Numerical methods for solving the Einstein equation in higher dimensions and in AdS with Cauchy evolution

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Numerical approaches to holography, quantum gravity and cosmology

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THE ROYAL SOCIETY

Astronphysical Numerical GR in 4D

- Understanding the strong field regime of GR
- Theoretical waveform models are routinely used for LIGO detections
- Challenges:
  - Simulating theories beyond GR
  - Neutron stars

Astronphysical Numerical GR in 4D

- Two main approaches:

  1. Generalised harmonic coordinates [Pretorius]

  2. BSSN formulation [Baumgarte&Shapiro;Shibata&Nakamura; Baker et al.; Campanelli et al.]

- Numerical methods

  - Pseudospectral/finite differences

  - Finite grid hierarchies/Adaptive mesh refinement (AMR)

Beyond Astronphysical Numerical GR

Beyond Astronphysical Numerical GR

- Higher dimensional asymptotically flat/Kaluza Klein spaces

→ understand fundamental aspects of gravity

- Asymptotically anti-de Sitter (AdS) spaces

→ holography

- Asymptotically de Sitter (dS) spaces

→ cosmology

Outline of the talk

- CCZ4 evolution
- Generalised harmonic evolution
- Conclusions and outlook

Disclaimer:

- The characteristic formulation has been very successful in (Poincare) AdS

See P. Chesler and L. Yaffe JHEP 1407 (2014) 086
[arXiv: 1309.1439]

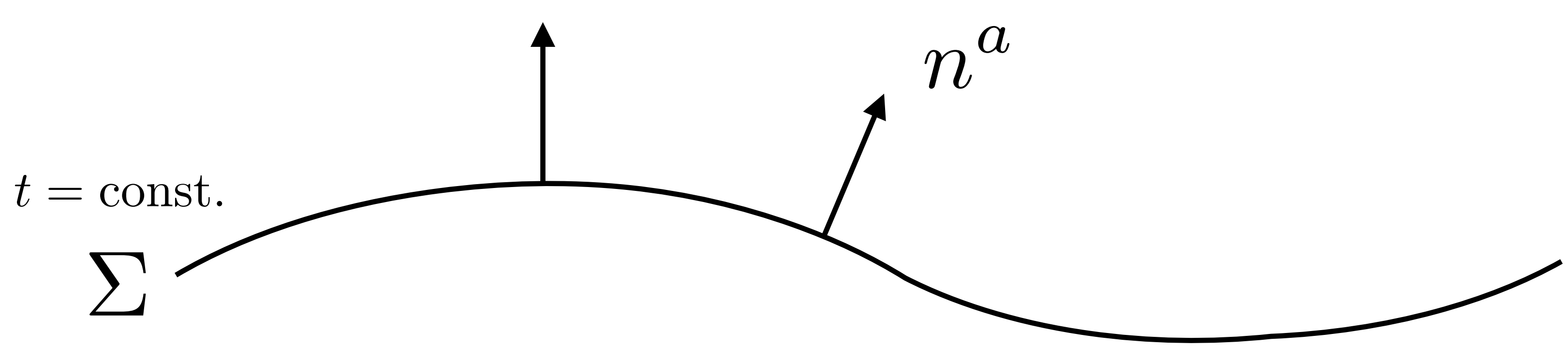
CCZ4 evolution

Time evolution in GR

- Use d+1 split:

$$ds^2 = -\alpha^2\, dt^2 + \gamma_{ij}(dx^i + \beta^i\, dt)(dx^j + \beta^j\, dt)$$

- Evolve γ_{ij} and $K_{ij} = -\frac{1}{2}\mathcal{L}_n \gamma_{ij}$
- Specify evolution equations for α and β^i



γ_{ij} : induced metric on Σ

$\mathbf{n} = \frac{1}{\alpha}\left(\partial_t - \beta^i \partial_i\right)$ normal vector to Σ

$K_{ij} = -\frac{1}{2}\mathcal{L}_{\mathbf{n}}\gamma_{ij}$: extrinsic curvature

- The ADM equations are only weakly hyperbolic!

BSSN

[Baumgarte, Shapiro, Shibata, Nakamura]

- Strongly hyperbolic variant of ADM
  - Separate out conformal factor and trace

$$(\gamma_{ij}, K_{ij}) \to (\chi, \tilde{\gamma}_{ij}, \tilde{A}_{ij}, K)$$

$$\chi = \gamma^{-1/d} \qquad \tilde{\gamma}_{ij} = \chi \gamma_{ij} \qquad \tilde{A}_{ij} = \chi K^{\mathrm{TF}}_{ij}$$

  - Evolve contracted connection separately

$$(\chi, \tilde{\gamma}_{ij}, \tilde{A}_{ij}, K, \tilde{\Gamma}^i) \qquad \tilde{\Gamma}^i = \tilde{\gamma}^{jk} \tilde{\Gamma}^i_{jk}$$

$$\begin{aligned} \partial_t \chi &= \\ \partial_t \tilde{\gamma}_{ij} &= \\ \partial_t \tilde{A}_{ij} &= \qquad \ldots \\ \partial_t K &= \\ \partial_t \tilde{\Gamma}^i &= \end{aligned}$$

CCZ4

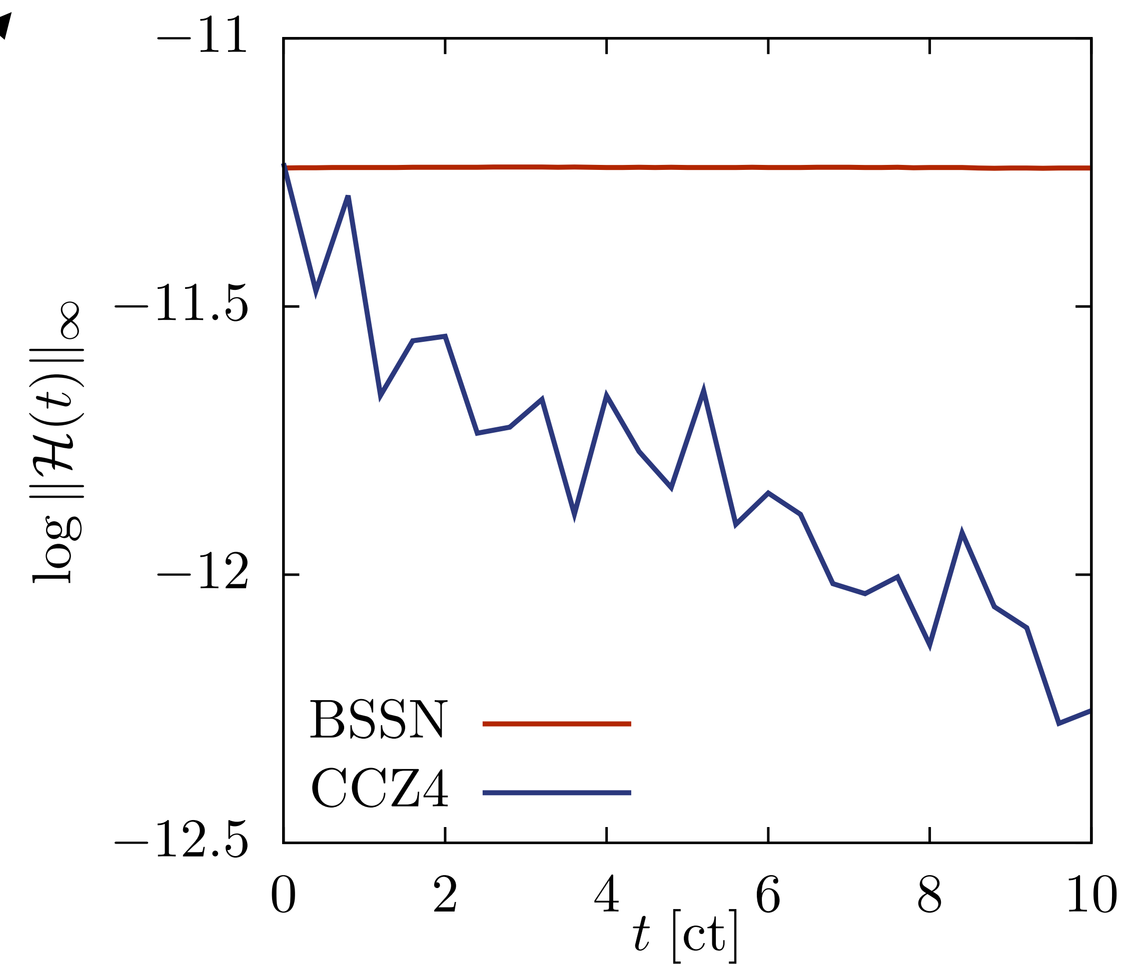
[Gundlach et al.; Hilditch et al.; Alic et al.]

- In BSSN the Hamiltonian constraint does NOT propagate
- The CCZ4 system gives to all the constraints a finite speed of propagation and damps them

$$R_{ab} + \nabla_a Z_b + \nabla_b Z_a - \kappa_1 \left[n_a Z_b + n_b Z_a - (1 + \kappa_2)\, g_{ab}\, n^c Z_c \right] = 0$$

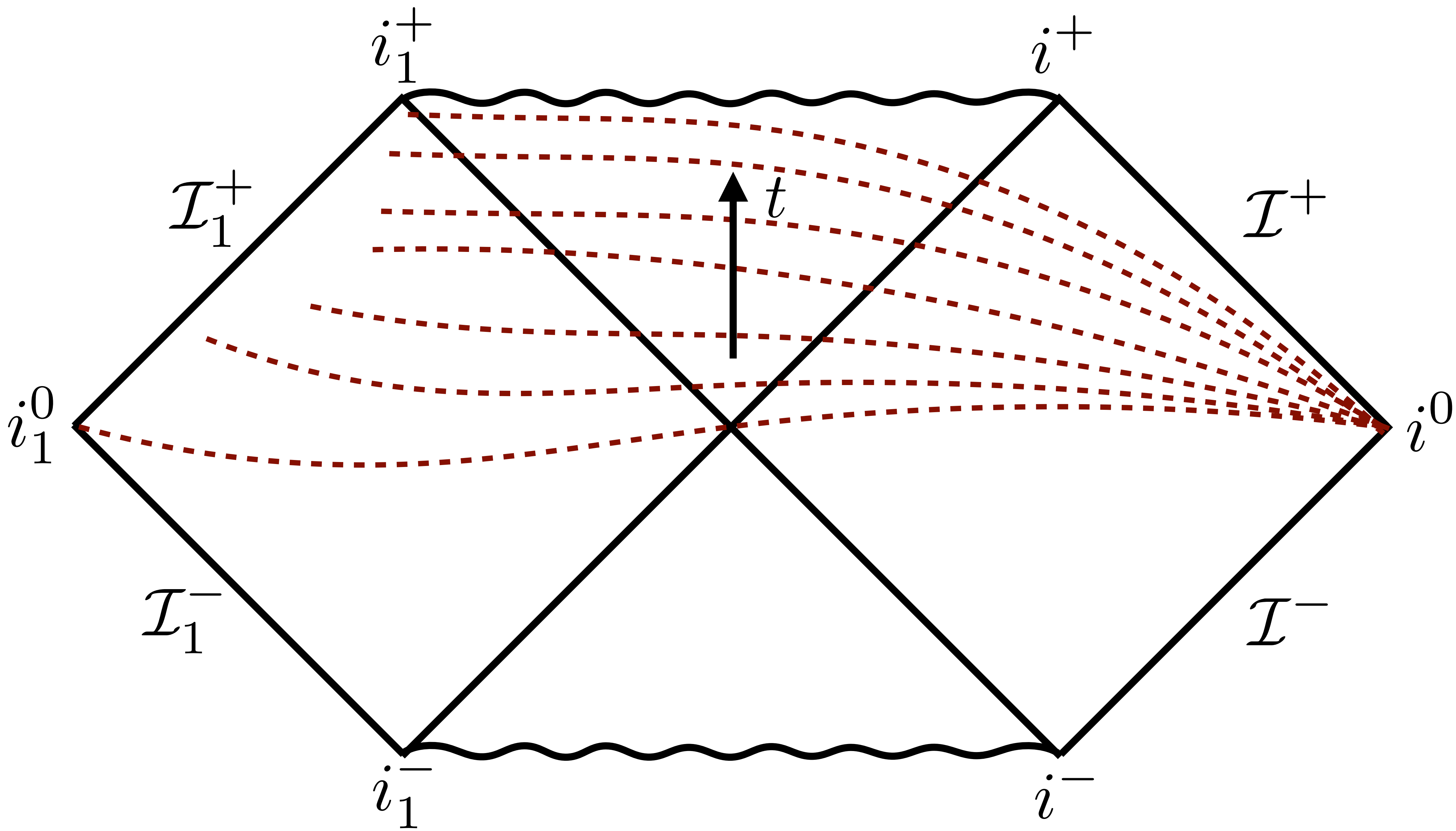
constraints propagate

constraints are exponentially damped



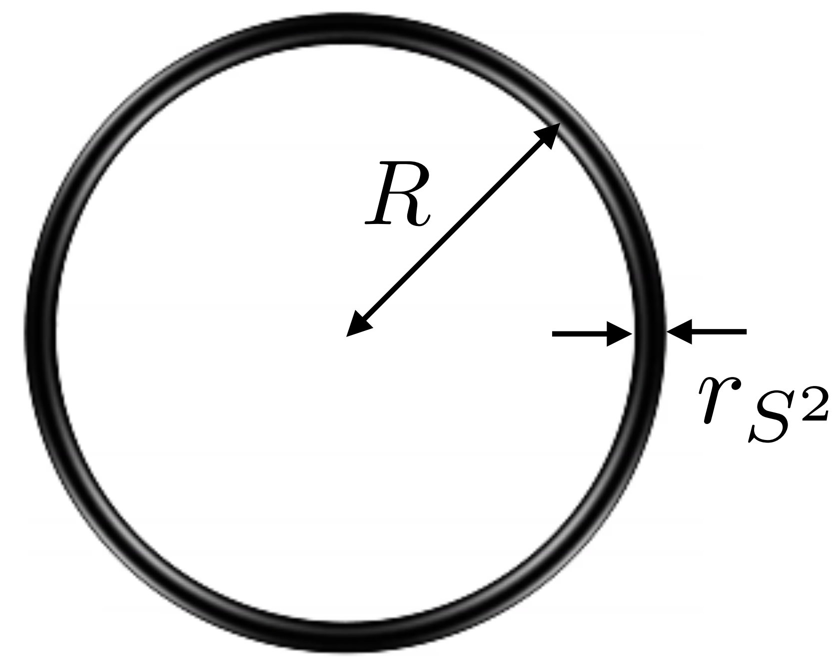
Puncture gauge

- Absorb some coordinate singularities in $\chi = \gamma^{-1/d}$
- Avoid physical singularities



Why simulations the astrophysical setting are harder?

- Separation of scales



- Extended, dynamic singularity
- Far from conformally flat

- No star-shaped AH

- Very expensive

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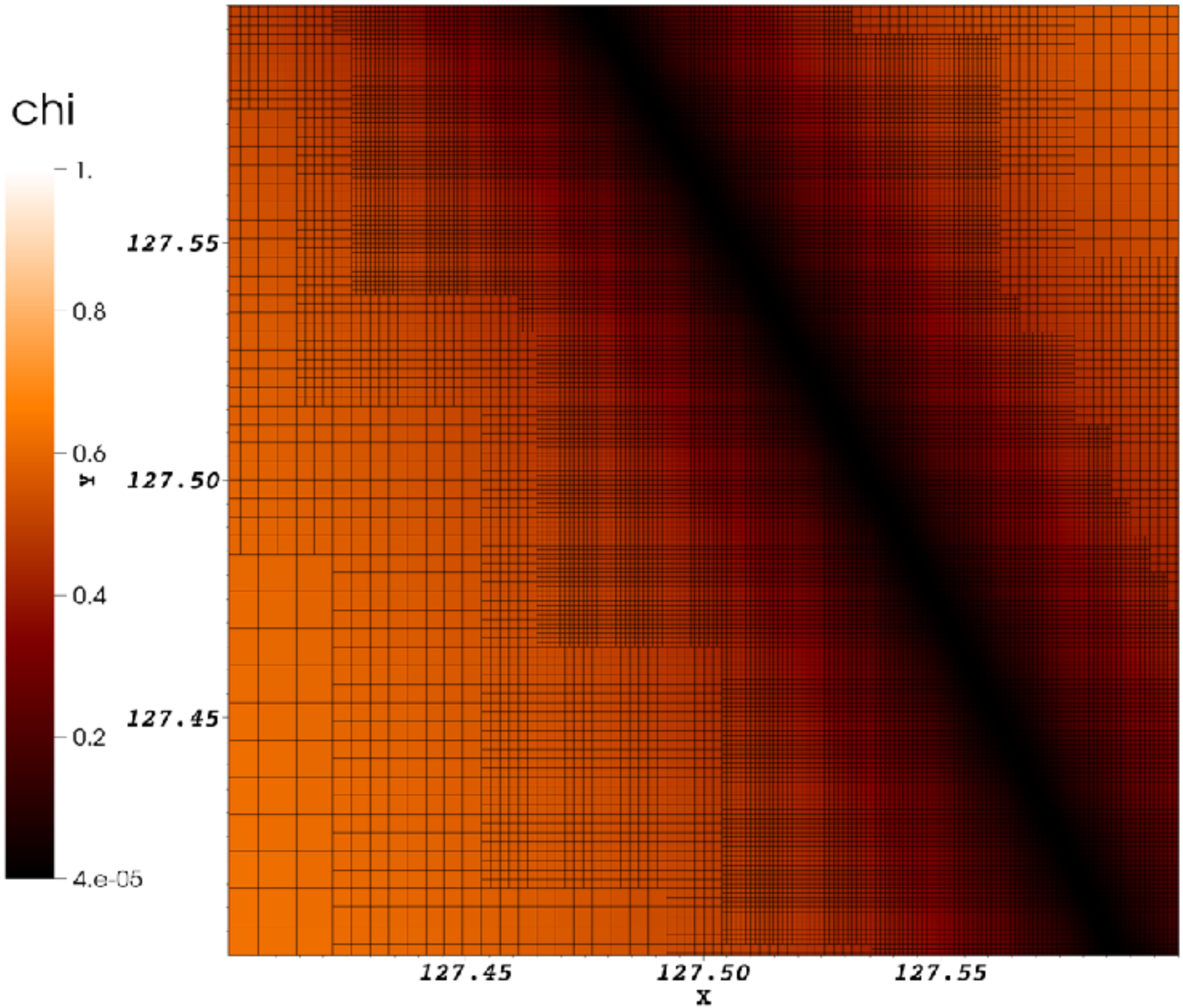
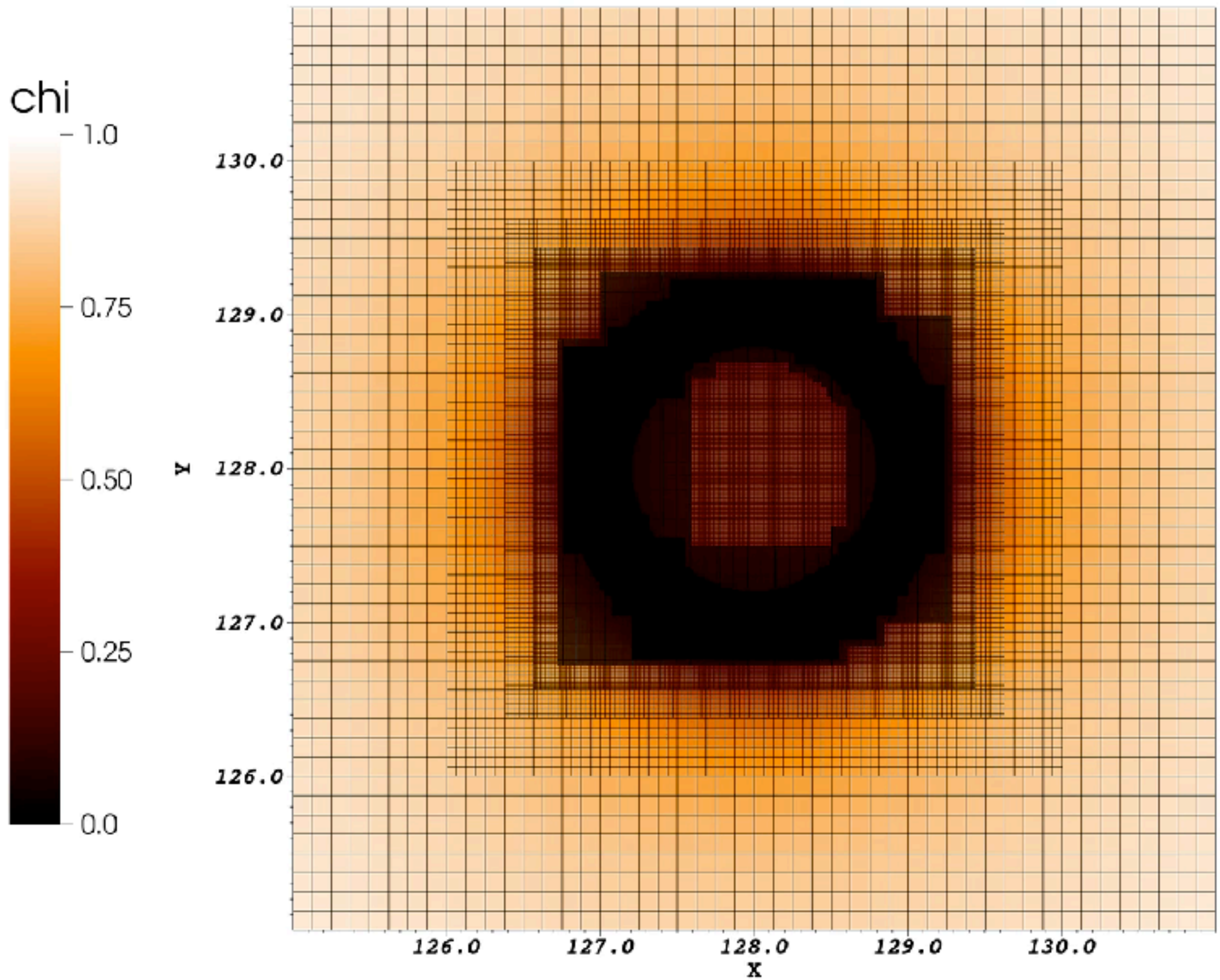
Adaptive mesh refinement



Adaptive mesh refinement



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➡ **New gauge, Shock capturing**

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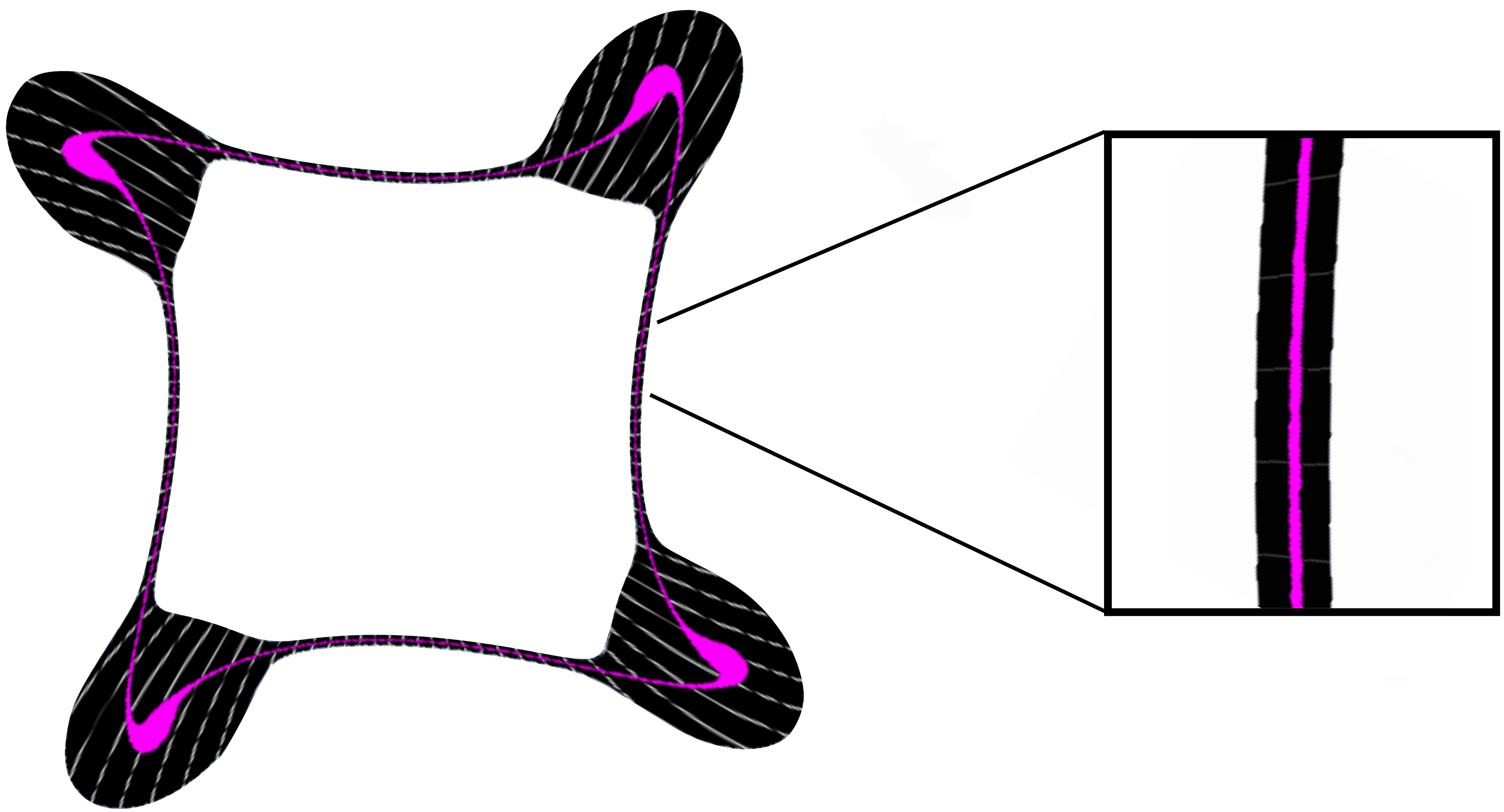
Shock capturing

- Features cannot be resolved
- Automatically triggered artificial viscosity



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- Separation of scales ➡ **New gauge, Adaptive mesh refinement**
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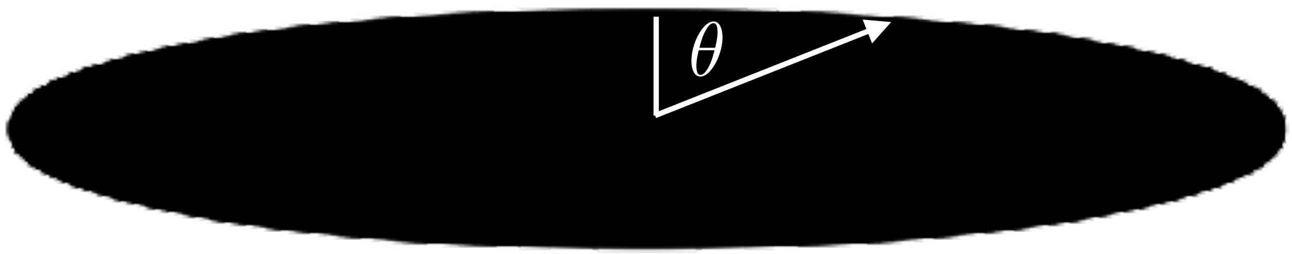
- No star-shaped AH ➡ **Parametric description of the AH**
- Very expensive

AH in numerical GR

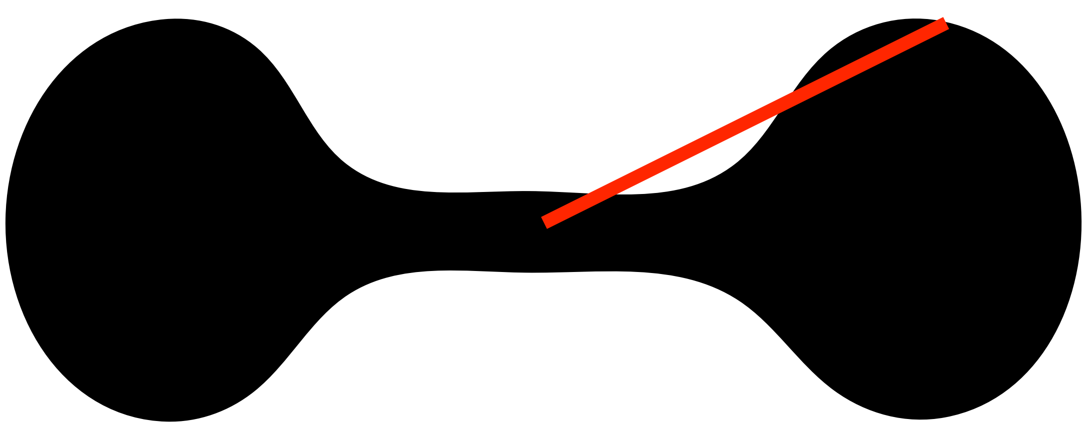
- Apparent horizons

- The traditional approach in numerical relativity is to assume that the AH satisfies:

$$r = R(\theta)$$



- This assumption fails in the during the non-linear stages of the evolution of certain instabilities



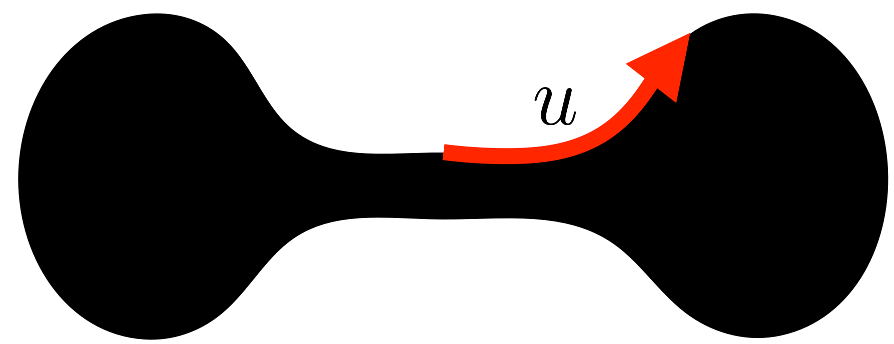
➡ $R(\theta)$ fails to be single-valued!

AH in numerical GR

- Apparent horizons

- Our solution: consider the AH as a parametric surface

$$(x(u), y(u), z(u))$$



- Equations to solve:

$$\Theta = (\gamma^{ab} - s^a\, s^b)(-k_{ab} - K_{ab}) = 0$$

$$\Delta_{\mathcal{H}} u = H(u)$$

$\Rightarrow$ manifestly elliptic

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- No star-shaped AH ➡ **Parametric description of the AH**

- Very expensive ➡ **Supercomputers**

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BSC Barcelona Supercomputing Center

HPC @ QMUL

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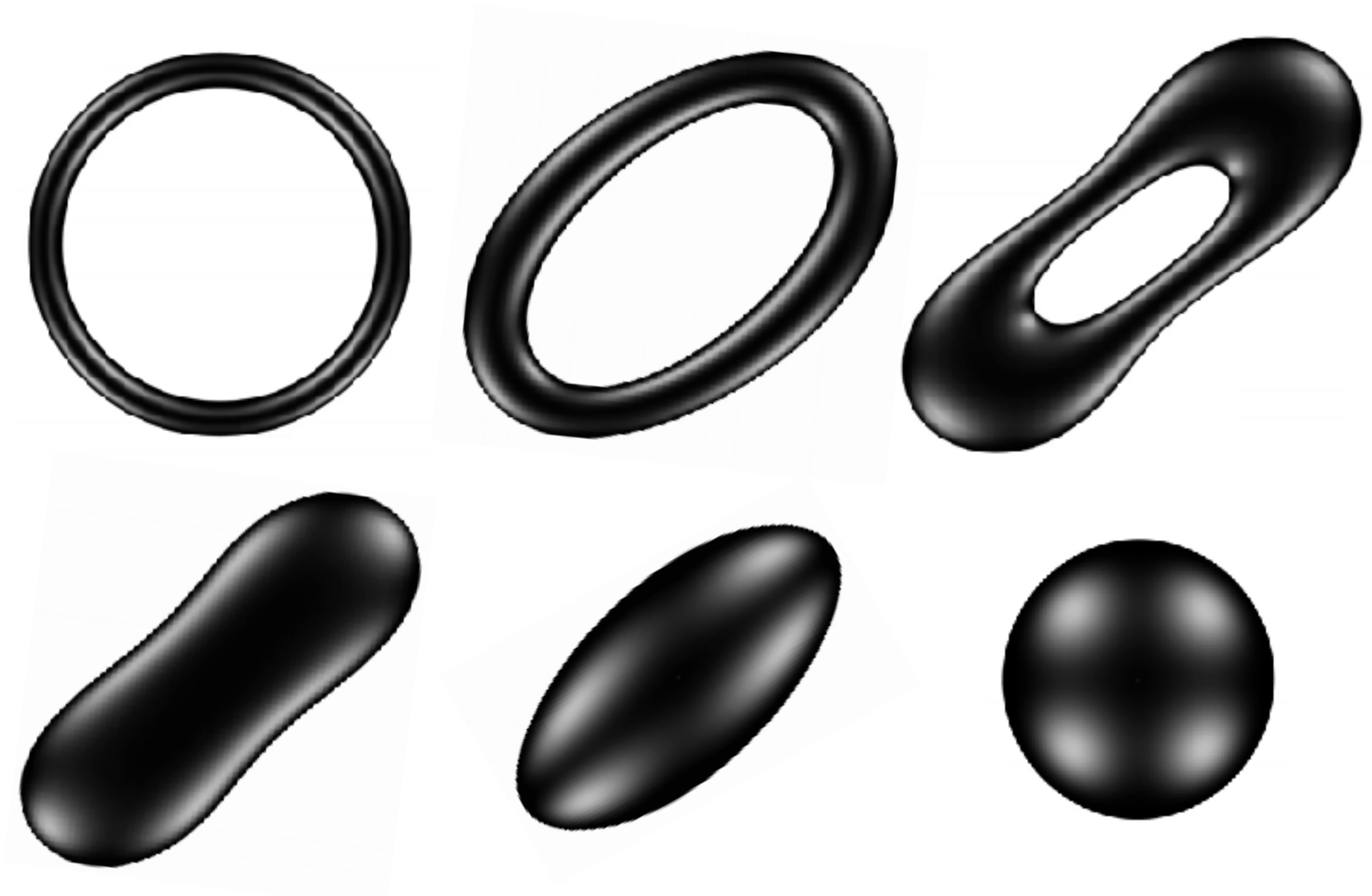
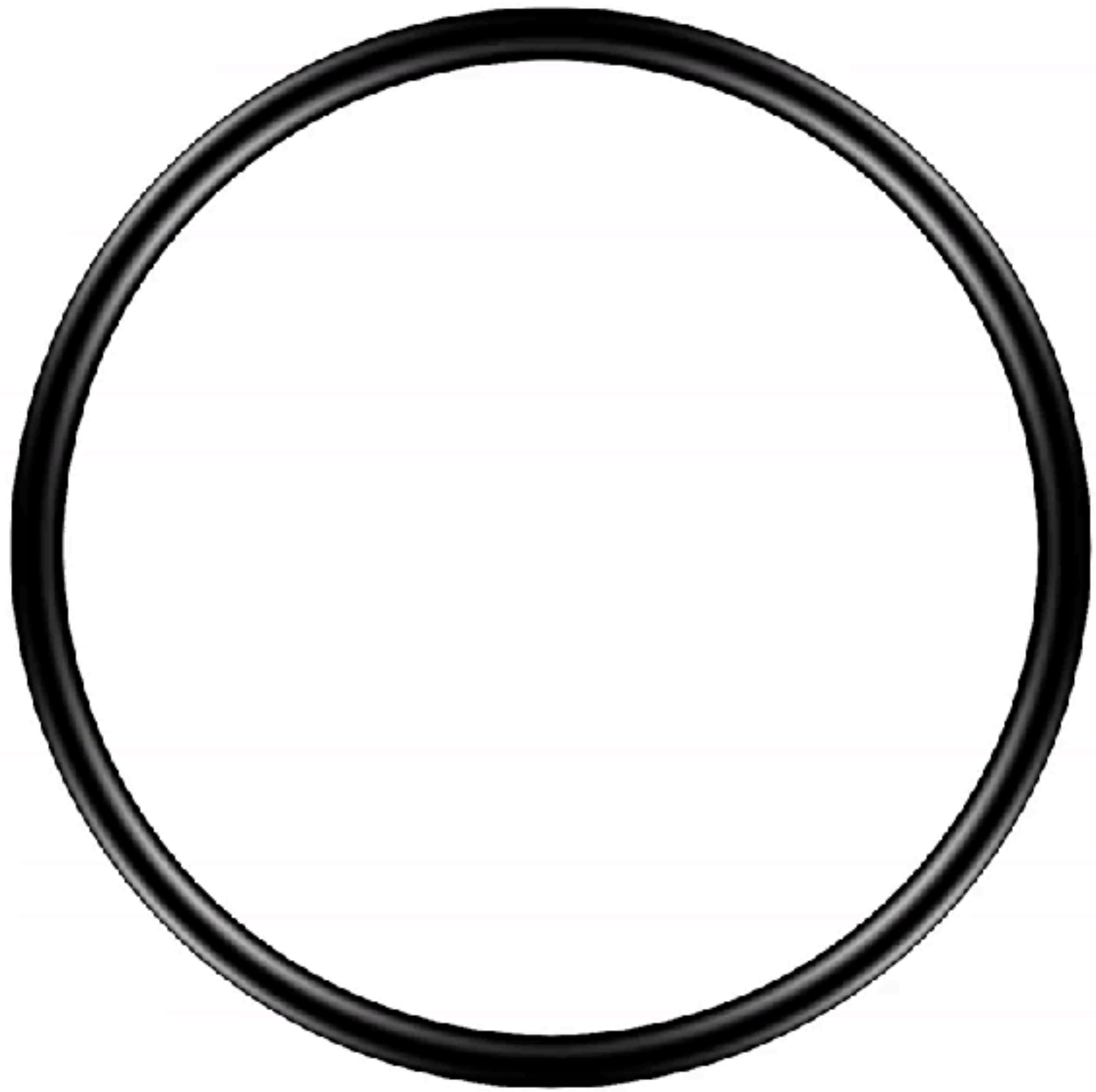
Argonne NATIONAL LABORATORY

Some results: black rings

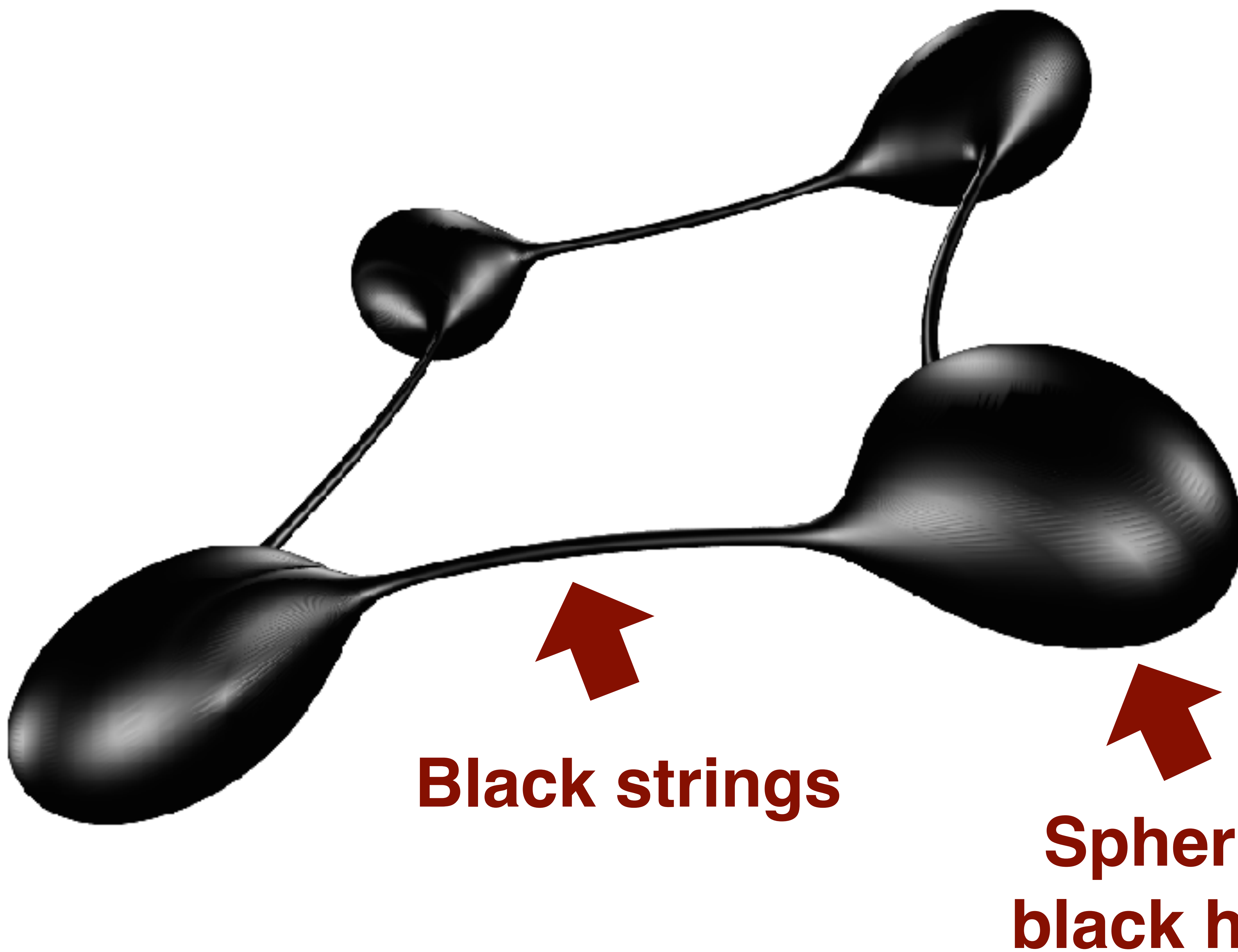
Rather thin rings

- Competition of GL and new “elastic” mode.
- Endpoint: Myers-Perry black hole

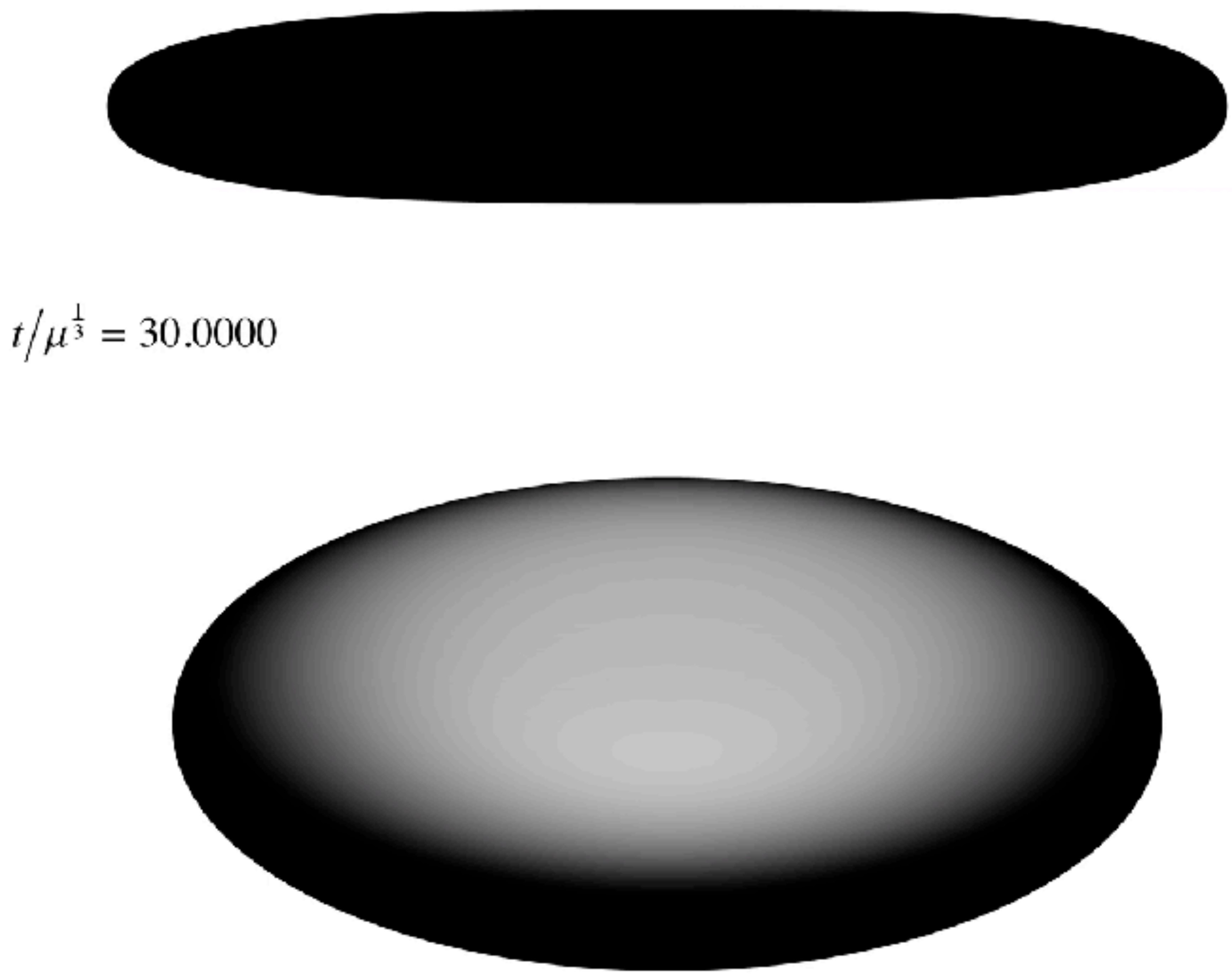




Very thin rings:



Some results: black holes


$t/\mu^{\frac{1}{3}} = 30.0000$

10,000 thinner than the original black hole!!!

Generalised harmonic evolution

Setup

- Evolution equations: generalised harmonic

$$
\begin{aligned}
0 = & -\frac{1}{2}\, g^{\alpha\beta}\, g_{\mu\nu,\alpha\beta} - g^{\alpha\beta}{}_{,(\mu} g_{\nu)\alpha,\beta} \\
& - H_{(\mu,\nu)} + H_\alpha\, \Gamma^\alpha{}_{\mu\nu} - \Gamma^\alpha{}_{\beta\mu}\, \Gamma^\beta{}_{\alpha\nu} \\
& - \kappa_1 (2\, n_{(\mu} C_{\nu)} - (1+\kappa_2)\, g_{\mu\nu}\, n^\alpha C_\alpha) \\
& - \frac{2\,\Lambda}{D-1}\, g_{\mu\nu} - 8\pi \left(T_{\mu\nu} - \frac{1}{D-1}\, T^\alpha{}_\alpha\, g_{\mu\nu} \right)
\end{aligned}
$$

$$H_\mu = f_\mu(g)$$

with $C_\mu = H_\mu - \Box x_\mu = 0$

Boundary conditions

- Decompose the metric into a pure AdS piece and a deviation:

$$g_{\mu\nu} = g_{\mu\nu}^{\mathrm{AdS}} + h_{\mu\nu}$$

- Poincare patch of AdS:

$$ds^2 = \frac{1}{z^2}(-dt^2 + dz^2 + d\vec{x}^2)$$

- Boundary conditions at z=0:

$$h_{zz} = z^{D-2}\, f_{zz}$$

$$h_{zi} = z^{D-1}\, f_{zi}$$

$$h_{ij} = z^{D-2}\, f_{ij}$$

$$\varphi = z^{D}\, f_{\varphi}$$

Gauge choice

- Expand the metric in a power series near z=0:

$$h_{\mu\nu} = z\, h^{(1)}_{\mu\nu} + z^2\, h^{(2)}_{\mu\nu} + \ldots$$

- Expand the field equations near z=0:

$$\tilde{\Box} h^{(1)}_{tt} = z^{-2}(-2\, h^{(1)}_{zz} + H^{(1)}_{z}) + \ldots$$
$$\tilde{\Box} h^{(1)}_{zz} = z^{-2}(4\, h^{(1)}_{tt} + 3\, h^{(1)}_{zz} - 4\, \Sigma_i h^{(1)}_{ii} - 2\, H^{(1)}_{z}) + \ldots$$
$$\tilde{\Box} h^{(1)}_{ij} = z^{-2}(2\, h^{(1)}_{zz} - H^{(1)}_{z}) + \ldots$$

- Expand constraint equations near z=0:

$$C^{(1)}_{z} = (-4\, h^{(1)}_{tt} - h^{(1)}_{zz} - \Sigma_i h^{(1)}_{ii} + H^{(1)}_{z}) + \ldots = 0$$

Comment: global AdS

- Consider asymptotically global AdS_5 spacetimes (with SO(3)) symmetry in Cartesian coordinates

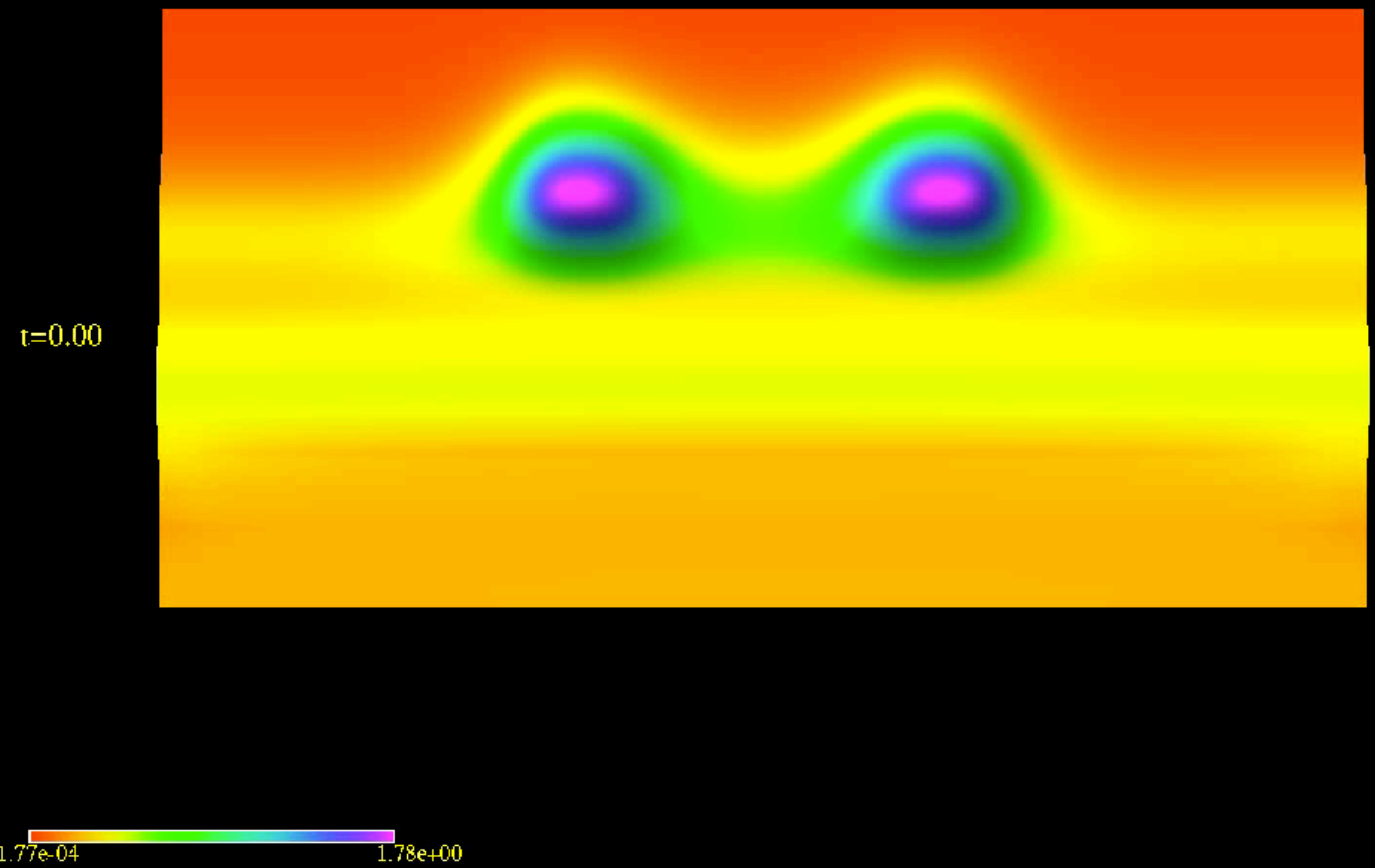
$$ds^2 = g_{tt}\,dt^2 + g_{xx}\,dx^2 + g_{yy}\,dy^2 + g_{\theta\theta}\,d\Omega_{(2)}^2 + 2(g_{tx}\,dt\,dx + g_{ty}\,dt\,dy + g_{xy}\,dx\,dy)$$

- AdS_5 in Cartesian coordinates: $x = \rho\cos\chi \quad y = \rho\sin\chi$

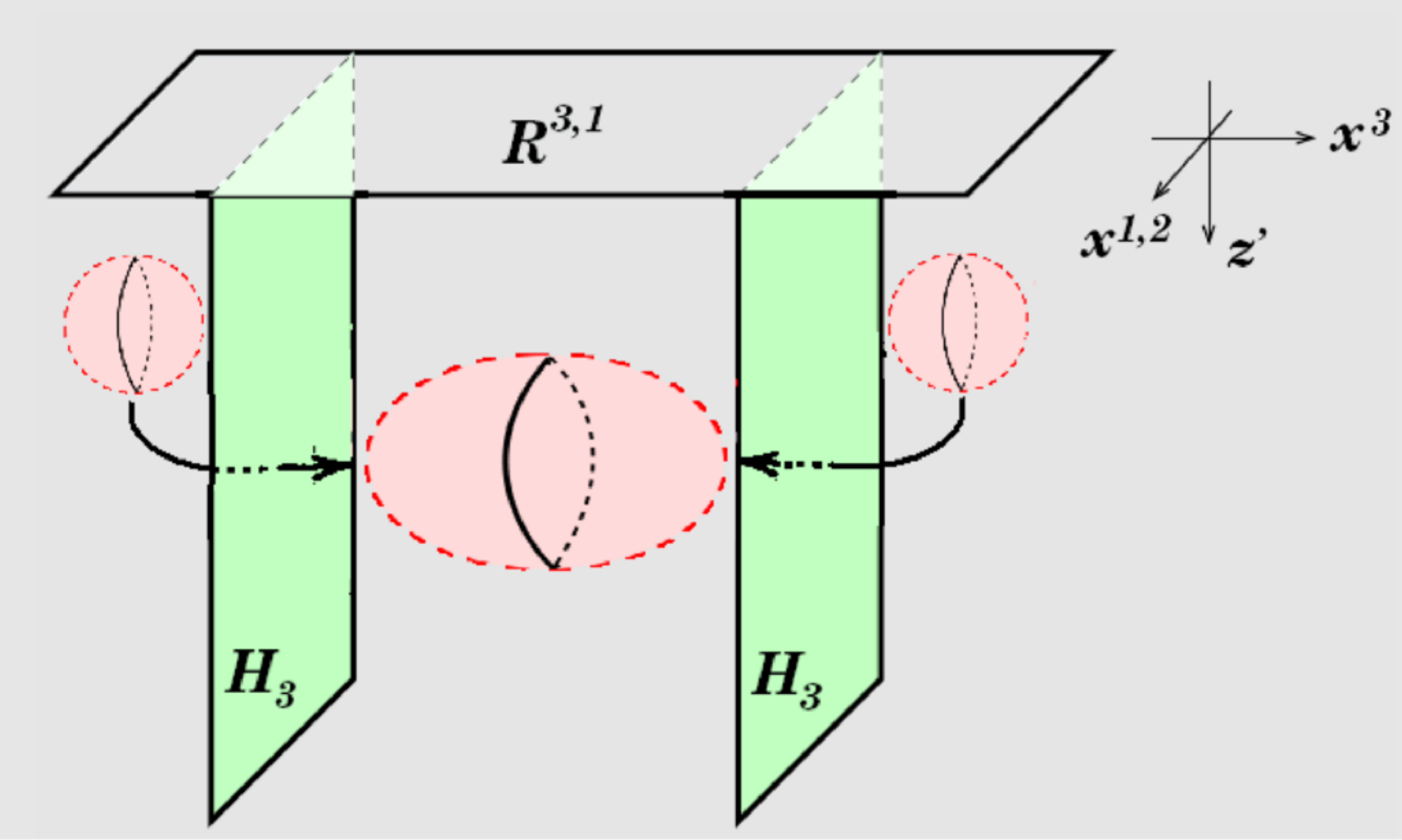
$$ds^2 = \frac{1}{(1-\rho^2)^2}[-f(\rho)dt^2 + 4(dx^2 + dy^2 + y^2 d\Omega_{(2)}^2] \qquad f(\rho) = (1-\rho^2)^2 + 4\rho^2$$

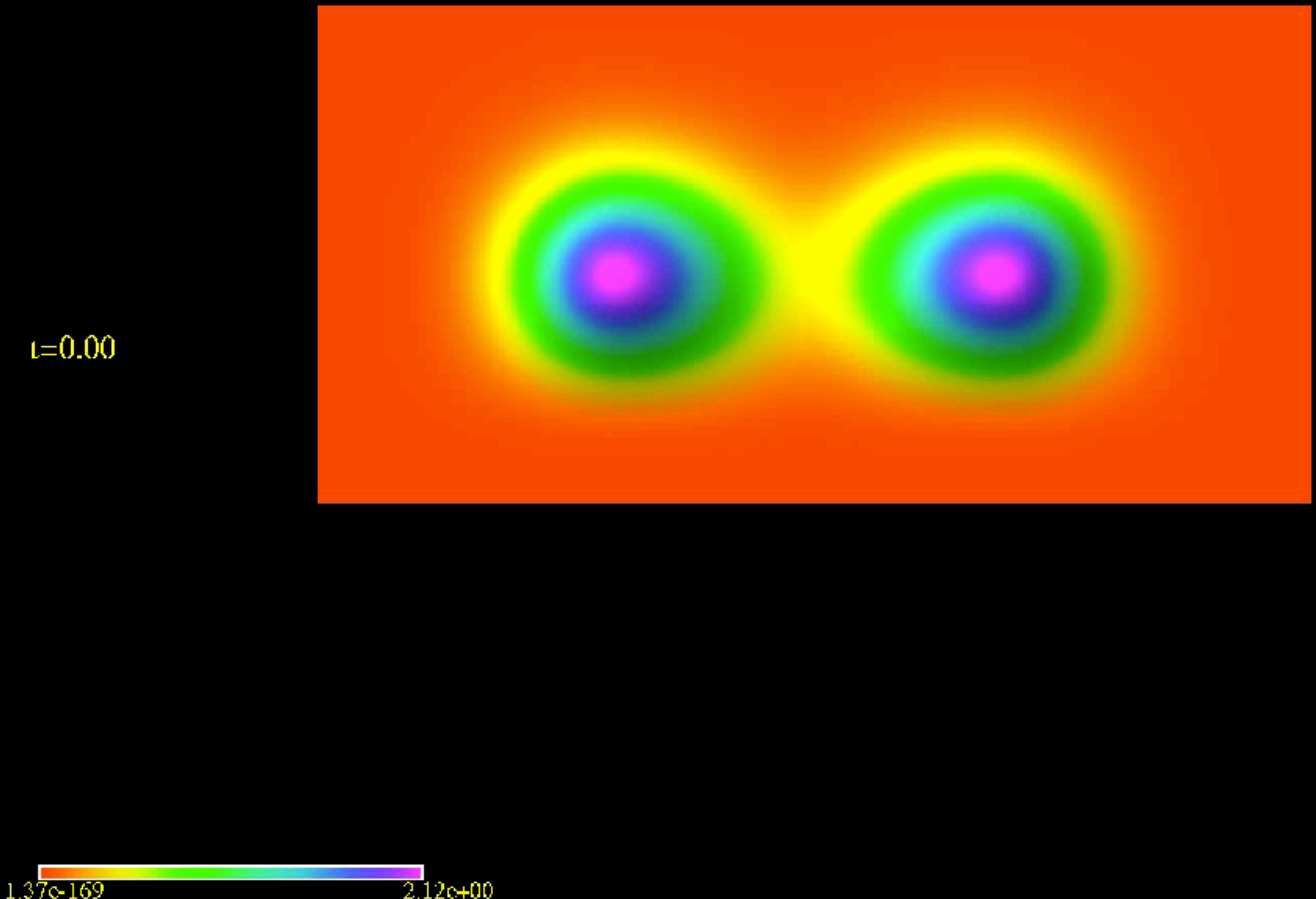
- Treat the sphere at infinity as a “Lego” sphere

Example: collapse into planar black hole

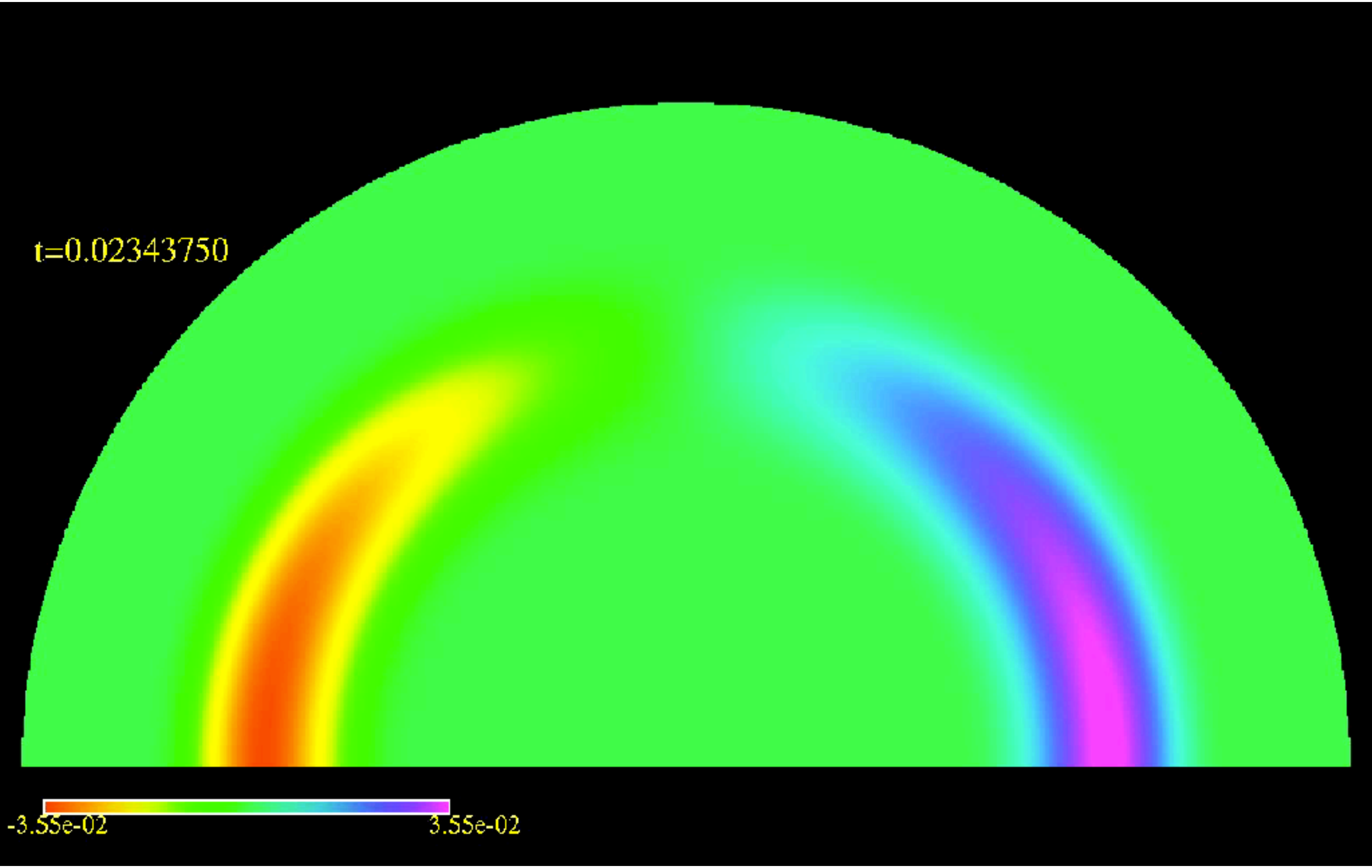

t=0.00
1.77e-04
1.78e+00

Example: Finite black hole collisions in Poincare AdS




t=0.00
1.37e-169
2.12e+00

Example: non-spherical collapse in global AdS


t=0.02343750
-3.55e-02
3.55e-02

Conclusions and outlook

- Numerical simulations in higher dimensions/AdS pose new challenges:
  - Multiple scales
  - Boundary conditions
  - Singularities

- One can reuse and expand the techniques/infrastructures developed in the traditional astrophysical setup

- Lots of open problems: black hole instabilities, collisions, turbulence…

Thank you for your attention!