Inria

MATHÉMATIQUES & INTERACTIONS

UNIVERSITÉ CÔTE D'AZUR

Funded by the European Union

European Research Council
Established by the European Commission

CIRCE project

New directions in Theoretical Physics 5 - Edinburgh - January 5th 2026

Blind identification of hidden (dynamo) equilibria

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with

Paul Mannix, Yannick Ponty, Calum Skene, Steve Tobias

   

The dynamo instability

Dynamo instability [Larmor 1919]: results from electromagnetic induction by the flow of an electrically conducting fluid.

$$\frac{\partial}{\partial t} b = \text{curl } (u \times b) + \eta \, \Delta b$$

$$\text{div } b = 0$$

$$\frac{\partial}{\partial t} u + u \cdot \text{grad}\, u = - \text{grad}\, p + \nu \Delta u + \text{curl}\, b \times b + f$$

$$\text{div } u = 0$$

with $b = \dfrac{B}{\sqrt{\rho\mu}}$ the magnetic field, u the velocity field, p the pressure field, f any additional forcing,

ν the kinematic viscosity and η the magnetic diffusivity.

Dynamo is defined to take place when $\exists\ \eta, b_0$ such that (*) has a solution $b(\cdot,0) = b_0$ and $\lim_{t \to +\infty} \|b(x,t)\|_{L^2} \neq 0$.

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$b = 0$ is always a (boring) equilibrium!.... **...but is it (linearly) stable?**

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$b = 0$ is always a (boring) equilibrium!.... ...but is it (linearly) stable?

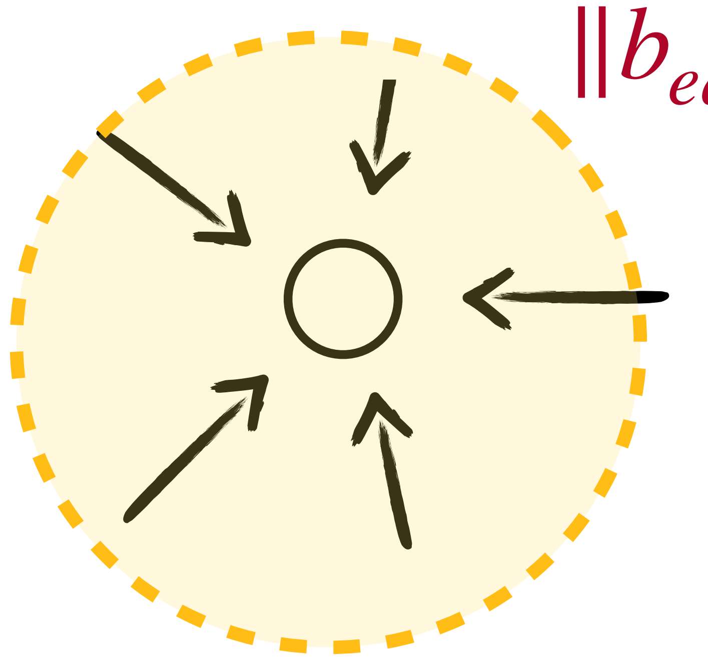
(Unfortunately when u is "too simple", the answer is generally yes.)

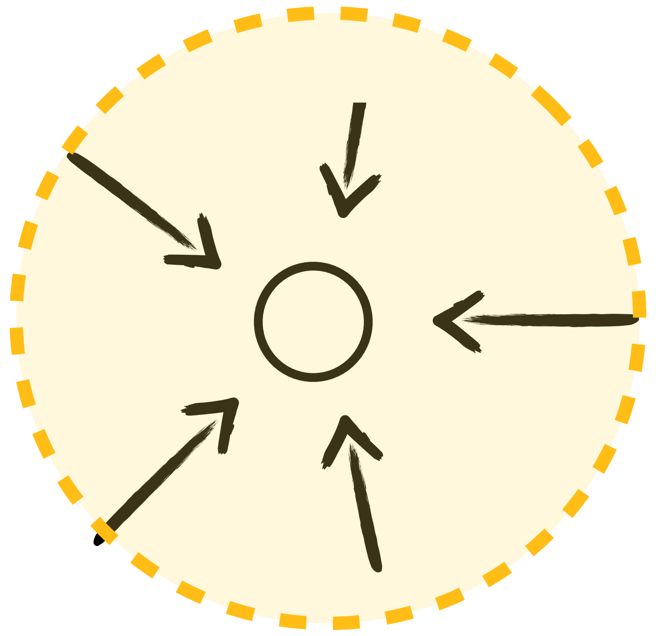
Search for subcritical dynamo equilibria

(Phase space)

Stable, magnetic (dynamo) equilibrium

$\|b_{eq}\|_2 \neq 0$



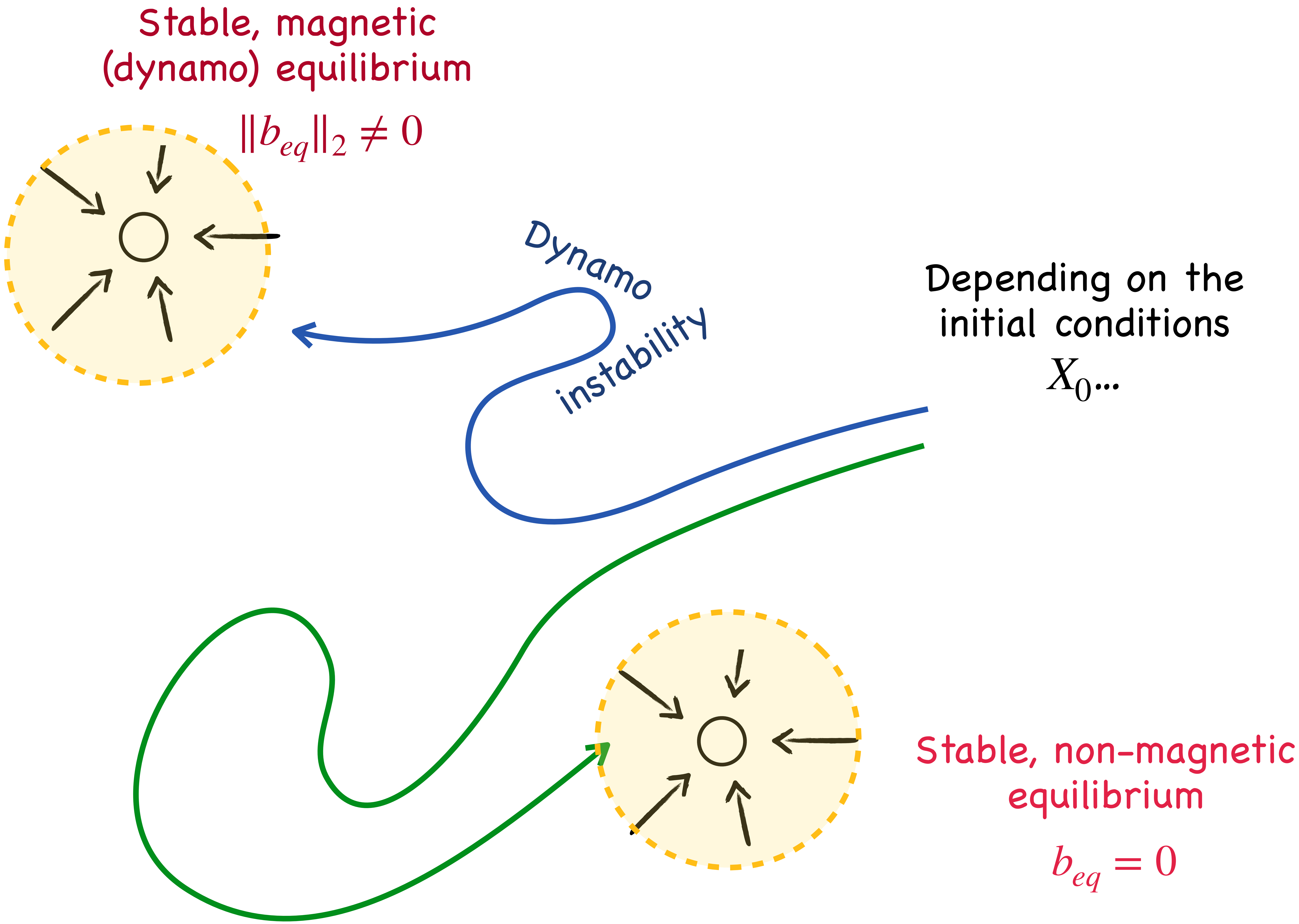


Stable, non-magnetic equilibrium

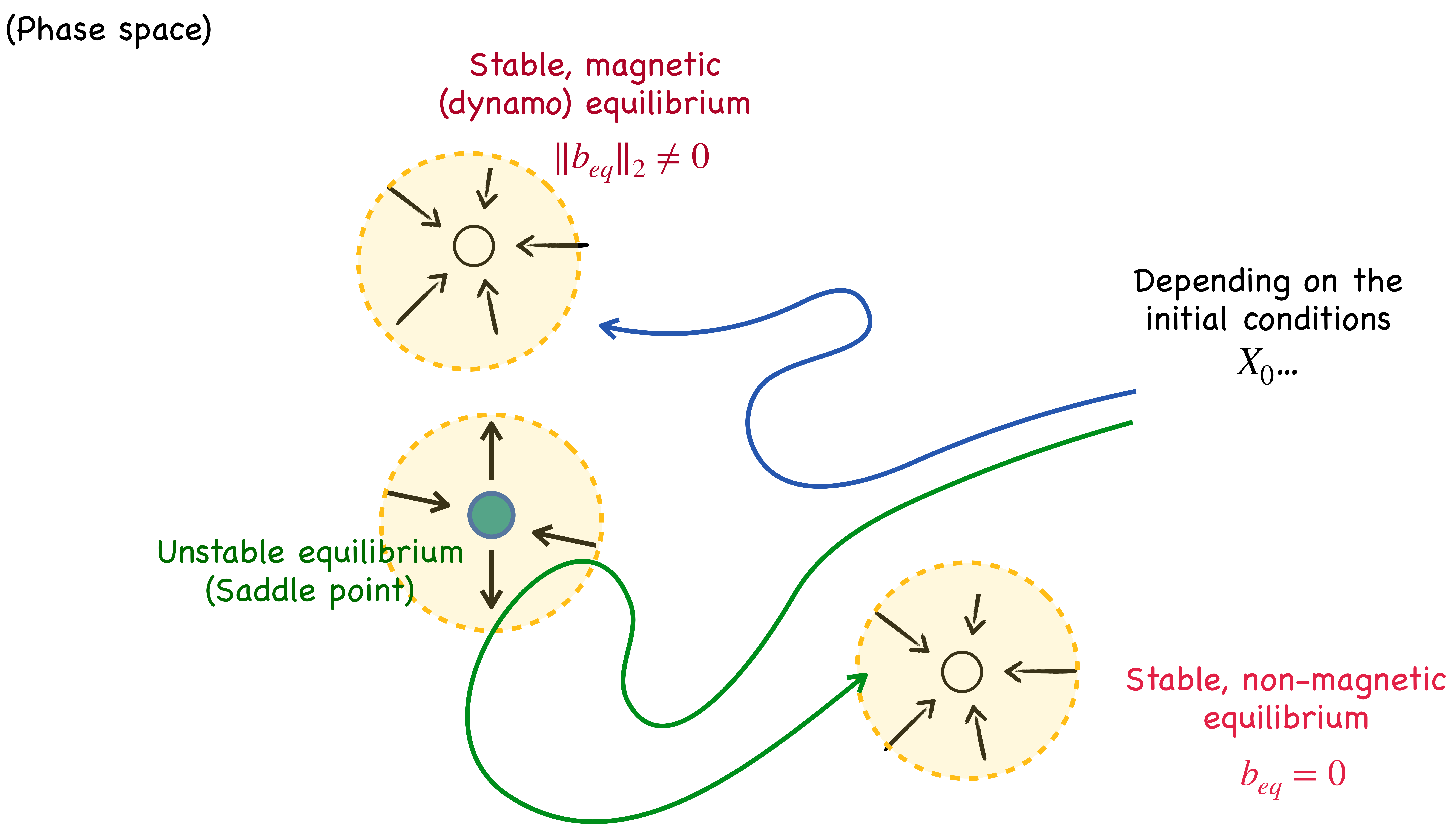
$b_{eq} = 0$

Search for subcritical dynamo equilibria

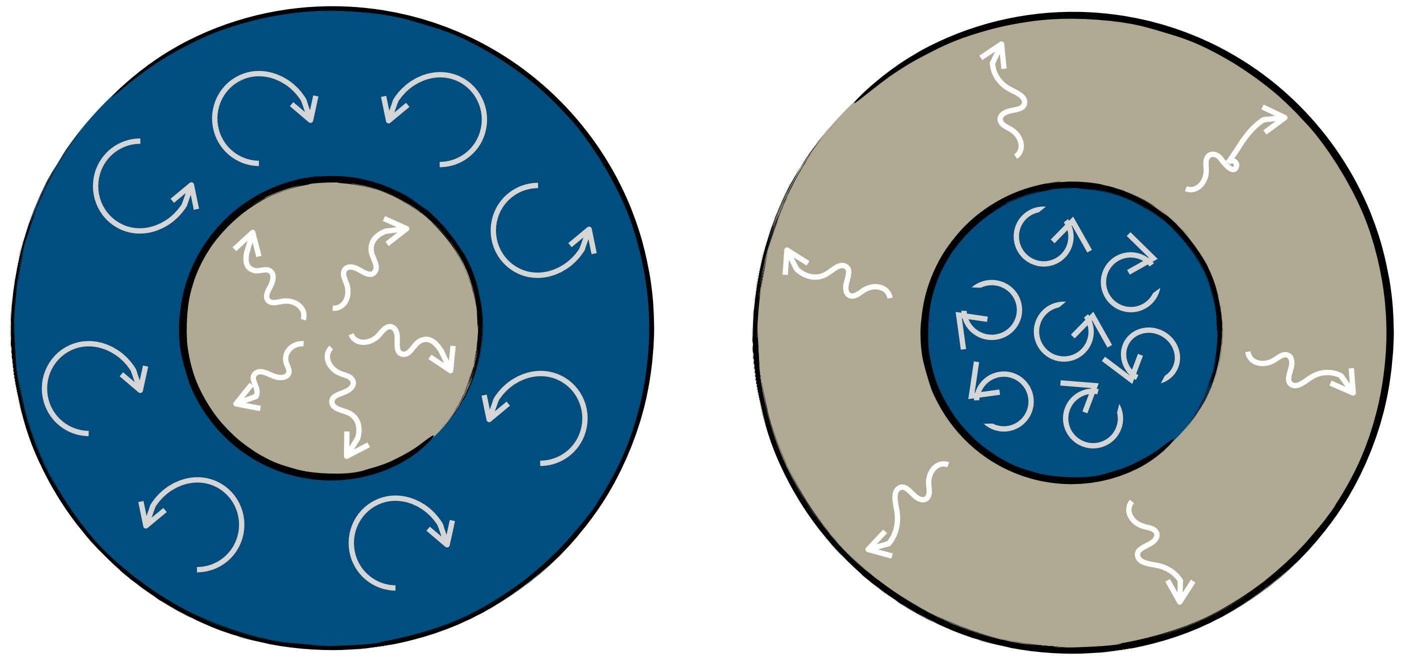
(Phase space)



Search for subcritical dynamo equilibria



Stars: radiative zone magnetism?



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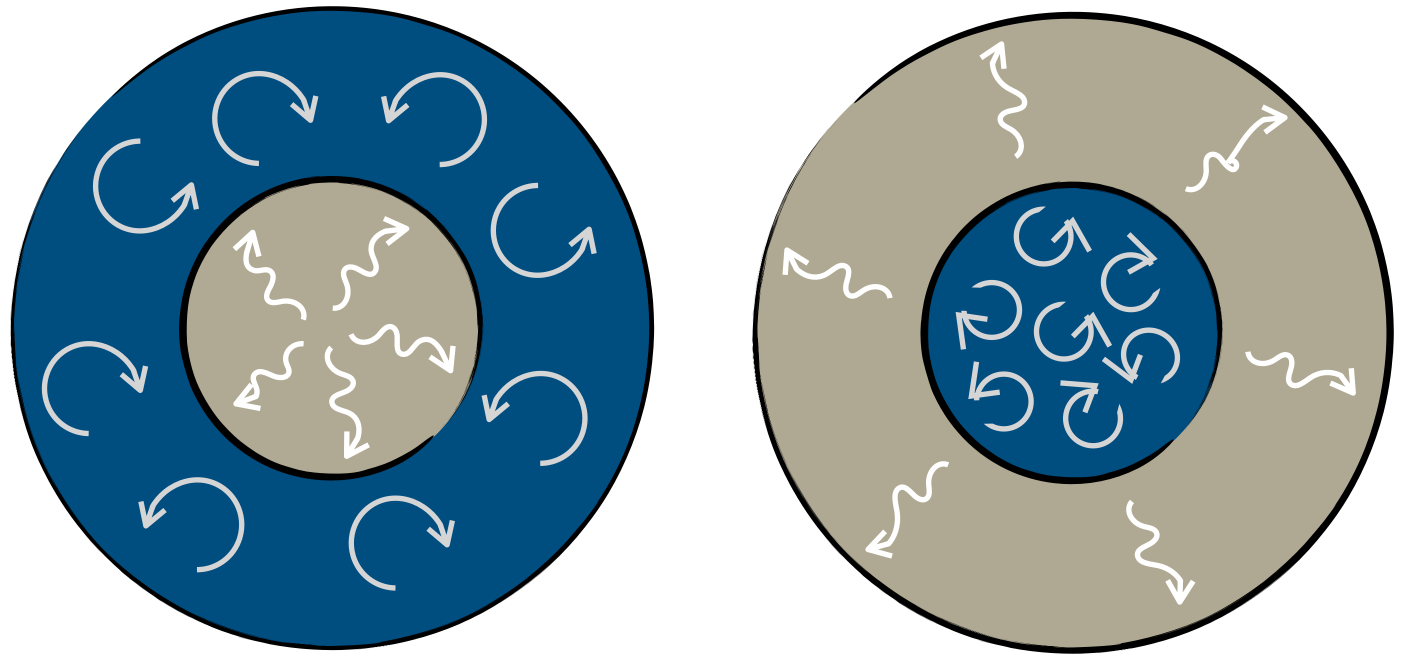
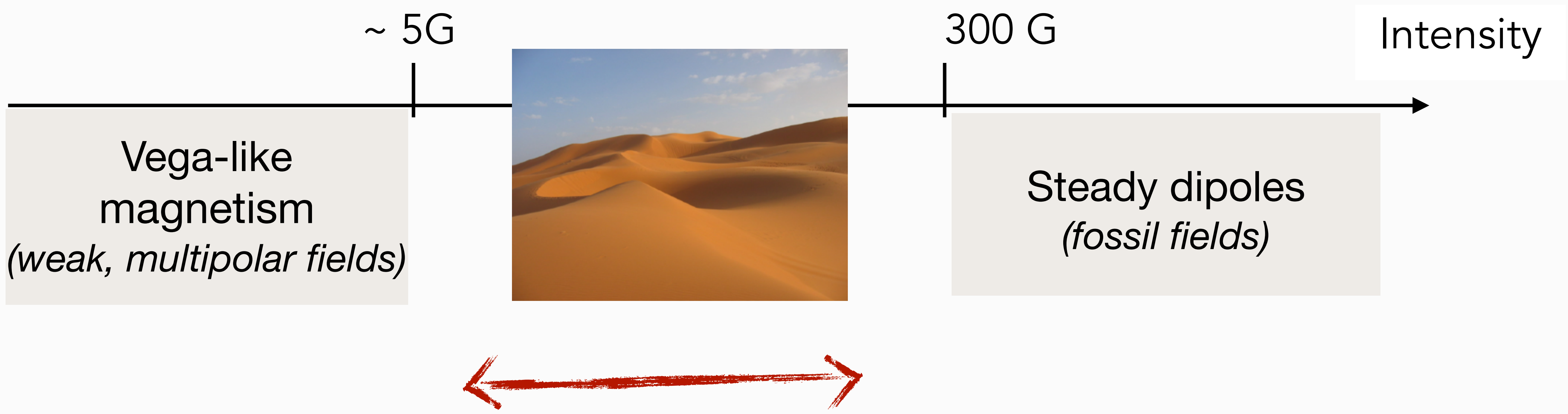


The "magnetic desert" of Ap-Bp stars [Lignières et al., Proc IAU 2013]

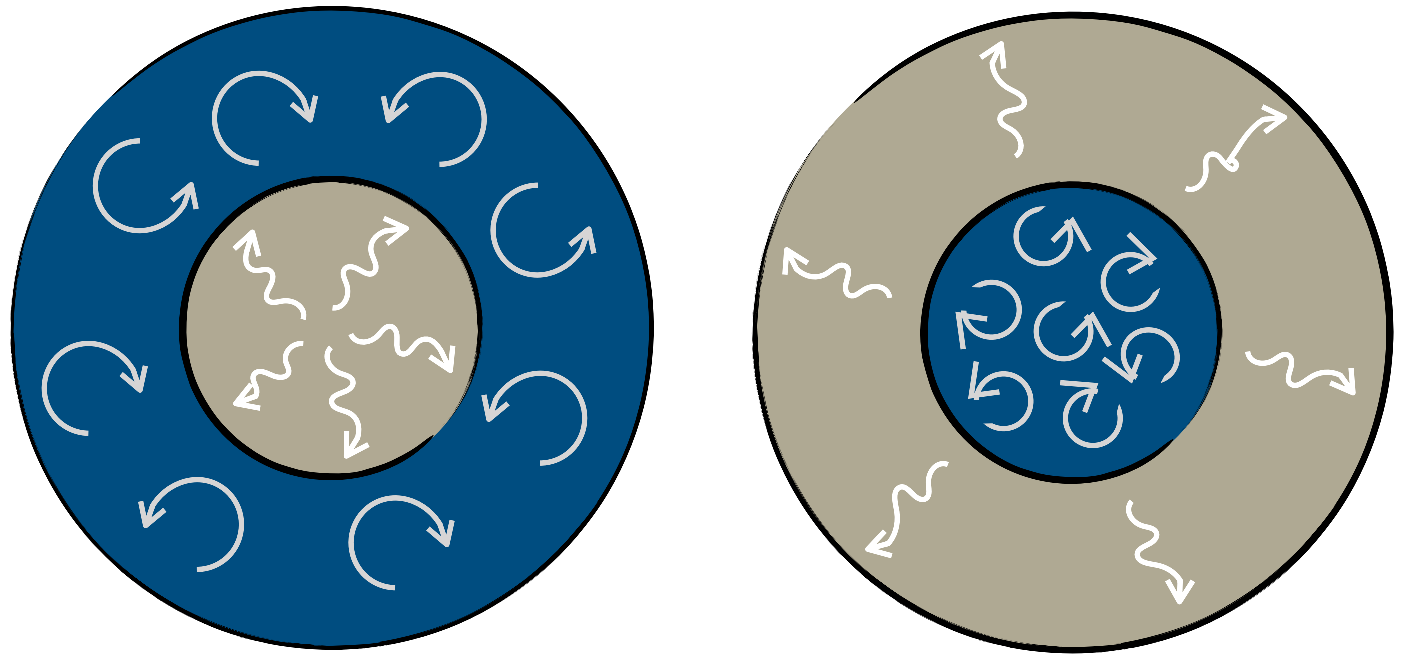


Same type of stars

Different evolution?

Some GAFD motivations: radiative and keplerian dynamos...

Stars: radiative zone magnetism?



The "magnetic desert" of Ap-Bp stars [Lignières et al., Proc IAU 2013]

~ 5G

300 G

Intensity

Vega-like magnetism
(weak, multipolar fields)

Steady dipoles
(fossil fields)

Same type of stars

Different evolution?

The Tayler-Spruit dynamo [Spruit, A&A 2001]

simulations from [Petitdemange, M. & Gissinger, Science 2023]:

Zero G

2×10^5 G

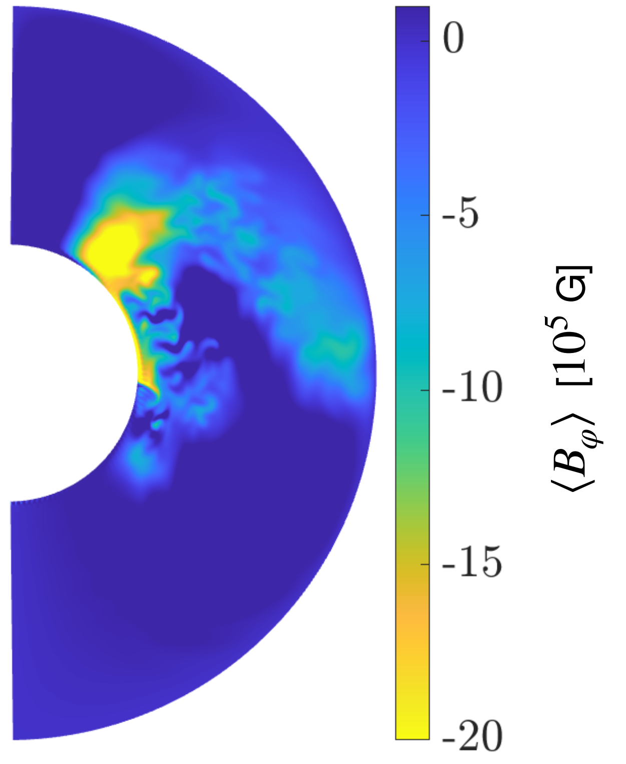
Intensity

No magnetism

Strong magnetism

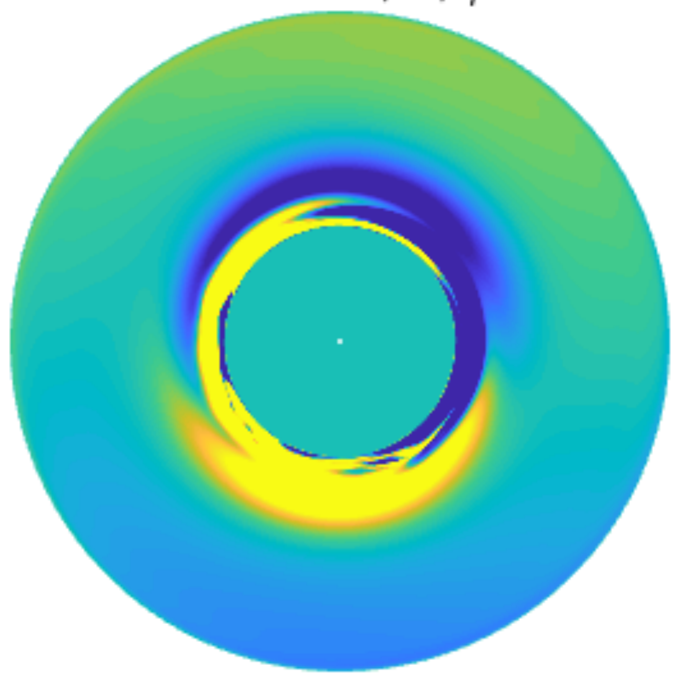
Same parameters

Different initial conditions!



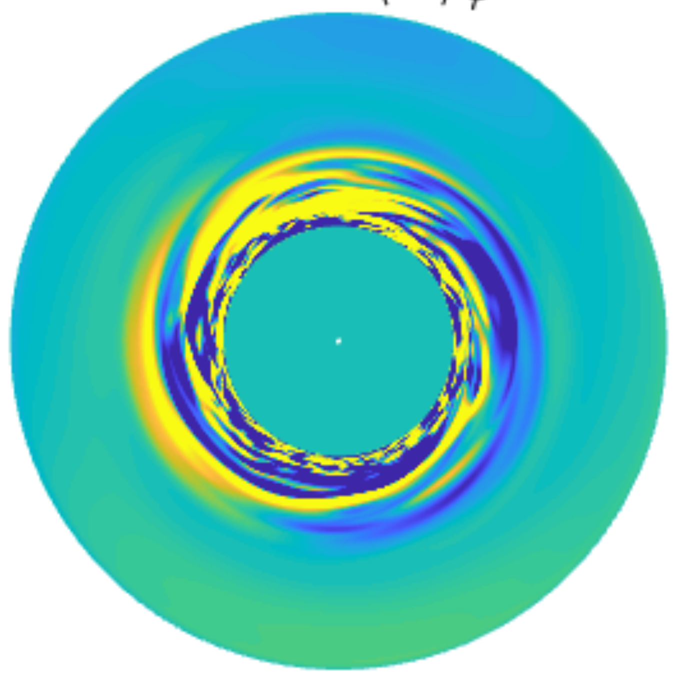
No turbulence

$\Omega - \langle \Omega \rangle_\varphi$

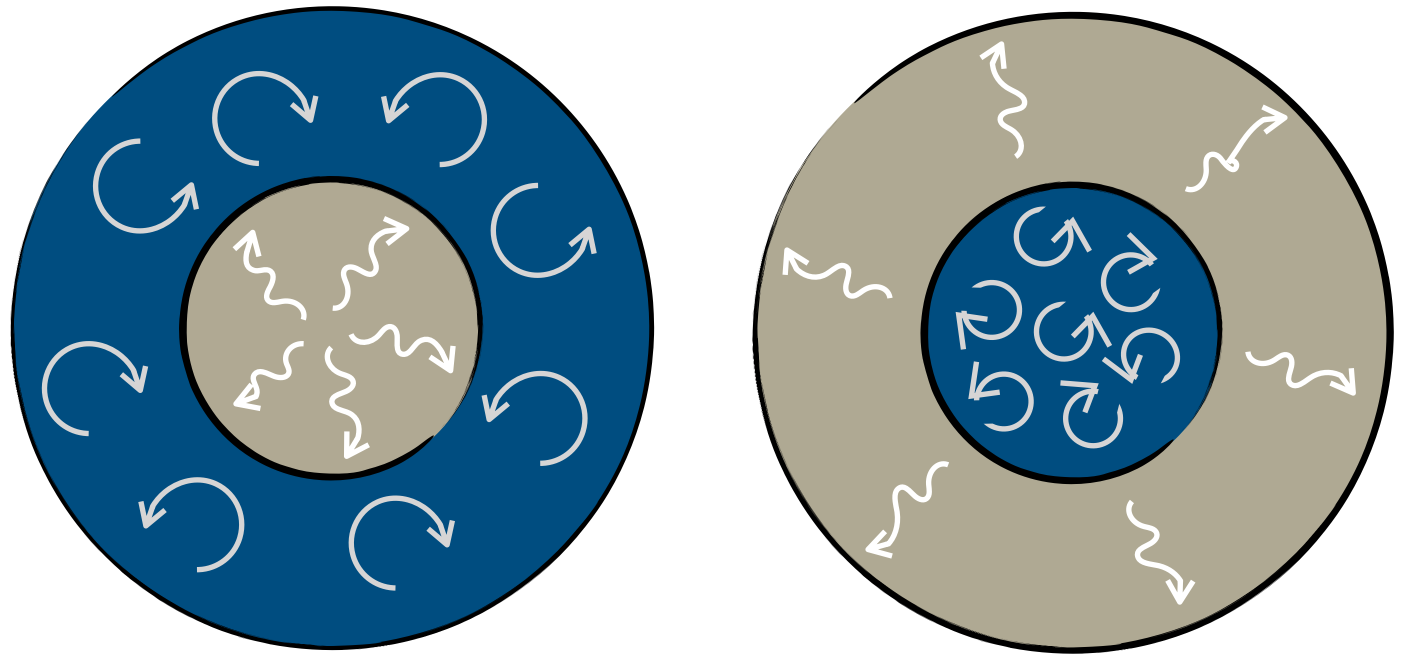


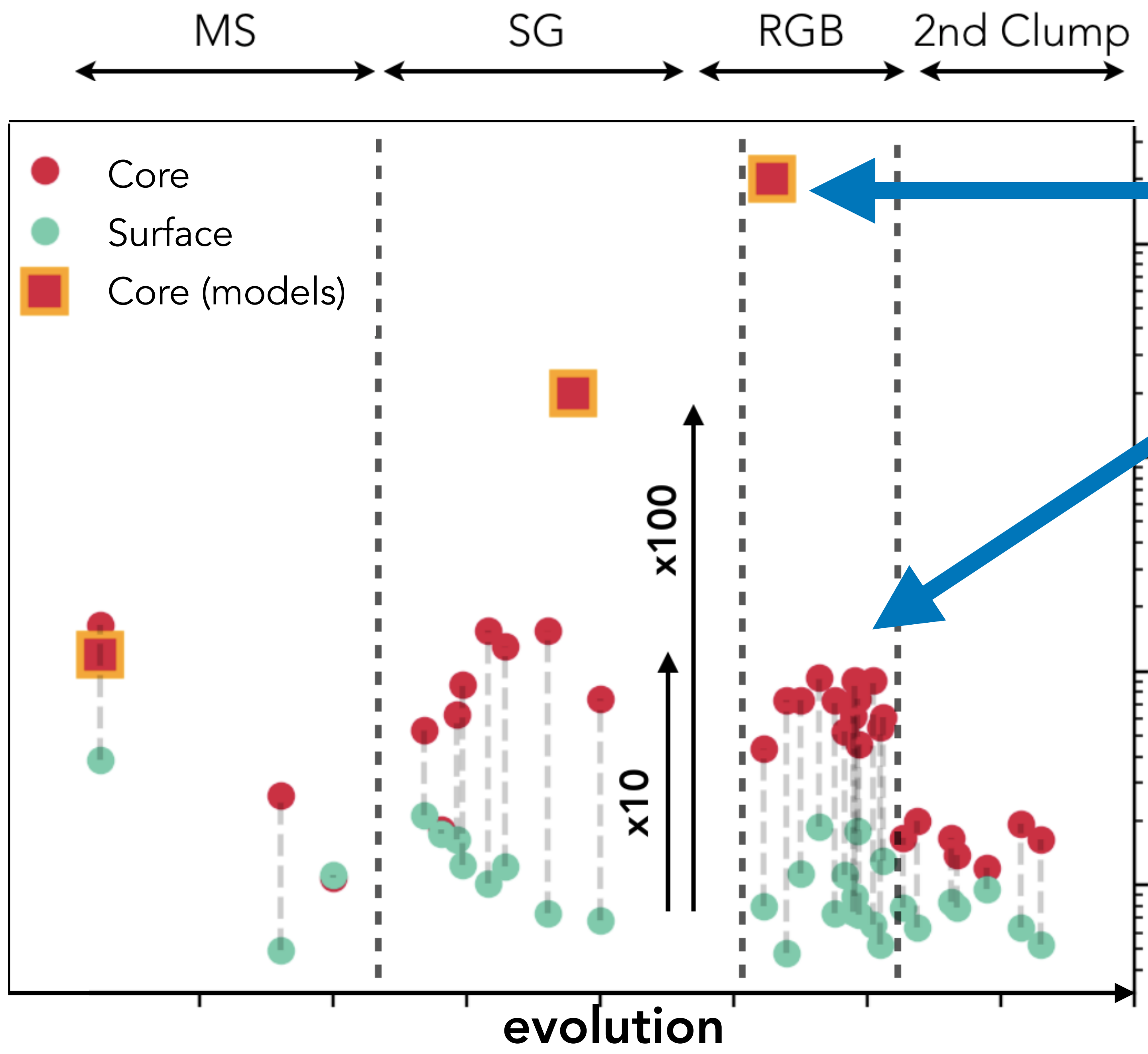
Turbulent flow

$\Omega - \langle \Omega \rangle_\varphi$



Stars: radiative zone magnetism?





[From Bugnet PhD thesis 2020]

Models

Observations

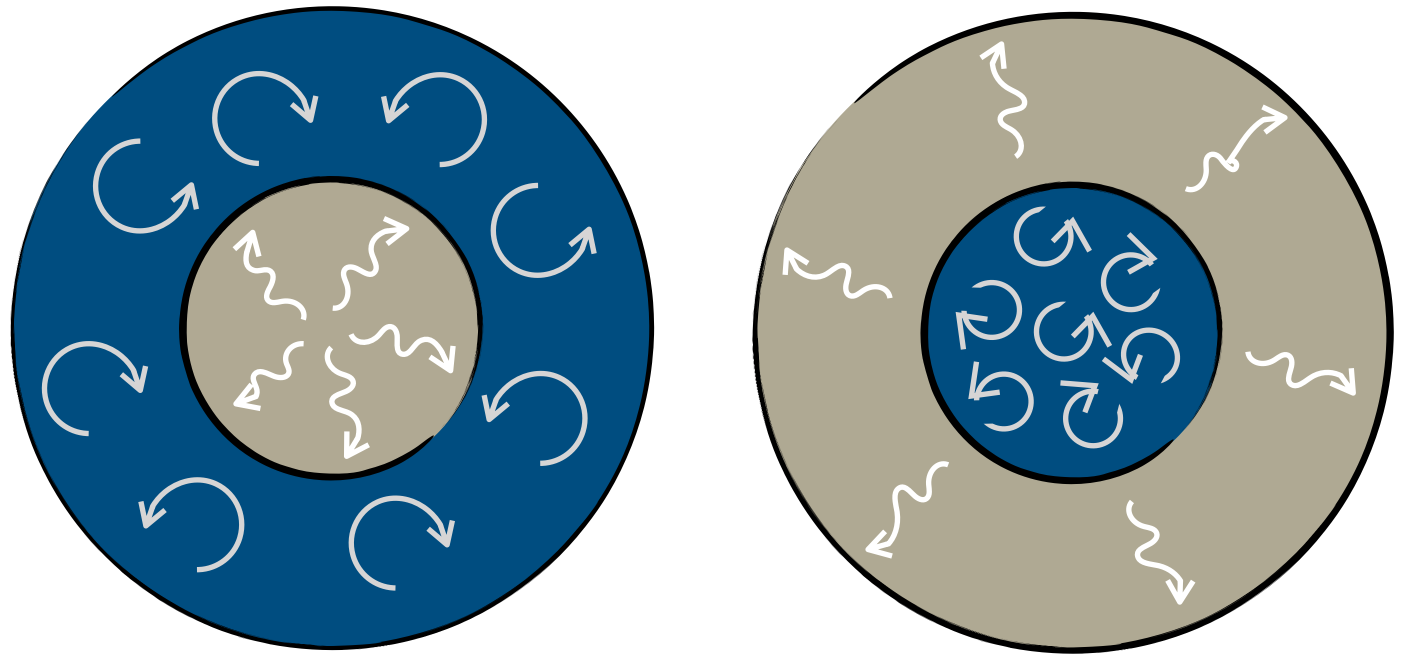
mismatch by 1 or 2 orders of magnitude in (radiative) cores

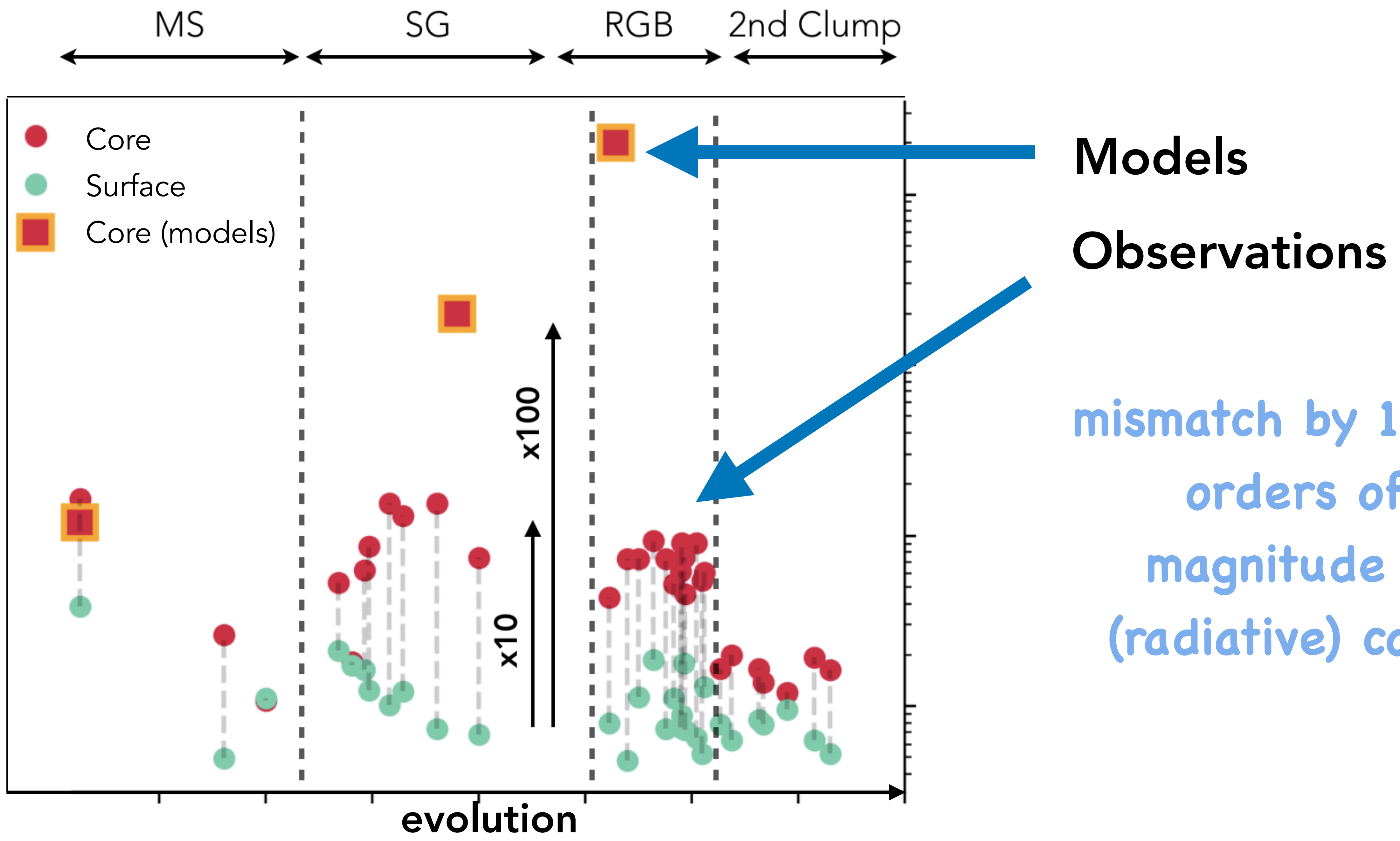
Need to parametrize magnetic stresses in stellar models:

$$\nu_{eff} = \frac{\langle b_r b_\varphi \rangle}{r\partial_r \langle \Omega \rangle} = ?$$

Some GAFD motivations: radiative and keplerian dynamos...

Stars: radiative zone magnetism?

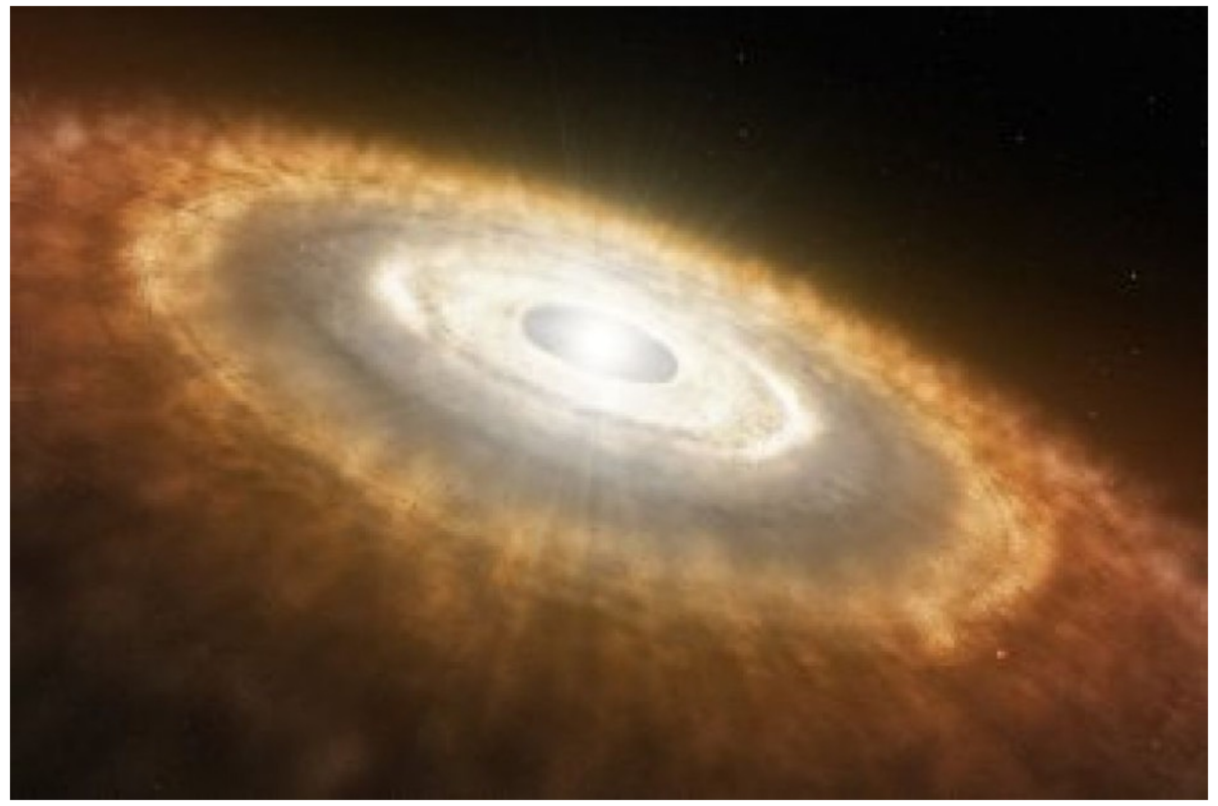




[From Bugnet PhD thesis 2020]

mismatch by 1 or 2 orders of magnitude in (radiative) cores

Keplerian discs dynamos?



Need to parametrize magnetic stresses in stellar models:

$$\nu_{eff} = \frac{\langle b_r b_\varphi \rangle}{r \partial_r \langle \Omega \rangle} = ?$$

Similar issue with protoplanetary discs:

ν_{eff} (inferred from observations) $\gg \nu$ (molecular).

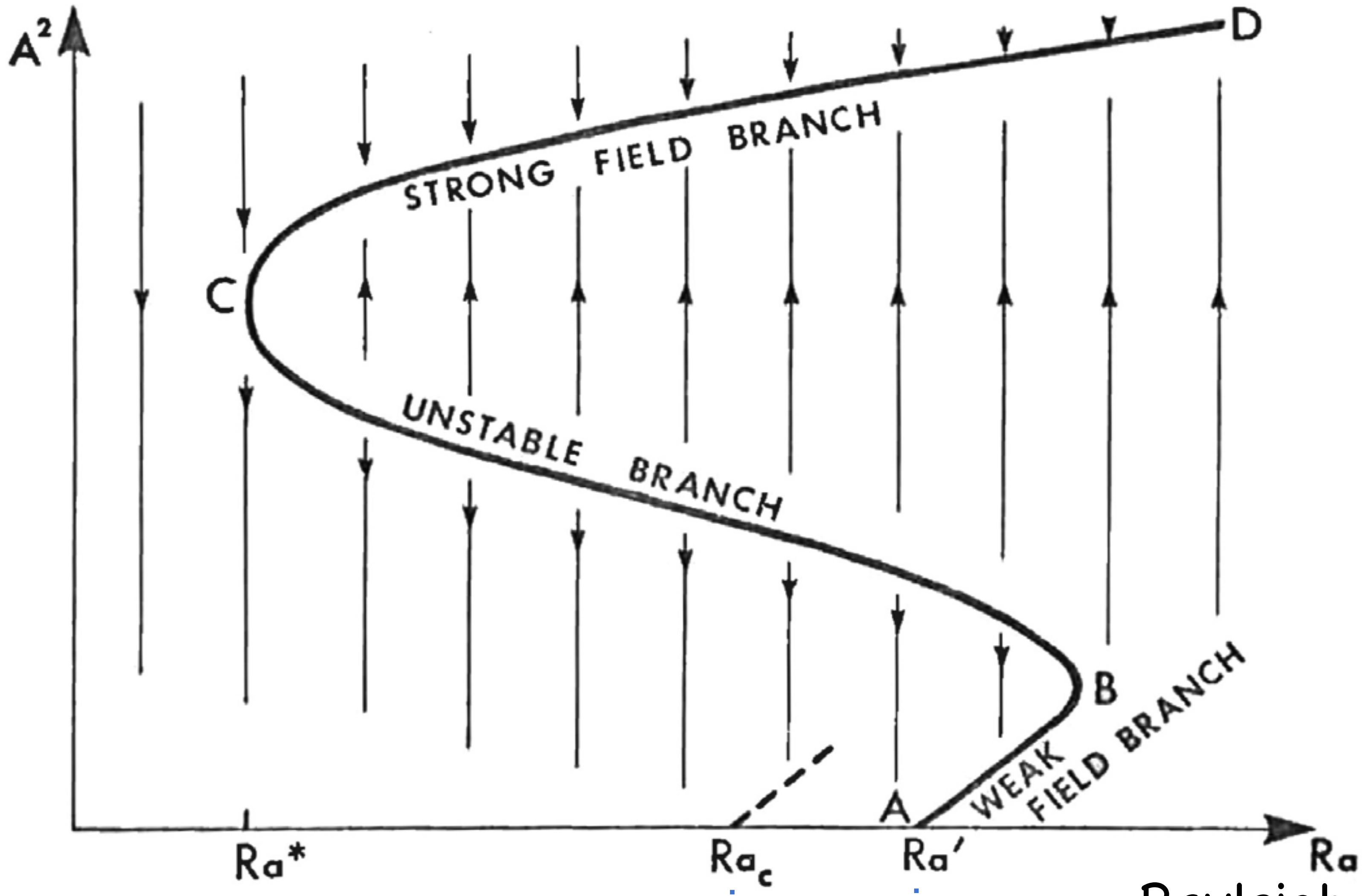
[e.g. Lin & Papaloizou, Ann Rev Astron Astrophys 1995].

—> Understand transport of angular momentum by dynamo magnetic fields?

Some GAFD motivations: the strong branch of the Geodynamo...

energy
of dynamo
equilibrium
$\|b_{eq}\|_2^2$
(conjecture!)

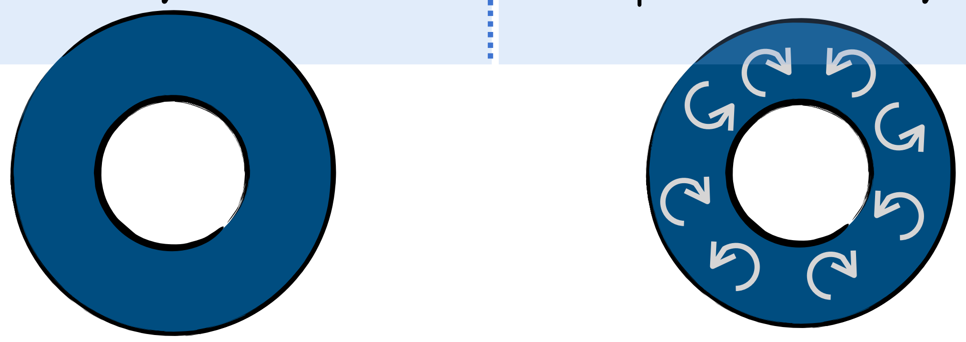
[from Roberts 1978]



Rayleigh number
(intensity of thermal driving)

b_{eq}=0 linearly stable | b_{eq}=0 unstable

u_{eq}=0 linearly stable | u_{eq}=0 linearly unstable



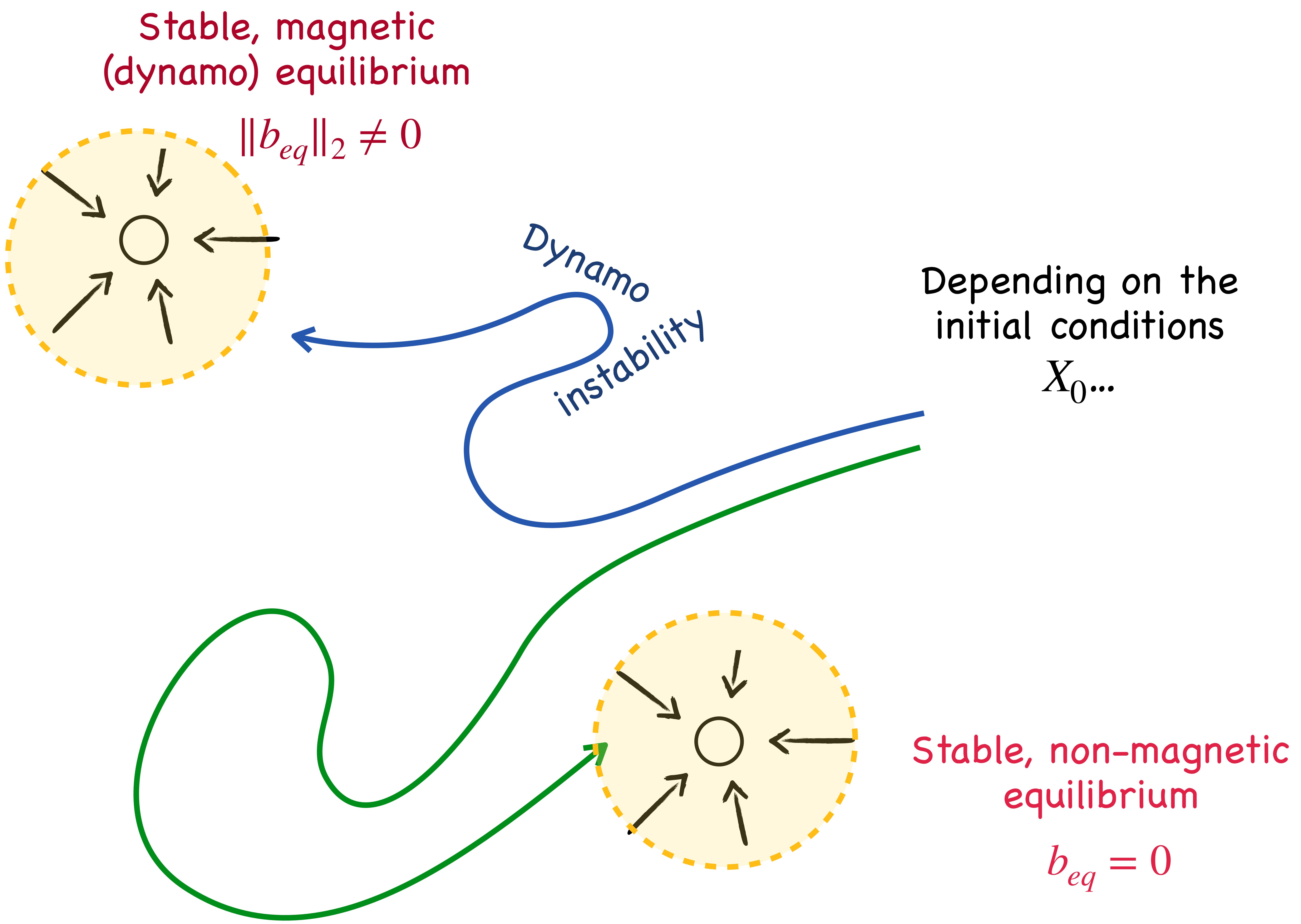
Earth's core:

$$E = \frac{\nu}{\Omega L^2} \approx 10^{-15} \quad ; \quad \mathrm{Ra} = ??$$

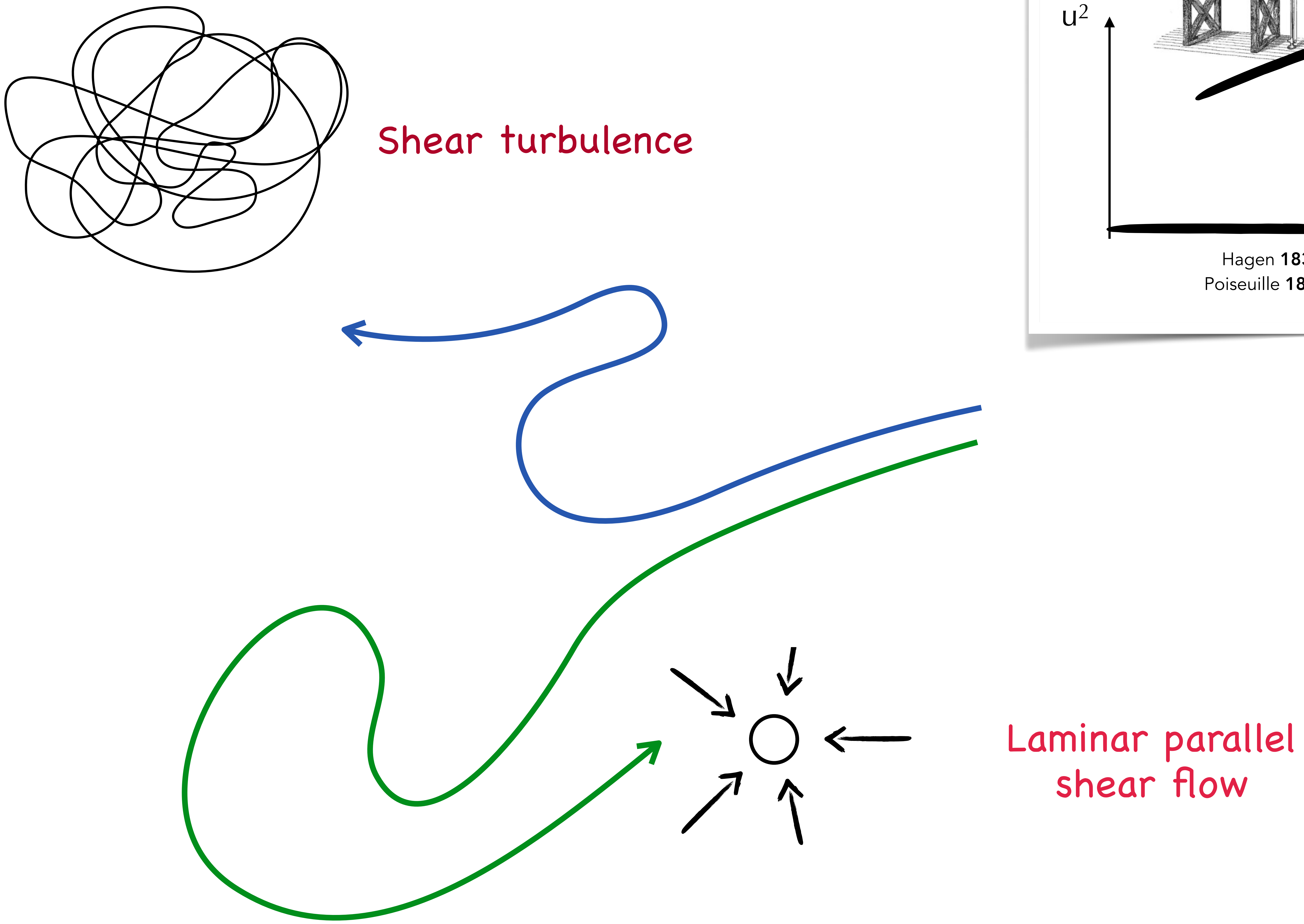
—> Does the strong branch even exist?
What does it look like?

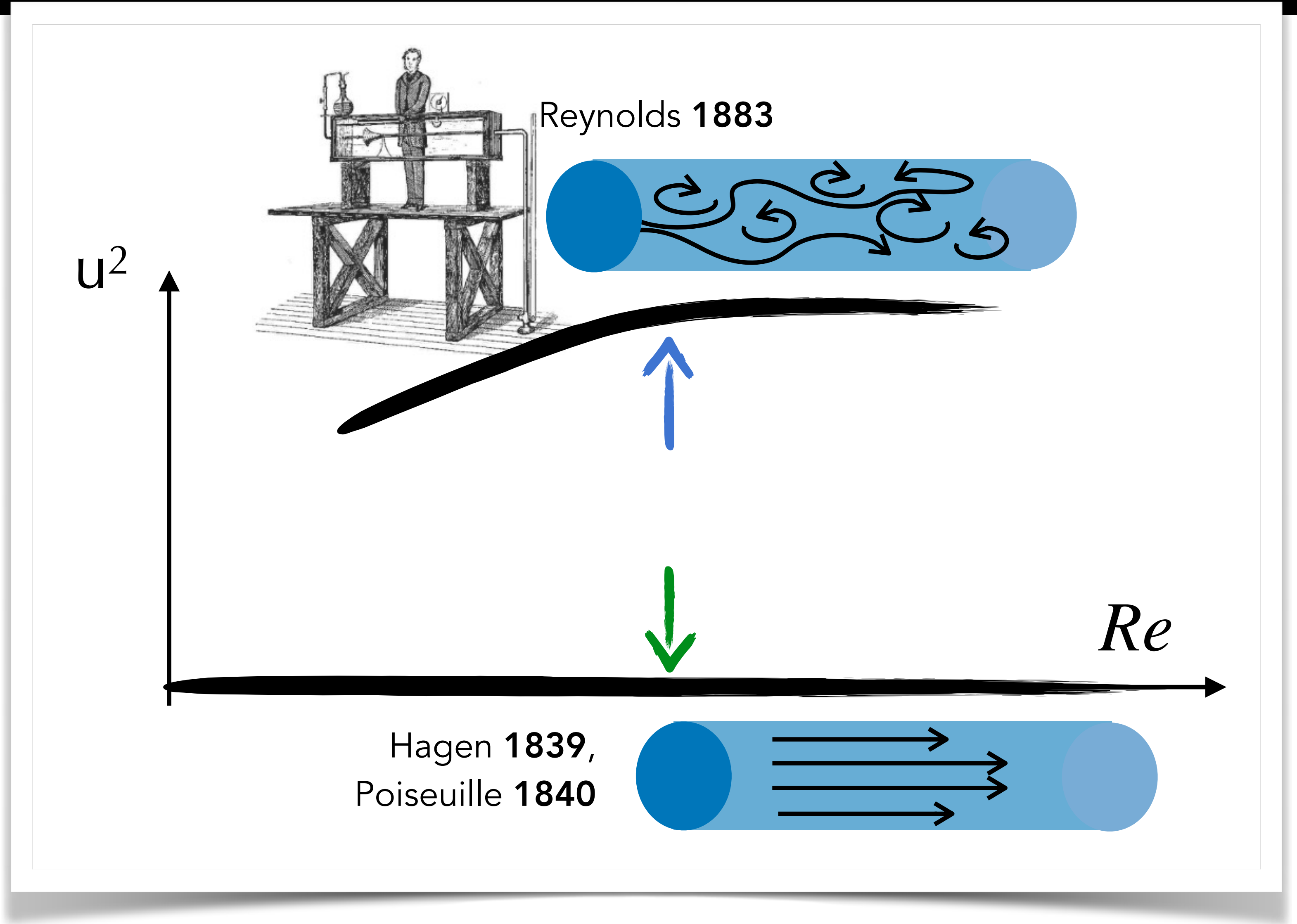
Could it drive convection for $Ra<Ra_c$?

Tracking subcritical (dynamo) equilibria



Turbulence in shear flows: basin boundaries and minimal seeds





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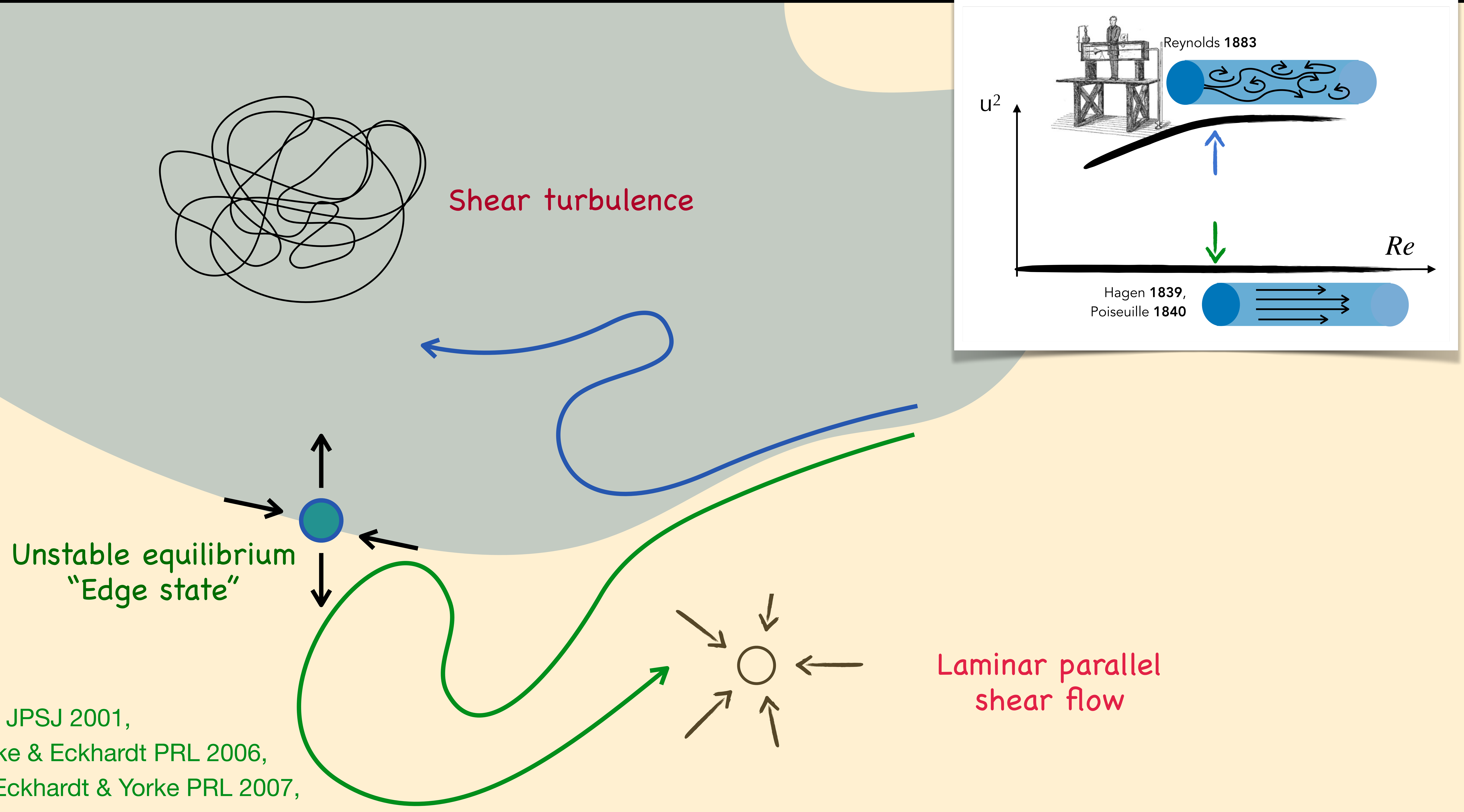

[Toh & Itano JPSJ 2001,
Skufca, Yorke & Eckhardt PRL 2006,
Schneider, Eckhardt & Yorke PRL 2007,
Duguet, Willis & Kerswell JFM 2008…]

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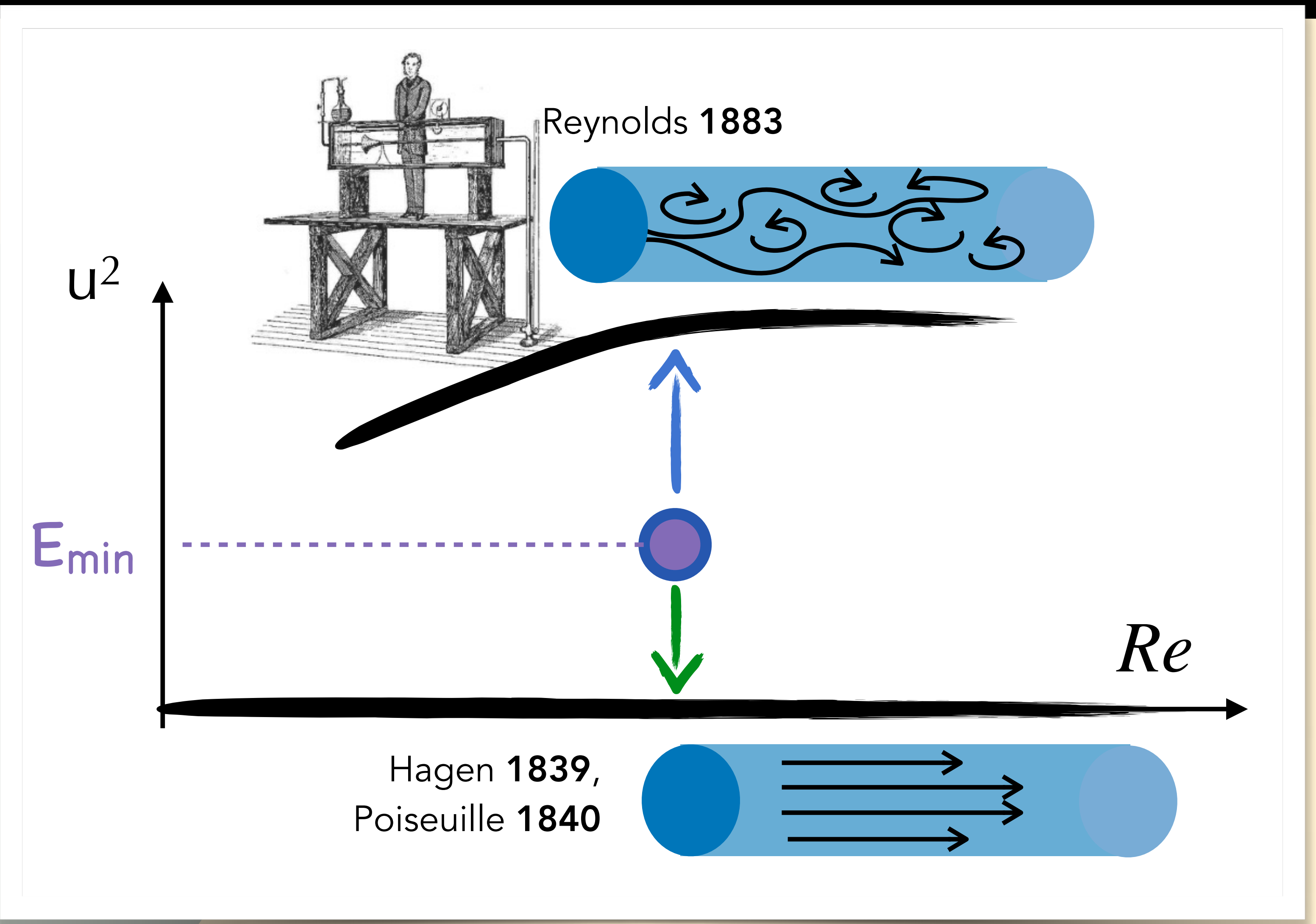

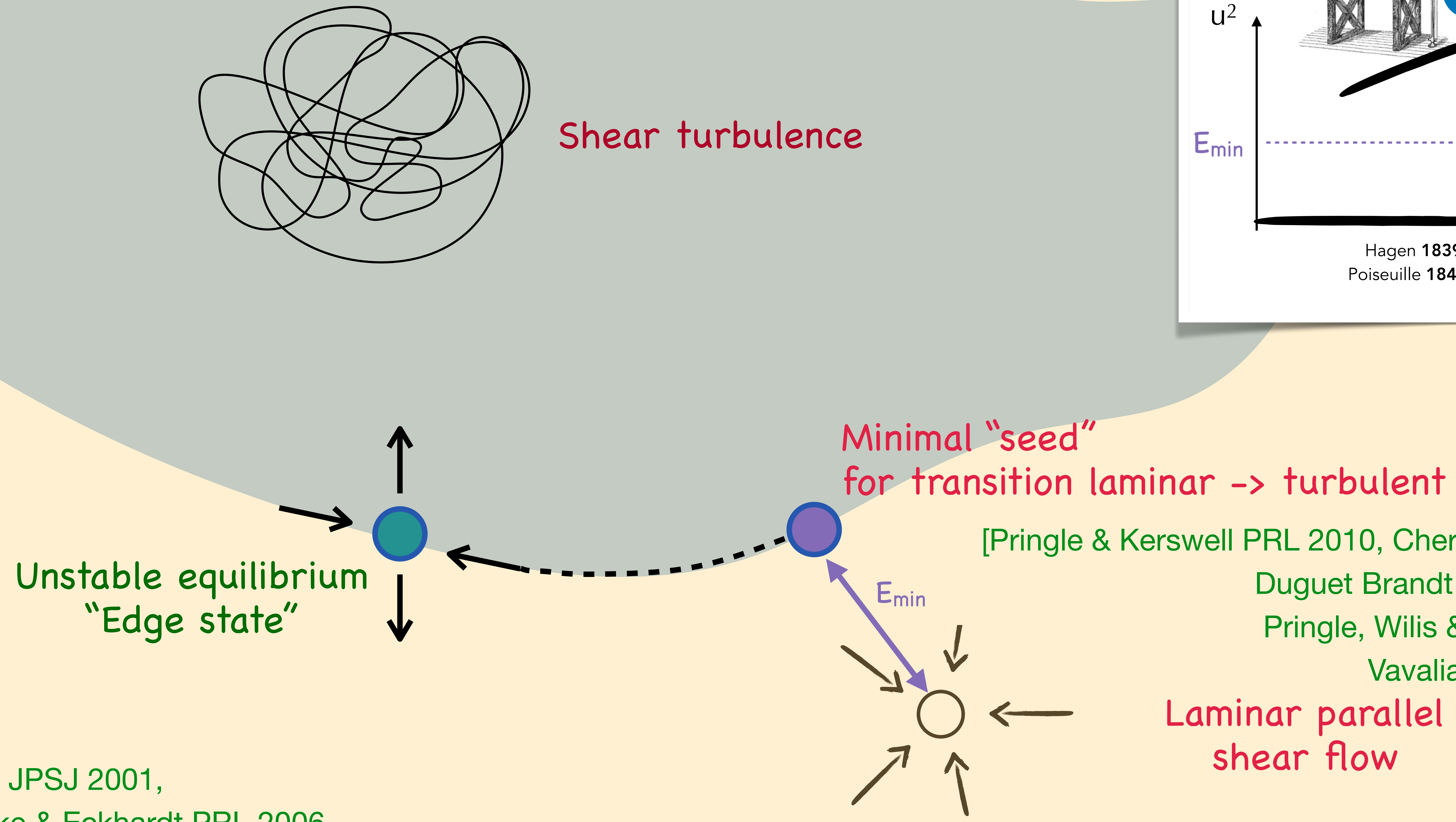

[Pringle & Kerswell PRL 2010, Cherubini et al. PRE 2010,
Duguet Brandt & Larsson PRE 2010,
Pringle, Wilis & Kerswell, JFM 2012,
Vavaliaris et al. PRF 2020…]

[Toh & Itano JPSJ 2001,
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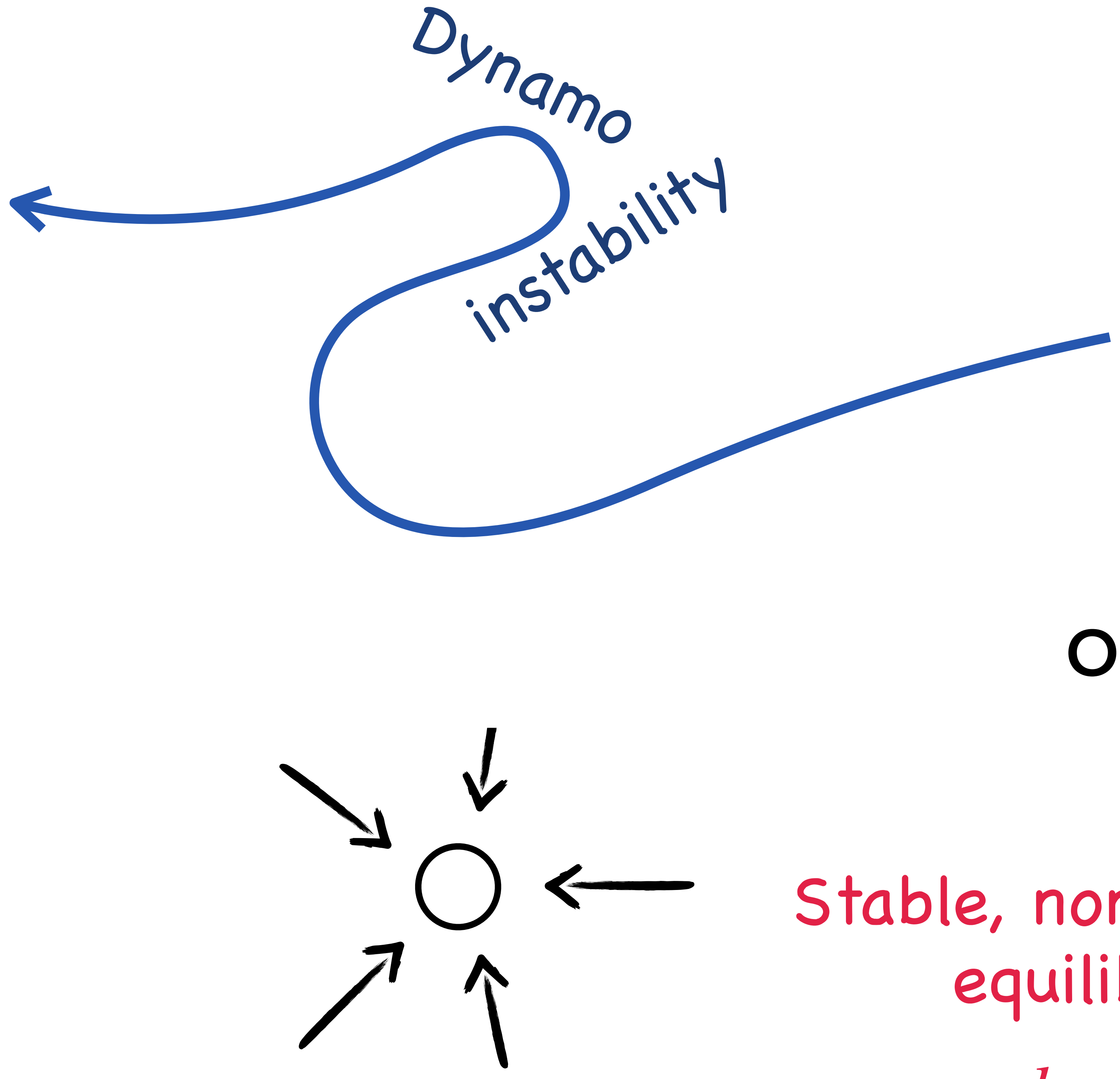
Problem: it is usually difficult to **blindly kickstart** a (nonlinear) dynamo!

Dynamo solution ?

$\|b_{eq}\|_2 \neq 0$



Option 0 : luck (or accident)

Stable, non-magnetic equilibrium

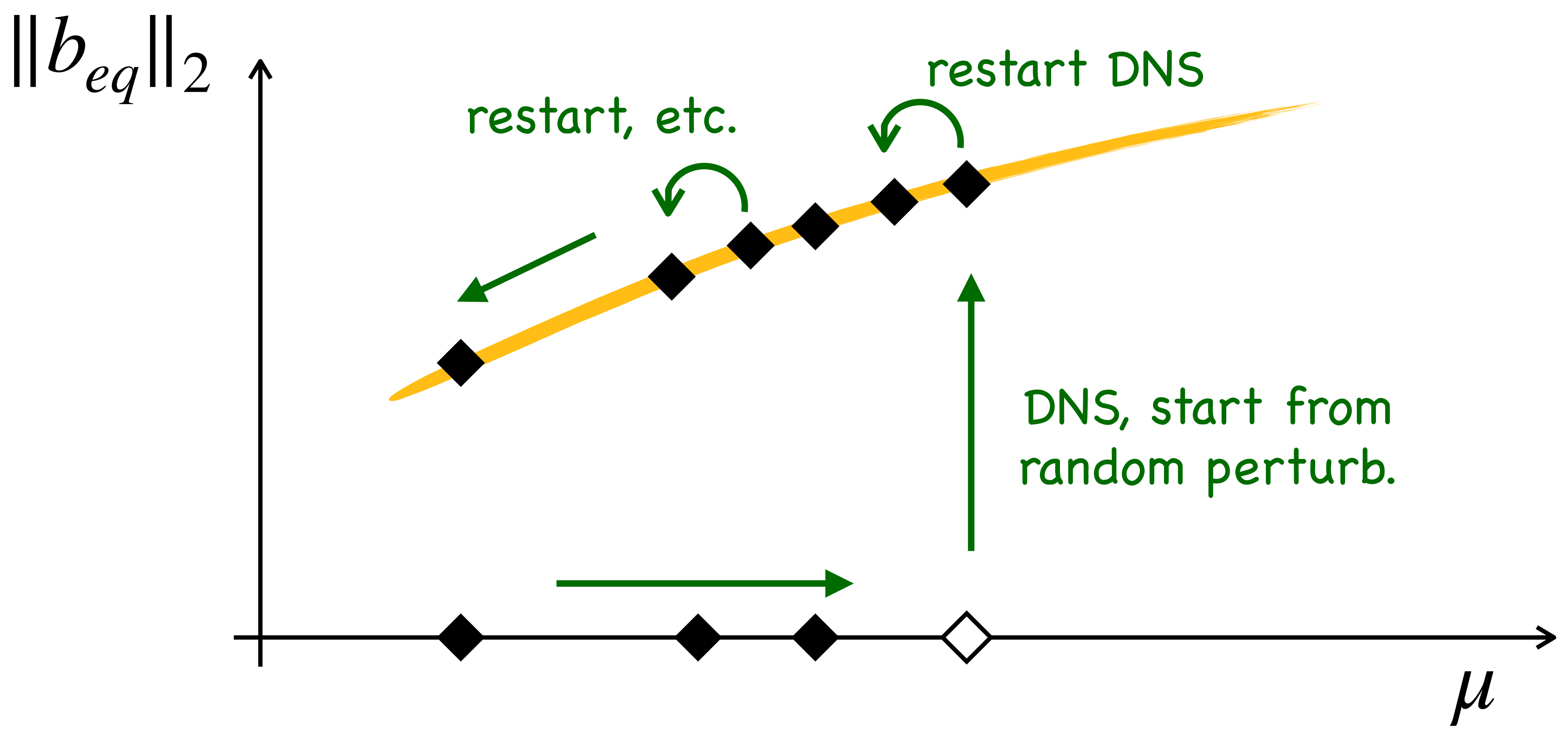
$b_{eq} = 0$

Problem: it is usually difficult to **blindly kickstart** a (nonlinear) dynamo!

Dynamo solution $\|b_{eq}\|_2 \neq 0$

?

Option 1: time-stepping = (re)start from a previous solution and follow stable branches



+ Can compute attracting turbulent states statistically steady

- Requires b=0 to become linearly unstable (at affordable μ!!)

- The path in parameter space can be complicated

Problem: it is usually difficult to **blindly kickstart** a (nonlinear) dynamo!

Dynamo solution $\|b_{eq}\|_2 \neq 0$

?

$$\frac{\partial}{\partial t} = 0$$

Option 2: Compute fixed points and use continuation

+ Can compute unstable equilibria!

[e.g. Rincon, Ogilvie & Proctor PRL 2007, Deguchi JFM 2019, 2020…]

- Constraining (steady, TW…)

- Requires a good guess for Newton to converge

Method: transient growth optimisation

Remember that **dynamo is defined to take place when $\exists\ b_0$ such that (*) has a solution with $\lim_{t\to+\infty} \|b(x,t)\|_{L^2} \neq 0$.**

Let an initial energy M_0, a (long-ish) time horizon T, and the set of admissible controls S_{M_0}:

$$S_{M_0} = \{b_0 : x \in \Omega \mapsto b_0(x) \in \mathbf{R}^3 \ : \ \mathrm{div}\, b_0 = 0 \quad \text{and} \quad \|b_0\|_2^2 = M_0\}\ .$$

We address the "dynamo stability question" through the following optimisation problem: find

$$\arg\max_{b_0 \in S_{M_0}} J(b_0, u, b) := \int_0^T \|b(x,t)\|_2^2 \, dt,$$

subject to the state equations

$$\left.\begin{aligned}
\partial_t u + u \cdot \nabla u + \nabla p - \nu \Delta u - \mathrm{curl}\, b \times b - f = 0, \\
\partial_t b - \eta \Delta b - \mathrm{curl}\ (u \times b) = 0, \\
\mathrm{div}\ u = 0, \\
\mathrm{div}\ b = 0, \\
b(\cdot, 0) = b_0, \\
+ \text{ suitable b.c. for } u, b \text{ on } \Gamma.
\end{aligned}\right\} \quad (\star)$$

Method: transient growth optimisation

Remember that **dynamo is defined to take place when $\exists\ b_0$ such that (*) has a solution with $\lim_{t\to+\infty} \|b(x,t)\|_{L^2} \neq 0$.**

Fix the parameters $(\nu, \eta, \ldots)$. Fix T.

Fix M_0. Determine the nonlinear 'optimal seed' $b_{opt}(M_0) = \arg\max_{b_0 \in S_{M_0}} J$.

Linear (kinematic) dynamo optimisations:
[Willis PRL 2012, Chen, Herreman & Jackson JFM 2015]…

Integrate up to $t \gg T$ the system ($\star$), using $b_0 = b_{opt}(M_0)$, until an equilibrium is reached.

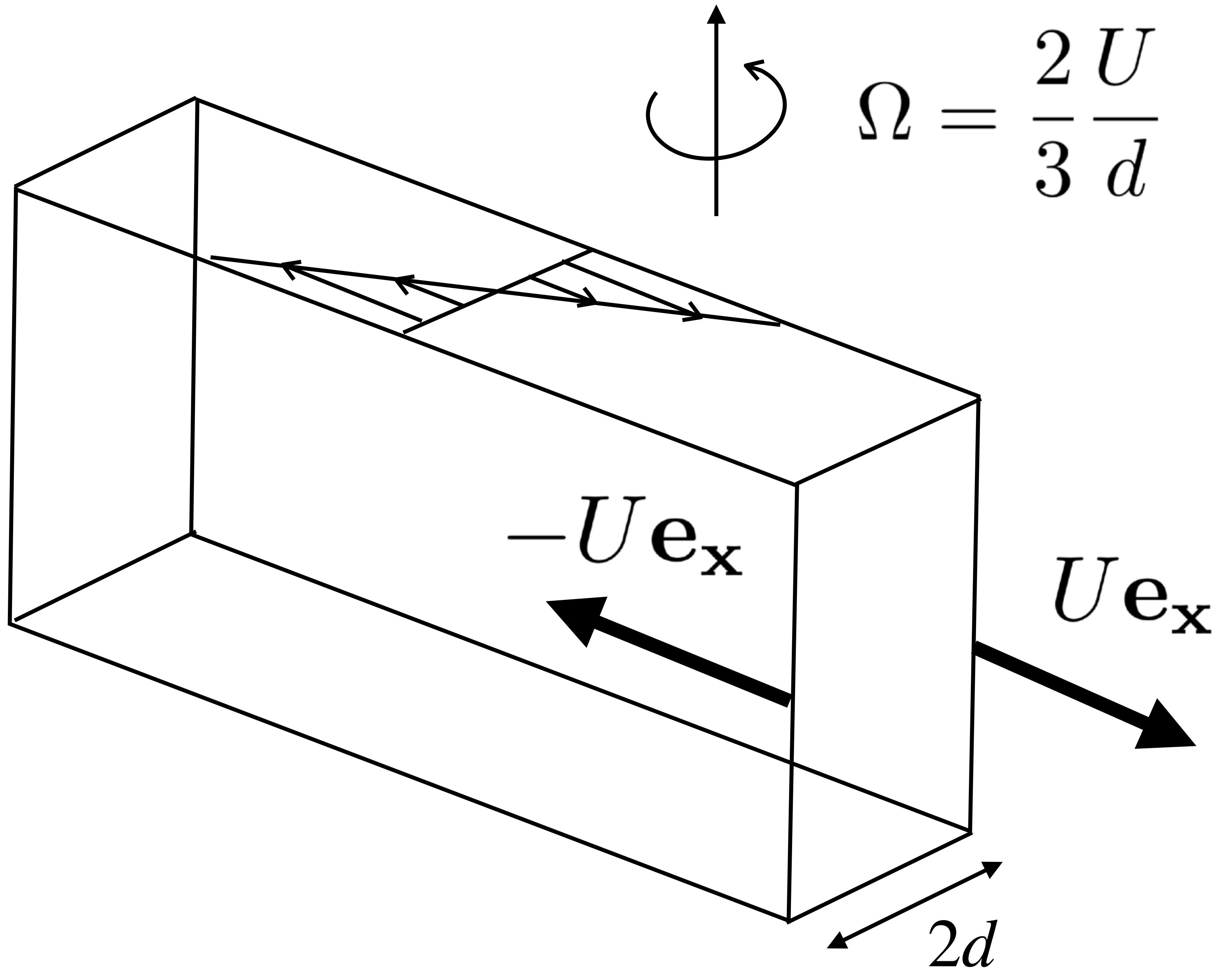
If the equilibrium is a dynamo: decrease M_0 and start over.
If not: keep faith, increase M_0 and start over.

The '**minimal dynamo seed**' corresponds to the **smallest-energy** b_{opt} that initiates a dynamo transition.

(No guarantee of global extremum!)

Important point: we do not place any assumption on b0, its symmetries, or the mechanisms at play…

Taylor-Green flow: from minimal seed to dynamo equilibrium - with P. Mannix & Y. Ponty



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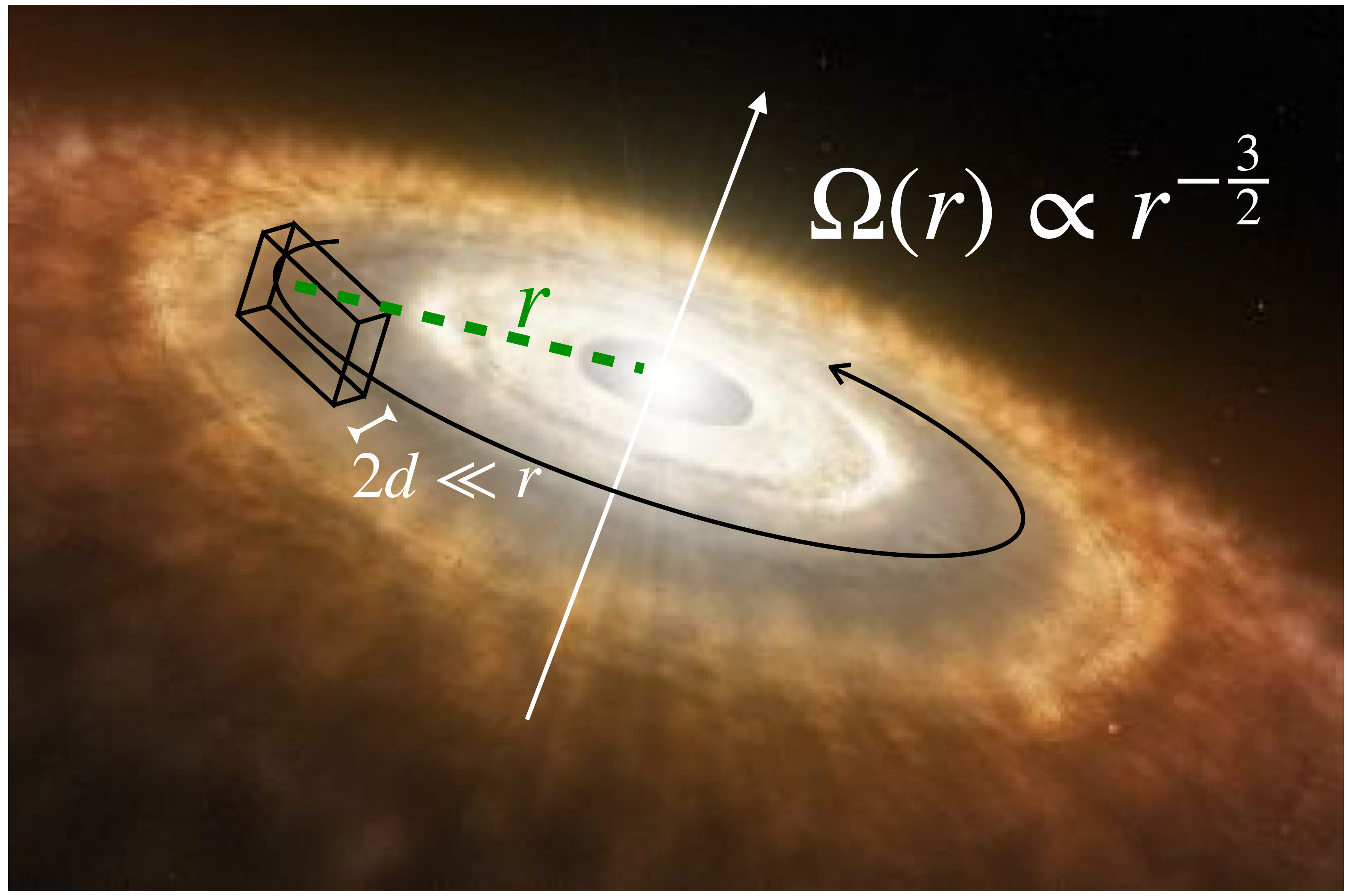


« Quasi-Keplerian » plane Couette flow: simplified local model of Keplerian flow.

Hydro: linearly stable... [Lezius & Jonhston, JFM 1976]

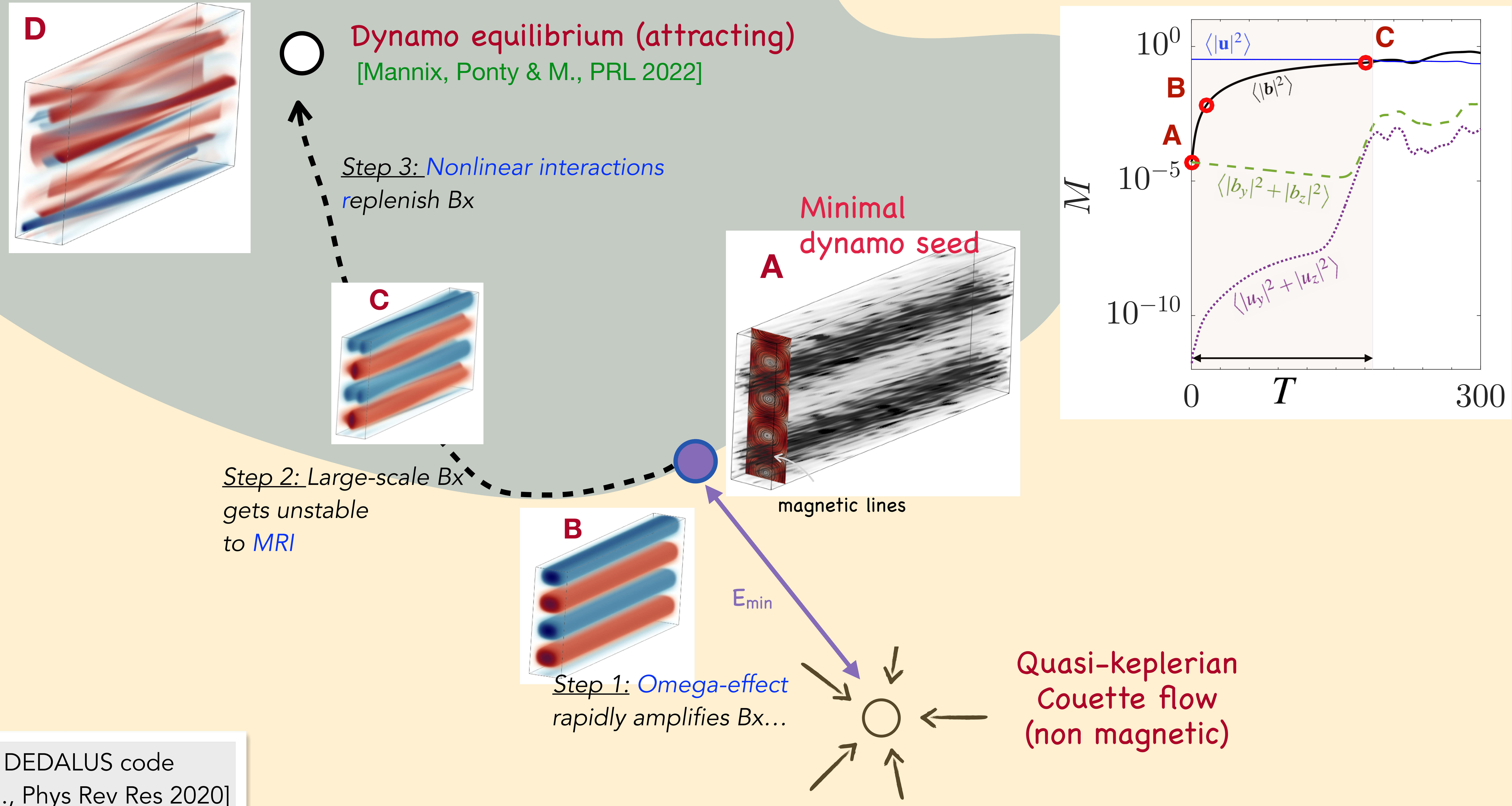
Planar base flow: linearly stable to dynamo... [Zeldovich, Sov. Phys JETP 1956]

BUT

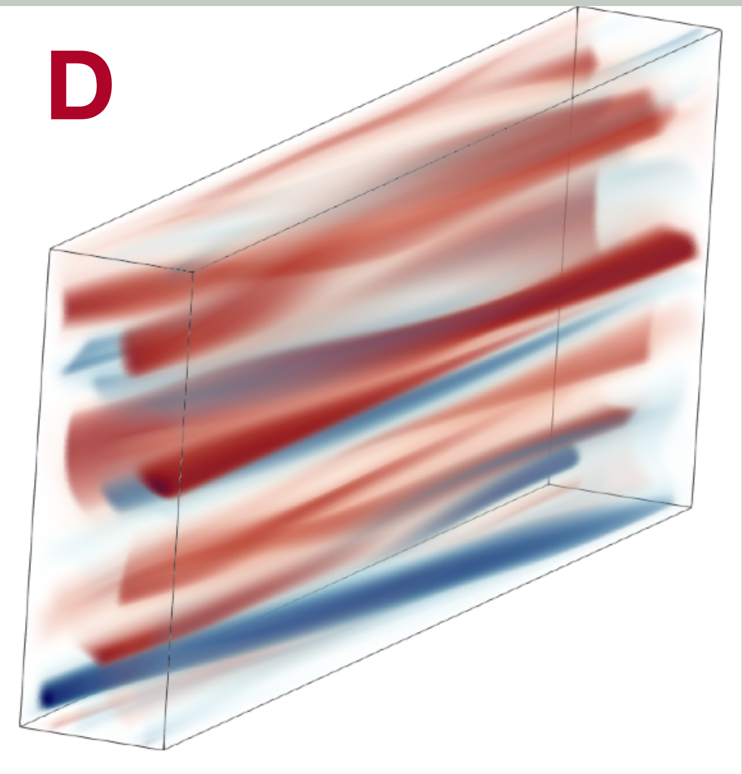
Unstable, steady MHD equilibrium found via homotopy method (assuming MRI dynamo)! [Rincon, Ogilvie & Proctor, PRL 2007]

Find stable (but possibly unsteady/turbulent) equilibrium?

Quasi-Keplerian PCF: from minimal seed to dynamo equilibrium - with P. Mannix & Y. Ponty



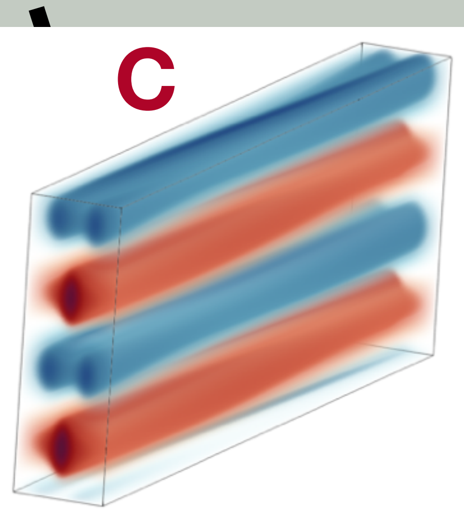
Using DEDALUS code
[Burns et al., Phys Rev Res 2020]



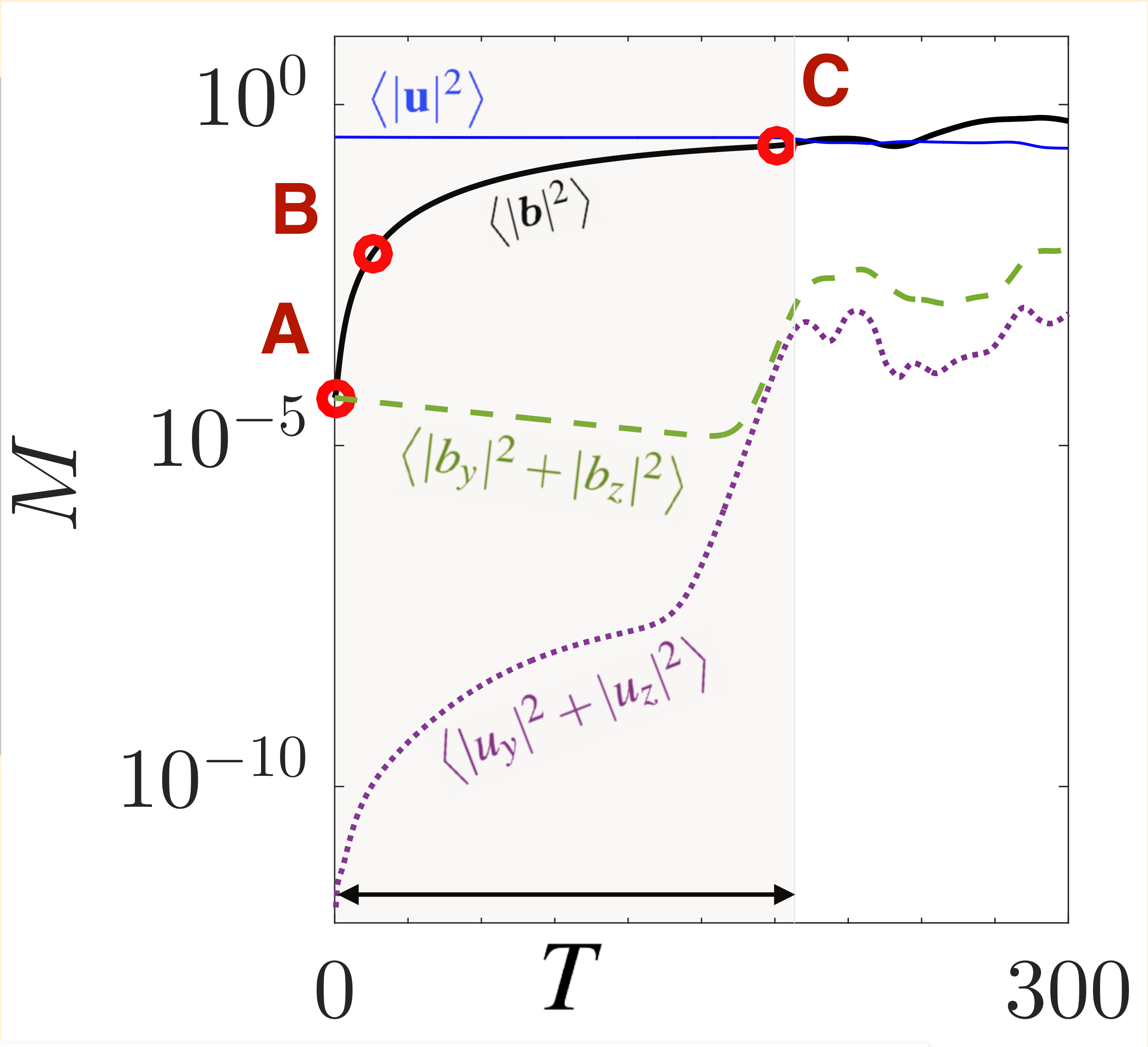
Dynamo equilibrium (attracting)

[Mannix, Ponty & M., PRL 2022]

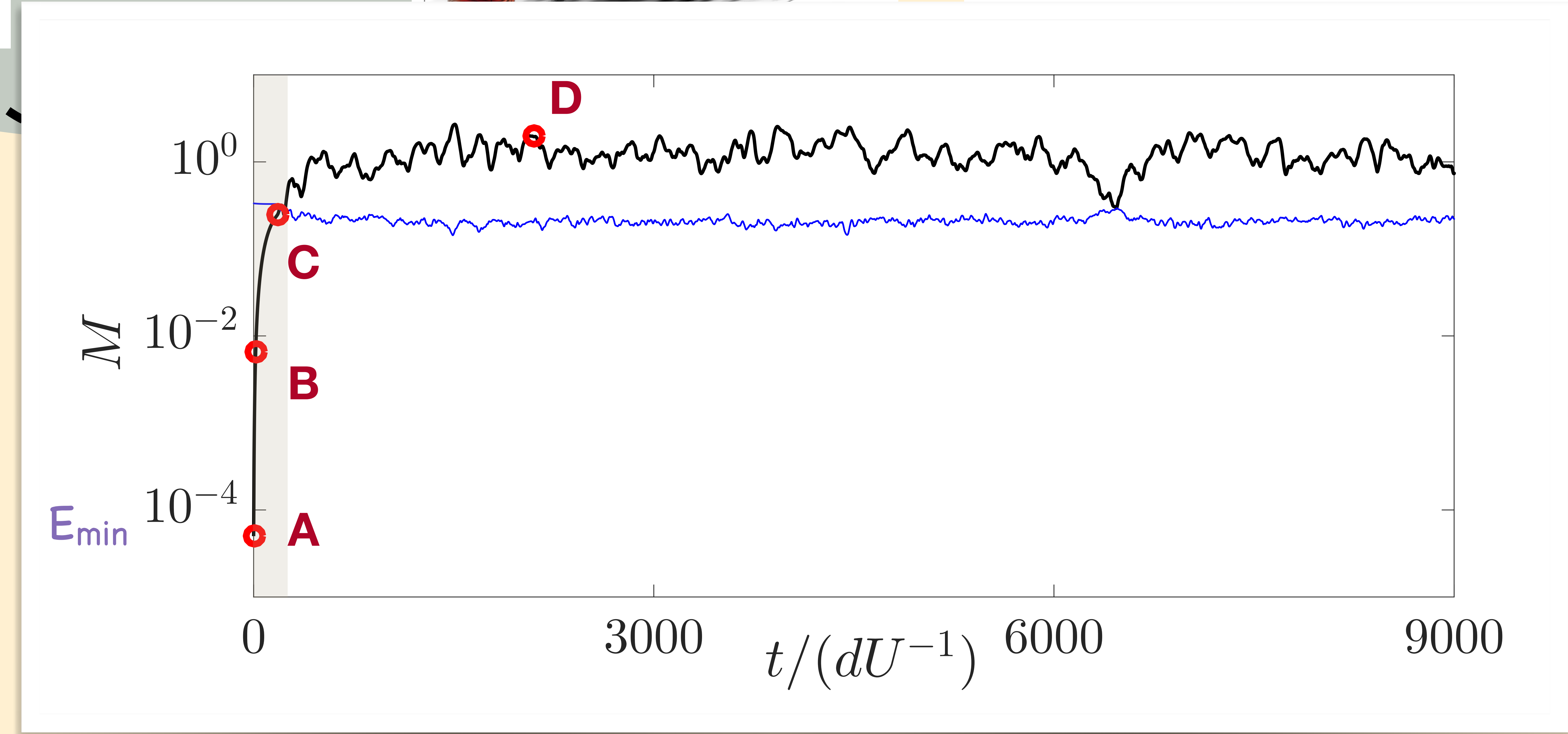
Step 3: Nonlinear interactions replenish Bx



Minimal dynamo seed



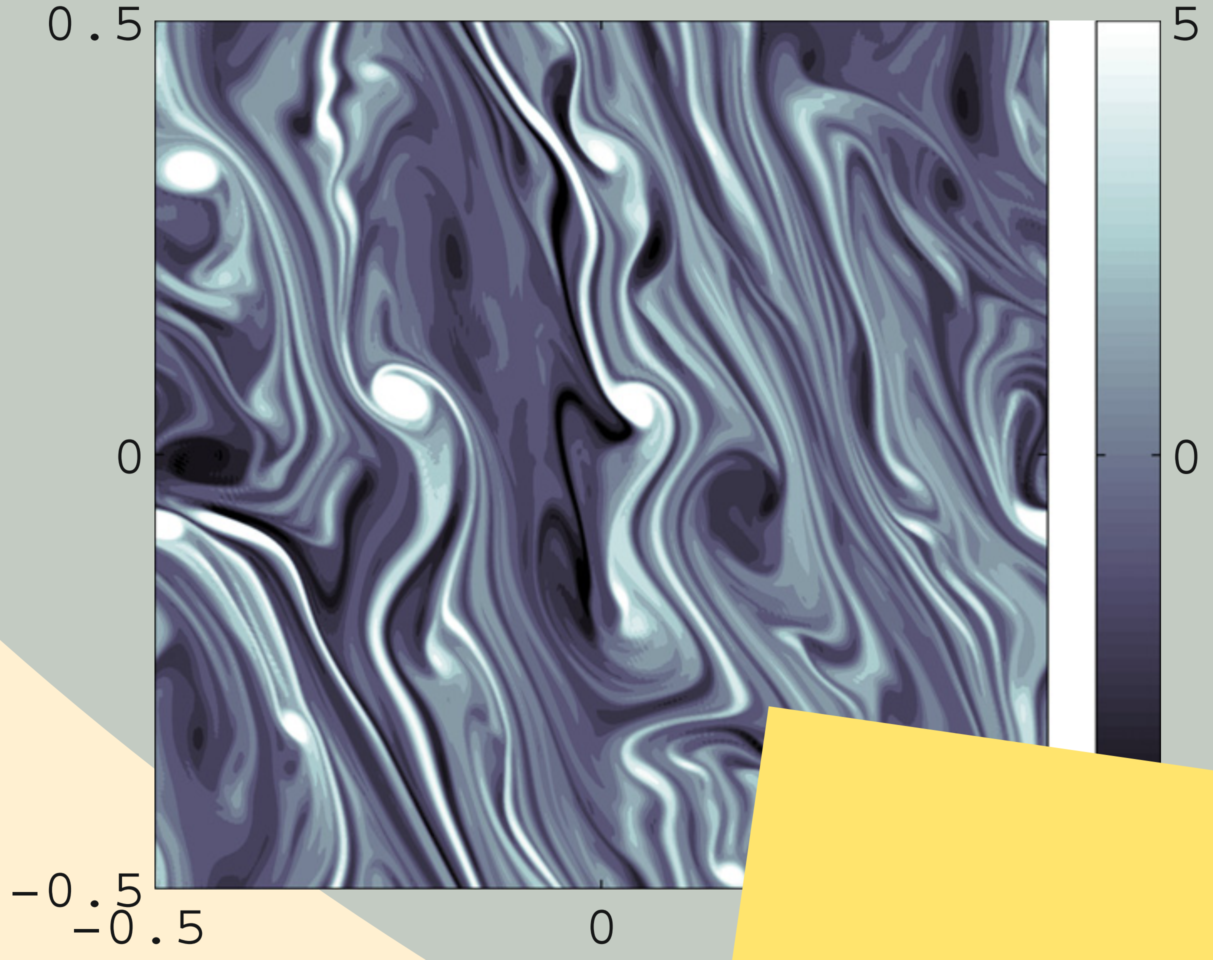
Step 2: Large-scale Bx gets unstable to MRI



Using DEDALUS code
[Burns et al., Phys Rev Res 2020]

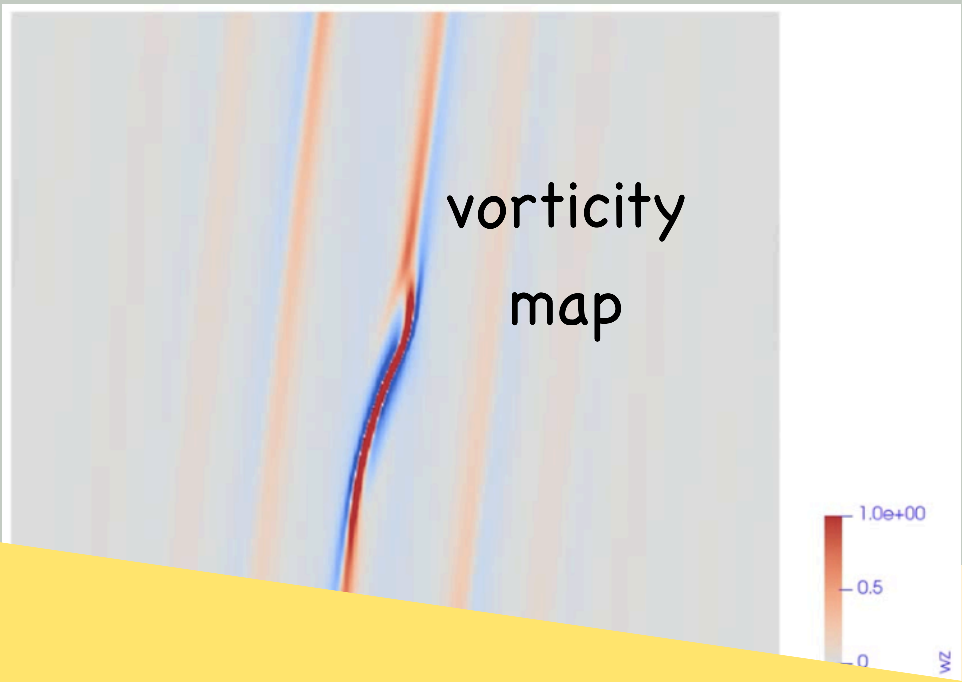
Subcritical baroclinic instability: from minimal seed to turbulent state - with Y. Ponty

Using SNOOPY code
[Lesur & Longaretti A&A 2005]



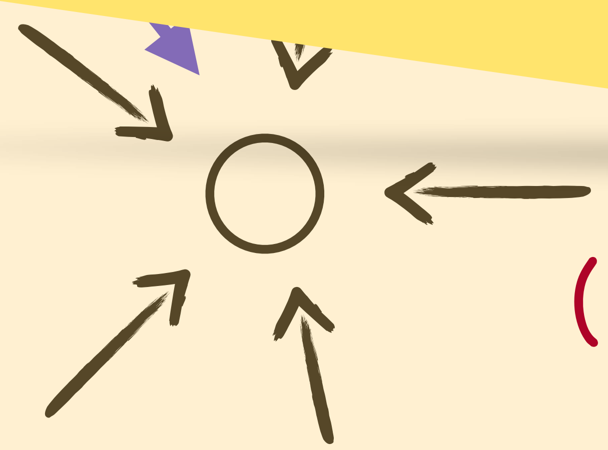
Subcritical baroclinic instability

[Lesur & Papaloizou, A&A 2010]



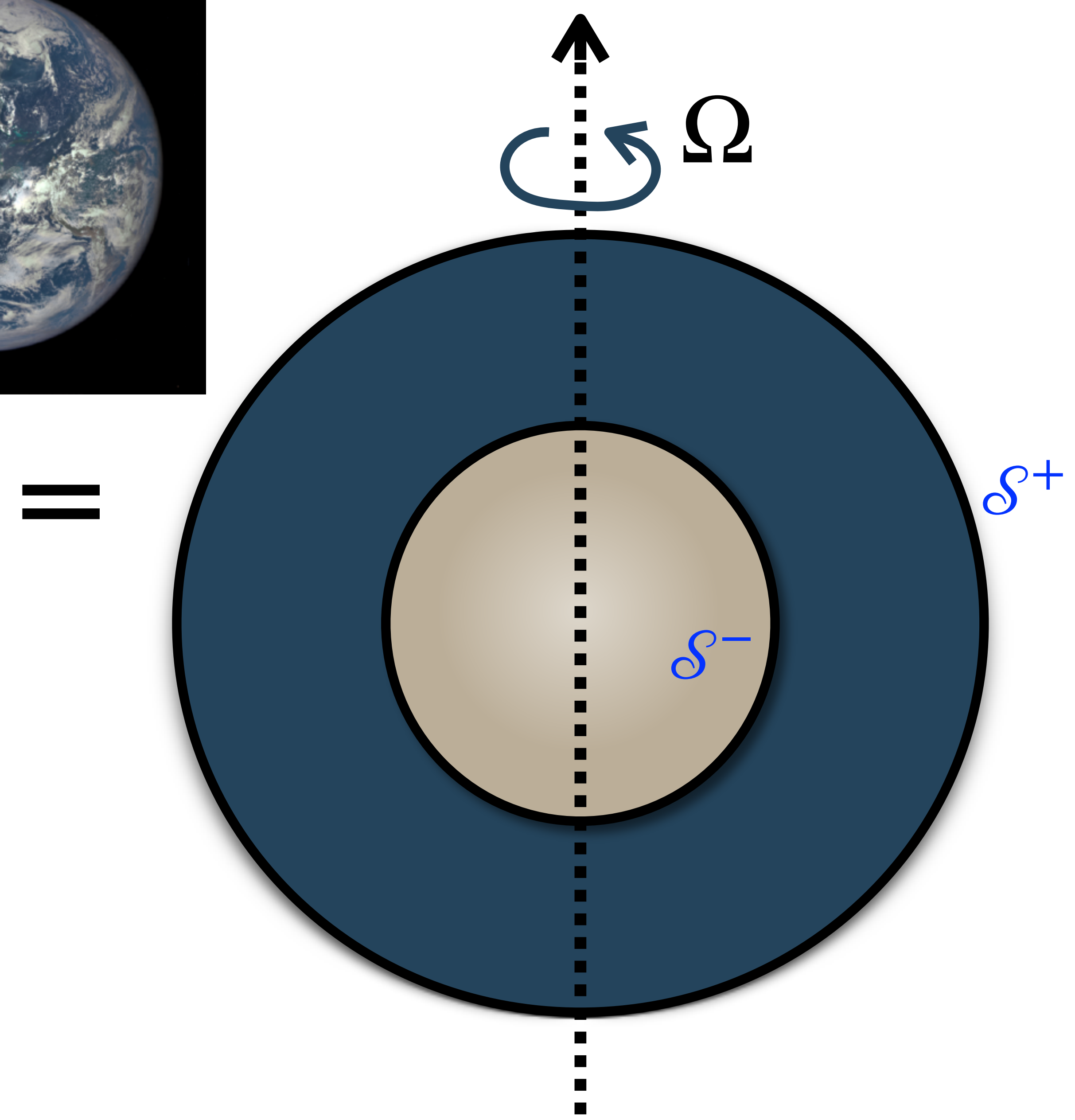
How to disrupt a Keplerian flow WITHOUT magnetic fields?

——> on-going PhD Clément Mariot



lerian
shear
(2D, stably stratified)

A first step: kickstart the subcritical solution from the numerical Geodynamo benchmark of [Christensen et al. 2001].



Christensen et al. 2001: « *…because non-magnetic convection is found stable against small magnetic perturbations [at these parameters] and because the dynamo solutions* ***seem to have only a small basin of attraction****, the initial state is of some concern.* »

the initial velocity is zero and the initial temperature is

$$T = \frac{r_o r_i}{r} - r_i + \frac{210A}{\sqrt{17920\pi}} \times (1 - 3x^2 + 3x^4 - x^6) \sin^4\theta \cos 4\phi \quad (9)$$

where θ is the colatitude, ϕ the longitude, and $x = 2r - r_i - r_o$. This describes a conductive state with a perturbation of harmonic degree and order four superimposed. The amplitude is set to $A = 0.1$. In case 1, the initial magnetic field is for $r_i \le r \le r_o$:

$$B_r = \frac{5}{8}\left(8r_o - 6r - 2\frac{r_i^4}{r^3}\right)\cos\theta \quad (10)$$

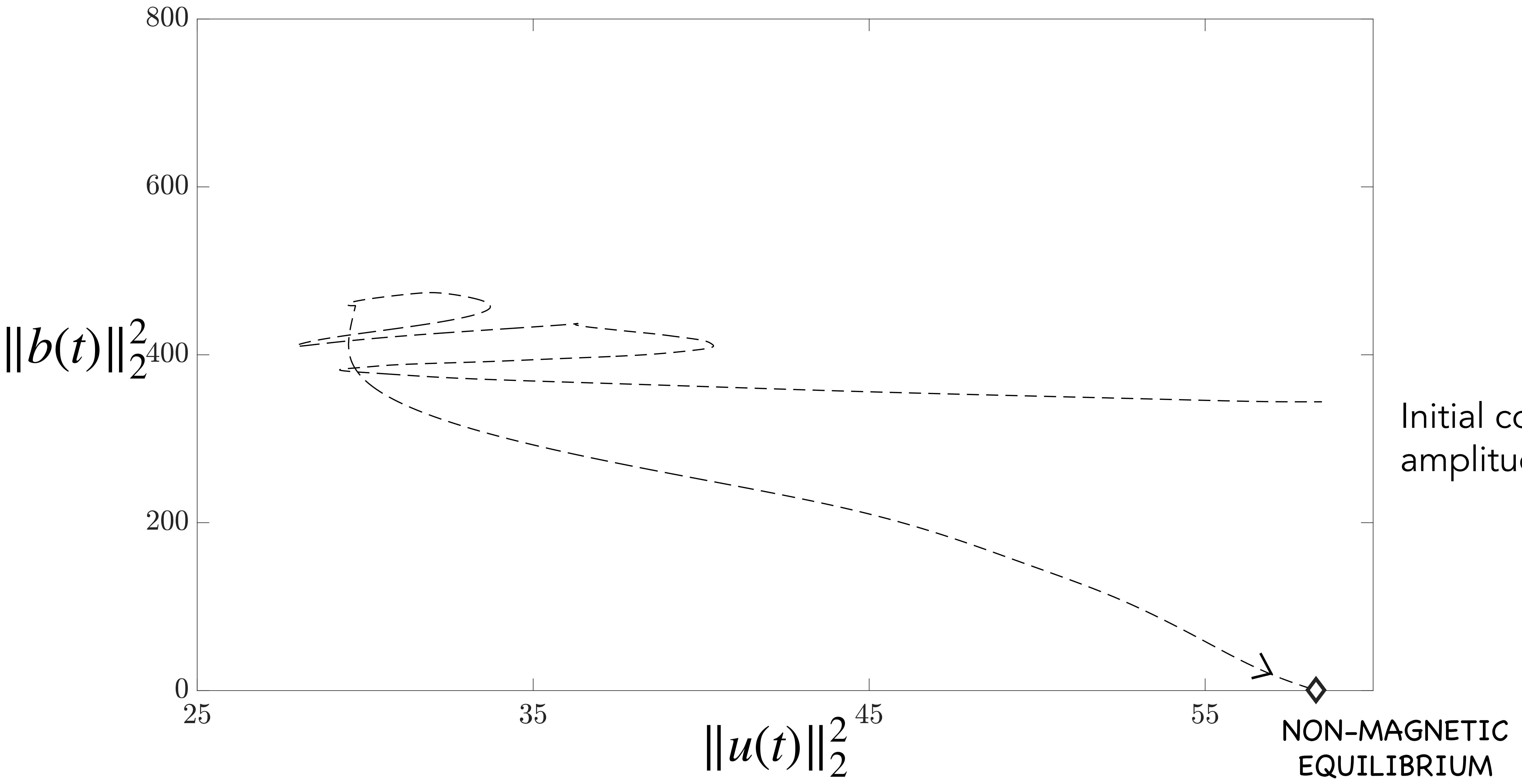
$$B_\theta = \frac{5}{8}\left(9r - 8r_o - \frac{r_i^4}{r^3}\right)\sin\theta \quad (11)$$

$$B_\phi = 5\sin(\pi(r - r_i))\sin 2\theta. \quad (12)$$

Boussinesq MHD, unstably stratified

No-slip BC for velocity u, temperature T fixed on $\mathcal{S}^\pm$ (unstably stratified)

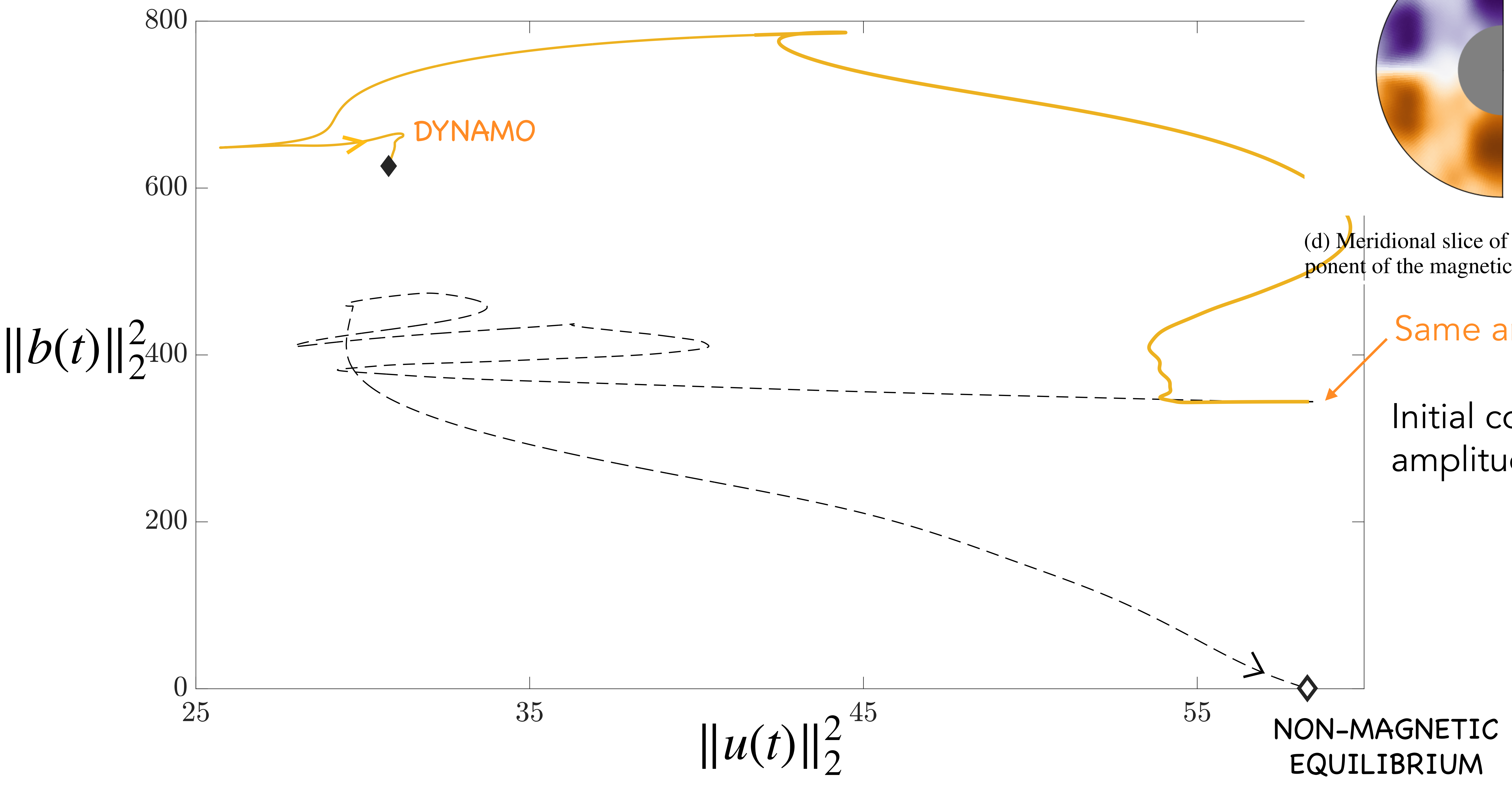
matching with potential magnetic field for B on $\mathcal{S}^\pm$

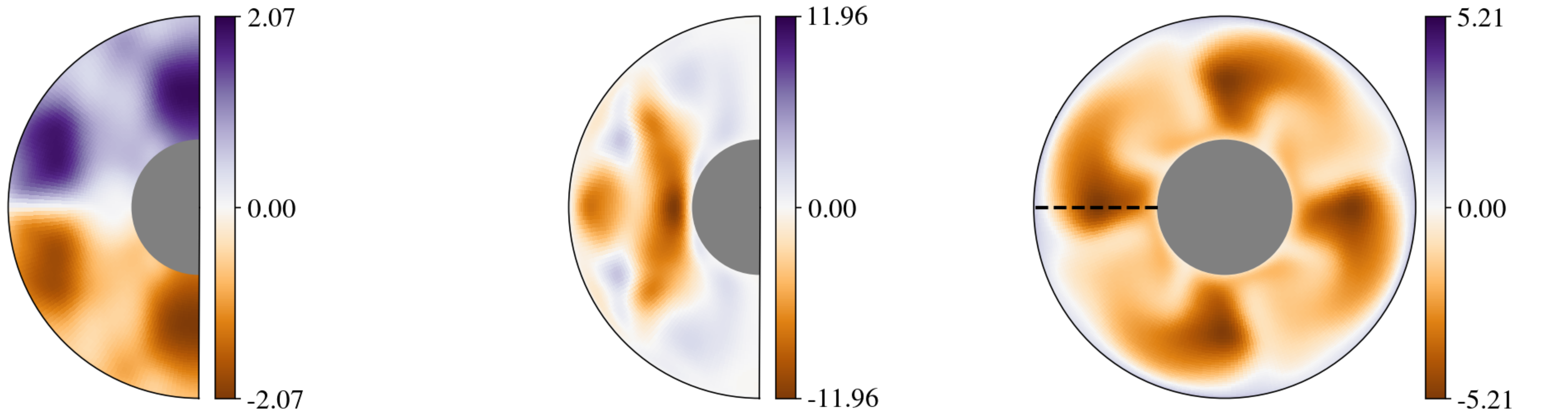


Initial condition from benchmark [Christensen et al. 2001], amplitude rescaled - - - > no dynamo

Using DEDALUS code
[Burns et al., Phys Rev Res 2020]

Geodynamo benchmark: dynamical landscape - with C. Skene & S. Tobias





(d) Meridional slice of the radial component of the magnetic field.

(e) Meridional slice of the radial component of the current.

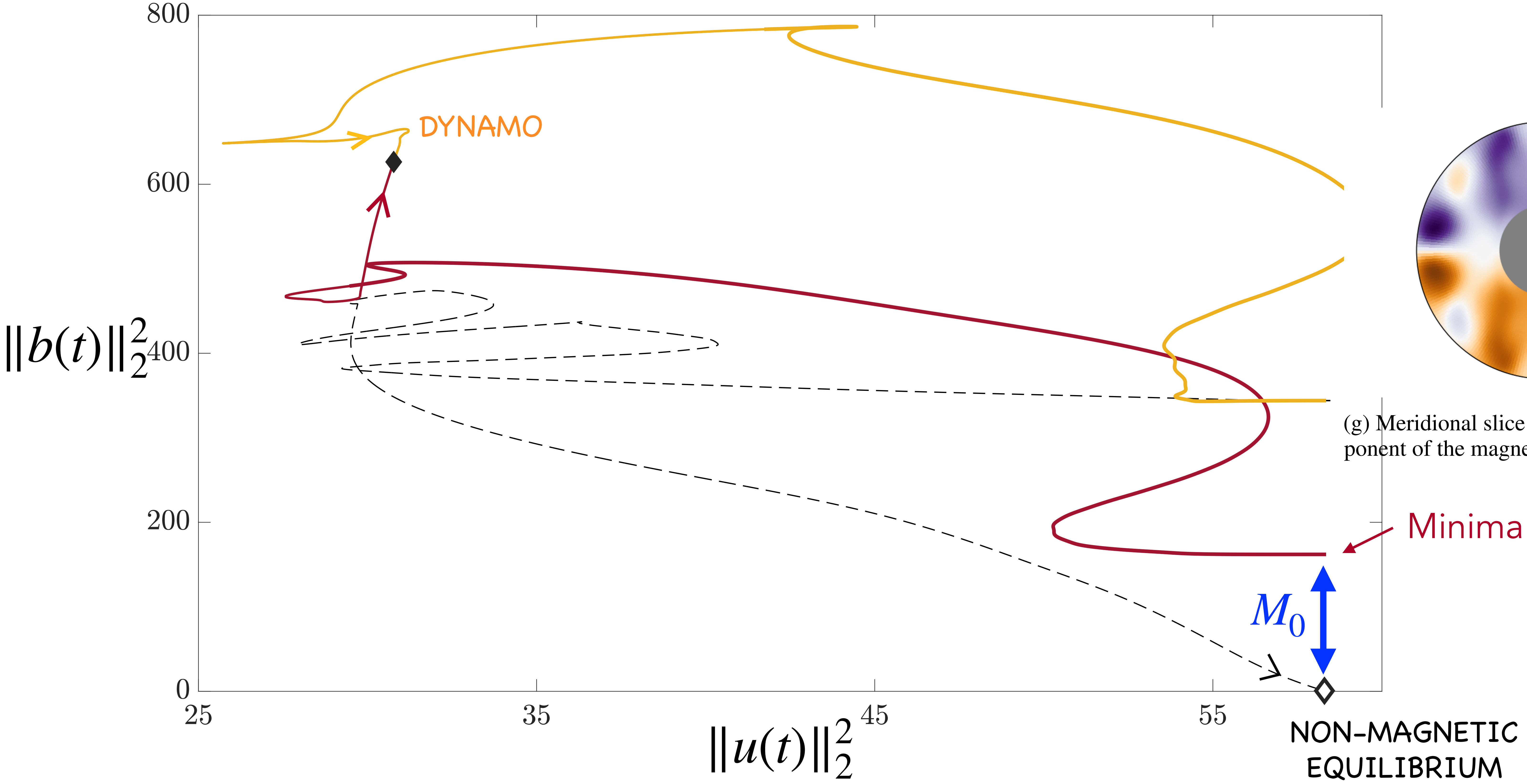
(f) Equatorial slice of the latitudinal cor the magnetic field.

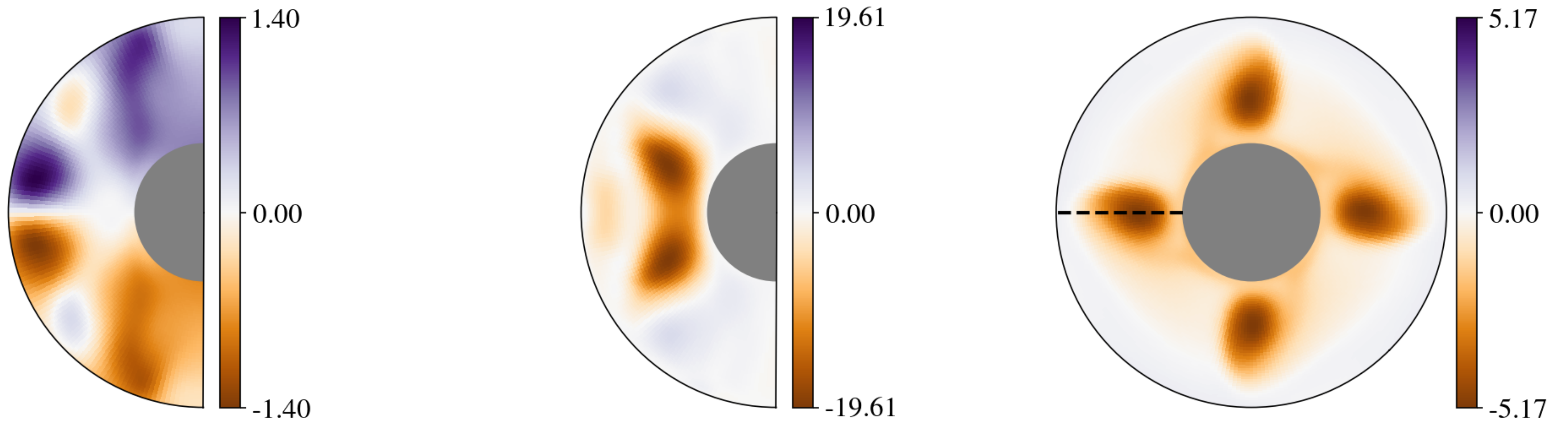
Same amplitude, but structure optimised

Initial condition from benchmark [Christensen et al. 2001], amplitude rescaled - - - > no dynamo

[Skene, M., Tobias, *JFM 2025*]

Using DEDALUS code
[Burns et al., Phys Rev Res 2020]





(g) Meridional slice of the radial component of the magnetic field.

(h) Meridional slice of the radial component of the current.

(i) Equatorial slice of the latitudinal c the magnetic field.

[Skene, M., Tobias, *JFM 2025*]

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Geodynamo benchmark: dynamical landscape - with C. Skene & S. Tobias

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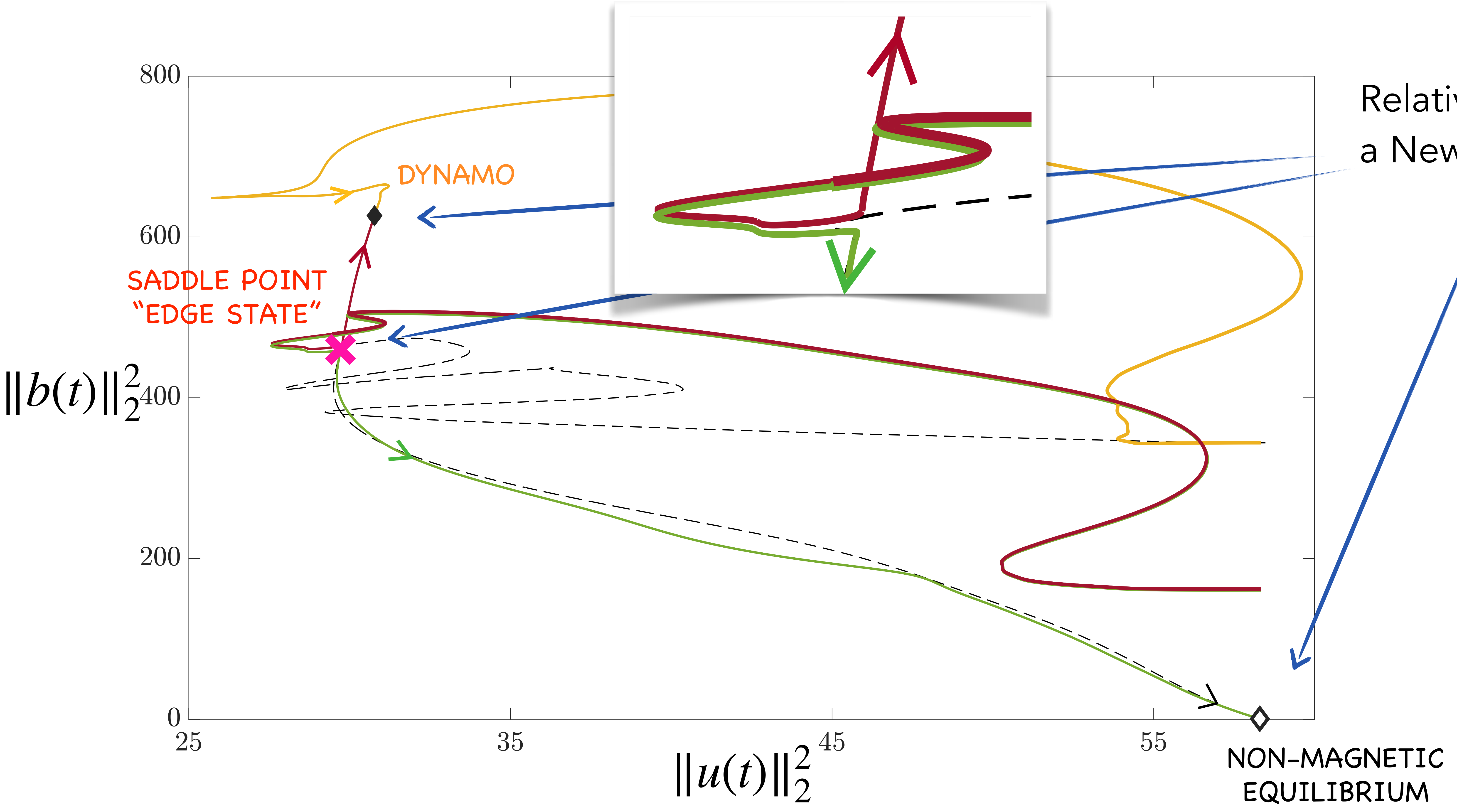

Relative equilibria (travelling waves), computed using a Newton hookstep method. [Willis 2019a,b]

[Skene, M., Tobias, *JFM 2025*]

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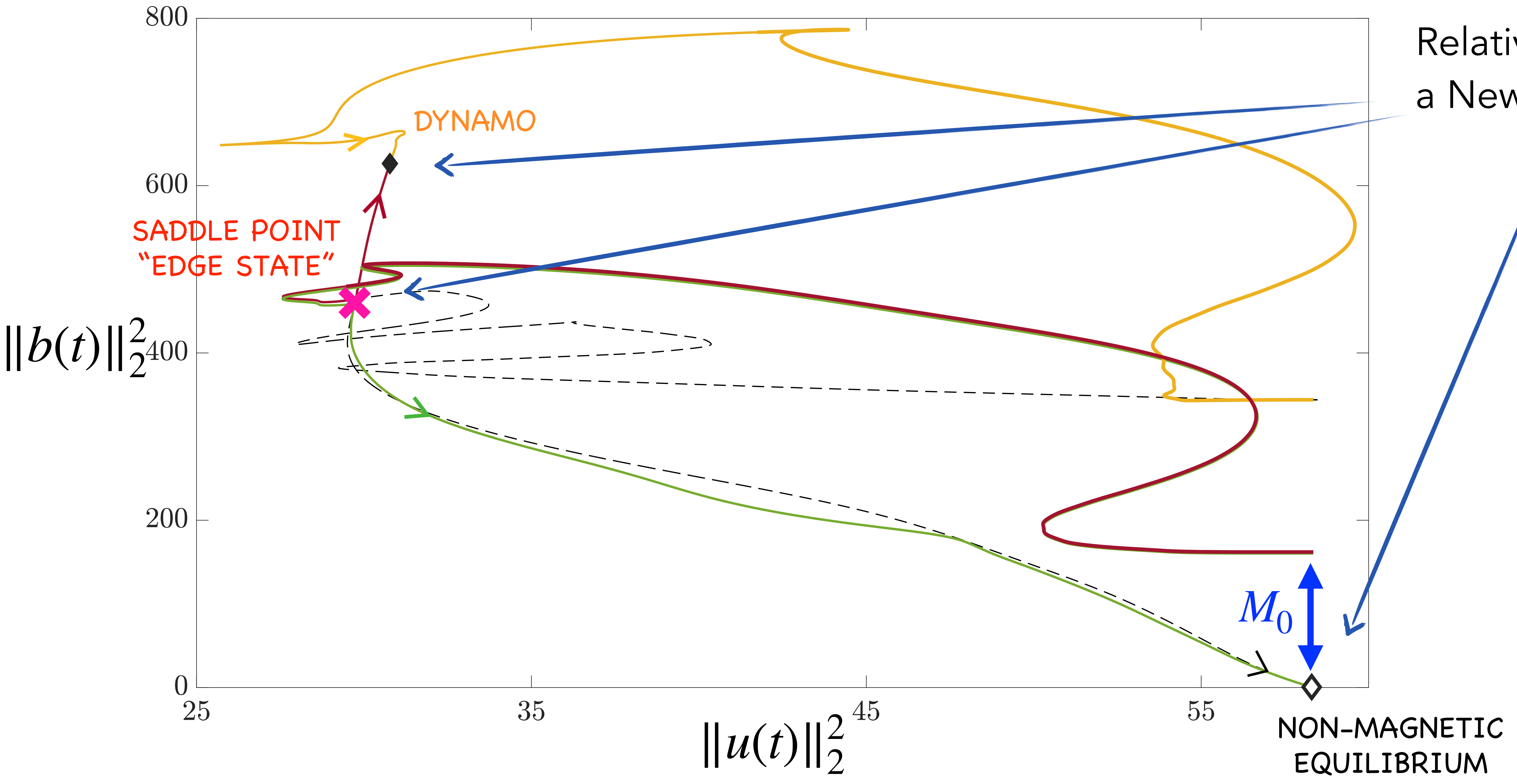
[Burns et al., Phys Rev Res 2020]



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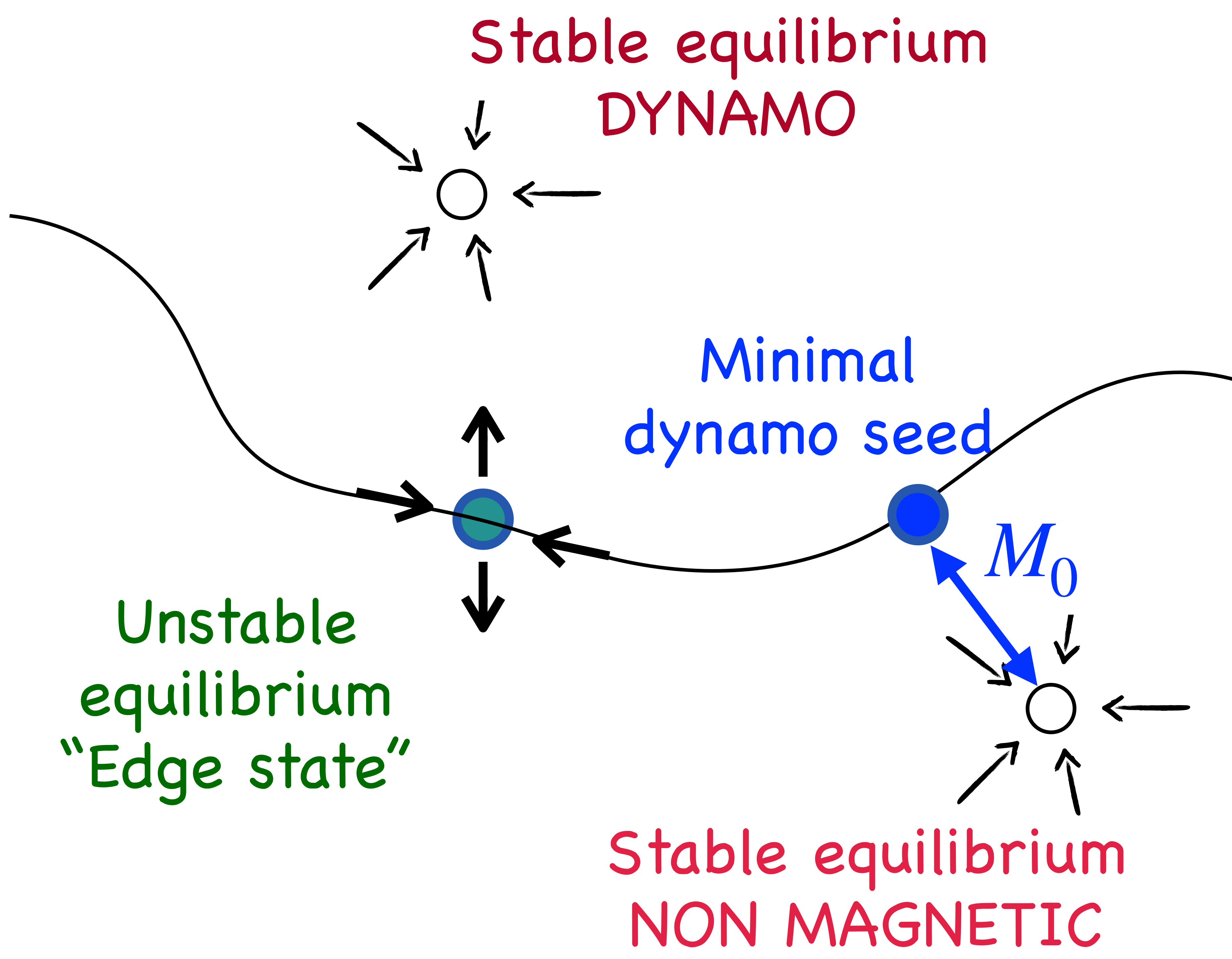

[Skene, M., Tobias, *JFM 2025*]

Using DEDALUS code
[Burns et al., Phys Rev Res 2020]

Some conclusions

Optimal control can identify stable, subcritical, possibly elusive magnetic equilibria.
It also identifies “minimal” dynamo seeds triggering nonlinear dynamo instability.

No prior assumptions (symmetries, time-dependency, dynamo mechanism…) are required, although for the purpose of finding a “minimal” dynamo seed, the choice of cost functional matters (finite target time!).

The minimal seeds-edge scenario of transition to turbulence in shear flows seems relevant for subcritical dynamo transition.

The identified states could be used as initialisation steps for continuation along branches!

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