mathcal{M}^{(0)}_{m+2}(p_i, p_r, \ldots)|^2 = 8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon}\frac{1}{s_{ir}}\hat{P}_{f_i f_r}(z_i, z_r, k_\perp; \epsilon) \otimes |\mathcal{M}^{(0)}_{m+1}(p_{ir}, \ldots)|^2$$

$$\mathbf{S}_r|\mathcal{M}^{(0)}_{m+2}(p_r, \ldots)|^2 = -8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon}\sum_{j,k}\frac{s_{jk}}{s_{jr}s_{kr}}|\mathcal{M}^{(0)}_{m+1,(i,k)}(\cancel{p_r}, \ldots)|^2$$

$$\mathbf{C}_{ir}\mathbf{S}_r|\mathcal{M}^{(0)}_{m+2}(p_i, p_r, \ldots)|^2 = 8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon}\frac{1}{s_{ir}}\frac{2z_i}{z_r}\boldsymbol{T}_i^2|\mathcal{M}^{(0)}_{m+1}(p_i, \ldots)|^2$$

- Must provide precise definitions of momenta entering factorized matrix elements
- Also of z_i, z_r and $k_\perp$

Single unresolved A_1 subtractions – extension

Definition of momenta entering factorized matrix elements: momentum mappings

$$\{p\}_{m+2} \to \{\tilde{p}\}_{m+1}$$

- Implement momentum conservation
- Mass-shell conditions conserved
- Lead to an exact factorization of the $m+2$ parton phase space
- Respect the structure of cancellations

Momentum mappings

- Separate momentum mappings for collinear and soft subtractions
- Recoil is distributed democratically (no spectator)
- Straightforward to generalize to any number of unresolved momenta

Momentum mappings

Collinear mapping

$$\tilde{p}_{ir}^{\mu} = \frac{1}{1-\alpha_{ir}}(p_i^{\mu} + p_r^{\mu} - \alpha_{ir} Q^{\mu})\,, \qquad \tilde{p}_n^{\mu} = \frac{1}{1-\alpha_{ir}} p_n^{\mu}\,, \quad n \neq i, r$$

$$\alpha_{ir} = \frac{1}{2}\left[y_{(ir)Q} - \sqrt{y_{(ir)Q}^2 - 4y_{ir}}\right]$$

- momentum conservation

$$\tilde{p}_{ir}^{\mu} + \sum_{n\neq i,r} \tilde{p}^{\mu} = p_i^{\mu} + p_r^{\mu} + \sum_{n\neq i,r} p^{\mu}$$

- phase space factorization

$$\mathrm{d}\phi_{m+2}(\{p\};Q) = \mathrm{d}\phi_{m+1}(\{\tilde{p}\}^{(ir)};Q)[\mathrm{d}p_{1,m+1}^{(ir)}(p_r,\tilde{p}_{ir};Q)]$$

$$[\mathrm{d}p_{1,m+1}^{(ir)}(p_r,\tilde{p}_{ir};Q)] = \mathrm{d}\alpha(1-\alpha)^{2m(1-\epsilon)-1}\frac{s_{\widetilde{ir}Q}}{2\pi}\mathrm{d}\phi_2(p_i,p_r;p_{(ir)})$$

Soft mapping

$$\tilde{p}_n^\mu = \Lambda^\mu_\nu[Q,(Q-p_r)/\lambda_r](p_n^\nu/\lambda_r)\,, \quad n \neq r\,, \qquad \lambda_r = \sqrt{1-y_{rQ}}\,,$$

$$\Lambda^\mu_\nu[K,\widetilde{K}] = g^\mu_\nu - \frac{2(K+\widetilde{K})^\mu(K+\widetilde{K})_\nu}{(K+\widetilde{K})^2} + \frac{2K^\mu\widetilde{K}_\nu}{K^2}$$

- momentum conservation

$$\sum_{n\neq r}\tilde{p}^\mu = p_r^\mu + \sum_{n\neq r} p^\mu$$

- phase space factorization

$$\mathrm{d}\phi_{m+2}(\{p\};Q) = \mathrm{d}\phi_{m+1}(\{\tilde{p}\}^{(r)};Q)[\mathrm{d}p^{(r)}_{1,m+1}(p_r;Q)]$$

$$[\mathrm{d}p^{(r)}_{1,m+1}(p_r;Q)] = \mathrm{d}y(1-y)^{m(1-\epsilon)-1}\frac{Q^2}{2\pi}\mathrm{d}\phi_2(p_r,K;Q)$$

Definitions of z_i, z_r and $k_\perp$

In the $\tilde{p}_{ir} \to p_i + p_r$ splitting we define

- Momentum fractions

$$z_i \to z_{i,r} = \frac{p_i \cdot Q}{(p_i + p_r) \cdot Q} \qquad \text{and} \qquad z_r \to z_{r,i} = \frac{p_r \cdot Q}{(p_i + p_r) \cdot Q}$$

  Q is the total incoming momentum

- Transverse momentum

$$k_\perp^\mu \to k_{\perp,i,r}^\mu = \zeta_{i,r} p_r^\mu - \zeta_{r,i} p_i^\mu + \zeta_{ir} \tilde{p}_{ir}^\mu \,,$$

$$\zeta_{i,r} = z_{i,r} - \frac{y_{ir}}{\alpha_{ir} y_{(ir)Q}} \,, \qquad \zeta_{i,r} = z_{i,r} - \frac{y_{ir}}{\alpha_{ir} y_{(ir)Q}} \,, \qquad \zeta_{i,r} = \frac{y_{ir}}{\alpha_{ir} y_{\widetilde{ir}\,Q}} (z_{r,i} - z_{i,r})$$

  We have $\tilde{p}_{ir} \cdot k_{\perp,i,r} = 0$ and $k_{\perp,i,r} \to 0$ in the collinear limit (no gauge term)

Single unresolved counterterms

The collinear and soft momentum mappings define extensions of the limit formulae over the full phase space

$$\mathbf{C}_{ir}|\mathcal{M}^{(0)}_{m+2}|^2 \longrightarrow \mathcal{C}^{(0,0)}_{ir}$$
$$\mathbf{S}_{r}|\mathcal{M}^{(0)}_{m+2}|^2 \longrightarrow \mathcal{S}^{(0,0)}_{r}$$
$$\mathbf{C}_{ir}\mathbf{S}_{r}|\mathcal{M}^{(0)}_{m+2}|^2 \longrightarrow \mathcal{C}_{ir}\mathcal{S}^{(0,0)}_{r}$$

- On the r.h.s. $\mathcal{C}^{(0,0)}_{ir}$, $\mathcal{S}^{(0,0)}_{r}$ and $\mathcal{C}_{ir}\mathcal{S}^{(0,0)}_{r}$ are functions of the original momenta that inherit the notation of the operators, but have nothing to do with taking limits
- Precise definitions of momenta, momentum fractions z_i, z_r and transverse momentum $k_\perp$ that appear in the AP functions are as above

The true subtraction term

$$\mathbf{A}_1|\mathcal{M}^{(0)}_{m+2}|^2 \longrightarrow \boldsymbol{\mathcal{A}}_1|\mathcal{M}^{(0)}_{m+2}|^2 = \sum_r \left[\sum_{i\neq r}\frac{1}{2}\mathcal{C}^{(0,0)}_{ir} + \mathcal{S}^{(0,0)}_{r} - \sum_{i\neq r}\mathcal{C}_{ir}\mathcal{S}^{(0,0)}_{r}\right]$$

The approximate cross section

$$\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} = \mathrm{d}\phi_{m+1}[\mathrm{d}p_1]\boldsymbol{\mathcal{A}}_1|\mathcal{M}^{(0)}_{m+2}|^2$$

Double unresolved A_2 subtractions – IR fractorization

Doubly-unresolved IR limits

- Triple collinear: $p_i || p_r || p_s$

$$\mathbf{C}_{irs} |\mathcal{M}^{(0)}_{m+2}(p_i, p_r, p_s \ldots)|^2 = (8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2 \frac{1}{s^2_{irs}} \hat{P}_{f_i f_r f_s} \otimes |\mathcal{M}^{(0)}_m(p_{irs}, \ldots)|^2$$

- Double collinear: $p_i || p_r$ and $p_j || p_s$

$$\mathbf{C}_{ir;js} |\mathcal{M}^{(0)}_{m+2}(p_i, p_r, p_j, p_s \ldots)|^2 = (8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2 \frac{1}{s_{ir} s_{js}} \hat{P}_{f_i f_r} \otimes \hat{P}_{f_j f_s} \otimes |\mathcal{M}^{(0)}_m(p_{ir}, p_{js}, \ldots)|^2$$

- Soft-collinear: $p_i || p_r$ and $p_s \to 0$

$$\mathbf{CS}_{ir;s} |\mathcal{M}^{(0)}_{m+2}(p_i, p_r, p_s \ldots)|^2 = -(8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2 \sum_{j,k} \frac{1}{2} \mathcal{S}_{jk}(s) \frac{1}{s_{ir}} \hat{P}_{f_i f_r} \otimes |\mathcal{M}^{(0)}_{m,(j,k)}(p_{ir}, \cancel{p_s}, \ldots)|^2$$

- Double soft: $p_r, p_s \to 0$

$$\mathbf{S}_r |\mathcal{M}^{(0)}_{m+2}(p_r, p_s \ldots)|^2 = (8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2 \Bigg[\sum_{i,k,j,l} \frac{1}{8} \mathcal{S}_{ik}(r) \mathcal{S}_{jl}(r) |\mathcal{M}^{(0)}_{m,(i,k),(j,l)}(\cancel{p_r}, \cancel{p_s}, \ldots)|^2 - \frac{1}{4} C_{\mathrm{A}} \sum_{i,k} \mathcal{S}_{ik}(r,s) |\mathcal{M}^{(0)}_{m,(i,k)}(\cancel{p_r}, \cancel{p_s}, \ldots)|^2 \Bigg]$$

Double unresolved A_2 subtractions – matching

The complete double unresolved subtraction term has the structure

$$\begin{aligned}\mathbf{A}_2|\mathcal{M}_{m+2}^{(0)}|^2 = \sum_r \sum_{s\neq r} \Bigg\{ \sum_{i\neq r,s} \Bigg[\frac{1}{6}\mathbf{C}_{irs} + \sum_{j\neq i,r,s} \frac{1}{8}\mathbf{C}_{ir;js} + \frac{1}{2}\mathbf{CS}_{ir;s}\Bigg] + \frac{1}{2}\mathbf{S}_{rs} \\ - \sum_{i\neq r,s} \Bigg[\frac{1}{2}\mathbf{C}_{irs}\mathbf{CS}_{ir;s} + \sum_{j\neq i,r,s} \frac{1}{2}\mathbf{C}_{ir;js}\mathbf{CS}_{ir;s} + \frac{1}{2}\mathbf{C}_{irs}\mathbf{S}_{rs} + \mathbf{CS}_{ir;s}\mathbf{S}_{rs} \\ + \sum_{j\neq i,r,s} \frac{1}{2}\boxed{\mathbf{C}_{ir;js}\mathbf{S}_{rs}}\Bigg] + \sum_{i\neq r,s} \Bigg[\mathbf{C}_{irs}\mathbf{CS}_{ir;s}\mathbf{S}_{rs} + \sum_{j\neq i,r,s} \boxed{\mathbf{C}_{ir;js}\mathbf{CS}_{ir;s}\mathbf{S}_{rs}}\Bigg]\Bigg\}|\mathcal{M}_{m+2}^{(0)}|^2\end{aligned}$$

- Must provide precise definitions of momenta entering factorized matrix elements
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Double unresolved A_2 subtractions – matching

The complete double unresolved subtraction term has the structure

$$\mathbf{A}_2|\mathcal{M}^{(0)}_{m+2}|^2 = \sum_r \sum_{s\neq r} \Bigg\{ \sum_{i\neq r,s} \bigg[\frac{1}{6}\mathbf{C}_{irs} + \sum_{j\neq i,r,s} \frac{1}{8}\mathbf{C}_{ir;js} + \frac{1}{2}\mathbf{CS}_{ir;s} \bigg] + \frac{1}{2}\mathbf{S}_{rs}$$
$$- \sum_{i\neq r,s} \bigg[\frac{1}{2}\mathbf{C}_{irs}\mathbf{CS}_{ir;s} + \sum_{j\neq i,r,s} \frac{1}{2}\mathbf{C}_{ir;js}\mathbf{CS}_{ir;s} + \frac{1}{2}\mathbf{C}_{irs}\mathbf{S}_{rs} + \mathbf{CS}_{ir;s}\mathbf{S}_{rs}$$
$$- \sum_{j\neq i,r,s} \frac{1}{2}\mathbf{C}_{ir;js}\mathbf{S}_{rs} - \mathbf{C}_{irs}\mathbf{CS}_{ir;s}\mathbf{S}_{rs} \bigg] \Bigg\} |\mathcal{M}^{(0)}_{m+2}|^2$$

- Must provide precise definitions of momenta entering factorized matrix elements
- Also of z_i, z_r and $k_\perp$

Double unresolved A_2 subtractions – extension

Definition of momenta entering factorized matrix elements: momentum mappings

$$\{p\}_{m+2} \to \{\tilde{p}\}_m$$

- Implement momentum conservation
- Mass-shell conditions conserved
- Lead to an exact factorization of the $m+2$ parton phase space
- Respect the structure of cancellations

Momentum mappings

- Separate momentum mappings for triple collinear, double collinear, soft-collinear and double soft subtractions
- Recoil is distributed democratically (no spectator)
- Simple generalizations of single unresolved mappings

Double unresolved counterterms

The various momentum mappings define extensions of the limit formulae over the full phase space

$$\mathbf{C}_{irs}|\mathcal{M}^{(0)}_{m+2}|^2 \longrightarrow \mathcal{C}^{(0,0)}_{irs}$$
$$\mathbf{C}_{ir;js}|\mathcal{M}^{(0)}_{m+2}|^2 \longrightarrow \mathcal{C}^{(0,0)}_{ir;js}$$
$$\mathbf{CS}_{ir;s}|\mathcal{M}^{(0)}_{m+2}|^2 \longrightarrow \mathcal{CS}^{(0,0)}_{ir;s}$$
$$\mathbf{S}_{rs}|\mathcal{M}^{(0)}_{m+2}|^2 \longrightarrow \mathcal{S}^{(0,0)}_{rs}$$
$$\vdots$$

- On the r.h.s. $\mathcal{C}^{(0,0)}_{ir}$, $\mathcal{S}^{(0,0)}_{r}$ and $\mathcal{C}_{ir}\mathcal{S}^{(0,0)}_{r}$ are functions of the original momenta that inherit the notation of the operators, but have nothing to do with taking limits.
- Precise definitions of momentum fractions and transverse momenta that appear in the AP functions are available, but not exhibited.

Double unresolved counterterms

The true subtraction term

$$\boldsymbol{\mathcal{A}}_2|\mathcal{M}_{m+2}^{(0)}|^2 = \sum_r \sum_{s\neq r} \Bigg\{ \sum_{i\neq r,s} \Bigg[\frac{1}{6}\mathcal{C}_{irs}^{(0,0)} + \sum_{j\neq i,r,s} \frac{1}{8}\mathcal{C}_{ir;js}^{(0,0)} + \frac{1}{2}\mathcal{CS}_{ir;s}^{(0,0)}\Bigg] + \frac{1}{2}\mathcal{S}_{rs}^{(0,0)}$$

$$- \sum_{i\neq r,s} \Bigg[\frac{1}{2}\mathcal{C}_{irs}\mathcal{CS}_{ir;s}^{(0,0)} + \sum_{j\neq i,r,s} \frac{1}{2}\mathcal{C}_{ir;js}\mathcal{CS}_{ir;s}^{(0,0)} + \frac{1}{2}\mathcal{C}_{irs}\mathcal{S}_{rs}^{(0,0)} + \mathcal{CS}_{ir;s}\mathcal{S}_{rs}^{(0,0)}$$

$$- \sum_{j\neq i,r,s} \frac{1}{2}\mathcal{C}_{ir;js}\mathcal{S}_{rs}^{(0,0)} - \mathcal{C}_{irs}\mathcal{CS}_{ir;s}\mathcal{S}_{rs}^{(0,0)}\Bigg]\Bigg\}$$

The approximate cross section

$$\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_2} = \mathrm{d}\phi_m[\mathrm{d}p_2]\boldsymbol{\mathcal{A}}_2|\mathcal{M}_{m+2}^{(0)}|^2$$

Rest of the unresolved counterterms

The remaining approximate cross sections are constructed in the same way

- In particular, it turns out that $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}}$ can be obtained from the single unresolved limit of $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2}$

$$\mathbf{A}_{12}|\mathcal{M}_{m+2}^{(0)}|^2 = \sum_r \left[\sum_{i\neq r} \frac{1}{2} \mathbf{C}_{ir}\mathbf{A}_2 + \mathbf{S}_r\mathbf{A}_2 - \sum_{i\neq r} \mathbf{C}_{ir}\mathbf{S}_r\mathbf{A}_2 \right] |\mathcal{M}_{m+2}^{(0)}|^2$$

- This construction relies on the fact that at the level of IR factorization formulae, A_1 has universal IR limits, and also on a certain compatibility between iterated single unresolved and strongly ordered double unresolved IR formulae.

- The extension of A_{12} must respect this compatibility, which puts certain constraints on the specific form in which some IR limit formulae are written, e.g.,

$$P^{\mu\nu} = -Ag^{\mu\nu} + Bk_\perp^\mu k_\perp^\nu \to -Ag^{\mu\nu} + B(-s_{ir}z_iz_r)\frac{k_\perp^\mu k_\perp^\nu}{k_\perp^2}$$

  are equivalent at the level of limits $(k_\perp^2 = -s_{ir}z_iz_r)$, but should use second form

- Use iterated single unresolved momentum mappings

Rest of the unresolved counterterms

The remaining approximate cross sections are constructed in the same way

- The real-virtual approximate cross section $\mathrm{d}\sigma_{m+1}^{\mathrm{RV},\mathrm{A}_1}$ is constructed exactly like $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1}$, only the specific IR limit formulae change
- The construction of $\left(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1}\right)^{\mathrm{A}_1}$ relies on the fact that $\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1}$ has universal IR behaviour and proceeds exactly like the building of $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1}$

Universal limits for subtraction terms

The existence of universal IR limits of approximate cross sections is (clearly) not guaranteed by QCD factorizataion.

- We do not specify which momenta can become unresolved, hence the single unresolved subtraction terms must themselves have universal IR limits
- In the real-virtual contribution, these terms appear in integrated form, and these forms again must have universal IR limits
- These are non-trivial constraints, since the (unintegrated and integrated) single soft factorization formula involves color-correlated matrix elements

$$\mathcal{S}_r^{(0,0)} \propto \sum_{i,k} \frac{s_{ik}}{s_{ir}s_{kr}} \langle \mathcal{M}_{m+1}^{(0)} | \boldsymbol{T}_i \boldsymbol{T}_k | \mathcal{M}_{m+1}^{(0)} \rangle$$

- In, say, the $p_j || p_s$ limit only the sum

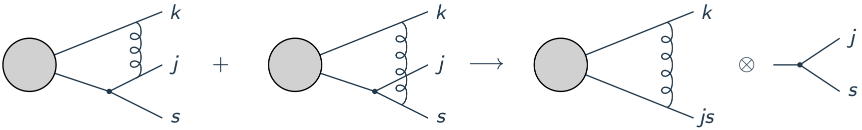
$$\langle \mathcal{M}_{m+1}^{(0)} | \boldsymbol{T}_j \boldsymbol{T}_k | \mathcal{M}_{m+1}^{(0)} \rangle + \langle \mathcal{M}_{m+1}^{(0)} | \boldsymbol{T}_s \boldsymbol{T}_k | \mathcal{M}_{m+1}^{(0)} \rangle$$

factorizes, due to soft gluon coherence, but not the two pieces separately

In, say, the $p_j || p_s$ limit only the sum

$$\langle \mathcal{M}^{(0)}_{m+1} | \boldsymbol{T}_j \boldsymbol{T}_k | \mathcal{M}^{(0)}_{m+1} \rangle + \langle \mathcal{M}^{(0)}_{m+1} | \boldsymbol{T}_s \boldsymbol{T}_k | \mathcal{M}^{(0)}_{m+1} \rangle$$

factorizes, due to soft gluon coherence, but not the two pieces separately



Universal limits for subtraction terms

Then we must make sure that in any collinear limit (for any i and r), the two appropriate terms from the soft formula actually go to the same limit

- The eikonal factors are homogeneous in p_j and p_s, so they go to the same limit (note no partial fraction decomposition)

$$\mathbf{C}_{js}\frac{s_{jk}}{s_{jr}s_{kr}} = \frac{z_j s_{(js)k}}{z_j s_{(js)r}s_{kr}} = \frac{s_{(js)k}}{s_{(js)r}s_{kr}} \qquad \text{and} \qquad \mathbf{C}_{js}\frac{s_{sk}}{s_{rs}s_{kr}} = \frac{z_s s_{(js)k}}{z_s s_{(js)r}s_{kr}} = \frac{s_{(js)k}}{s_{(js)r}s_{kr}}$$

- But we must also have that the mapped momenta that appear in the factorized matrix elements in

$$\langle\mathcal{M}^{(0)}_{m+1}|\boldsymbol{T}_j\boldsymbol{T}_k|\mathcal{M}^{(0)}_{m+1}\rangle \qquad \text{and} \qquad \langle\mathcal{M}^{(0)}_{m+1}|\boldsymbol{T}_s\boldsymbol{T}_k|\mathcal{M}^{(0)}_{m+1}\rangle$$

  also go to the same limit.

- Constrains the soft momentum mapping. A trivial way of satisfying this constraint is to use the same mapped momenta in all terms in the soft formula $\Leftrightarrow$ dipole picture.

Momentum mappings used to define the counterterms

$$\{p\}_{n+p} \xrightarrow{R} \{\tilde{p}\}_n \quad \Rightarrow \quad \mathrm{d}\phi_{n+p}(\{p\}; Q) = \mathrm{d}\phi_n(\{\tilde{p}\}_n^{(R)}; Q)[\mathrm{d}p_{p,n}^{(R)}]$$

- lead to exact factorization of phase space
- different collinear and soft mappings (R labels precise limit)

Counterterms are products (in color and spin space) of

- factorized ME's independent of variables in $[\mathrm{d}p_{p,n}^{(R)}]$
- singular factors (AP functions, soft currents), to be integrated over $[\mathrm{d}p_{p,n}^{(R)}]$

$$\mathcal{X}_R(\{p\}_{n+p}) = \left(8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon}\right)^p \, \mathit{Sing}_R(p_p^{(R)}) \otimes |\mathcal{M}_n^{(0)}(\{\tilde{p}\}_n^{(R)})|^2$$

Can compute once and for all the integral over unresolved partons

$$\int_p \mathcal{X}_R(\{p\}_{n+p}) = \left(8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon}\right)^p \left[\int_p \mathit{Sing}_R(p_p^{(R)})\right] \otimes |\mathcal{M}_n^{(0)}(\{\tilde{p}\}_n^{(R)})|^2$$

Solving the integrals

Strategy for computing the integrals: direct integration

1. write phase space in terms of angles and energies
2. angular integrals in terms of Mellin-Barnes representations
3. resolve the ϵ poles by analytic continuation
4. MB integrals to Euler-type integrals, pole coefficients are finite parametric integrals

1. choose explicit parametrization of phase space
2. write the parametric integral representation in chosen variables
3. resolve the ϵ poles by sector decomposition
4. pole coefficients are finite parametric integrals

5. evaluate the parametric integrals in terms of multiple polylogs
6. simplify result (optional)

$$\Downarrow$$

Poles and logs of the finite parts known fully analytically, regular pieces of finite parts computed numerically on a grid

Pole cancellation in the VV contribution

After adding all integrated approximate cross sections the double virtual contribution must be finite in ϵ.

$$\sigma_m^{\mathrm{NNLO}} = \int_m \left\{ \mathrm{d}\sigma_m^{\mathrm{VV}} + \int_2 \left[\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_2}} - \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_{12}}} \right] + \int_1 \left[\mathrm{d}\sigma_{m+1}^{\mathrm{RV,A_1}} + \left(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_1}} \right)^{\mathrm{A_1}} \right] \right\} J_m$$

- After summing over unobserved flavors, all integrated approximate cross sections can be written as products (in color space) of various insertion operators with lower point cross sections.
- Have checked the cancellation of the $\frac{1}{\epsilon^4}$ and $\frac{1}{\epsilon^3}$ poles analytically for any number of jets (i.e., with m symbolic).
- Have checked $m = 2$ $(e^+e^- \to q\bar{q},\ H \to b\bar{b})$ explicitly and we find that all poles cancel.
- Have checked $m = 3$ $(e^+e^- \to q\bar{q}g)$ explicitly and we find that all poles cancel.

Example: $H \to b\bar{b}$

The double virtual contribution has the following pole structure ($\mu^2 = m_H^2$)

$$\begin{aligned}
\mathrm{d}\sigma^{\mathrm{VV}}_{H\to b\bar{b}} = \left(\frac{\alpha_{\mathrm{s}}(\mu^2)}{2\pi}\right)^2 \mathrm{d}\sigma^{\mathrm{B}}_{H\to b\bar{b}} \Bigg\{ \frac{2C_{\mathrm{F}}^2}{\epsilon^4} + \left(\frac{11\,C_{\mathrm{A}}\,C_{\mathrm{F}}}{4} + 6C_{\mathrm{F}}^2 - \frac{C_{\mathrm{F}}\,n_{\mathrm{f}}}{2}\right)\frac{1}{\epsilon^3} \\
+ \left[\left(\frac{8}{9} + \frac{\pi^2}{12}\right) C_{\mathrm{A}}\,C_{\mathrm{F}} + \left(\frac{17}{2} - 2\pi^2\right) C_{\mathrm{F}}^2 - \frac{2\,C_{\mathrm{F}}\,n_{\mathrm{f}}}{9}\right]\frac{1}{\epsilon^2} \\
+ \left[\left(-\frac{961}{216} + \frac{13\zeta_3}{2}\right) C_{\mathrm{A}}\,C_{\mathrm{F}} + \left(\frac{109}{8} - 2\pi^2 - 14\zeta_3\right) C_{\mathrm{F}}^2 + \frac{65\,C_{\mathrm{F}}\,n_{\mathrm{f}}}{108}\right]\frac{1}{\epsilon} \Bigg\}
\end{aligned}$$

(Anastasiou, Herzog, Lazopoulos, `arXiv:0111.2368`)

The double virtual contribution has the following pole structure ($\mu^2 = m_H^2$)

$$\mathrm{d}\sigma^{\mathrm{VV}}_{H\to b\bar{b}} = \left(\frac{\alpha_\mathrm{s}(\mu^2)}{2\pi}\right)^2 \mathrm{d}\sigma^{\mathrm{B}}_{H\to b\bar{b}} \left\{ \frac{2C_\mathrm{F}^2}{\epsilon^4} + \left(\frac{11 C_\mathrm{A} C_\mathrm{F}}{4} + 6C_\mathrm{F}^2 - \frac{C_\mathrm{F} n_\mathrm{f}}{2}\right)\frac{1}{\epsilon^3} \right.$$
$$+ \left[\left(\frac{8}{9} + \frac{\pi^2}{12}\right) C_\mathrm{A} C_\mathrm{F} + \left(\frac{17}{2} - 2\pi^2\right) C_\mathrm{F}^2 - \frac{2C_\mathrm{F} n_\mathrm{f}}{9}\right]\frac{1}{\epsilon^2}$$
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(Anastasiou, Herzog, Lazopoulos, arXiv:0111.2368)

The sum of the integrated approximate cross sections gives ($\mu^2 = m_H^2$)

$$\sum \int \mathrm{d}\sigma^{\mathrm{A}} = \left(\frac{\alpha_\mathrm{s}(\mu^2)}{2\pi}\right)^2 \mathrm{d}\sigma^{\mathrm{B}}_{H\to b\bar{b}} \left\{ \frac{-2C_\mathrm{F}^2}{\epsilon^4} + \left(-\frac{11 C_\mathrm{A} C_\mathrm{F}}{4} - 6C_\mathrm{F}^2 + \frac{C_\mathrm{F} n_\mathrm{f}}{2}\right)\frac{1}{\epsilon^3} \right.$$
$$+ \left[\left(-\frac{8}{9} - \frac{\pi^2}{12}\right) C_\mathrm{A} C_\mathrm{F} + \left(-\frac{17}{2} + 2\pi^2\right) C_\mathrm{F}^2 + \frac{2C_\mathrm{F} n_\mathrm{f}}{9}\right]\frac{1}{\epsilon^2}$$
$$\left. + \left[\left(\frac{961}{216} - \frac{13\zeta_3}{2}\right) C_\mathrm{A} C_\mathrm{F} + \left(-\frac{109}{8} + 2\pi^2 + 14\zeta_3\right) C_\mathrm{F}^2 - \frac{65 C_\mathrm{F} n_\mathrm{f}}{108}\right]\frac{1}{\epsilon} \right\}$$

(Del Duca, Duhr, GS, Tramontano, Trócsányi, arXiv:1501.07226)

Example: $e^+e^- \to 3$ jets

The double virtual contribution has the following pole structure ($\mu^2 = s$)

$$\mathrm{d}\sigma_3^{\mathrm{VV}} = \mathcal{P}oles\Big(A_3^{(2\times 0)} + A_3^{(1\times 1)}\Big) + \mathcal{F}inite\Big(A_3^{(2\times 0)} + A_3^{(1\times 1)}\Big)$$

where

$$\begin{aligned}\mathcal{P}oles\Big(A_3^{(2\times 0)} + A_3^{(1\times 1)}\Big) = 2\Bigg[&- \Big(\boldsymbol{I}_{q\bar{q}g}^{(1)}(\epsilon)\Big)^2 - \frac{\beta_0}{\epsilon}\boldsymbol{I}_{q\bar{q}g}^{(1)}(\epsilon) \\ &+ e^{-\epsilon\gamma}\frac{\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\left(\frac{\beta_0}{\epsilon} + K\right)\boldsymbol{I}_{q\bar{q}g}^{(1)}(2\epsilon) + \boldsymbol{H}_{q\bar{q}g}^{(2)}\Bigg] A_3^0(1_q, 3_g, 2_{\bar{q}}) \\ &+ 2\boldsymbol{I}_{q\bar{q}g}^{(1)}(\epsilon) A_3^{1\times 0}(1_q, 3_g, 2_{\bar{q}})\end{aligned}$$

with

$$\begin{aligned}\boldsymbol{H}_{q\bar{q}g}^{(2)} = \frac{e^{\epsilon\gamma}}{4\epsilon\Gamma(1-\epsilon)}\Bigg[&\left(4\zeta_3 + \frac{589}{432} - \frac{11\pi^2}{72}\right)N_{\mathrm{c}} + \left(-\frac{1}{2}\zeta_3 - \frac{41}{54} - \frac{\pi^2}{48}\right) \\ &+ \left(-3\zeta_3 - \frac{3}{16} + \frac{\pi^2}{4}\right)\frac{1}{N_{\mathrm{c}}} + \left(-\frac{19}{18} + \frac{\pi^2}{36}\right)N_{\mathrm{c}} n_{\mathrm{f}} + \left(-\frac{1}{54} - \frac{\pi^2}{24}\right)\frac{n_{\mathrm{f}}}{N_{\mathrm{c}}} + \frac{5}{27}n_{\mathrm{f}}^2\Bigg]\end{aligned}$$

(Gehrmann-De Ridder, Gehrmann, Glover, Heinrich, `arXiv:0710.0346`)

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Adding the sum of the integrated approximate cross sections gives

$$\mathcal{P}oles\Big(A_3^{(2\times0)} + A_3^{(1\times1)}\Big) + \mathcal{P}oles\sum\int \mathrm{d}\sigma^{\mathrm{A}} = 117\text{k terms}$$

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- zero numerically in any phase space point

```
In[35]:= N[PolesVV3 /. {y13 → 2 / 10, y23 → 3 / 10}, 40]
```

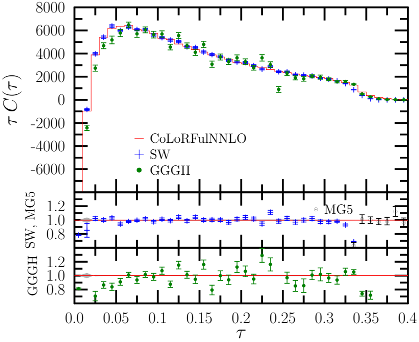
Out[35]=

$$\frac{0.\times10^{-387} + \frac{0.\times10^{-388}}{\mathrm{Nc}^2} + 0.\times10^{-388}\,\mathrm{Nc}^2 + \frac{0.\times10^{-438}\,\mathrm{nf}}{\mathrm{Nc}} + 0.\times10^{-439}\,\mathrm{Nc}\,\mathrm{nf}}{\mathrm{e}^2} + \frac{1}{\mathrm{e}}\left(\left(0.\times10^{-384} + 0.\times10^{-438}\,i\right) + \frac{0.\times10^{-385}}{\mathrm{Nc}^2} + \left(0.\times10^{-384} + 0.\times10^{-438}\,i\right)\mathrm{Nc}^2 + \frac{0.\times10^{-437}\,\mathrm{nf}}{\mathrm{Nc}} + 0.\times10^{-437}\,\mathrm{Nc}\,\mathrm{nf}\right) + \mathrm{O}[\mathrm{e}]^0$$

- zero analytically after simplification using symbol technology (C. Duhr)

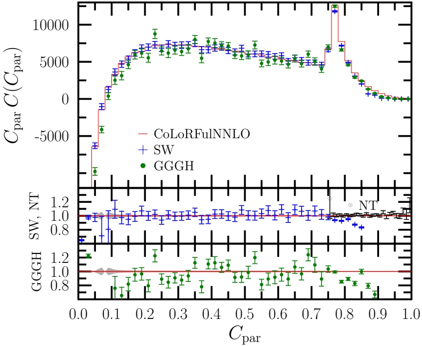
NNLO coefficients ($\mathcal{O}(\alpha_s^3)$ parts) for thrust and the C-parameter

- Thrust



$$\tau = 1 - \max_{\vec{n}} \left(\frac{\sum_i |\vec{n} \cdot \vec{p}_i|}{\sum_i |\vec{p}_i|} \right)$$

- C-parameter



$$C_{\rm par} = \frac{3}{2} \frac{\sum_{i,j} |\vec{p}_i||\vec{p}_j| \sin^2\theta_{ij}}{\left(\sum_i |\vec{p}_i|\right)^2}$$

Constrained subtractions

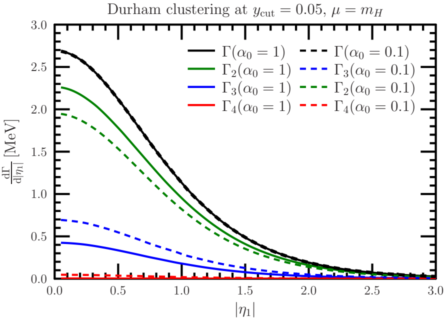
We can constrain subtractions to near singular regions: $\alpha_0 \in (0,1]$

- poles cancel numerically ($\alpha_0 = 0.1$)

$$\mathrm{d}\sigma^{\mathrm{VV}}_{H\to b\bar{b}} + \sum \int \mathrm{d}\sigma^{\mathrm{A}} = \frac{5.4\times 10^{-8}}{\epsilon^4} + \frac{3.9\times 10^{-5}}{\epsilon^3} + \frac{3.3\times 10^{-3}}{\epsilon^2} + \frac{6.7\times 10^{-3}}{\epsilon} + \mathcal{O}(1)$$

$$Err\left(\sum \int \mathrm{d}\sigma^{\mathrm{A}}\right) = \frac{3.1\times 10^{-5}}{\epsilon^4} + \frac{5.0\times 10^{-4}}{\epsilon^3} + \frac{8.1\times 10^{-3}}{\epsilon^2} + \frac{7.7\times 10^{-2}}{\epsilon} + \mathcal{O}(1)$$

- Pesudorapidity of leading jet in $H \to b\bar{b}$



Constrained subtractions

We can constrain subtractions to near singular regions: $\alpha_0 \in (0, 1]$

- improved efficiency

α_0	1	0.1
timing (rel.)	1	0.40
$\langle N_{\mathrm{sub}} \rangle$	52	14.5

$\langle N_{\mathrm{sub}} \rangle$ is the average number of subtraction terms computed

Towards processes with hadronic initial states

The NNLO cross section with initial states

Overall structure unchanged, but must include (known) mass factorization counterterms

$$\sigma^{\mathrm{NNLO}}[J] = \int_{m+2} \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} + \int_{m+1} \mathrm{d}\sigma^{\mathrm{RV}}_{m+1} J_{m+1} + \int_{m} \mathrm{d}\sigma^{\mathrm{VV}}_{m} J_{m} + \int_{m+1} \mathrm{d}\sigma^{\mathrm{C}_1}_{m+1} J_{m+1} + \int_{m} \mathrm{d}\sigma^{\mathrm{C}_2}_{m} J_{m}$$

"No new conceptual issues, but lots of tedious details to work out."

Morally true ✔

- IR factorization formulae known from crossing and/or direct computation
- Principles of matching, extension unchanged (only more terms to catalog)

But ✘

- Need new mappings for initial-final collinear limits
- Mappings not suited for processes with massive particles (e.g., V, H)
- Naive crossing of momentum fractions z and transverse momenta $k_\perp$ will not work

IR factorization from crossing

The symbolic operators $\mathbf{C}_{ar}$ and $\mathbf{S}_r$ denote taking the single collinear and single soft limits

- Collinear: $p_a || p_r \; (p_r \to (1 - x_a) p_a + k_\perp + \mathcal{O}(k_\perp^2))$

$$\mathbf{C}_{ar} |\mathcal{M}^{(0)}_{m+2}(p_r, \ldots; p_a + p_b)|^2 = 8\pi\alpha_s \mu^{2\epsilon} \frac{1}{x_a} \frac{1}{s_{ar}} \\ \times \hat{P}_{f_{ar} f_r}(x_a, x_r, k_\perp; \epsilon) \otimes |\mathcal{M}^{(0)}_{m+1}(\cancel{p_r}, \ldots; p_{(ar)} + p_b)|^2$$

where the initial-final AP kernel is related to the final-final one by crossing

$$\hat{P}_{f_{ar} f_r}(x_a, x_r, k_\perp; \epsilon) = -(-1)^{F(f_a) + F(f_{ar})} x_a \hat{P}_{f_a \bar{f}_r}(1/x_a, -x_r/x_a, k_\perp; \epsilon)$$

- Soft: $p_r \to 0$

$$\mathbf{S}_r |\mathcal{M}^{(0)}_{m+2}(p_r, \ldots; p_a + p_b)|^2 = -8\pi\alpha_s \mu^{2\epsilon} \sum_{j,k} \frac{s_{jk}}{s_{jr} s_{kr}} |\mathcal{M}^{(0)}_{m+1,(i,k)}(\cancel{p_r}, \ldots; p_a + p_b)|^2$$

In order to avoid double subtraction overlapping limits must be identified and removed

All quantities (momenta, momentum fractions x_a, x_r, transverse momentum $k_\perp$) must be unambiguously defined over the full phase space

New momentum mappings

The recoil is redistributed to the initial momenta

Collinear mapping for initial-final configurations

$$\tilde{p}_a^\mu = \xi_{a,r} p_a^\mu \,, \qquad \tilde{p}_b^\mu = p_b^\mu \,, \qquad \tilde{p}_n^\mu = \Lambda(Q, \tilde{Q})^\mu_\nu \, p_n^\nu \,, \quad n \neq a, r$$

$$\xi_{a,r} = 1 - \frac{2 p_r \cdot p_{(ab)}}{p_{(ab)}^2}$$

- momentum conservation

$$p_a^\mu + p_b^\mu = p_r^\mu + \sum_{n \neq r} p^\mu \,, \qquad \tilde{p}_a^\mu + \tilde{p}_b^\mu = \sum_{n \neq r} \tilde{p}^\mu$$

- phase space convolution

$$\mathrm{d}\phi_{m+2}(\{p\}; p_{(ab)}) = \int_{\xi_{\min}}^{\xi_{\max}} \mathrm{d}\xi \, \mathrm{d}\phi_{m+1}(\{\tilde{p}\}^{(ar)}; \xi p_a + p_b) \frac{p_{(ab)}^2}{2\pi} \, \mathrm{d}\phi_2(Q, p_r; p_{(ab)})$$

New momentum mappings

The recoil is redistributed to the initial momenta

New collinear mapping for final-final configurations

$$\tilde{p}_a^\mu = (1-\alpha_{ir})p_a^\mu\,, \qquad \tilde{p}_b^\mu = (1-\alpha_{ir})p_b^\mu\,, \qquad \tilde{p}_{ir}^\mu = p_i^\mu + p_r^\mu - \alpha_{ir} Q^\mu\,,$$
$$\tilde{p}_n^\mu = p_n^\mu\,, \qquad n \neq i, r\,,$$
$$\alpha_{ir} = \frac{1}{2}\left[y_{(ir)Q} - \sqrt{y_{(ir)Q}^2 - 4y_{ir}}\right]$$

- momentum conservation

$$p_a^\mu + p_b^\mu = p_i^\mu + p_r^\mu + \sum_{n\neq i,r} p^\mu\,, \qquad \tilde{p}_a^\mu + \tilde{p}_b^\mu = \tilde{p}_{ir}^\mu + \sum_{n\neq i,r} \tilde{p}^\mu$$

- phase space convolution

$$\mathrm{d}\phi_{m+2}(\{p\};p_{(ab)}) = \int_{\alpha_{\min}}^{\alpha_{\max}} \mathrm{d}\alpha\,\mathrm{d}\phi_{m+1}(\{\tilde{p}\}^{(ir)};(1-\alpha)p_{(ab)})\frac{2\tilde{p}_{ir}\cdot p_{(ab)}}{2\pi}\,\mathrm{d}\phi_2(p_i,p_r;\tilde{p}_{ir}+\alpha p_{(ab)})$$

New momentum mappings

The recoil is redistributed to the initial momenta

New soft mapping

$$\tilde{p}_a^\mu = \sqrt{\lambda_r} p_a^\mu \,, \qquad \tilde{p}_b^\mu = \sqrt{\lambda_r} p_b^\mu \,, \qquad \tilde{p}_n^\mu = \Lambda(Q, \tilde{Q})^\mu_\nu \, p_n^\nu \,, \qquad n \neq r \,,$$
$$\lambda_r = 1 - \frac{2 p_r \cdot p_{(ab)}}{p_{(ab)}^2}$$

- momentum conservation

$$p_a^\mu + p_b^\mu = p_r^\mu + \sum_{n \neq i, r} p^\mu \,, \qquad \tilde{p}_a^\mu + \tilde{p}_b^\mu = \sum_{n \neq r} \tilde{p}^\mu$$

- phase space convolution

$$\mathrm{d}\phi_{m+2}(\{p\}; p_{(ab)}) = \int_{\lambda_{\min}}^{\lambda_{\max}} \mathrm{d}\lambda \, \mathrm{d}\phi_{m+1}(\{\tilde{p}\}^{(r)}; \sqrt{\lambda} p_{(ab)}) \frac{p_{(ab)}^2}{2\pi} \, \mathrm{d}\phi_2(Q, p_r; p_{(ab)})$$

The recoil is redistributed to the initial momenta

New soft mapping

$$\tilde{p}_a^\mu = \sqrt{\lambda_r}p_a^\mu \,, \qquad \tilde{p}_b^\mu = \sqrt{\lambda_r}p_b^\mu \,, \qquad \tilde{p}_n^\mu = \Lambda(Q,\tilde{Q})^\mu_\nu \, p_n^\nu \,, \qquad n \neq r \,,$$
$$\lambda_r = 1 - \frac{2p_r \cdot p_{(ab)}}{p_{(ab)}^2}$$

- momentum conservation

$$p_a^\mu + p_b^\mu = p_r^\mu + \sum_{n\neq i,r} p^\mu \,, \qquad \tilde{p}_a^\mu + \tilde{p}_b^\mu = \sum_{n\neq r} \tilde{p}^\mu$$

- phase space convolution

$$\mathrm{d}\phi_{m+2}(\{p\};p_{(ab)}) = \int_{\lambda_{\min}}^{\lambda_{\max}} \mathrm{d}\lambda \, \mathrm{d}\phi_{m+1}(\{\tilde{p}\}^{(r)};\sqrt{\lambda}p_{(ab)}) \frac{p_{(ab)}^2}{2\pi} \, \mathrm{d}\phi_2(Q,p_r;p_{(ab)})$$

Double unresolved mappings are straightforward generalizations of the above

Definitions of x_a, x_r and $k_\perp$

Define momentum fractions from crossing?

- Single collinear ✔

$$x_a = \frac{1}{z_{i,r}}\bigg|_{p_i \to -p_a} = \frac{(p_i + p_r)\cdot Q}{p_i \cdot Q}\bigg|_{p_i \to -p_a} = 1 - \frac{p_r \cdot Q}{p_a \cdot Q} = \xi_{a,r}$$

Same as $\xi_{a,r}$ appearing in the collinear mapping, clearly $x_a \in [0,1]$.

- Triple collinear ✘

$$x_a \stackrel{?}{=} \frac{1}{z_{i,rs}}\bigg|_{p_i \to -p_a} = \frac{(p_i + p_r + p_s)\cdot Q}{p_i \cdot Q}\bigg|_{p_i \to -p_a} = 1 - \frac{p_r \cdot Q}{p_a \cdot Q} - \frac{p_s \cdot Q}{p_a \cdot Q}$$

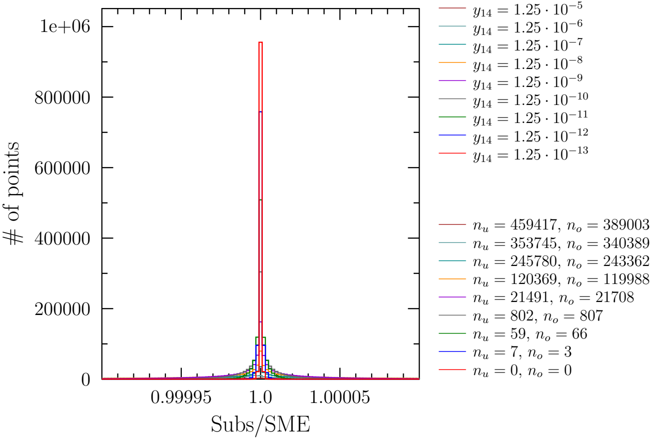
But we find that $0 \notin [0,1]$! In fact, x_a can vanish at "ordinary" points inside the double real phase space.

Momentum fractions for initial-final collinear splitting cannot be defined by naive crossing.

Have tentative definitions for momentum fractions and transverse momenta for all single and double limits

Does it work?

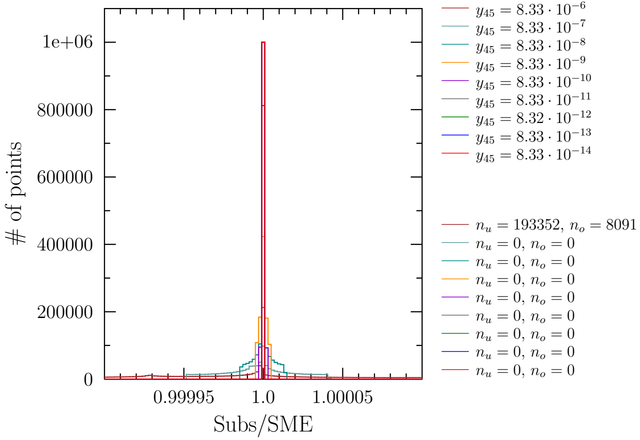
Check that the ratio of the double real emission matrix element to the the sum of all subtractions tends to one for all IR limits. E.g., $u(p_1) + \bar{d}(p_2) \to W^-(p_3) + g(p_4) + g(p_5)$



$\mathbf{C}_{14}$ limit

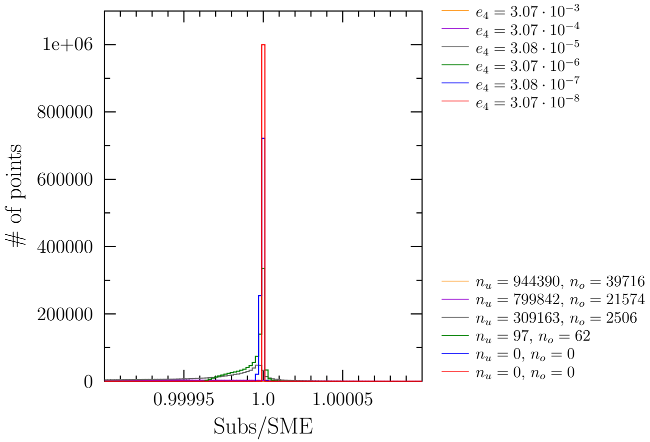
Does it work?

Check that the ratio of the double real emission matrix element to the the sum of all subtractions tends to one for all IR limits. E.g., $u(p_1) + \bar{d}(p_2) \to W^-(p_3) + g(p_4) + g(p_5)$



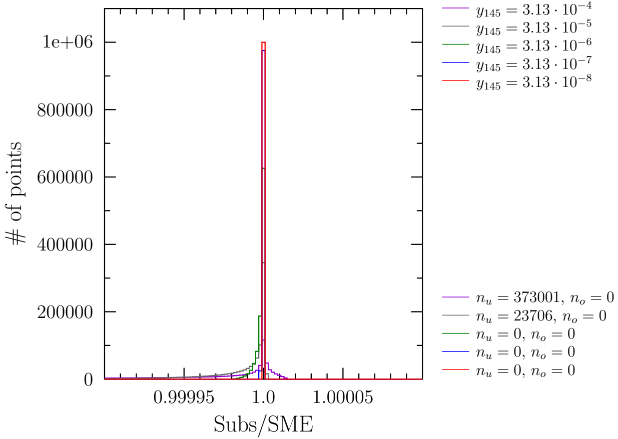
$\mathbf{C}_{45}$ limit

Check that the ratio of the double real emission matrix element to the the sum of all subtractions tends to one for all IR limits. E.g., $u(p_1) + \bar{d}(p_2) \to W^-(p_3) + g(p_4) + g(p_5)$



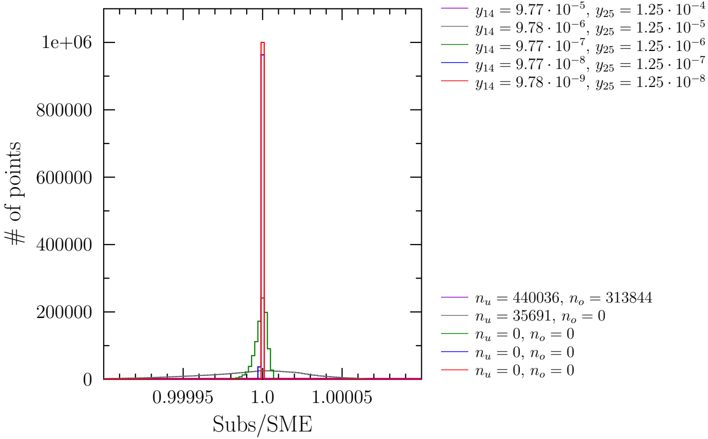
$\mathbf{S}_4$ limit

Check that the ratio of the double real emission matrix element to the the sum of all subtractions tends to one for all IR limits. E.g., $u(p_1) + \bar{d}(p_2) \to W^-(p_3) + g(p_4) + g(p_5)$



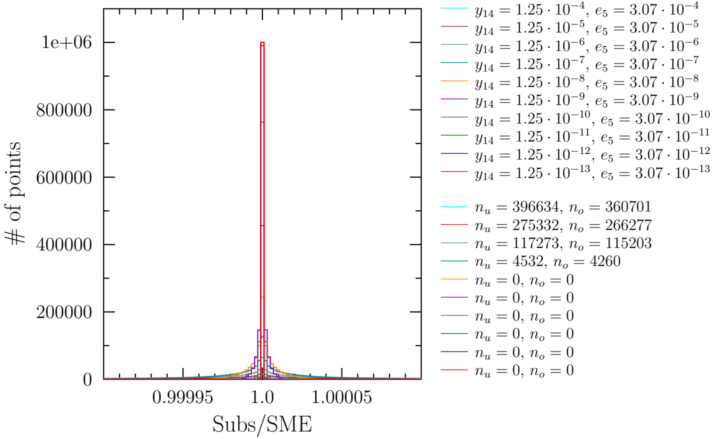
$\mathbf{C}_{145}$ limit

Check that the ratio of the double real emission matrix element to the the sum of all subtractions tends to one for all IR limits. E.g., $u(p_1) + \bar{d}(p_2) \to W^-(p_3) + g(p_4) + g(p_5)$



$\mathbf{C}_{14;25}$ limit

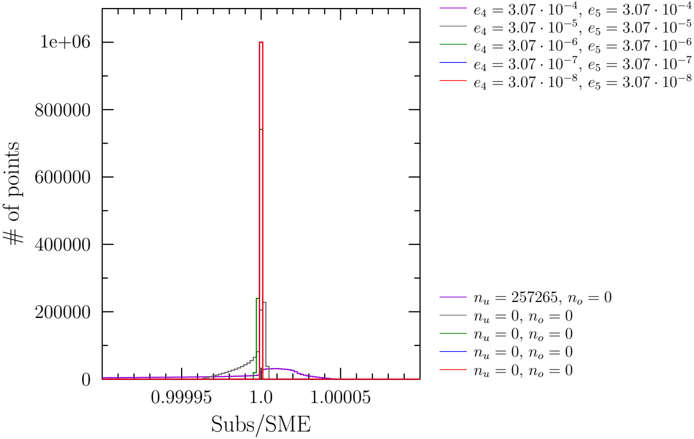
Check that the ratio of the double real emission matrix element to the the sum of all subtractions tends to one for all IR limits. E.g., $u(p_1) + \bar{d}(p_2) \to W^-(p_3) + g(p_4) + g(p_5)$



$\mathbb{CS}_{14;5}$ limit

Does it work?

Check that the ratio of the double real emission matrix element to the the sum of all subtractions tends to one for all IR limits. E.g., $u(p_1) + \bar{d}(p_2) \to W^-(p_3) + g(p_4) + g(p_5)$

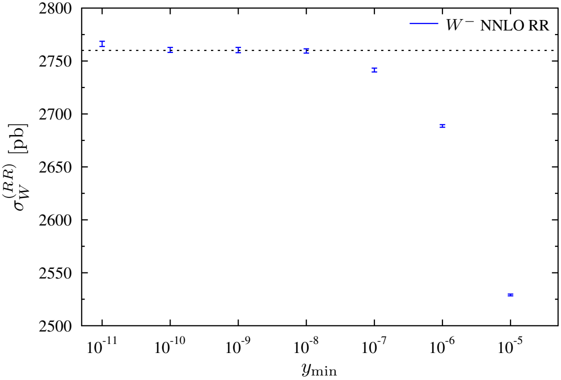


$\mathbf{S}_{45}$ limit

Does it work?

Subtractions work as designed in all limits, so try to integrate

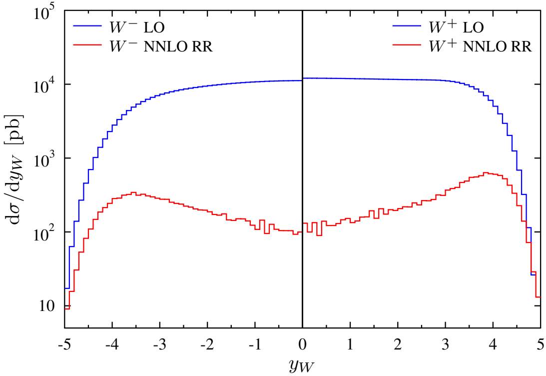
- Cutoff dependence of subtracted RR contribution to total cross section for $pp \to W^-$ ($2p_i \cdot p_j > y_{\rm min}\,\hat{s}$)



Does it work?

Subtractions work as designed in all limits, so try to integrate

- Subtracted RR contribution to rapidity distribution of the W in $pp \to W^\pm$



Conclusions and outlook

Conclusions and outlook

CoLoRFulNNLO framework

- Completely Local subtRactions for Fully differential NNLO
- Construction of subtraction terms based on IR limit formulae
- Analytic integration of subtraction terms feasible with modern techniques
- Demonstrated cancellation of poles for $m = 2$ and $m = 3$
- Worked out in full detail for processes with no colored particles in the initial state
- Good numerical convergence and stability for $e^+e^- \to 3$ jets

Conclusions and outlook

Extension to hadronic initial states on the way

- Subtraction terms for double real radiation defined for generic processes
- Subtraction terms for real-virtual radiation tentatively defined for generic processes

TODO:

- Subtraction terms for mass factorization counterterms (NLO complexity)
- Some integrals done, but many more to do