Anomalous dimensions in conformal systems non-perturbative gradient flow approach

Anna Hasenfratz
University of Colorado Boulder

Numerical approaches to
holography, quantum gravity and cosmology
June 23 2018, Edinburgh

Lattice QCD

Ken Wilson's 1974 paper
"Confinement of Quarks"

PHYSICAL REVIEW D VOLUME 10, NUMBER 8 15 OCTOBER 1974

Confinement of quarks*

Kenneth G. Wilson
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14850

started Lattice QCD

Wilson Renormalization Group

LQCD/LQFT calculations would be meaningless without understanding universality of critical behavior.

THE RENORMALIZATION GROUP AND THE ϵ EXPANSION*

Kenneth G. WILSON

Institute for Advanced Study, Princeton, N.J. 08540, USA
and Laboratory of Nuclear Studies, Cornell University, Ithaca, N.Y. 14850, USA†

and

J. KOGUT‡

Institute for Advanced Study, Princeton, N.J. 08540, USA

Received 2 July 1973

Wilson and Kogut’s review is still a treasure trove

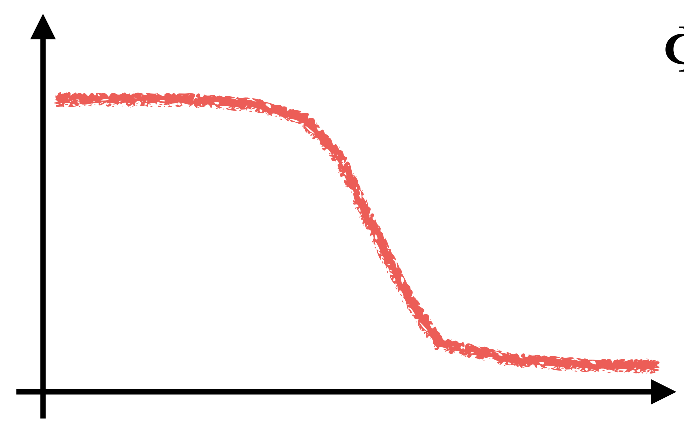
2 steps of Wilson Renormalization Group

Start with the action $S(\phi, g_0)$;

Step 1: Introduce "blocked fields"

position space: [1]

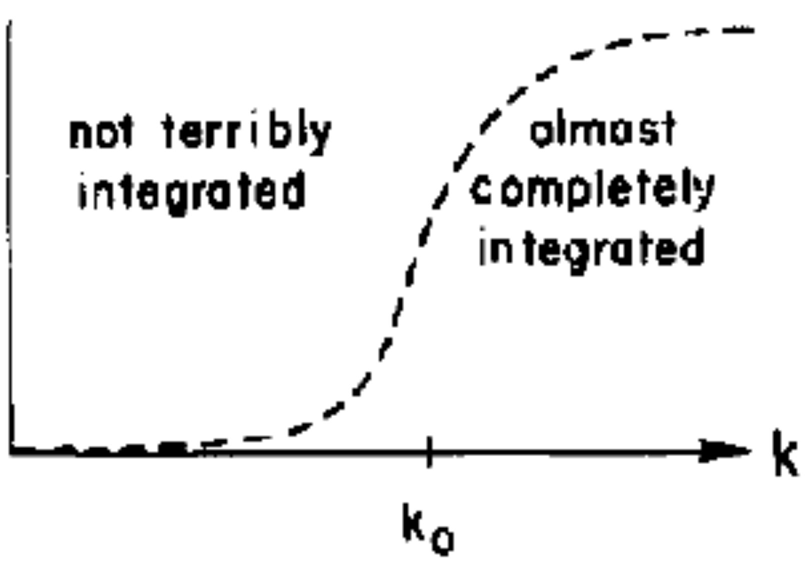
$$\Phi(x) = \sum_s f(s)\phi(x+s)$$



momentum space: [1]

$$\Phi(p) = f(p)\phi(p) \sim e^{-p^2/t}\phi(p)$$

The goal is, as described by Wilson-Kogut in Ch. 11, is to remove high momentum modes



11. Exact renormalization group equations in differential form

[1] Keep the normalization the blocked fields the same as the original

2 steps of Wilson Renormalization Group

Step 2: rescale $\Lambda_{cutoff} \rightarrow \Lambda_{cutoff}/b$

$$\Phi_b(x_b) = \Phi(bx) \qquad\qquad \Phi_b(p_b) = \Phi(p/b)$$

Integrate out the original fields. The partition function is unchanged

$$Z = \int \mathcal{D}\phi e^{-S(\phi,g_0)} = \int \mathcal{D}\phi \mathcal{D}\Phi_b \prod \delta(\Phi_b - \Phi) e^{-S(\phi,g_0)} = \int \mathcal{D}\Phi_b e^{-S(\Phi_b,g')}$$

but the action changes

$$S(\phi,g_0) \rightarrow S(\phi,g')$$

The lattice correlation length $\xi \rightarrow \xi/b$

- the the RG flow either runs to the $\xi=0$ trivial FP or
- stays on the critical $\xi=\infty$ surface where it can run to a FP

The topology of the action space

W-K, Ch 12

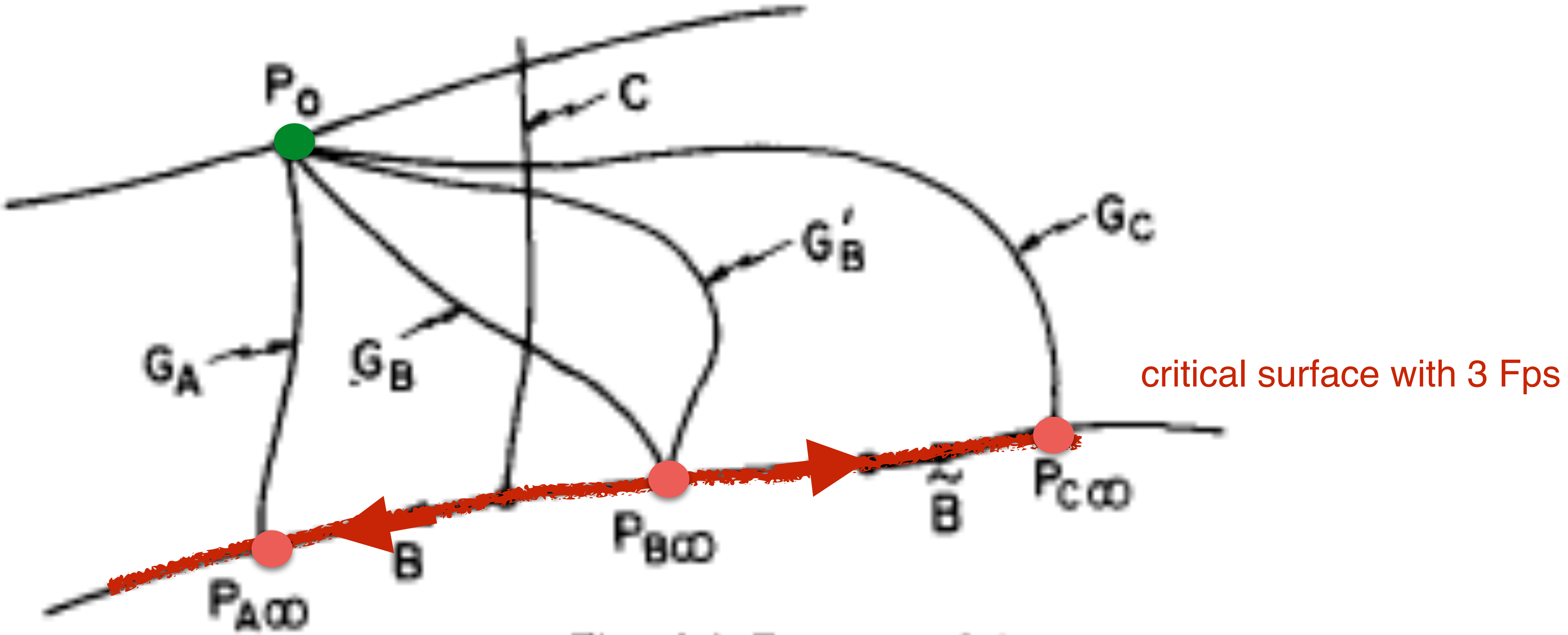


Fig. 12.8. Topology of the renormalization group in a region having three fixed points $P_{A\infty}$, $P_{B\infty}$ and $P_{C\infty}$ on the critical surface. $P_{A\infty}$ and $P_{C\infty}$ are once-unstable. $P_{B\infty}$ is twice-unstable.

The topology of the action space

W-K, Ch 12

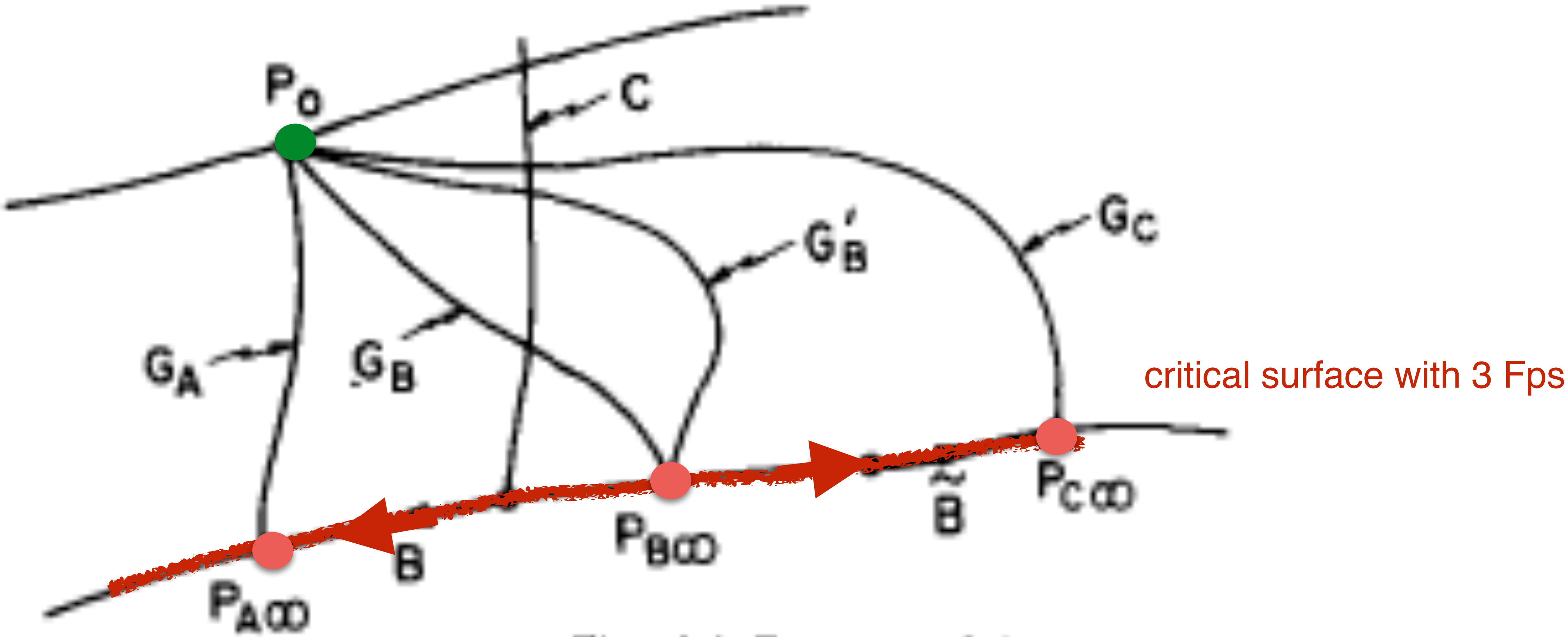


Fig. 12.8. Topology of the renormalization group in a region having three fixed points $P_{A\infty}$, $P_{B\infty}$ and $P_{C\infty}$ on the critical surface. $P_{A\infty}$ and $P_{C\infty}$ are once-unstable. $P_{B\infty}$ is twice-unstable.

Are there FPs of the RG transformation?
How many?
What is universal?

The exponent η

A linear RG transformation does not have a FP unless the fields are normalized with the exponent η (wave function renormalization)

position space:

$$\Phi(x) = \sum_s f(s)\phi(x+s)$$

momentum space

$$\Phi(p) = f(p)\phi(p) \sim e^{-p^2/t}\phi(p)$$

Correlation function on the critical surface (ξ = ∞)

$$\phi(x)\phi(x+s) \propto s^{-(d-2+\eta)} \longrightarrow \Phi_b(\frac{x}{b})\Phi_b(\frac{x}{b}+\frac{s}{b}) \propto s^{-(d-2+\eta)} b^{(d-2+\eta}$$

Satisfied only if

$$\Phi_b(x_b) = \Phi(bx) b^{-\eta/2}$$

$$\Phi_b(p_b) = \Phi(p/b) b^{-d_\phi+\eta/2}$$

Without the η terms there is no FP!

[1] Keep the normalization the blocked fields the same as the original

Linear RG:

position space:

$$\Phi_b(x_b) = \Phi(bx) b^{-\eta/2}$$

momentum space

$$\Phi_b(p_b) = \Phi(p/b) b^{-d_\phi + \eta/2}$$

Linear RG:

position space:

$$\Phi_b(x_b) = \Phi(bx)b^{-\eta/2}$$

momentum space

$$\Phi_b(p_b) = \Phi(p/b)b^{-d_\phi+\eta/2}$$

Goal: find the FP action and study its properties analytically

-W-K discusses this in Ch. 11
-Polchinski's exact RG is similar (but no rescaling)[1]
-Functional or exact RG studies

1 The sources in the functional integral are restricted to p/b, providing effective rescaling

Linear RG:

position space:

$$\Phi_b(x_b) = \Phi(bx)b^{-\eta/2}$$

Goal: verify the existence of the FP and study critical exponent numerically

- R. Swendsen's MCRG

PhysRevLett.42.859,1979

momentum space

$$\Phi_b(p_b) = \Phi(p/b)b^{-d_\phi+\eta/2}$$

Goal: find the FP action and study its properties analytically

- W-K discusses this in Ch. 11
- Polchinski's exact RG is similar (but no rescaling)[1]
- Functional or exact RG studies

1 The sources in the functional integral are restricted to p/b, providing effective rescaling

Monte Carlo Renormalization Group

PhysRevLett.42.859,1979

Action — Configuration ensemble

$$\begin{tikzcd}
S(\phi, g_0) \arrow[r, "\text{MC}"] \arrow[d, "\text{RG}"] & \{\phi\} \arrow[d, "\text{block}"] \\
S(\phi, g') \arrow[r, "\text{MC}"] & \{\Phi_b\}
\end{tikzcd}$$

RG transformed expectation values can be calculated without explicit knowledge of the blocked action

$$\langle O(\phi) \rangle_{S'} = \langle O(\Phi_b(\phi)) \rangle_S$$

Monte Carlo Renormalization Group

Swendsen calculated the linearized RG matrix from simple expectation values

PhysRevLett.42.859,1979

$$\frac{\partial\langle S_\gamma{}^{(n)}\rangle}{\partial K_\beta{}^{(n-1)}}=\sum_\alpha \frac{\partial K_\alpha{}^{(n)}}{\partial K_\beta{}^{(n-1)}}\frac{\partial\langle S_\gamma{}^{(n)}\rangle}{\partial K_\alpha{}^{(n)}},$$

$$\frac{\partial\langle S_\gamma{}^{(n)}\rangle}{\partial K_\beta{}^{(n-1)}}=\langle S_\gamma{}^{(n)}S_\beta{}^{(n-1)}\rangle-\langle S_\gamma{}^{(n)}\rangle\langle S_\beta{}^{(n-1)}\rangle$$

$$\frac{\partial\langle S_\gamma{}^{(n)}\rangle}{\partial K_\alpha{}^{(n)}}=\langle S_\gamma{}^{(n)}S_\alpha{}^{(n)}\rangle-\langle S_\gamma{}^{(n)}\rangle\langle S_\alpha{}^{(n)}\rangle$$

K_α : coupling
S_α : operator

The method did not become popular:
- systematical errors are hard to control
- at every RG step L → L / 2 : dof drop rapidly
  only a few discrete RG steps are possible

Monte Carlo Renormalization Group

Swendsen calculated the linearized RG matrix from simple expectation values

PhysRevLett.42.859,1979

$$\frac{\partial\langle S_\gamma^{(n)}\rangle}{\partial K_\beta^{(n-1)}} = \sum_\alpha \frac{\partial K_\alpha^{(n)}}{\partial K_\beta^{(n-1)}} \frac{\partial\langle S_\gamma^{(n)}\rangle}{\partial K_\alpha^{(n)}},$$

$$\frac{\partial\langle S_\gamma^{(n)}\rangle}{\partial K_\beta^{(n-1)}} = \langle S_\gamma^{(n)} S_\beta^{(n-1)}\rangle - \langle S_\gamma^{(n)}\rangle\langle S_\beta^{(n-1)}\rangle$$

K_α : coupling
S_α : operator

$$\frac{\partial\langle S_\gamma^{(n)}\rangle}{\partial K_\alpha^{(n)}} = \langle S_\gamma^{(n)} S_\alpha^{(n)}\rangle - \langle S_\gamma^{(n)}\rangle\langle S_\alpha^{(n)}\rangle$$

The method did not become popular:

- systematical errors are hard to control
- at every RG step $L \rightarrow L/2$: dof drop rapidly
  only a few discrete RG steps are possible

Gradient flow : a new approach

Luscher Comm.Math Phys 293, 899 (2010)

GF is a continuous smoothing that removes short distance fluctuations

For scalar model :

$$\partial_t \phi_t = -(\partial_\phi S(\phi_t))\phi_t\,, \quad \phi_{t=0} = \phi$$

From now on 't' stands for gradient flow time;
space-time is denoted by x_0

Gradient flow : a new approach

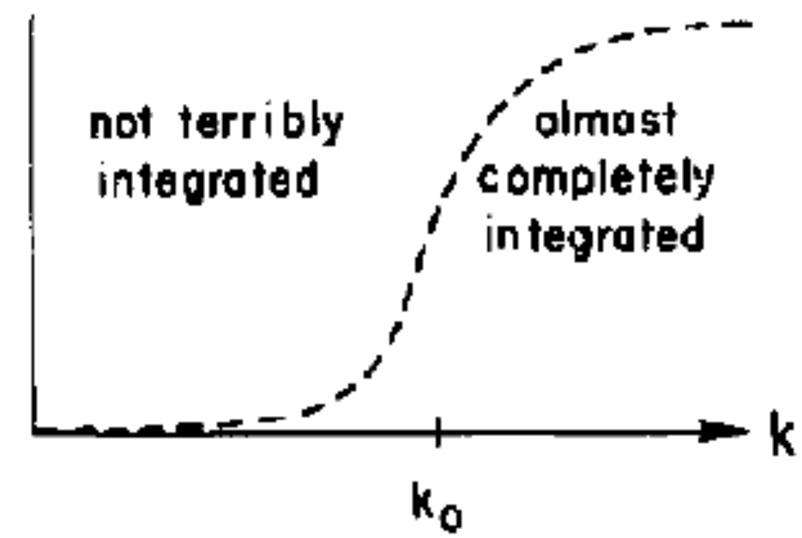
Luscher Comm.Math Phys 293, 899 (2010)

GF is a continuous smoothing that removes short distance fluctuations
For scalar model :

$$\partial_t \phi_t = -(\partial_\phi S(\phi_t))\phi_t\,, \quad \phi_{t=0} = \phi$$

free flow : $\Phi(p) = \mathcal{N}\, e^{-p^2/t}\phi(p)$

That looks just like Ch. 11 of Wilson-Kogut!



From now on ’t’ stands for gradient flow time;
space-time is denoted by x_0

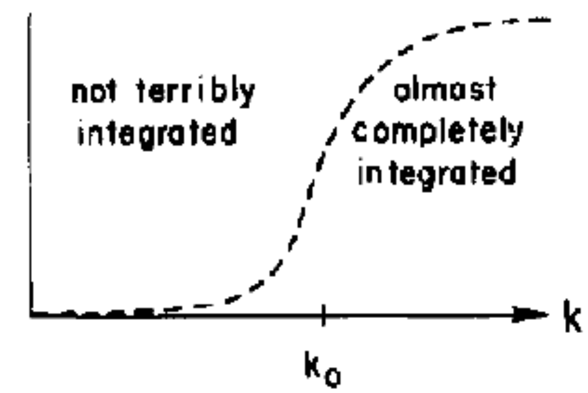
Gradient flow : a new approach

Luscher Comm.Math Phys 293, 899 (2010)

GF is a continuous smoothing that removes short distance fluctuations
For scalar model :

$$\partial_t \phi_t = -(\partial_\phi S(\phi_t))\phi_t \,, \quad \phi_{t=0} = \phi$$

free flow : $\Phi(p) = \mathcal{N}\, e^{-p^2/t} \phi(p)$



That looks just like Ch. 11 of Wilson-Kogut!

Luscher JHEP 04 123 (2013)

Gauge flow: $\partial_t V_t = -(\partial S_W[V_t])V_t \,, \quad V_0 = U$
Fermions evolve on the gauge background: $\partial_t \chi_t = \Delta[V_t]\chi_t \,, \quad \chi_0 = \psi$

The flow action does not have to match the model

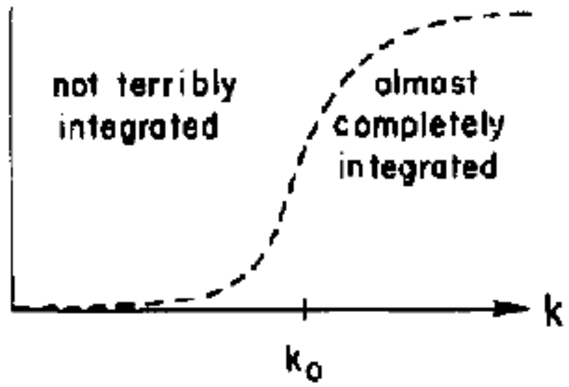
Gradient flow : a new approach

Luscher Comm.Math Phys 293, 899 (2010)

GF is a continuous smoothing that removes short distance fluctuations
For scalar model :

$$\partial_t \phi_t = -(\partial_\phi S(\phi_t))\phi_t\,, \quad \phi_{t=0} = \phi$$

free flow : $\Phi(p) = \mathcal{N}\, e^{-p^2/t}\phi(p)$



That looks just like Ch. 11 of Wilson-Kogut!

Luscher JHEP 04 123 (2013)

Gauge flow: $\partial_t V_t = -(\partial S_W[V_t])V_t\,, \quad V_0 = U$
Fermions evolve on the gauge background: $\partial_t \chi_t = \Delta[V_t]\chi_t\,, \quad \chi_0 = \psi$

The flow action does not have to match the model

Is GF an RG transformation?

GF vs RG

GF misses two important attributes of an RG transformation:

- there is no rescaling $\Lambda_{\text{cut}} \rightarrow \Lambda_{\text{cut}}/b$
- linear transformation does not include the term $b^{-\eta/2}$

Both issues can be solved

Polchinski Nucl. Phys. B231, 1984

Polchinski's formulation in momentum space never had rescaling

GF vs RG

GF misses two important attributes of an RG transformation:

- there is no rescaling $\Lambda_{\text{cut}} \to \Lambda_{\text{cut}}/b$
- linear transformation does not include the term $b^{-\eta/2}$

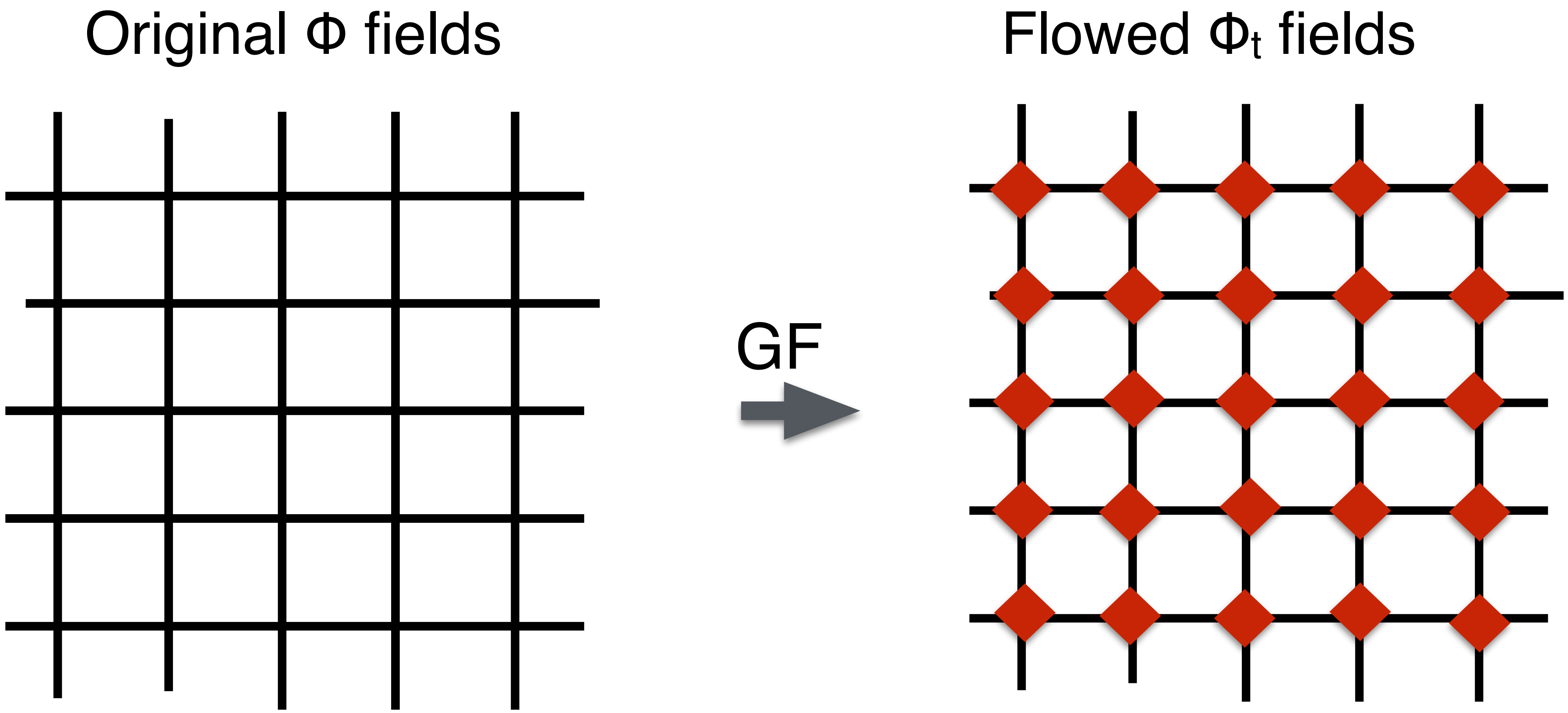
Both issues can be solved

Polchinski Nucl. Phys. B231, 1984

Polchinski's formulation in momentum space never had rescaling

Could GF be used as an RG transformation?

GF vs RG



- Rescale with *b : x → x/b*

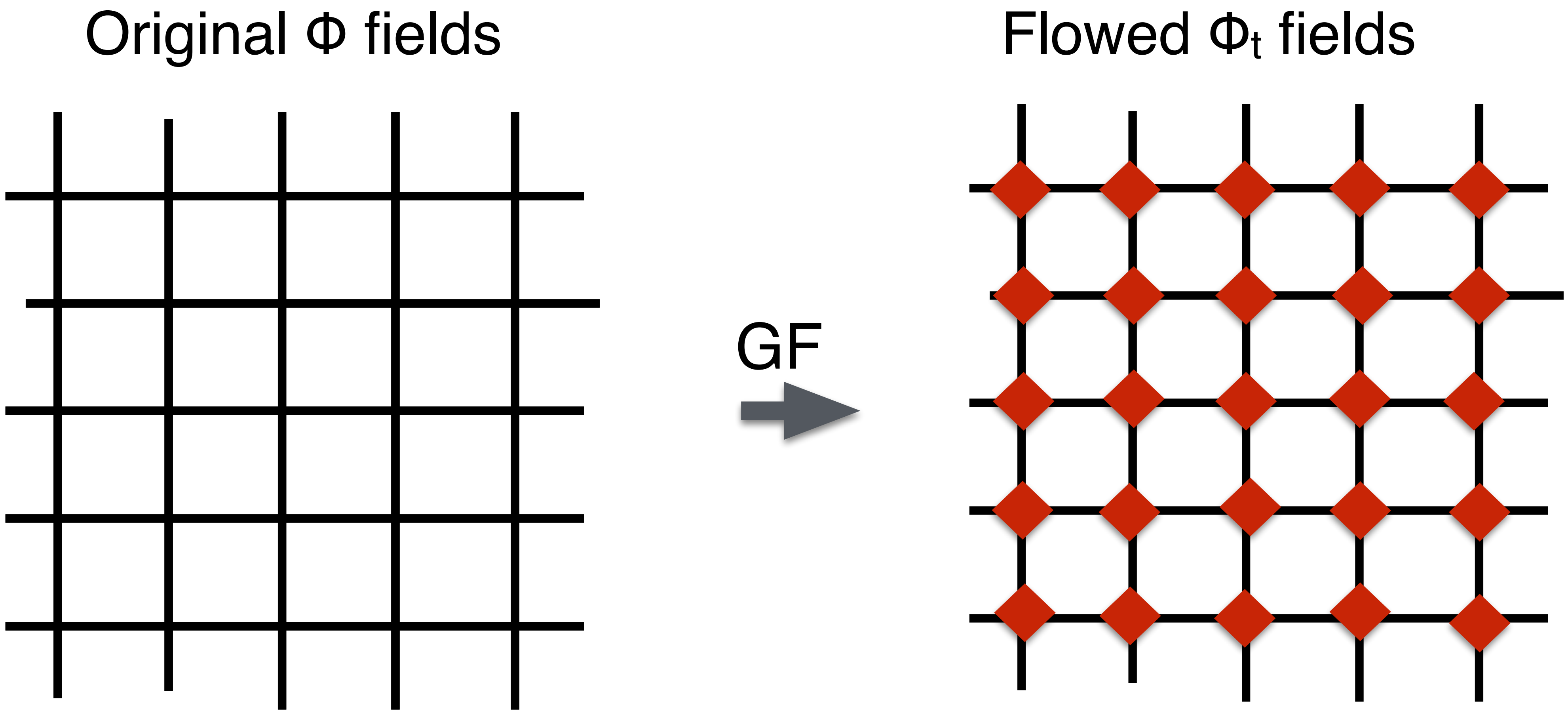
we have not even introduced *b !*

b is independent of flow time but natural $b^2 \sim t$

- Correct normalization: $\Phi_b = b^{-\eta/2}\phi_t$

Do we need to 'decimate' or drop the unwanted the degrees of freedom?

GF vs RG



Long distance correlators do not need decimation:

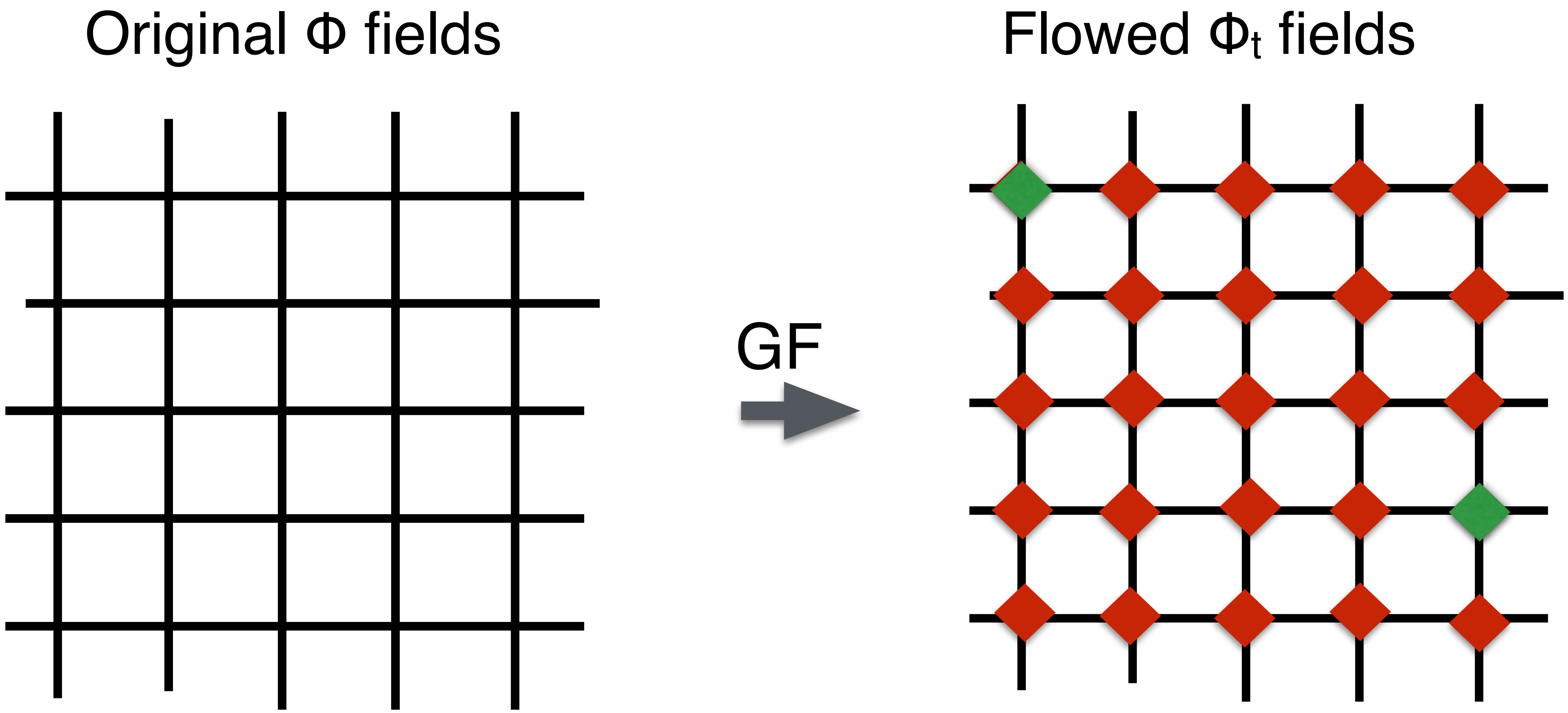
$$\langle O(0,\Phi_b)O(x/b,\Phi_b)\rangle = \langle O(0,\phi_t)O(x,\phi_t)\rangle b^{-n_O\eta}$$

At the level of expectation values this is a proper RG transformation

Gain: t and b are continuous and there is no loss of dof

Natural choice is $b^2 \sim t$; different choices will lead to different Fps but same IR physics - except pathological cases

GF vs RG



Long distance correlators do not need decimation:

$$\langle O(0,\Phi_b)O(x/b,\Phi_b)\rangle = \langle O(0,\phi_t)O(x,\phi_t)\rangle b^{-n_O\eta}$$

At the level of expectation values this is a proper RG transformation

Gain: t and b are continuous and there is no loss of dof

Natural choice is $b^2 \sim t$; different choices will lead to different Fps but same IR physics - except pathological cases

Why long distance correlator?

A correlator of an operator decays exponentially at large distances

$$C(x_0;g_i;a) = \langle O(0)O(x_0)\rangle_{g_i} \to Ae^{-M_O x_0}, \quad x_0 \to \infty$$

Under an RG transformation of scale change b:

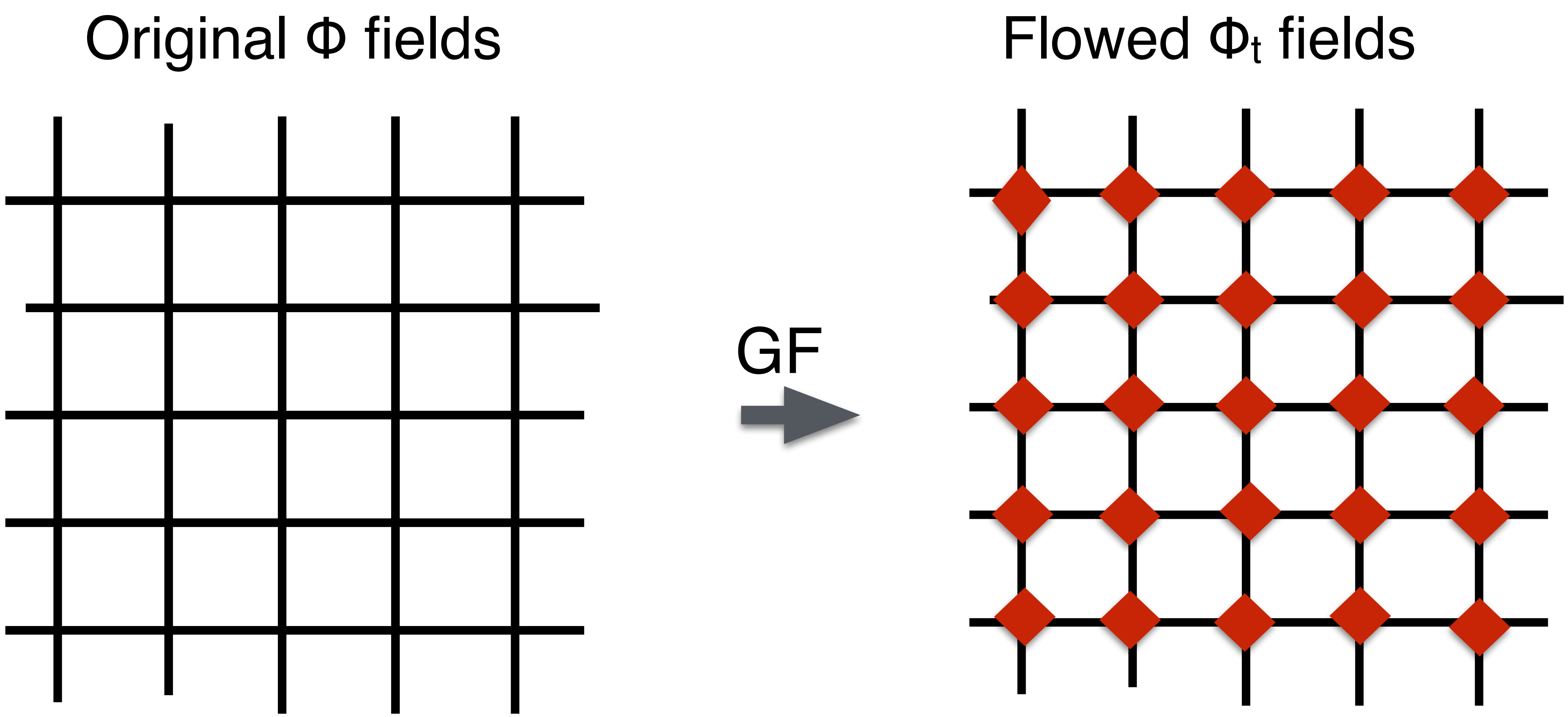
$$C_b(x_0/b;g_i') = b^{2\Delta_O} C(x_0;g_i) \to A' e^{-(bM_O)(x_0/b)}$$

where $\Delta_O = d_O + \gamma_O$

Only the amplitudes change!

The form of the correlator is valid for any b - integer or not -
when the excited states have dies out

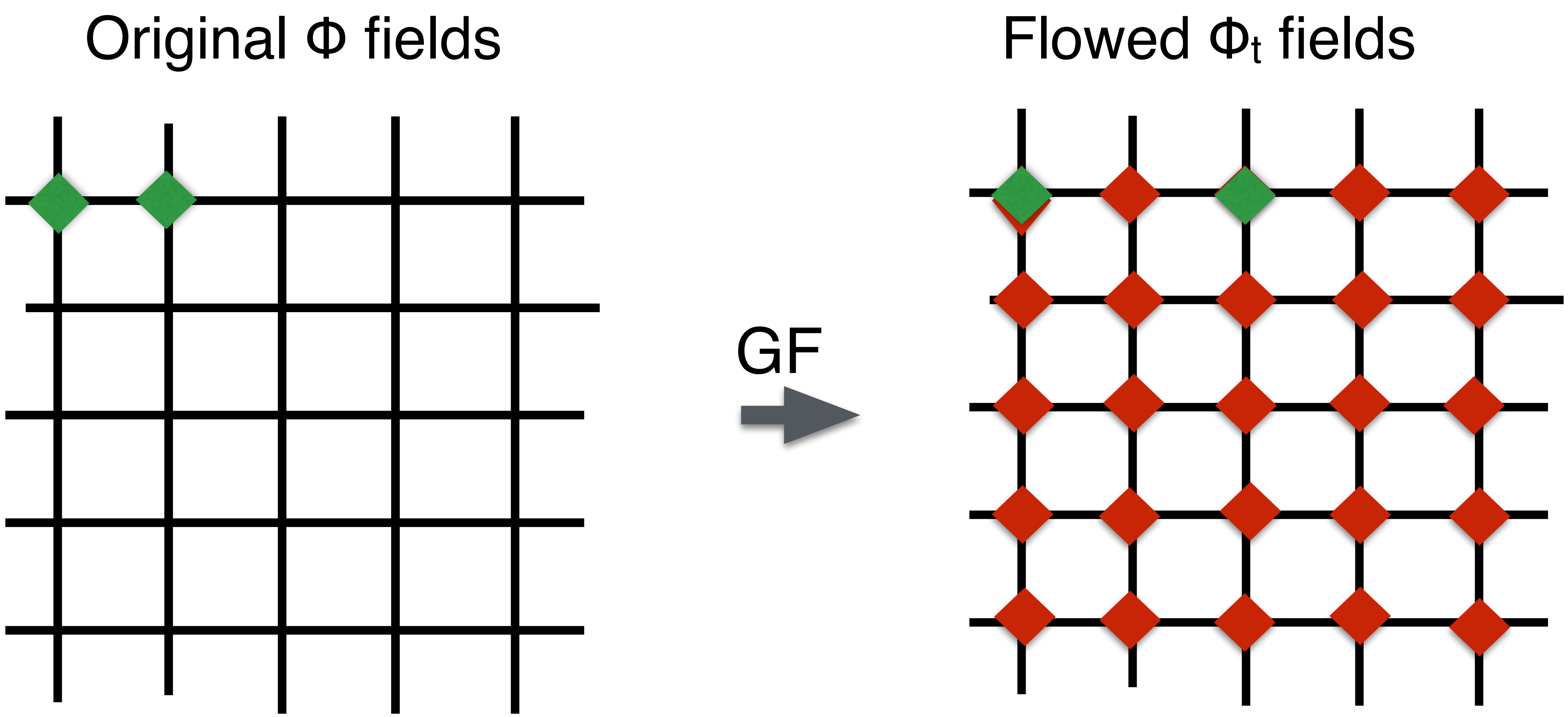
GF vs RG



At the level of expectation values GF is a proper RG transformation for long distance quantities

It won't work for Swendsen's method, that requires short distance operators

GF vs RG



At the level of expectation values GF is a proper RG transformation for long distance quantities

It won't work for Swendsen's method, that requires short distance operators

Application - I : anomalous dimensions

A correlator of an operator decays exponentially at large distances

$$C(x_0;g_i) = \langle O(0)O(x_0)\rangle_{g_i} \to A e^{-M_O x_0}, \quad x_0 \to \infty$$

Under an RG transformation with scale change b:

$$C_b(x_0/b;g_i') = b^{2\Delta_O} C(x_0;g_i) \to A' e^{-(bM_O)(x_0/b)}$$

The ratio is independent of x_0 and predicts the scaling dimension

$$R^O = \frac{C_b(x_0/b;g_i';a)}{C(x_0;g_i;a)} = \frac{A'}{A} = b^{2\Delta_O}$$

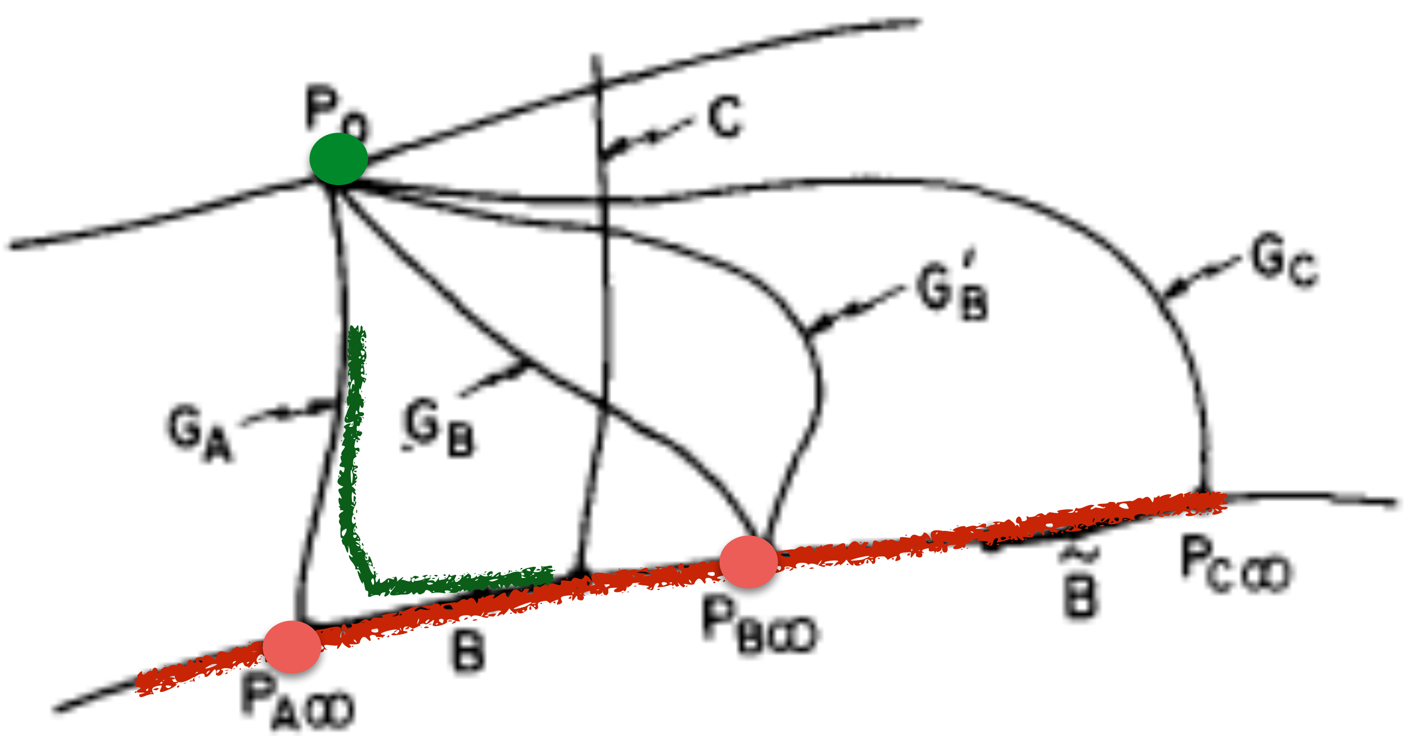
and predicts the scaling dimension $\Delta_O = d_O + \gamma_O$

How about relevant operators? They better be zero, otherwise the RG flow runs away from the FP — anomalous dimensions become non-universal

Application - I : anomalous dimensions

$$R^{O} = \frac{C_b(x_0/b; g_i'; a)}{C(x_0; g_i; a)} = \frac{A'}{A} = b^{2\Delta_O} \qquad \Delta_O = d_O + \gamma_O$$

How about relevant operators?
They better be zero, otherwise the RG flow runs away from the FP
— anomalous dimensions become non-universal



Application - I : anomalous dimensions

Flowed correlators contain the η exponent

$$R^{O}{}_{t} = \frac{C_t(x_0)}{C(x_0)} = b^{\eta n_O} \frac{C_b(x_0/b; g_i'; a)}{C(x_0; g_i; a)} = b^{2\Delta_O + \eta n_O} = t^{\Delta_O + n_O \eta/2}$$

The η exponent is determined by requiring there is a FP:

If an operator has no anomalous dimension, its correlator ratio is

$$R_t = b^{d_O + n_O \eta/2}$$

This could be

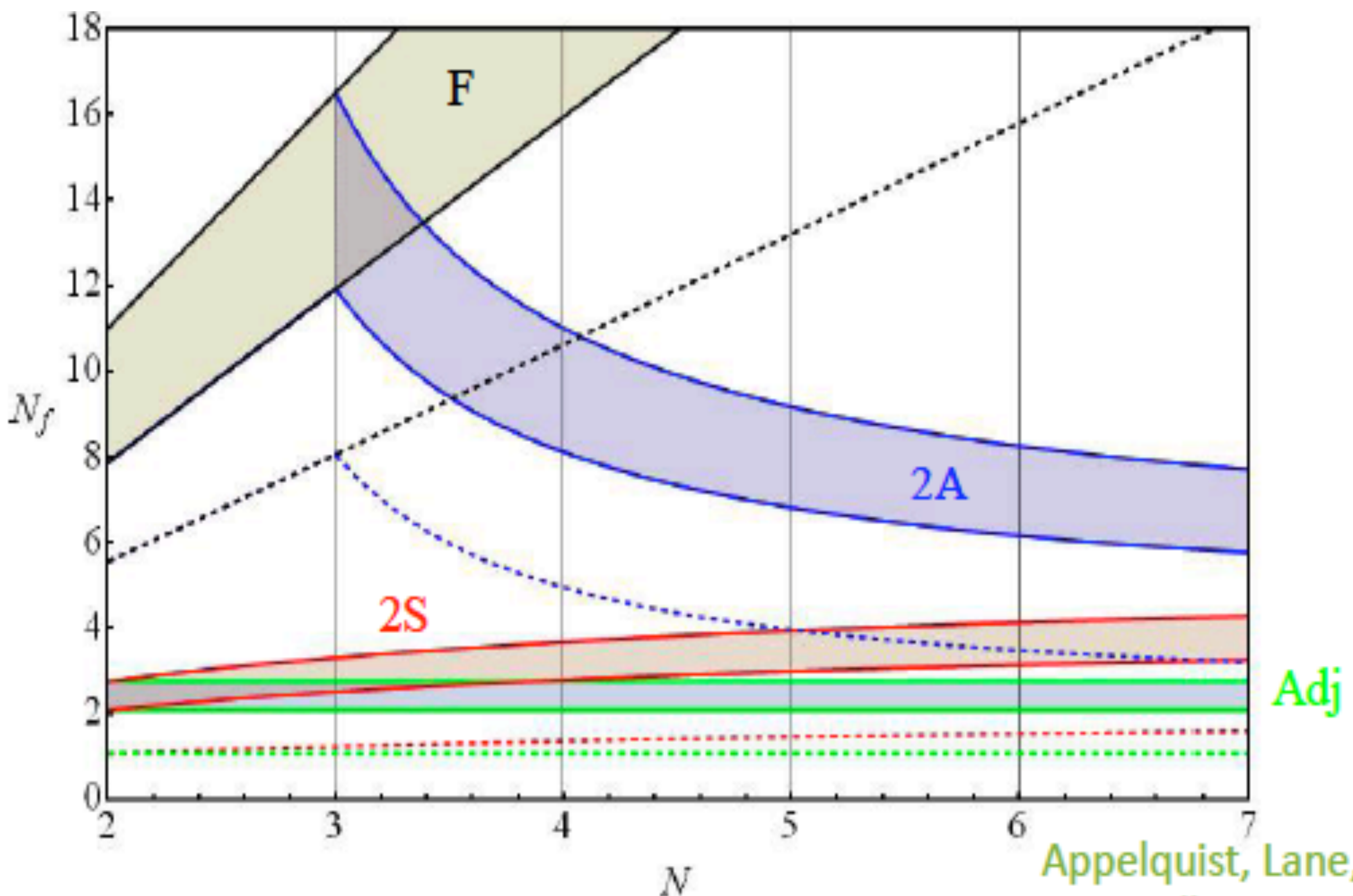
- non - composite operator : $O = \phi(x)$
- symmetry-protected operator: axial charge $A_0 = \bar{\psi}\gamma_5\gamma_0\psi$ or conserved vector

Numerical example:

SU(3) gauge with 12 fundamental fermions

The model is controversial but there is growing evidence that it is conformal

Dietrich, Sannino



Appelquist, Lane, Mahanta, Wijewardhana, Cohen, Georgi, Yamawaki, Shrock, Dietrich, Sannino, Tuominen

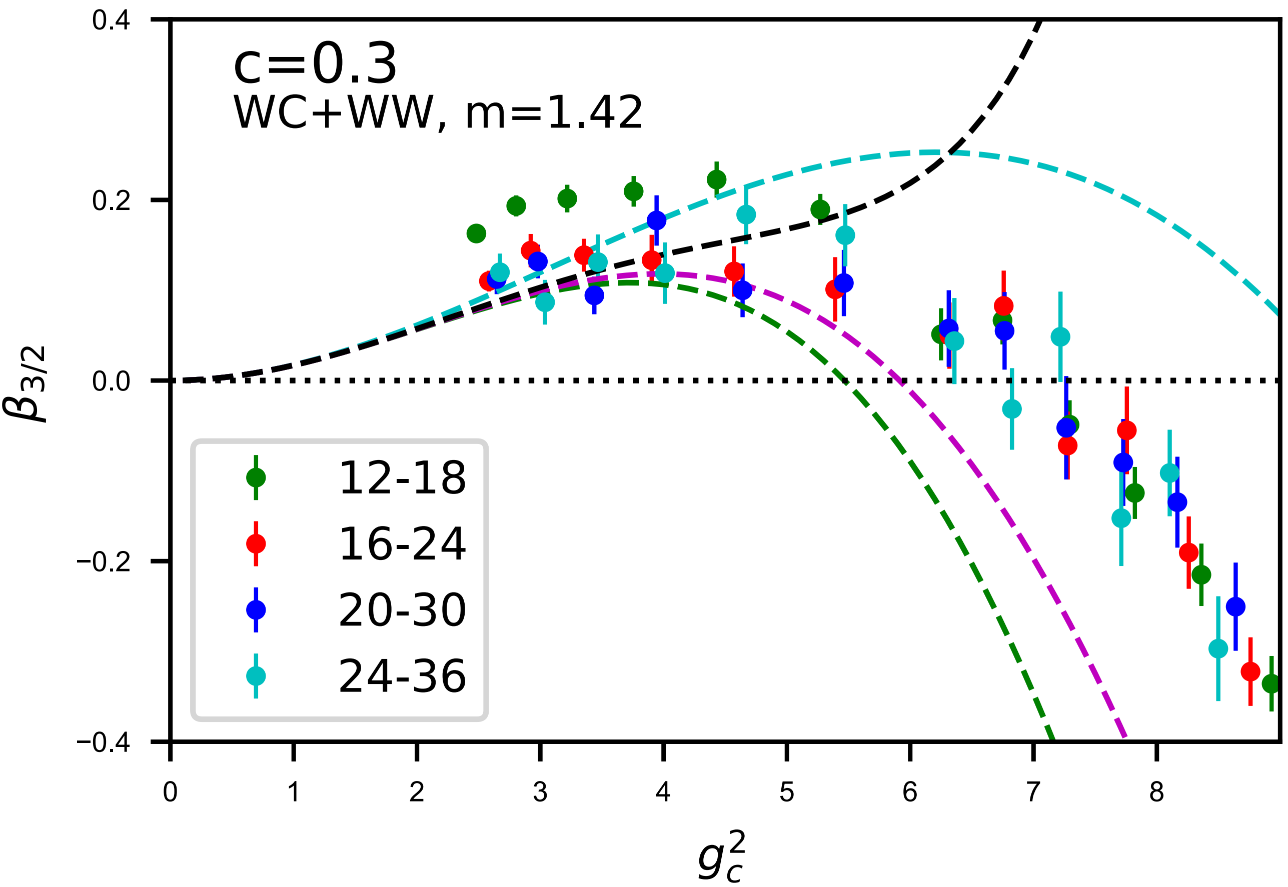
Schwinger-Dyson argumenst suggest the conformal window opens around N_f=12
Numerical results put the conformal boundary lower

Cheng PhysRevD.90.014509
Cheng JHEP07(2013)061
Lombardo JHEP12(2014)183
AH,D.Schaich JHEP02(2018)132
Fodor PhysRevD.94.091501

SU(3) gauge with N_f=12 fundamental flavors

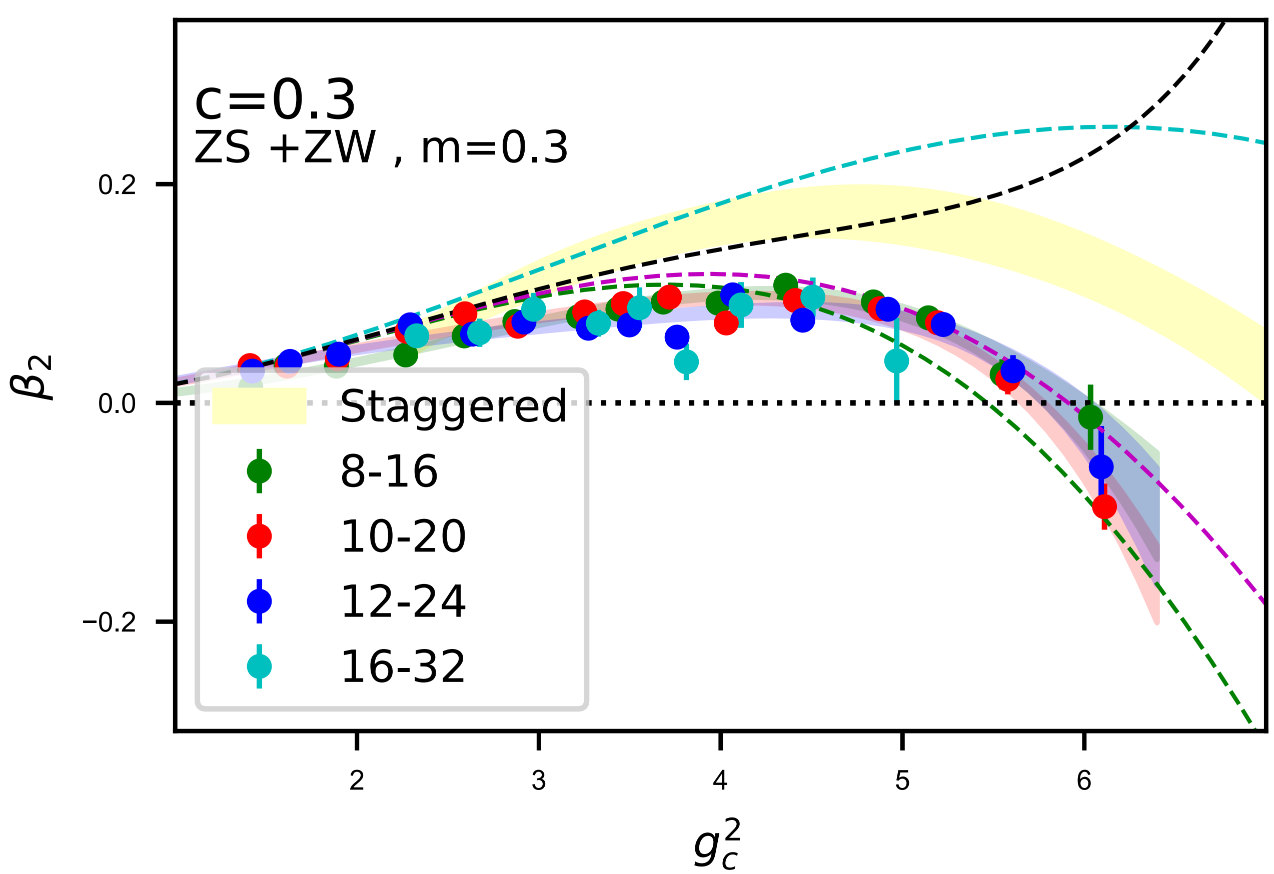
RG β function (step scaling function)

Staggered



AH,D.Schaich JHEP02(2018)132

Domain wall



AH, C. Rebbi, O. Witzel

SU(3) gauge with N_f=12 fundamental flavors

- mass anomalous dimension γ_m =0.23-0.25 from perturbation theory, FSS numerical studies, Dirac eigenmodes
- the gauge coupling walks very slow - substantial scaling violation effects are expected
- baryon and tensor anomalous dimension would be interesting but no predictions exists

N_f=12 numerical study

We use staggered fermions

A. Carosso, AH, E. Neil

- configurations with m=0.0025 (practically m=0) exist on $24^3\times 48$ and $32^3\times 64$ volumes from FSS; gauge coupling β=4.0,5.0,5.5,5.75, 6.0

Cheng PhysRevD.90.014509

- chiral symmetry protects the axial charge: $A_0=\bar{\psi}\gamma_0\gamma_5\psi$
  it has no anomalous dimension → its correlator predicts η

- numerically much easier to calculate a correlator where only the source is flowed (Luscher JHEP 04 123 (2013))

- the **ratio of ratios** predict γ_O

$$\mathcal{R}_t^O(x_0)=\frac{\langle O(0)O_t(x_0)\rangle}{\langle O(0)O(x_0)\rangle}\left(\frac{\langle A(0)A(x_0)\rangle}{\langle A(0)A_t(x_0)\rangle}\right)^{n_O/n_A}=t^{\gamma_O}$$

N_f=12 numerical study

Unfortunately we use staggered fermions

$$S_{st} = \frac{1}{2}\sum_{n,\mu}\bar{\chi}(n)(-1)^{\Sigma_{\nu<\mu} n_\nu}[U_{n,\mu}\chi(n+\mu) - U_{n,-\mu}\chi(n-\mu)] + m\sum_n \bar{\chi}(n)\chi(n)$$

chiral symmetry is different from continuum (might even be in different universality class)

fermion operators couple to their parity partner: $\bar{\psi}\Gamma\psi$ and $\bar{\psi}\Gamma\gamma_0\gamma_5\psi$ are in the same channel

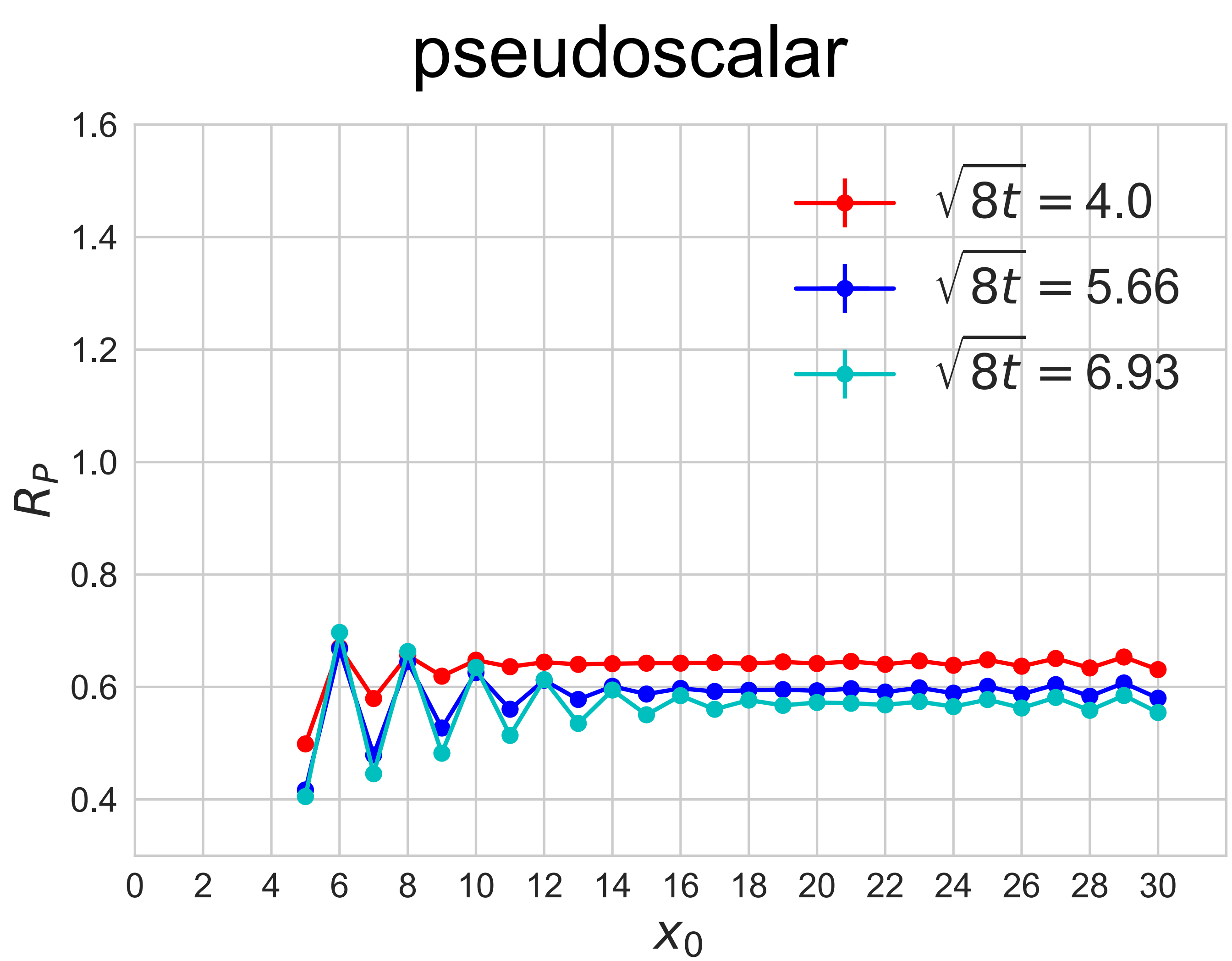
- these states have the same anomalous dimension and (finite volume) mass
- the ratios oscillate (much harder to determine anomalous dimensions)

axial charge still protected, $A_0 = \bar{\psi}\gamma_0\gamma_5\psi$ has no anomalous dim.

Ratio of ratios - pseudo scalar

$$\mathcal{R}_t^{O}(x_0) = \frac{\langle O(0) O_t(x_0)\rangle}{\langle O(0) O(x_0)\rangle}\left(\frac{\langle A(0) A(x_0)\rangle}{\langle A(0) A_t(x_0)\rangle}\right)^{n_O/n_A} = t^{\gamma_O}$$

has no x_0 dependence asymptotically



Oscillation due to excited states and operator overlap

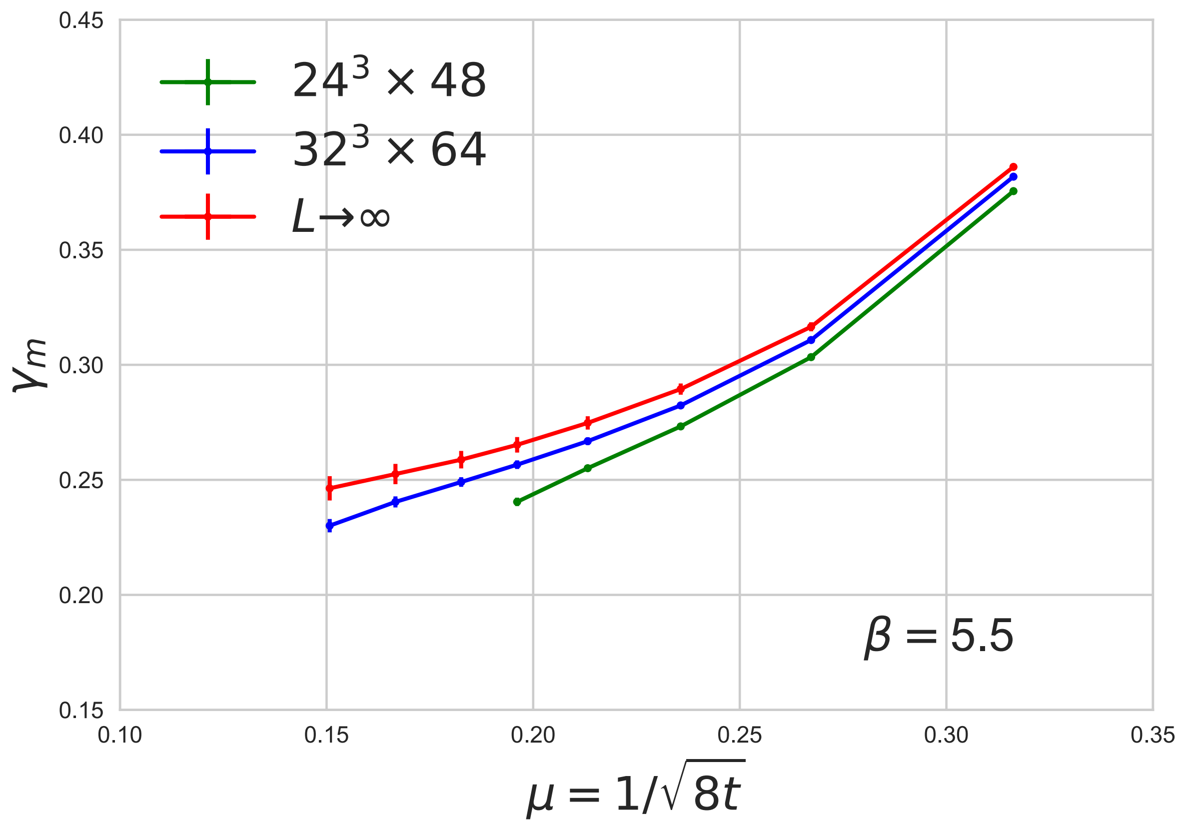
—> limits max t

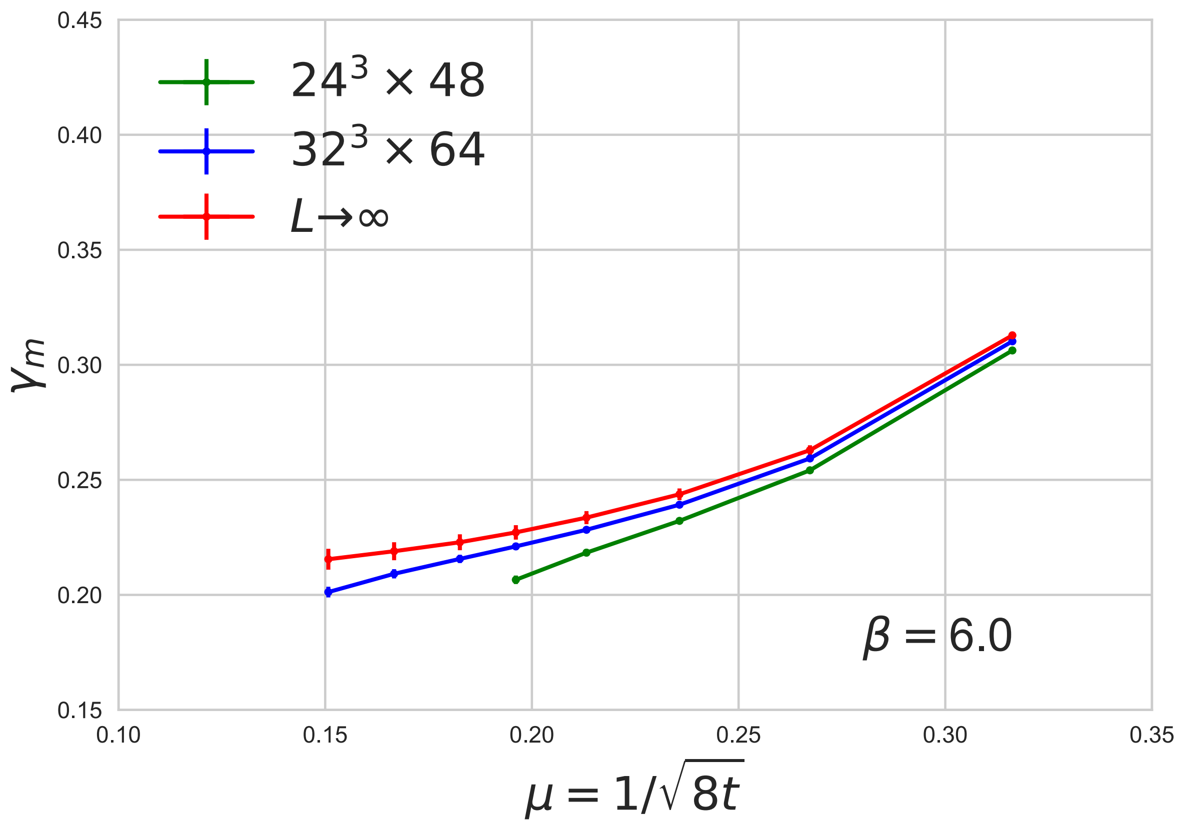
flow time dependence: anomalous dimension

Volume corrections - PS

Finite volume corrections are significant;
- approximate correction based on RG arguments (dependence is t/L^2)

$$R(g,s^2t,s^2L) = R(g,s^2t,sL) + s^{-\gamma_0}\Big(R(g,t,sL) - R(g,t,L)\Big) + \text{h.o.}$$





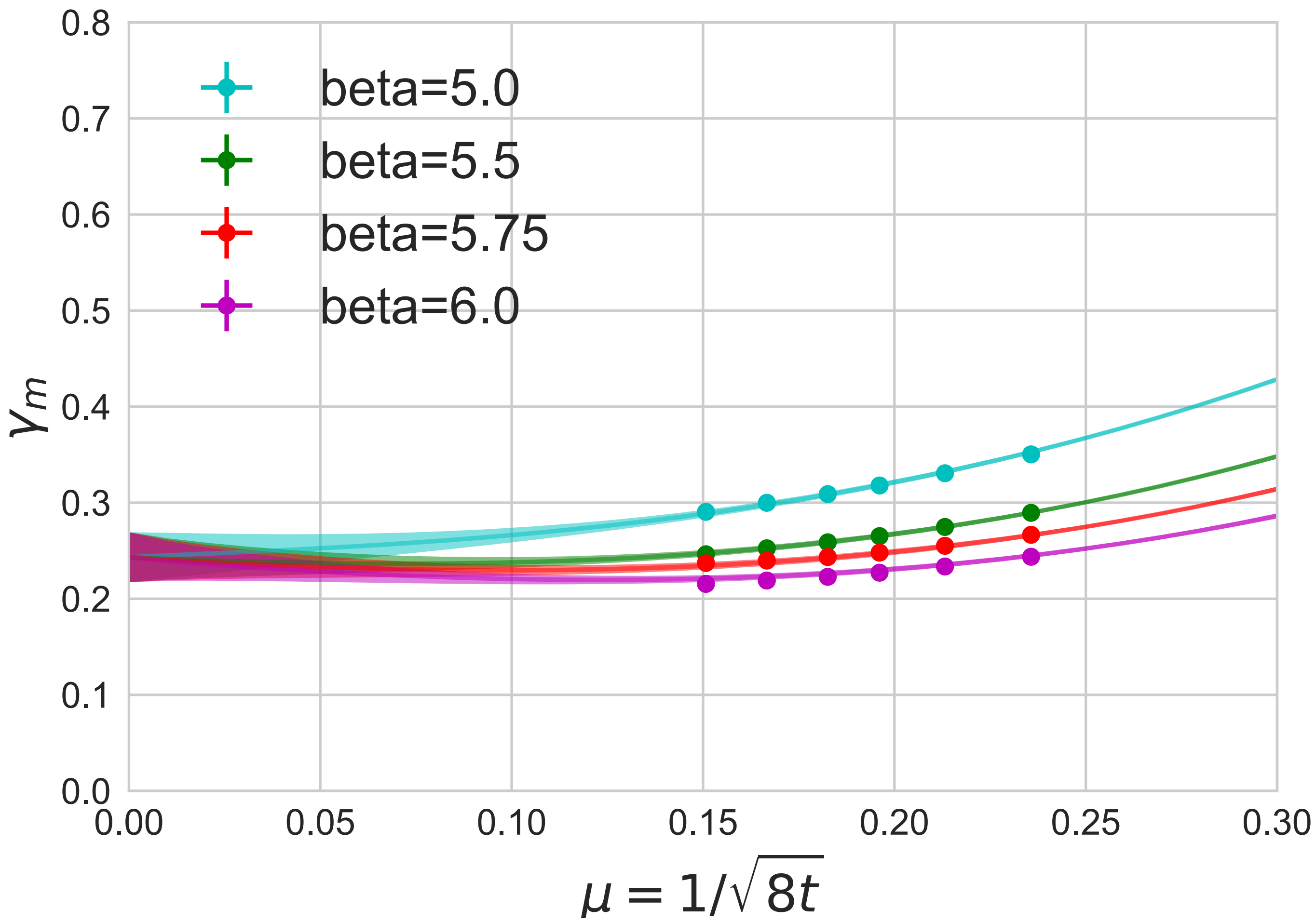
Flow time dependence indicates slowly running gauge coupling

Flow time and bare coupling corrections - PS

The flow has to reach the IRFP to predict the universal exponent
At finite flow time (away from the IRFP) there are corrections.
Extrapolation:

$$\gamma_0 + \sum_{\beta}(a_\beta \mu^{\omega_1} + b_\beta \mu^{\omega_2})$$

2 exponents (leading and nlo)
amplitudes depend on β



As t→∞

$$\gamma_m = 0.24(3)$$

error: systematic + statistical
consistent with other methods

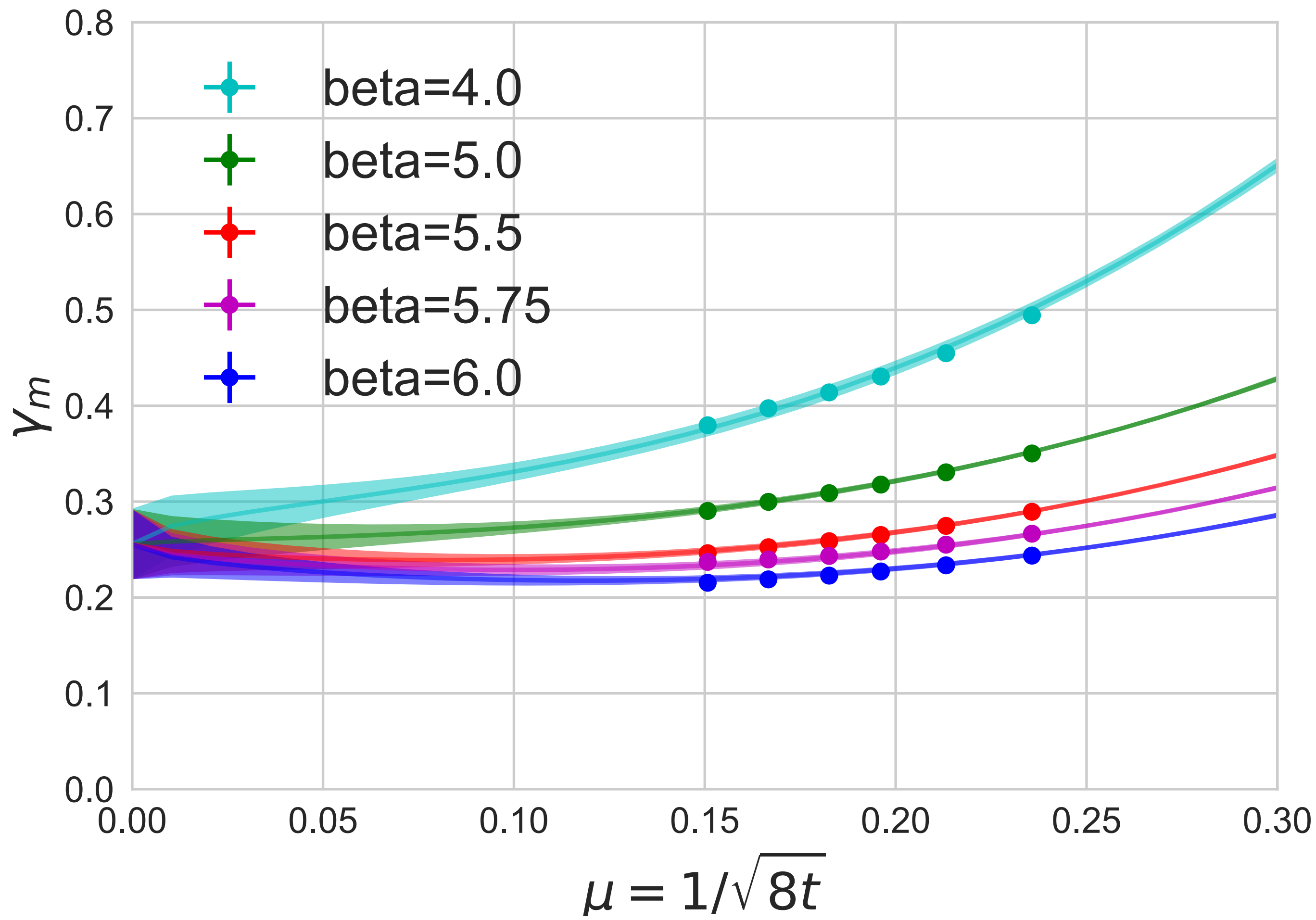
Systematical errors - PS

To check systematical erros:

- Add/remove first/last flow time,
- add/remove β,
- change fit form:

  γ_{m} is stable

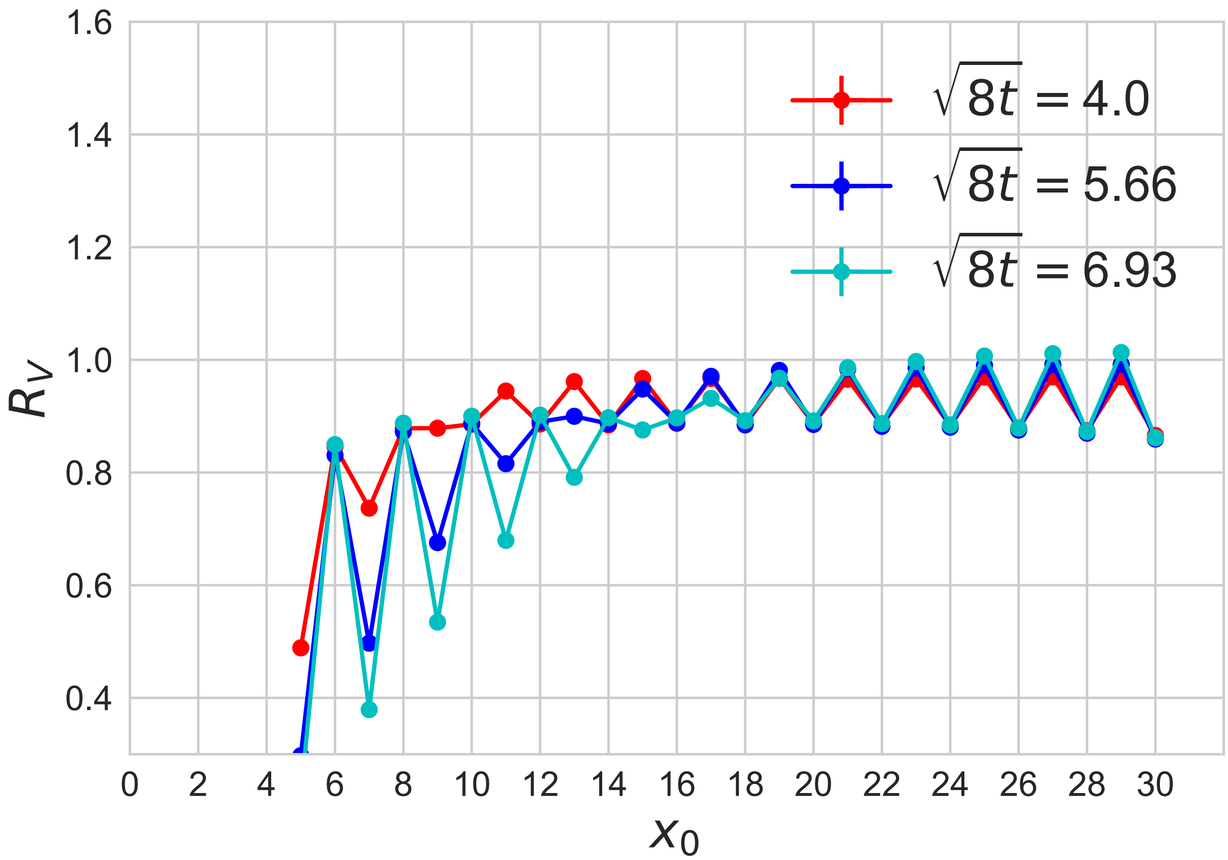
As t→∞ $\gamma_m = 0.24(3)$



strong gauge coupling dependence - shows slow running

Vector channel

vector - tensor



Oscillation pronounced but little flow time dependence

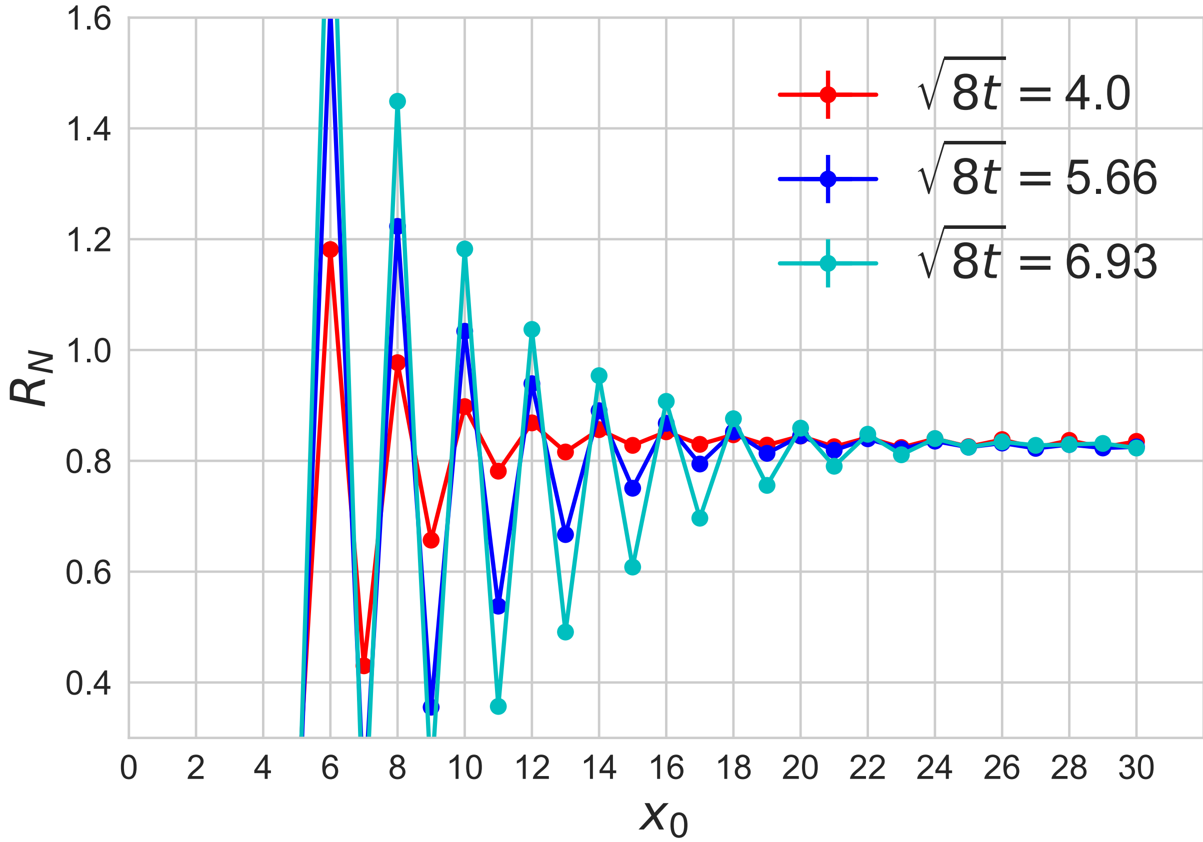
Fit as

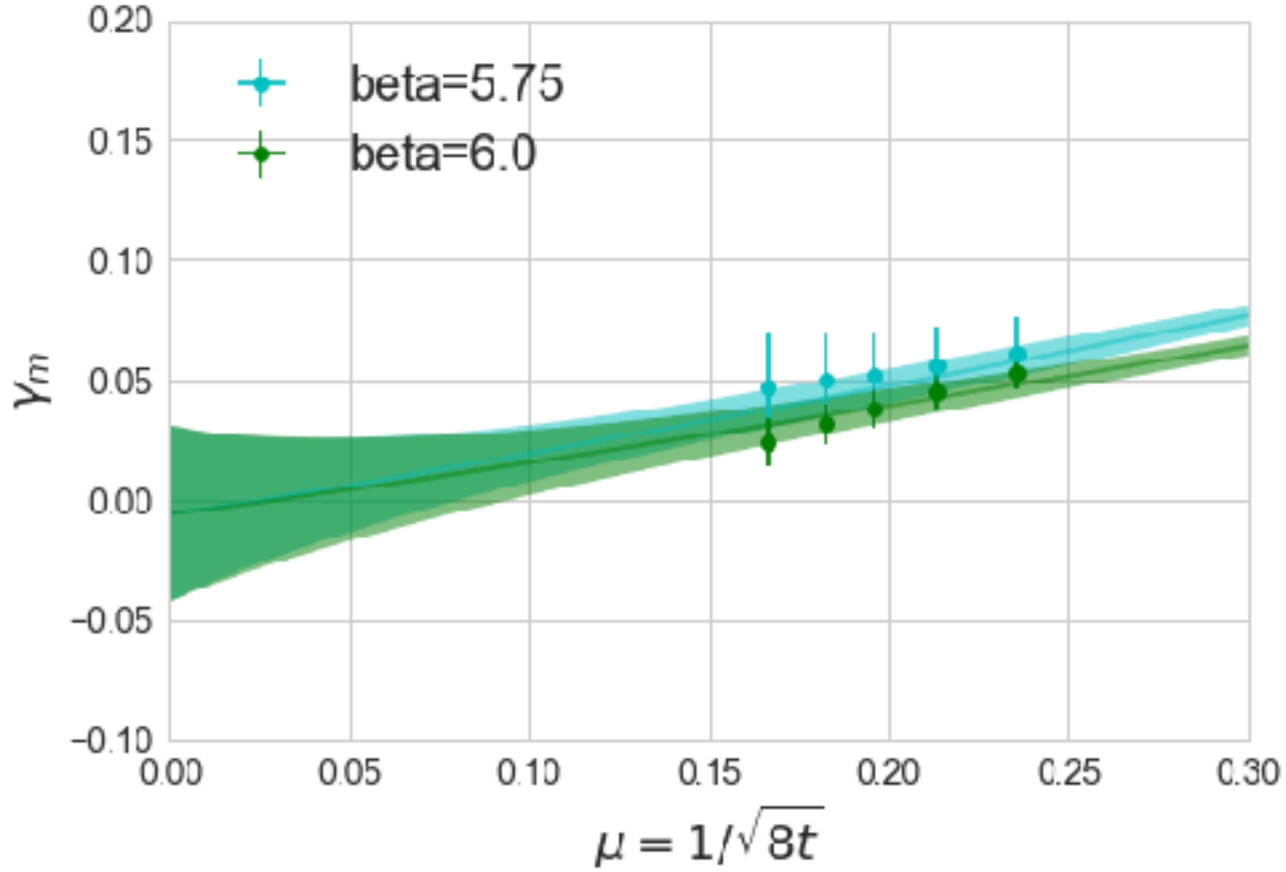
$$\frac{A_t e^{-m1x_0} + B_t e - m2x_0}{Ae^{-m1x_0} + Be - m2x_0} = \frac{A_t}{A}\frac{1 + B_t / A_t e^{-\Delta m x_0}}{1 + B / Ae^{-\Delta m x_0}}$$

2 anomalous dimensions, from A_t/A and B_t/B both vanish within errors

Nucleon channel

nucleon - Lambda





Oscillation suppressed,
little flow time dependence,
limited x_0 range

Anomalous dimension is small
$\gamma_N = 0.006(6)$
(perturbative: $\gamma_N = 0.09$)

Will not work for partial composite models ….

Things to do:

- Finite volume effects deserve more attention (i.e. larger volumes, longer time direction)
- Staggered fermions are a poor choice here: DW or Wilson are more promising
- N_f=10 flavors could have larger anomalous dimensions (DW or Wilson needed!)
- Anyone with existing conformal configurations: try it! (but need massless or nearly massless configs)
- SYM ?
- N_f=2 or 4 : reproduce perturbative mass anomalous dimension, try nucleon

- 3D scalar model: might not compete with FSS but can predict anomalous dimension of irrelevant operators

Summary

Gradient flow can be used as an RG transformation if

- linear fields are properly normalized
- rescaling of dimensional variables are taken care of in defining expectation values
- GF can be flowed with any (reasonable) action
- $b^2 \sim t$ to connect flow time and RG scale - continuous!

Powerful approach to measure anomalous dimensions;

Other applications where physics can be extracted from long distance correlators?