The Non-perturbative Phase Diagram of the BMN Matrix Model

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Numerical approaches to holography, quantum gravity and cosmology

Edinburgh, 21-24th May 2018

Y. Asano, V. Filev, S. Kováčik and D.O'C [arXiv:1805.05314]
V. Filev and D.O'C. [arXiv:1506.01366 and 1512.02536]

- Introduction
- From Membranes to Matrices
- The BFSS model,
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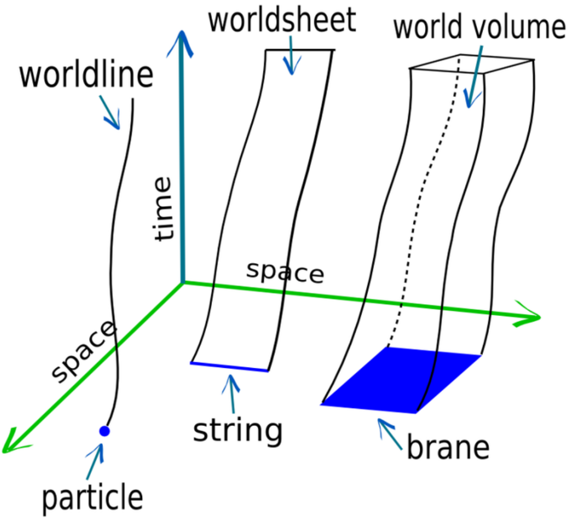
- Where to go from here.

A particle, a string and a membrane



The action functional

$$S_{particle} = -m \int d\tau_{proper} = -m \int dt \sqrt{1 - v^2}.$$

For small (non-relativistic) velocities this gives

$$S_{particle} = -m \int dt + \int dt \frac{mv^2}{2}$$

Movement in a generic background

Nambu-Goto particle action

$$S_{particle} = -m \int dt \sqrt{-g_{\mu\nu} \frac{dX^\mu}{dt} \frac{dX^\nu}{dt}}$$

(signature $\{-, +, \cdots, +\}$). The equations of motion give us the geodesic equation.

Using the Lagrange multiplier h we have

Polyakov particle action

$$S = \frac{1}{2} \int dt \{ h^{-1} g_{\mu\nu} \frac{dX^\mu}{dt} \frac{dX^\nu}{dt} - hm^2 \}$$

Eliminating h with its saddle point, $h = \sqrt{-g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu}/m$, recovers the Nambu form.

Coupling to an electromagnetic field

$$S_{charged-particle} = -m\int dt\sqrt{-g_{\mu\nu}\frac{dX^{\mu}}{dt}\frac{dX^{\nu}}{dt}} - q\int \frac{dX^{\mu}}{dt}A_{\mu}dt$$

We can repeat this exercise for a string

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int d\sigma d\tau \sqrt{-detG} \qquad G_{\mu\nu} = \partial_\mu X^M \partial_\nu X^N g_{MN}$$

or the Polyakov form, with the Lagrange multiplier metric $h_{\mu\nu}$,

$$S_P = -\frac{1}{4\pi\alpha'}\int_\Sigma d\sigma d\tau \sqrt{-h} h^{\mu\nu} G_{\mu\nu}$$

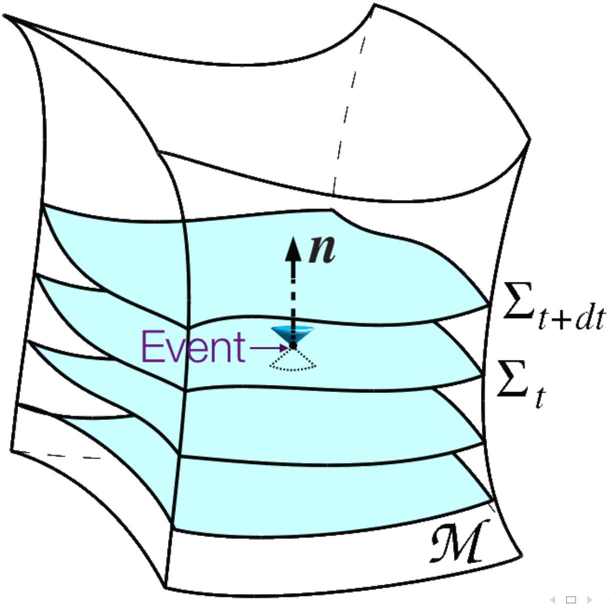
The string is very special in that it is a conformally invariant action. Again one can couple the string to e.g. an RR 2-form to get the

$$S_{NG} - q\int \partial_\mu X^M \partial_\nu X^N \epsilon^{\mu\nu} B_{MN}$$

Quantisation

We can quantise the particle or string in either a path integral or Hamiltonian formulation and the results are well appreciated. Both can be generalised to supersymmetric versions with the string leading to string theory and conformal field theory.

Membrane propagating in spacetime



Membrane Actions

Nambu Goto—the simplest

$S_{NG} = \int_{\mathcal{M}} d^{p+1}x\sqrt{-detG} \qquad G_{\mu\nu} = \partial_\mu X^M \partial_\nu X^N g_{MN}$

Higher form gauge field on the world volume

$$S_{p-form} = -\int_{\mathcal{M}} \frac{1}{(p+1)!}\epsilon^{\mu_1\dots\mu_{p+1}} C_{\mu_1\dots\mu_{p+1}}$$

$$C_{\mu_1\dots\mu_{p+1}} = \partial_{\mu_1} X^{M_1} \dots \partial_{\mu_{p+1}} X^{M_{p+1}} C_{M_1\dots M_{p+1}}$$

We can add

- an anti-symmetric part to $G_{\mu\nu}$ to get a Dirac-Born-Infeld action.
- extrinsic curvature terms.

Supersymmetric S_{NG} exist only in 4, 5, 7 and 11 dim-spacetime.

From Membranes to Matrices (à la Hoppe)

The Membrane action, Polyakov form

$$S = -\frac{T}{2}\int_{\mathcal{M}} d^3\sigma\sqrt{-h}\left(h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - \Lambda\right)$$

Choose $\Lambda = 1$ (rescale X^a and T), and for membrane topology $\mathbb{R}\times\Sigma$ use the gauge $h_{0i} = 0$ and $h_{00} = -\frac{4}{\rho}\det(h_{ij})$.

The action becomes

$$S = \frac{T\rho}{4}\int dt \int_\Sigma d^2\sigma \left(\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} - \frac{4}{\rho^2}\det(h_{ij})\right)$$

Noting that

$$\det(\partial_i X^a \partial_j X^b h_{ab}) =$$
$$\frac{1}{p!}\{X^{a_1}, X^{a_2} \ldots, X^{a_p}\}\{X^{b_1}, X^{b_2} \ldots, X^{b_p}\} h_{a_1 b_1} h_{a_2 b_2} \ldots h_{a_p b_p}$$
$$\{X^{a_1}, X^{a_2} \ldots, X^{a_p}\} := \epsilon^{j_1, j_2, \ldots, j_p} \partial_{j_1} X^{a_1} \partial_{j_2} X^{a_2} \ldots \partial_{j_p} X^{a_p}$$

$$S = \frac{T\rho}{4} \int dt \int_\Sigma d^2\sigma \left(\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} - \frac{4}{\rho^2} \det(h_{ij}) \right)$$

becomes

$$S = \frac{T\rho}{4} \int dt \int_\Sigma d^2\sigma \left(\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} - \frac{4}{p!\rho^2} \{X^{a_1}, X^{a_2} \ldots, X^{a_p}\}^2 \right)$$

In 2-dim $det(h_{ij})$ can be rewritten using $\{f, g\} = \epsilon^{ij}\partial_i f \partial_j g$ as

$$S = \frac{T\rho}{4}\int dt \int_\Sigma d^2\sigma \left(\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} - \frac{4}{\rho^2}\{X^\mu, X^\nu\}^2\right)$$

and the constraints become

$$\dot{X}^\mu \partial_i X_\mu = 0 \implies \{\dot{X}^\mu, X_\mu\} = 0$$
$$\text{and} \qquad \dot{X}^\mu \dot{X}_\mu = -\frac{2}{\rho^2}\{X^\mu, X^\nu\}\{X_\mu, X_\nu\}\,.$$

Using lightcone coordinates with $X^\pm = (X^0 \pm X^{D-1})/\sqrt{2}$ with $X^+ = \tau$ we can solve the constraint for $\dot{X}^-$ and Legendre transform to the Hamiltonian to find

$$S = -T\int\sqrt{-G} \longrightarrow H = \int_\Sigma (\frac{1}{\rho T}P^a P^a + \frac{T}{2\rho}\{X^a, X^b\}^2)$$

With the remaining constraint $\{P^a, X^a\} = 0$.

In this scheme functions are approximated by $N \times N$ matrices, $f \to \mathrm{F}$, and $\int_\Sigma f \to \mathrm{TrF}$. The Hamiltonian becomes

$$\mathrm{H} = -\frac{1}{2}\nabla^2 - \frac{1}{4}\sum_{i,j=1}^{d} \mathrm{Tr}[X^i, X^j]^2$$

and describes a "fuzzy" relativistic membrane in $d+1$ dimensions. Note: Much of the classical topology and geometry are lost in the quantum theory.

Once we have the Hamiltonian H we can consider thermal ensembles of membranes whose partition function is given by

$$Z = \mathrm{Tr}_{Phys}(\mathrm{e}^{-\beta H})$$

where the physical constraint means the states are $U(N)$ invariant.

Gauss law constraint

The projection onto physical states is implementing the Gauss law constraint.— Use gauge field.

The simplest example of a quantum mechanical model with Gauss Law constraint in this class is a family of p gauged Gaussians. Their Euclidean actions are

$$N \int_0^{\beta} \mathrm{Tr}(\frac{1}{2}(\mathcal{D}_\tau X^i)^2 + \frac{1}{2} m^2 (X^i)^2)$$

$\mathcal{D}_\tau X^i = \partial_\tau X^i - i[A, X^i]$.

Properties of gauge gaussian models

- The eigenvalues of X^i have a Wigner semi-circle distribution.
- At $T = 0$, we can gauged A away, while for large T we get a pure matrix model with A one of the matrices.
- The entry of A as an additional matrix in the dynamics signals a phase transition. In the Gaussian case with p scalars it occurs at

$$T_c = \frac{m}{\ln p}$$

  The transition can be observed as centre symmetry breaking in the Polyakov loop.

Bosonic matrix membranes are approximately gauge gaussian models V. Filev and D.O'C. [1506.01366 and 1512.02536]. Note they are the zero volume limit of Yang-Mills compactified on T^3 and on closer inspection they exhibit two phase transitions, very close in temperature.

Quantum Gravity

At short distances it is expected [Doplicher, Fredenhagen and Roberts, 1995] that spatial co-ordinates, X^a should not commute $[X^a, X^b] \neq 0$ in analogy with $[x, p] = i\hbar$ in phase space, but $[X^a, X^b] = i\theta^{ab}$ breaks rotational invariance.

We only need the coordinates to commute at low energies.

Hand waving à la Polchinski, 2014 (arXiv:1412.5704):
Take each X^a to be an $N \times N$ matrix and try

$$H_0 = \mathrm{Tr}(\frac{1}{2}\sum_{a=1}^{p} \dot{X}^a \dot{X}^a - \frac{1}{4}\sum_{a,b=1}^{p}[X^a, X^b][X^a, X^b])$$

The model describes membranes, Hoppe 1982.

$$S = -T\int\sqrt{-G} \longrightarrow H = \int(\frac{1}{\rho T}P^a P^a + \frac{T}{2\rho}\{X^a, X^b\}^2)$$

With the remaining constraint $\{P^a, X^a\} = 0$.

At low energy, or the bottom of the potential $[X^a, X^b] = 0$.

The BFSS model

$$S_{SMembrane} = \int \sqrt{-G} - \int C + \quad \text{Fermionic terms}$$

The susy version only exists in $4, 5, 7$ and 11 spacetime dimensions.

BFFS Model — The supersymmetric membrane à la Hoppe

$$\mathrm{H} = \mathrm{Tr}(\frac{1}{2}\sum_{a=1}^{9} P^a P^a - \frac{1}{4}\sum_{a,b=1}^{9}[X^a, X^b][X^a, X^b] + \frac{1}{2}\Theta^T\gamma^a[X^a, \Theta])$$

The model is claimed to be a non-perturbative 2nd quantised formulation of M-theory.

A system of N interacting D0 branes.

Note the flat directions.

Finite Temperature Model

The partition function and Energy of the model at finite temperature is

$$Z = Tr_{Phys}(\mathrm{e}^{-\beta\mathcal{H}}) \qquad \text{and} \qquad E = \frac{Tr_{Phys}(\mathcal{H}\mathrm{e}^{-\beta\mathcal{H}})}{Z} = \langle\mathcal{H}\rangle$$

The 16 fermionic matrices $\Theta_\alpha = \Theta_{\alpha A} t^A$ are quantised as

$$\{\Theta_{\alpha A}, \Theta_{\beta B}\} = 2\delta_{\alpha\beta}\delta_{AB}$$

The $\Theta_{\alpha A}$ are $2^{8(N^2-1)}$ and the Fermionic Hilbert space is

$$\mathcal{H}^F = \mathcal{H}_{256} \otimes \cdots \otimes \mathcal{H}_{256}$$

with $\mathcal{H}_{256} = \mathbf{44} \oplus \mathbf{84} \oplus \mathbf{128}$ suggestive of
the graviton ($\mathbf{44}$), anti-symmetric tensor ($\mathbf{84}$) and gravitino ($\mathbf{128}$)
of $11 - d$ SUGRA.

For an attempt to find the ground state see: J. Hoppe et al
arXiv:0809.5270

Lagrangian formulation

.

The BFSS matrix model is also the dimensional reduction of ten dimensional supersymmetric Yang-Mills theory down to one dimension:

$$
\begin{aligned}
S_M &= \frac{1}{g^2}\int dt\,\mathrm{Tr}\left\{\frac{1}{2}(\mathcal{D}_0 X^i)^2 + \frac{1}{4}[X^i, X^j]^2 \right. \\
&\qquad \left. -\frac{i}{2}\Psi^T C_{10}\,\Gamma^0 D_0\Psi + \frac{1}{2}\Psi^T C_{10}\,\Gamma^i[X^i,\Psi]\right\} \ ,
\end{aligned}
$$

where Ψ is a thirty two component Majorana–Weyl spinor, Γ^μ are ten dimensional gamma matrices and C_{10} is the charge conjugation matrix satisfying $C_{10}\Gamma^\mu C_{10}^{-1} = -\Gamma^{\mu\,T}$.

$$\Delta S_\mu = -\frac{1}{2g^2}\int_0^\beta d\tau \,\mathbf{Tr}\left((\frac{\mu}{6})^2 (X^a)^2 + (\frac{\mu}{3})^2 (X^i)^2 + \frac{2\mu}{3} i\epsilon_{ijk} X^i X^j X^k + \frac{\mu}{4}\Psi^T \gamma^{123}\Psi \right)$$

The gravity dual and its geometry

Gauge/gravity duality predicts that the strong coupling regime of the theory is described by II_A supergravity, which lifts to 11-dimensional supergravity.

The bosonic action for eleven-dimensional supergravity is given by

$$S_{11D} = \frac{1}{2\kappa_{11}^2}\int [\sqrt{-g}R - \frac{1}{2}F_4 \wedge *F_4 - \frac{1}{6}A_3 \wedge F_4 \wedge F_4]$$

where $2\kappa_{11}^2 = 16\pi G_N^{11} = \frac{(2\pi l_p)^9}{2\pi}$.

The relevant solution to eleven dimensional supergravity for the dual geometry to the BFSS model corresponds to N coincident $D0$ branes in the IIA theory. It is given by

$$ds^2 = -H^{-1}dt^2 + dr^2 + r^2 d\Omega_8^2 + H(dx_{10} - Cdt)^2$$

with $A_3 = 0$

The one-form is given by $C = H^{-1} - 1$ and $H = 1 + \frac{\alpha_0 N}{r^7}$ where $\alpha_0 = (2\pi)^2 14\pi g_s l_s^7$.

Including temperature

The idea is to include a **black hole** in the gravitational system.

The Hawking termperature provides the temperature of the system.

Hawking radiation

We expect difficulties at low temperatures, as the system should Hawking radiate. It is argued that this is related to the flat directions and the propensity of the system to leak into these regions.

The black hole geometry

$$ds_{11}^2 = -H^{-1}Fdt^2 + F^{-1}dr^2 + r^2 d\Omega_8^2 + H(dx_{10} - Cdt)^2$$

Set $U = r/\alpha'$ and we are interested in $\alpha' \to \infty$
$H(U) = \frac{240\pi^5\lambda}{U^7}$ and the black hole time dilation factor
$F(U) = 1 - \frac{U_0^7}{U^7}$ with $U_0 = 240\pi^5\alpha'^5\lambda$. The temperature

$$\frac{T}{\lambda^{1/3}} = \frac{1}{4\pi\lambda^{1/3}} H^{-1/2} F'(U_0) = \frac{7}{2^4 15^{1/2} \pi^{7/2}} \left(\frac{U_0}{\lambda^{1/3}}\right)^{5/2}.$$

From black hole entropy we obtain the prediction for the Energy

$$S = \frac{A}{4G_N} \sim \left(\frac{T}{\lambda^{1/3}}\right)^{9/2} \implies \frac{E}{\lambda N^2} \sim \left(\frac{T}{\lambda^{1/3}}\right)^{14/5}$$

Checks of the predictions

We found excellent agreement with this prediction V. Filev and D.O'C. [1506.01366 and 1512.02536].
The best current results (Berkowitz et al 2016) consistent with gauge gravity give

$$\frac{1}{N^2}\frac{E}{\lambda^{1/3}} = 7.41\left(\frac{T}{\lambda^{1/3}}\right)^{\frac{14}{5}} - (10.0 \pm 0.4)\left(\frac{T}{\lambda^{1/3}}\right)^{\frac{23}{5}} + (5.8 \pm 0.5)\, T^{\frac{29}{5}} + \ldots - \frac{5.77\, T^{\frac{2}{5}} + (3.5 \pm 2.0)\, T^{\frac{11}{5}}}{N^2} + \ldots$$

Probing the dual geometry.

We need a suitable probe to test the dual geometry further.

Using D_4 branes as probes (these adds new fundamental matter).

See: M. Berkooz and M. R. Douglas, "Five-branes in M(atrix) theory," [hep-th/9610236].
In IIA string theory this describes a $D0 - D4$ system.

D4-brames as probes

We use D_4 branes. These adds new fundamental matter fields to the Hamiltonian.

M. Berkooz and M. R. Douglas, “Five-branes in M(atrix) theory,” [hep-th/9610236].
In IIA string theory this describes a $D0 - D4$ system.

The more general framework involves $Dp - D(p+4)$ systems.

Add new bosonic degrees of freedom Φ_α as two complex $N \times N_f$ matrices

$$\mathcal{S}_\Phi^{\mathrm{E}} = \frac{1}{g^2} \int_0^\beta d\tau \, \mathrm{tr} \Big(D_\tau \bar{\Phi}^\rho D_\tau \Phi_\rho - \bar{\Phi}^\alpha (\sigma^A)_\alpha^{\ \beta} J^A_{ab} [X^a, X^b] \Phi_\beta$$

$$+ \bar{\Phi}^\rho (X^i)^2 \Phi_\rho - \frac{1}{2} \bar{\Phi}^\alpha \Phi_\beta \bar{\Phi}^\beta \Phi_\alpha + \bar{\Phi}^\alpha \Phi_\alpha \bar{\Phi}^\beta \Phi_\beta \Big) \ .$$

J^A and K^A are the $SO(3)$ generators

$$J^A_{ab} = \frac{1}{2} (L_{A4})_{ab} + \frac{1}{4} \varepsilon^{ABC} (L_{BC})_{ab} \ , \ K^A_{ab} = -\frac{1}{2} (L_{A4})_{ab} + \frac{1}{4} \varepsilon^{ABC} (L_{BC})_{ab}$$

equivalent to the

$$SO(4) \text{ generators} \quad (L_{ab})_{cd} = i(\delta_{ad}\delta_{bc} - \delta_{ac}\delta_{bd})$$

$$
\begin{aligned}
S_\chi &= \frac{1}{g^2}\int \mathrm{tr}\Big(i\chi^\dagger D_0\chi + \bar{\chi}\gamma^a X^a\chi \\
&\qquad + \sqrt{2}\, i\,\varepsilon_{\alpha\beta}\,\bar{\chi}\lambda_\alpha\Phi_\beta - \sqrt{2}\, i\,\varepsilon_{\alpha\beta}\,\bar{\Phi}^\alpha\bar{\lambda}_\beta\chi\Big) \ .
\end{aligned}
$$

where $\lambda_\alpha = P^\iota_\alpha\psi_\iota$.

The full model is

$$
S_{BD} = S_{BFSS} + S_\Phi + S_\chi \, .
$$

The lattice discretisation is again delicate but works!

The Bosonic model (and ADHM Data)

$$S_{\mathrm{bos}} = N\int_0^\beta d\tau \Bigg[\mathrm{Tr}\left(\frac{1}{2} D_\tau X^a D_\tau X^a + \frac{1}{2} D_\tau \bar{X}^{\rho\dot\rho} D_\tau X_{\rho\dot\rho} - \frac{1}{4}[X^a, X^b]^2 + \frac{1}{2}[X^a, \bar{X}^{\rho\dot\rho}][X^a, X_{\rho\dot\rho}] \right) + \mathrm{tr}\left(D_\tau \bar{\Phi}^\rho D_\tau \Phi_\rho + \bar{\Phi}^\rho (X^a - m^a)^2 \Phi_\rho \right) + \frac{1}{2}\mathrm{Tr}\sum_{A=1}^{3} \mathcal{D}^A \mathcal{D}^A \Bigg].$$

where $X^a = 0$ with

$$\mathcal{D}^A = \sigma_\rho^{A\,\sigma}\left(\frac{1}{2}[\bar{X}^{\rho\dot\rho}, X_{\sigma\dot\rho}] - \Phi_\sigma \bar{\Phi}^\rho \right) = 0$$

specify ADHM data for Yang-Mills instantons on $\mathbb{R}^4$.

$1/D$ expansion of the bosonic model.

Partition Function

$$Z = \int [dX][dA] \mathrm{e}^{-N \int_0^\beta dt \, \mathrm{Tr}\left(\frac{1}{2}(\mathcal{D}_t X^i)^2\right) - \frac{N}{4}\lambda^{abcd} \int_0^\beta dt \, X_a^i X_b^i X_c^j X_d^j} \,.$$

The commutator square term can be written as:

$$\mathrm{Tr}[X^i, X^j]^2 = \mathrm{Tr}\left([t^a, t^c][t^b, t^d]\right) X_a^i X_b^i X_c^j X_d^j = \lambda^{abcd} X_a^i X_b^i X_c^j X_d^j \,, \tag{1}$$

where t^a are $SU(N)$ generators.

$$Z = \int [dX][dA][dk] \mathrm{e}^{-\frac{N}{2} \int_0^\beta dt \left\{\mathrm{Tr}(\mathcal{D}_t X^i)^2 + k^{ab} X_a^i X_b^i\right\} + \frac{N}{4}\mu_{abcd} \int_0^\beta dt \, k^{ab} k^{cd}} \,. \tag{2}$$

The saddle point approximation for k^{ab} gives $k^{ab} = p^{2/3}\delta^{ab}$. A detailed $1/D$ analysis of the membrane model shows there are in fact two phase transitions.

For many purposes the effective dynamics of the Bosonic membrane is given by the action

Gauged Gaussian model

$$S_{\text{eff}} \approx N \int_{-\infty}^{\infty} dt\, \mathrm{Tr}\left(\frac{1}{2}(\mathcal{D}_t X^i)^2 - \frac{1}{2}m^2(X^i)^2\right) .$$

Perturbing around the gauged gaussian model with

$$m^2 \simeq p^{2/3}$$

the bosonic membrane model can be largely solved analytically.

Results for the flavoured bosonic model

With $N_f << N$ the fundamental fields act as probes on the adjoint background. The $SO(9)$ symmetry has been broken to $SO(5) \times SO(4)$. In the low temperature phase the system is well described by a gaussian model with three masses $m_A^t = 1.964 \pm 0.003$, $m_A^l = 2.001 \pm 0.003$ and $m_f = 1.463 \pm 0.001$, the adjoint longitudinal and transverse masses and the mass of the fundamental fields respectively.

Yuhma Asano, Veselin G. Filev, Samuel Kováčik and D. O'C. arXiv 1605.05597

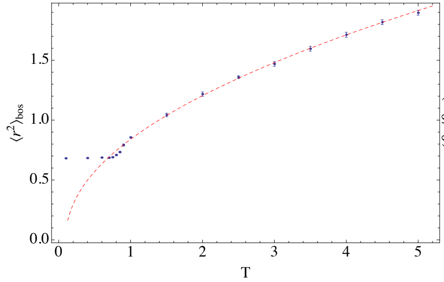
Two new observables

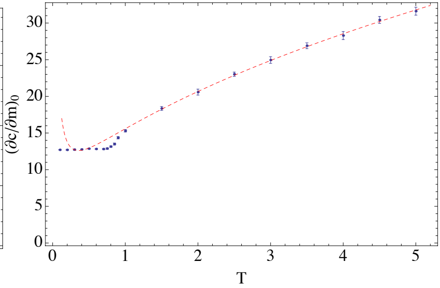
$$r^2 = \frac{1}{\beta N_f} \int_0^\beta d\tau \ \mathrm{tr}\, \bar{\Phi}^\rho \Phi_\rho$$

and the condensate defined as

$$c^a(m) = \frac{\partial}{\partial m^a} \left(-\frac{1}{N\beta} \log Z \right)$$

$N_f = 1$ and $N = 10$:





With $X^a \to X^a - m^a$ and for $m^a = 0$ we can look at the condensate susceptibility:

$$\left(\frac{\partial c}{\partial m}\right)_0 = \frac{2}{\beta} \int_0^\beta d\tau \, \mathrm{tr} \, \bar{\Phi}^\rho \Phi_\rho - \frac{N}{5\beta} \left(\int_0^\beta d\tau \, \mathrm{tr} \, 2\bar{\Phi}^\rho X^a \Phi_\rho \right)^2 .$$

The D4-brane as a probe of the geometry.

The dual adds N_f D4 probe branes. In the probe approximation $N_f \ll N_c$, their dynamics is governed by the Dirac-Born-Infeld action:

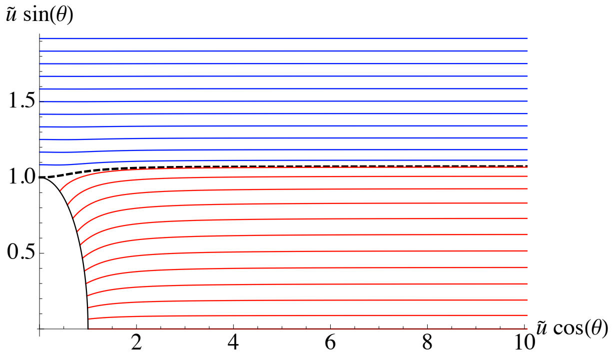
$$S_{\mathrm{DBI}} = -\frac{N_f}{(2\pi)^4\,\alpha'^{5/2}\,g_s}\int d^4\xi\, e^{-\Phi}\,\sqrt{-\mathrm{det}||G_{\alpha\beta} + (2\pi\alpha')F_{\alpha\beta}||}\ ,$$

where $G_{\alpha\beta}$ is the induced metric and $F_{\alpha\beta}$ is the $U(1)$ gauge field of the D4-brane. For us $F_{\alpha\beta} = 0$.

$$d\Omega_8^2 = d\theta^2 + \cos^2\theta\, d\Omega_3^2 + \sin^2\theta\, d\Omega_4^2$$

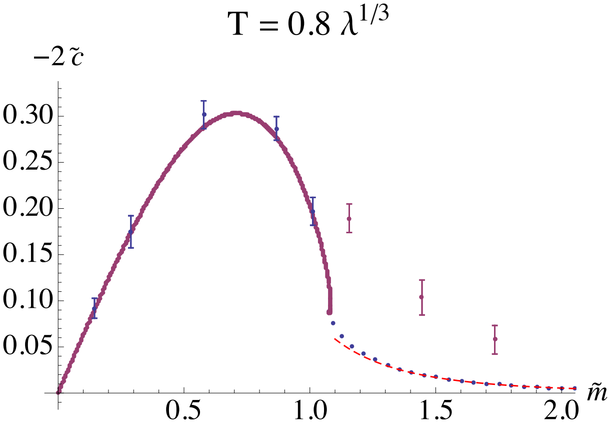
and taking a D4-brane embedding extended along: t, u, Ω_3 with a non-trivial profile $\theta(u)$.

Embeddings



$$\tilde{u}\sin\theta = m + \frac{\tilde{c}}{\tilde{u}^2} + \dots .$$

The condensate and the dual prediction



V. Filev and D. O'C. arXiv 1512.02536.

The data overlaps surprisingly well with the gravity prediction in the region where the $D4$ brane ends in the black hole.

Membranes on other backgrounds

There are many options for background geometries:

PP-Wave backgrounds

Two options that lead to massive deformations of the BFSS model

N=1*

Breaks susy down to 4 remaining.

BMN model

Preserves all 16 susys and has $SU(4|2)$ symmetry.

The BMN or PWMM

The supermembrane on the maximally supersymmetric plane wave spacetime

$$ds^2 = -2dx^+dx^- + dx^a dx^a + dx^i dx^i - dx^+dx^+((\frac{\mu}{6})^2(x^i)^2 + (\frac{\mu}{3})^2(x^a)^2)$$

with

$$dC = \mu dx^1 \wedge dx^2 \wedge dX^3 \wedge dx^+$$

so that $F_{123+} = \mu$. This leads to the additional contribution to the Hamiltonian

$$\Delta \mathrm{H}_\mu = \frac{N}{2}\mathbf{Tr}\Big((\frac{\mu}{6})^2(X^a)^2 + (\frac{\mu}{3})^2(X^i)^2 + \frac{2\mu}{3} i\epsilon_{ijk}X^iX^jX^k + \frac{\mu}{4}\Theta^T\gamma^{123}\Theta\Big)$$

The BMN model

The BMN action

$$
\begin{aligned}
S_{BMN} \quad = \quad & \frac{1}{2g^2}\int dt\,\mathrm{Tr}\Big\{(\mathcal{D}_0X^i)^2 - (\frac{\mu}{6})^2(X^a)^2 - (\frac{\mu}{3})^2(X^i)^2 \\
& \quad -i\Psi^T C_{10}\,\Gamma^0 D_0\Psi - \frac{\mu}{4}\Psi^T\gamma^{123}\Psi \\
& \quad +\frac{1}{4}[X^i,X^j]^2 - \frac{2\mu}{3}i\epsilon_{ijk}X^iX^jX^k - \frac{1}{2}\Psi^T C_{10}\,\Gamma^i[X^i,\Psi]\Big\} \quad ,
\end{aligned}
$$

Large mass expansion

For large μ the model becomes the supersymmetric Gaussian model

Finite temperature Euclidean Action

$$S_{BMN} = \frac{1}{2g^2}\int_0^\beta d\tau\,\mathrm{Tr}\Big\{(\mathcal{D}_\tau X^i)^2 + (\frac{\mu}{6})^2(X^a)^2 + (\frac{\mu}{3})^2(X^i)^2$$
$$\Psi^T D_\tau \Psi + \frac{\mu}{4}\Psi^T\gamma^{123}\Psi\Big\}$$

This model has a phase transition at $T_c = \frac{\mu}{12\ln 3}$

Perturbative expansion in large μ.

Three loop result of *Hadizadeh, Ramadanovic, Semenoff and Young* [hep-th/0409318]

$$T_c = \frac{\mu}{12 \ln 3} \left\{ 1 + \frac{2^6 \times 5}{3^4} \frac{\lambda}{\mu^3} - \left(\frac{23 \times 19927}{2^2 \times 3^7} + \frac{1765769 \ln 3}{2^4 \times 3^8} \right) \frac{\lambda^2}{\mu^6} + \cdots \right\}$$

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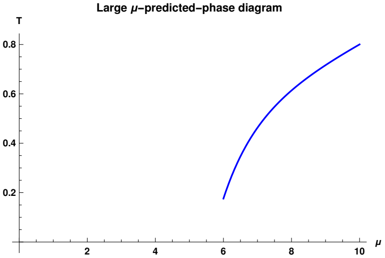


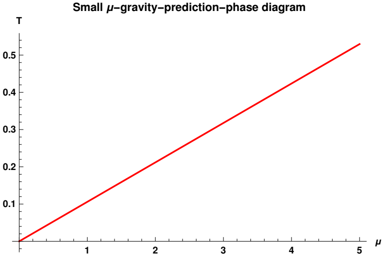
Passes through zero at $\mu = 13.4$.

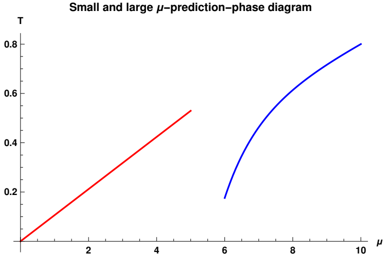
Gravity prediction at small μ

Costa, Greenspan, Penedones and Santos, [arXiv:1411.5541]

$$\lim_{\frac{\lambda}{\mu^2}\to\infty} \frac{T_c^{\mathrm{SUGRA}}}{\mu} = 0.105905(57)\,.$$

The prediction is for low temperatures and small μ the transition temperature approaches zero linearly in μ.





Padé approximant prediction of T_c

$$T_c = \frac{\mu}{12\ln 3}\left\{1 + r_1\frac{\lambda}{\mu^3} + r_2\frac{\lambda^2}{\mu^6} + \cdots\right\}$$

with $r_1 = \frac{2^6\times 5}{3}$ and $r_2 = -(\frac{23\times 19927}{2^2\times 3} + \frac{1765769\ln 3}{2^4\times 3^2})$

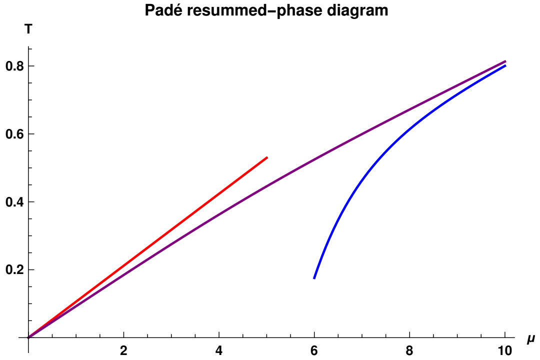
Using a Padé Approximant: $1 + r_1 g + r_2 g^2 + \cdots \rightarrow 1 + \frac{1+r_1 g}{1-\frac{r_2}{r_1}g}$

$$\implies T_c^{\text{Padé}} = \frac{\mu}{12\ln 3}\left\{1 + \frac{r_1\frac{\lambda}{\mu^3}}{1 - \frac{r_2}{r_1}\frac{\lambda}{\mu^3}}\right\}$$

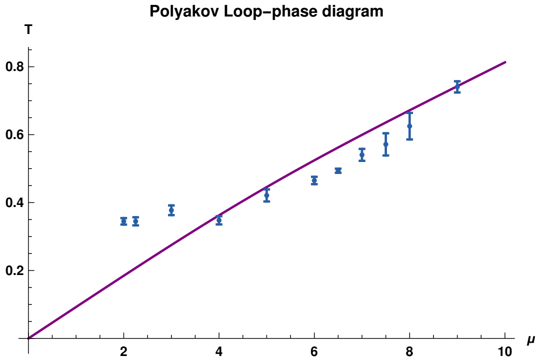
Now we can take the small μ limit

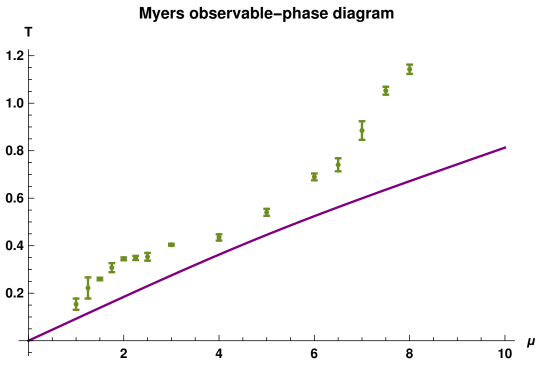
$$\lim_{\frac{\lambda}{\mu^2}\to\infty} \frac{T_c^{\text{Padé}}}{\mu} \simeq \frac{1}{12\ln 3}\left(1-\frac{r_1^2}{r_2}\right) = 0.0925579$$

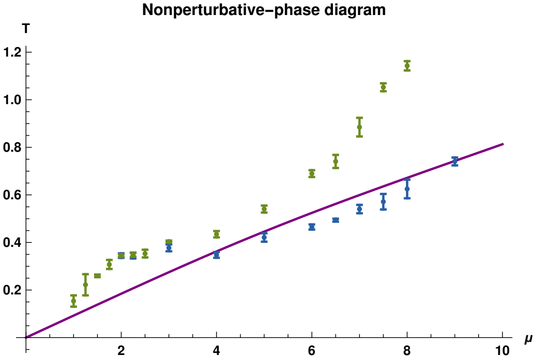
$$\lim_{\frac{\lambda}{\mu^2}\to\infty} \frac{T_c^{\text{SUGRA}}}{\mu} = 0.105905(57)\,.$$



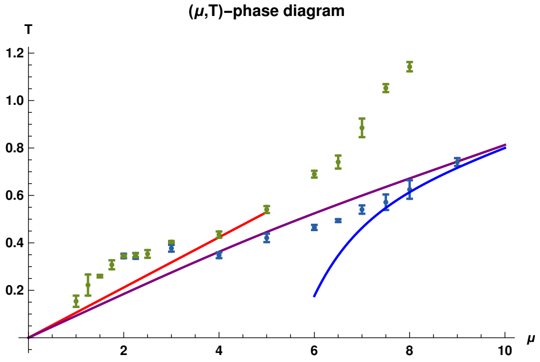
A non-perturbative phase diagram from the Polyalov Loop.




Myers observable–phase diagram
T
1.2
1.0
0.8
0.6
0.4
0.2
2
4
6
8
10
μ



Green Myers transition
Blue Polyakov loop transition
Purple Padé prediction for the transition


(μ,T)–phase diagram
T
1.2
1.0
0.8
0.6
0.4
0.2
2
4
6
8
10
μ

4-parameter Lattice discretisation

The bosonic lattice Laplacian

$$\Delta_{Bose} = \Delta + r_b a^2 \Delta^2 \,, \quad \text{where} \quad \Delta = \frac{2 - e^{aD_\tau} - e^{-aD_\tau}}{a^2} \,.$$

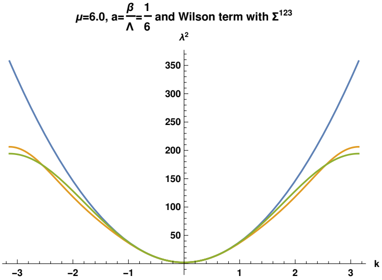
Lattice Dirac operator

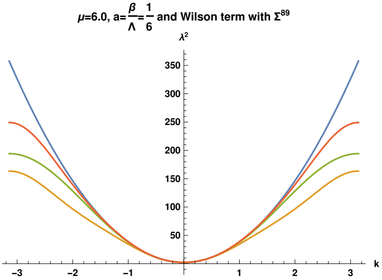
$$D_{Lat} = K_a \mathbf{1}_{16} - i\frac{\mu}{4}\gamma^{567} + \Sigma^{123} K_w \,, \quad \text{where} \quad \Sigma^{123} = i\gamma^{123} \,.$$

$$K_a = (1-r)\frac{e^{aD_\tau} - e^{-aD_\tau}}{2a} + r\frac{e^{2aD_\tau} - e^{-2aD_\tau}}{4a} \qquad \text{lattice derivative}$$

$$K_w = r_{1f}\, a\Delta + r_{2f}\, a^3 \Delta^2 \qquad \text{the Wilson term}$$

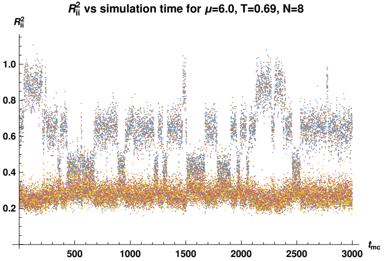
Lattice Dispersion relations

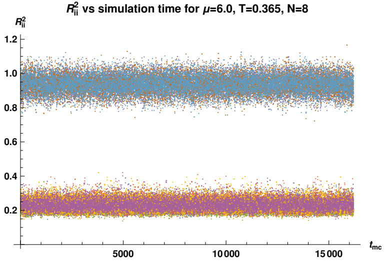


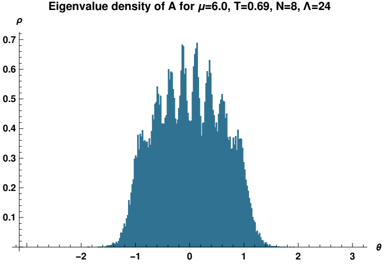


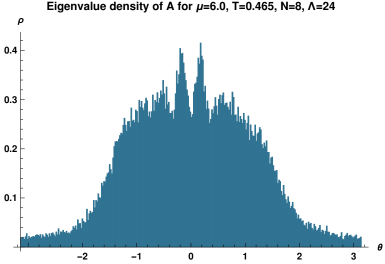
Eigenvalues $k^2 + \frac{\mu^2}{4}$ (blue parabola),
$\Delta_{Bose} + \frac{\mu^2}{4}$ light green,
Σ^{89} splitting red and orange curves.

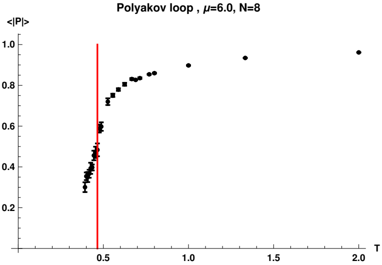
Observables

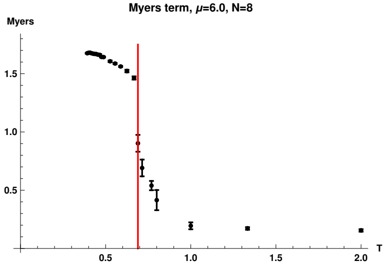


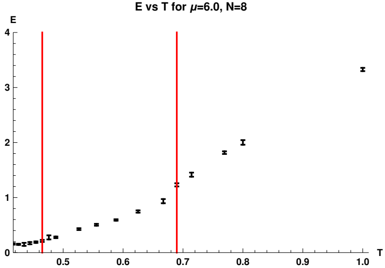






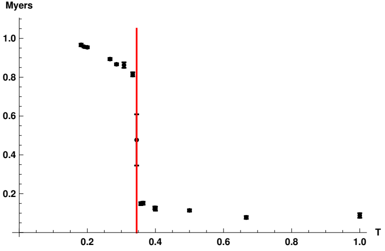

Polyakov loop , μ=6.0, N=8
<|P|>
T


Myers term, μ=6.0, N=8
Myers
T

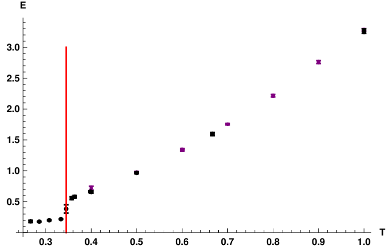

E vs T for μ=6.0, N=8
E
T

Small μ

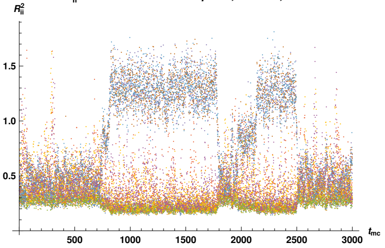
Myers term, μ=2.0, N=8, Λ=24



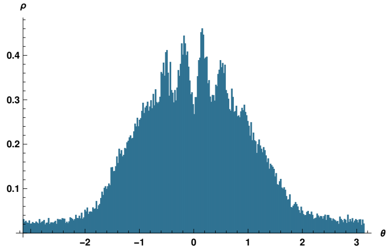
E vs T for μ=2.0, N=8



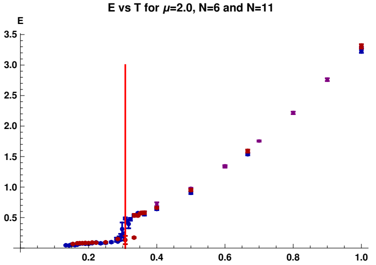
R^2_{ii} vs simulation time for μ=6.0, T=0.345, N=8

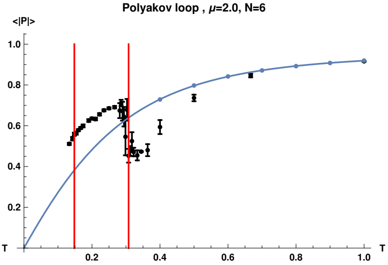


Eigenvalue density of A for μ=2.0, T=0.345, N=8, Λ=24



Non-monotonic Polyakov loop





Where do we go from here

- Study the bosonic BMN model—its phase diagram, theoretical predictions.
- Implications of $SU(4|2)$ symmetry.
- M2-branes.
- Probe BMN with D4-branes—already coded.
- $N = 1^*$ model — at coding stage.
- $N = 2$ models.
- Black dual geometries?
- M5-brane matrix models?