

# Waves of Topological Origin in the Fluid Earth System and Beyond

Brad Marston  
Brown University

Antoine Venaille  
ENS Lyon

Rosa Xu  
Princeton

Steve Tobias  
Edinburgh

Pierre Delplace  
ENS Lyon

Jung-Eun Lee  
Brown

Jeff Parker  
Google Climate

Chris Li  
Rice

Baylor Fox-Kemper  
Brown

Ziyan (Zoe) Zhu  
Boston College

Science **358**, 1075 (2017)

PRL **124**, 195001 (2020)

PRR **2**, 033425 (2020)

PRR **5**, 033191 (2023)

JAS **81**, 923 (2024)

# Topological Protection

(Laughlin 1981, Halperin 1982 & Thouless *et al.* 1982)

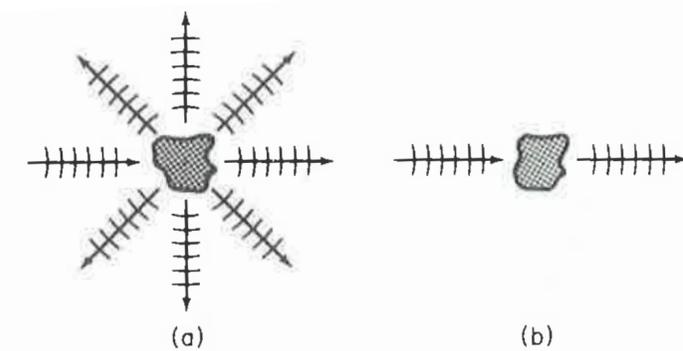
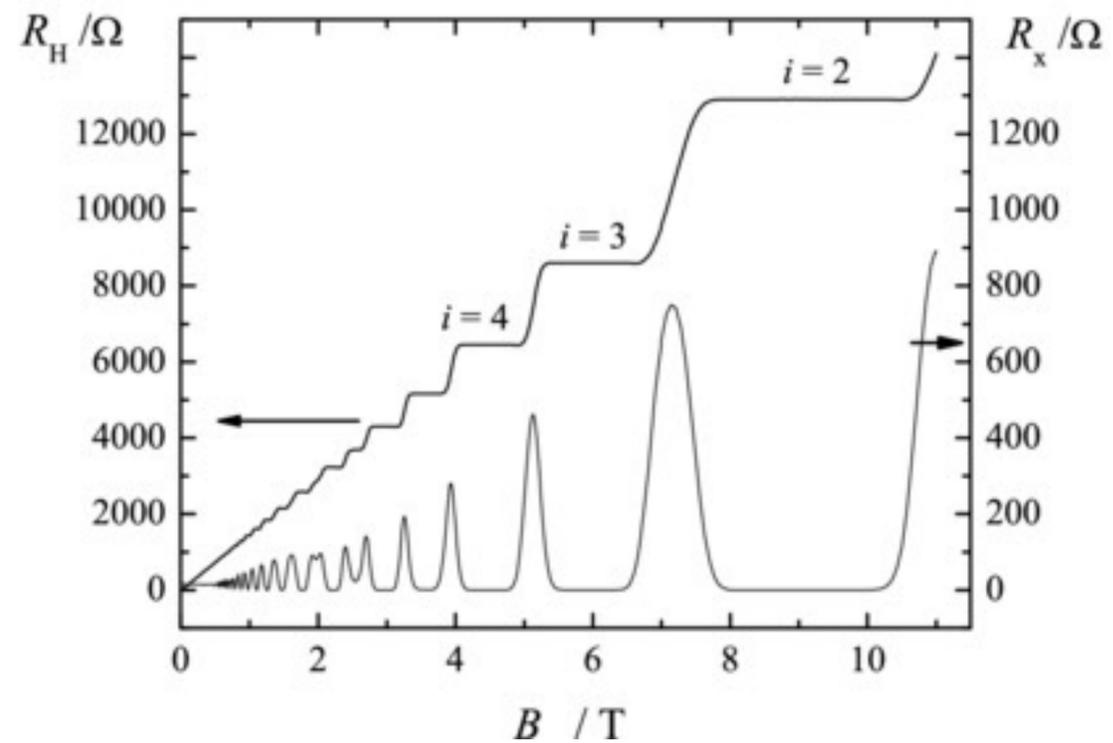
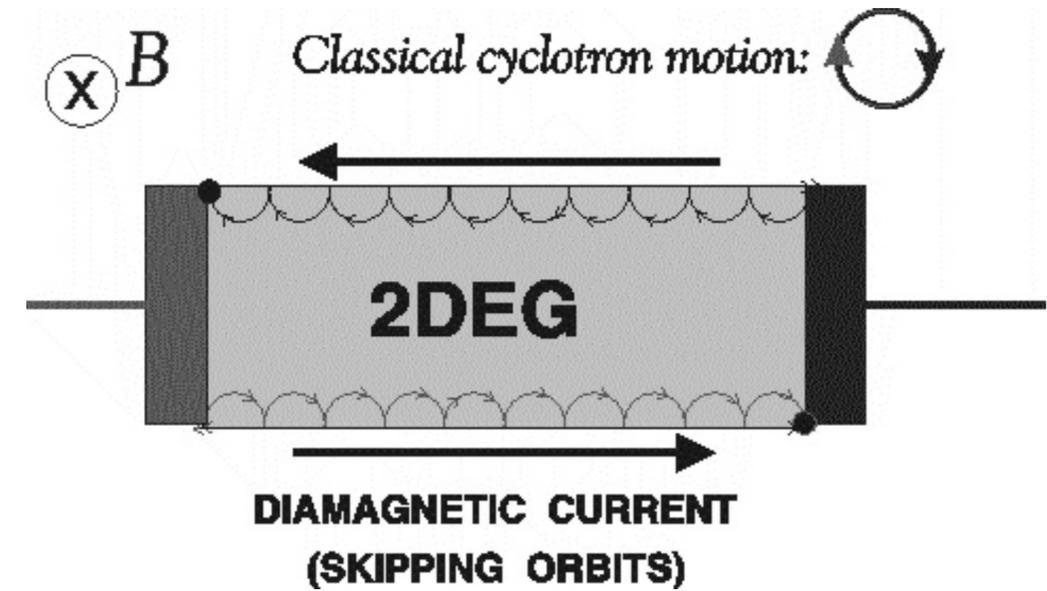
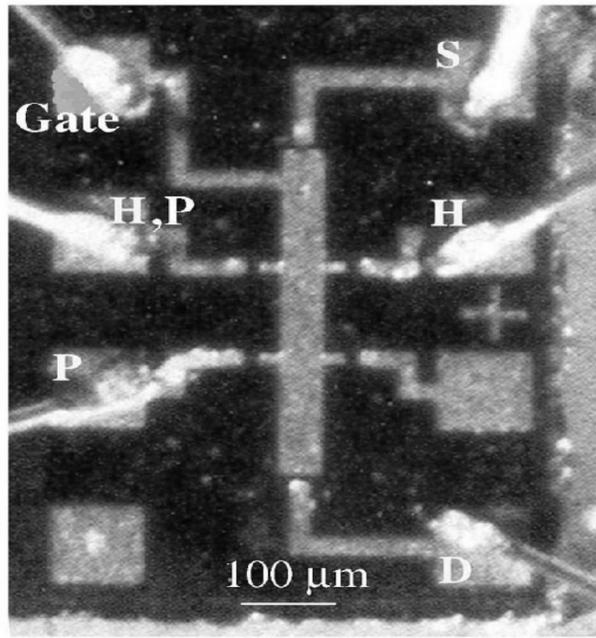


Figure 3.2. Two dimensional scattering in a) zero field, b) strong field.

R. E. Prange (1990)

# Quantized Hall Conductance in a Two-Dimensional Periodic Potential

D. J. Thouless, M. Kohmoto,<sup>(a)</sup> M. P. Nightingale, and M. den Nijs

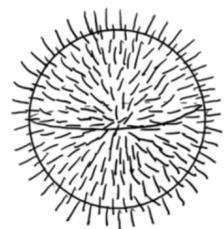
*Department of Physics, University of Washington, Seattle, Washington 98195*

(Received 30 April 1982)

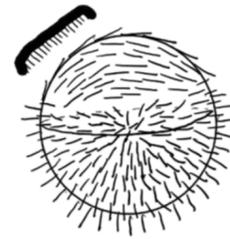
The Hall conductance of a two-dimensional electron gas has been studied in a uniform magnetic field and a periodic substrate potential  $U$ . The Kubo formula is written in a form that makes apparent the quantization when the Fermi energy lies in a gap. Explicit expressions have been obtained for the Hall conductance for both large and small  $U/\hbar\omega_c$ .

$$\begin{aligned}\sigma_H &= \frac{ie^2}{2\pi h} \sum \int d^2k \int d^2r \left( \frac{\partial u^*}{\partial k_1} \frac{\partial u}{\partial k_2} - \frac{\partial u^*}{\partial k_2} \frac{\partial u}{\partial k_1} \right) \\ &= \frac{ie^2}{4\pi h} \sum \oint dk_j \int d^2r \left( u^* \frac{\partial u}{\partial k_j} - \frac{\partial u^*}{\partial k_j} u \right),\end{aligned}$$

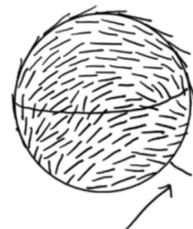
# Topology



A ball with stiff, straight porcupine-like quills emanating out from it

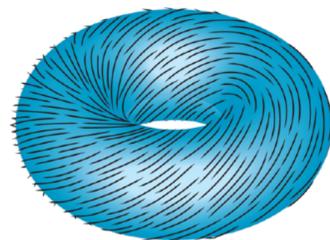


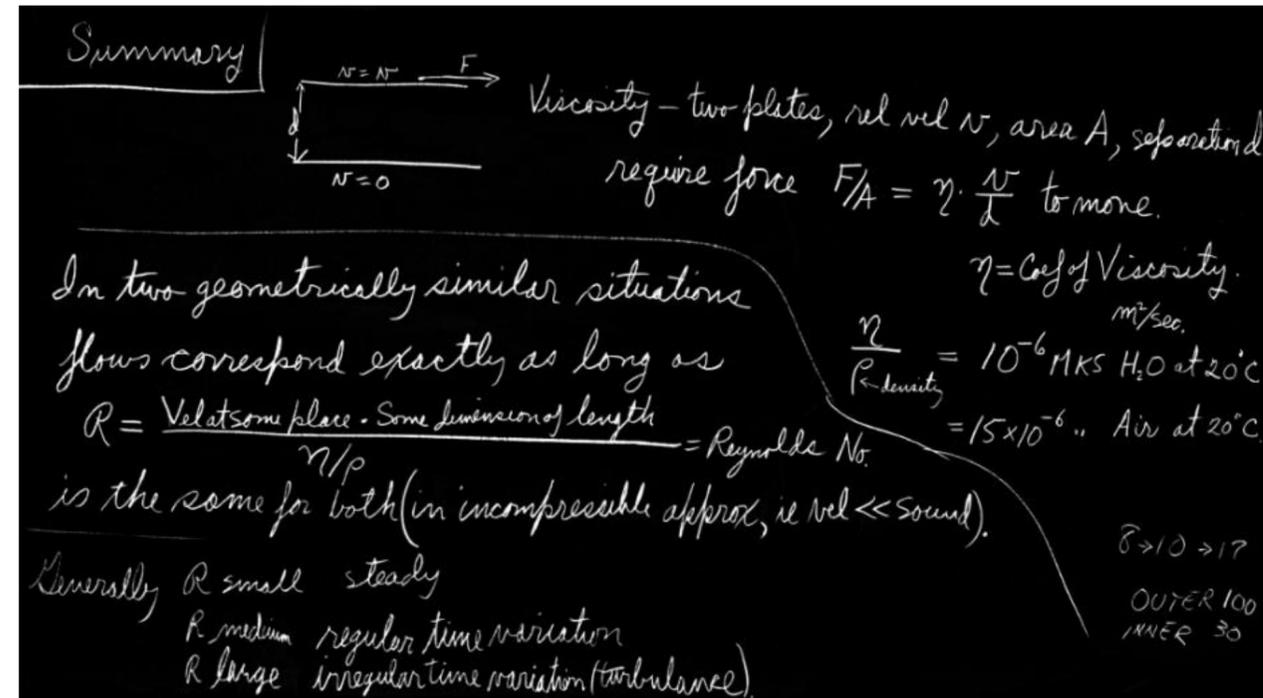
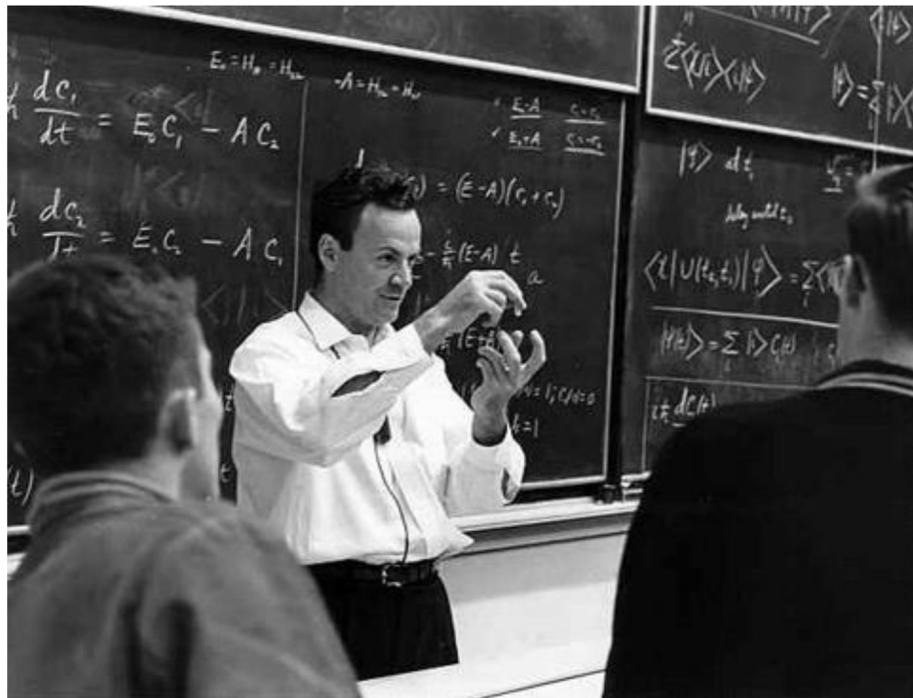
A start at combing the ball so that the quills lie flat against the ball.



Yikes! One quill sticks out.

@susan\_dagostino



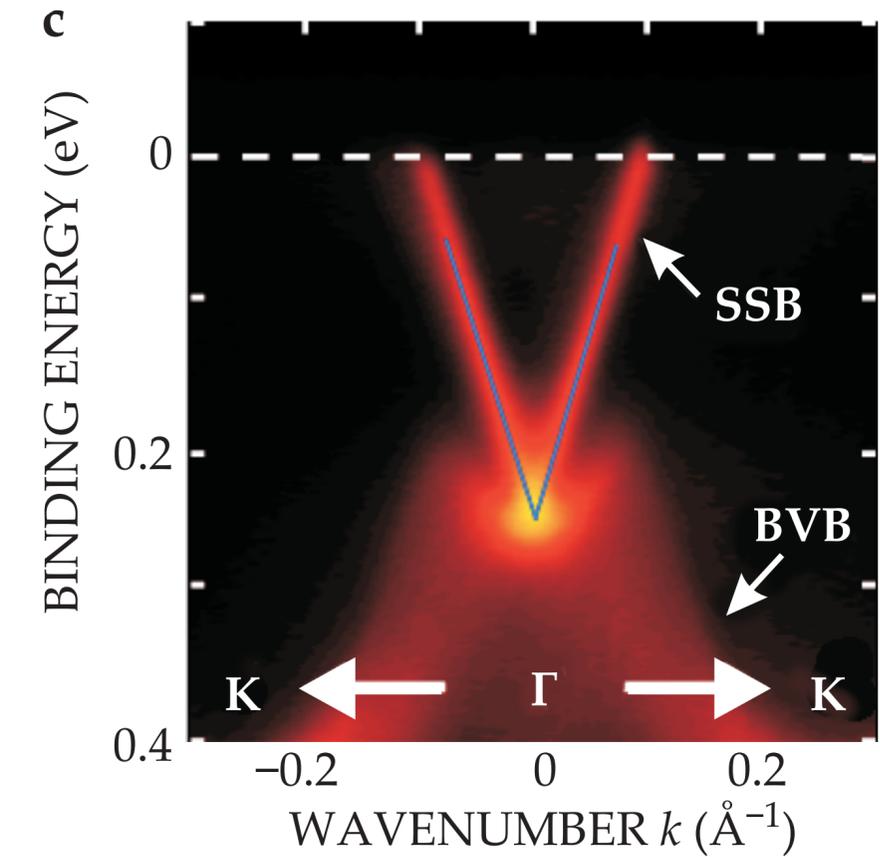
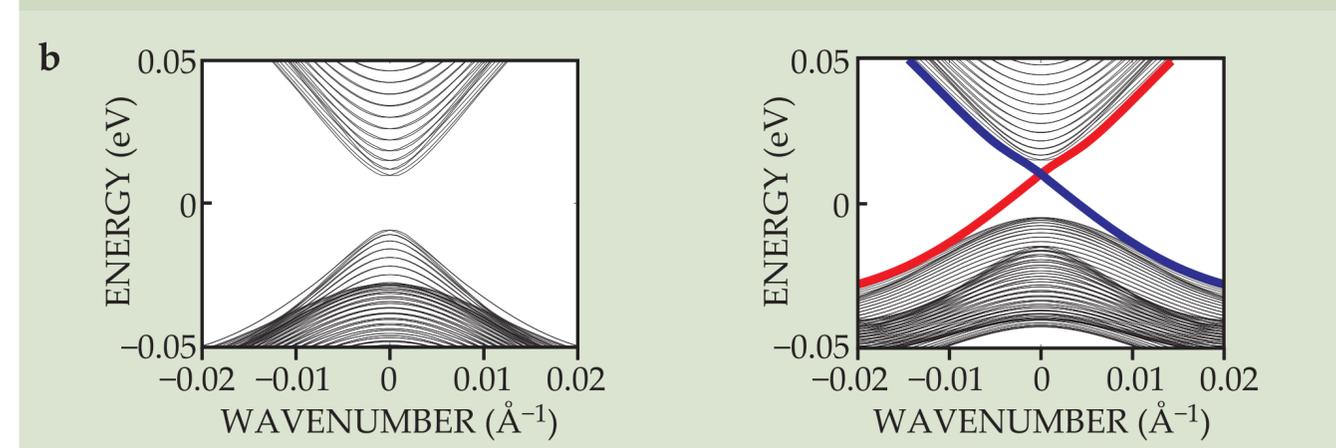
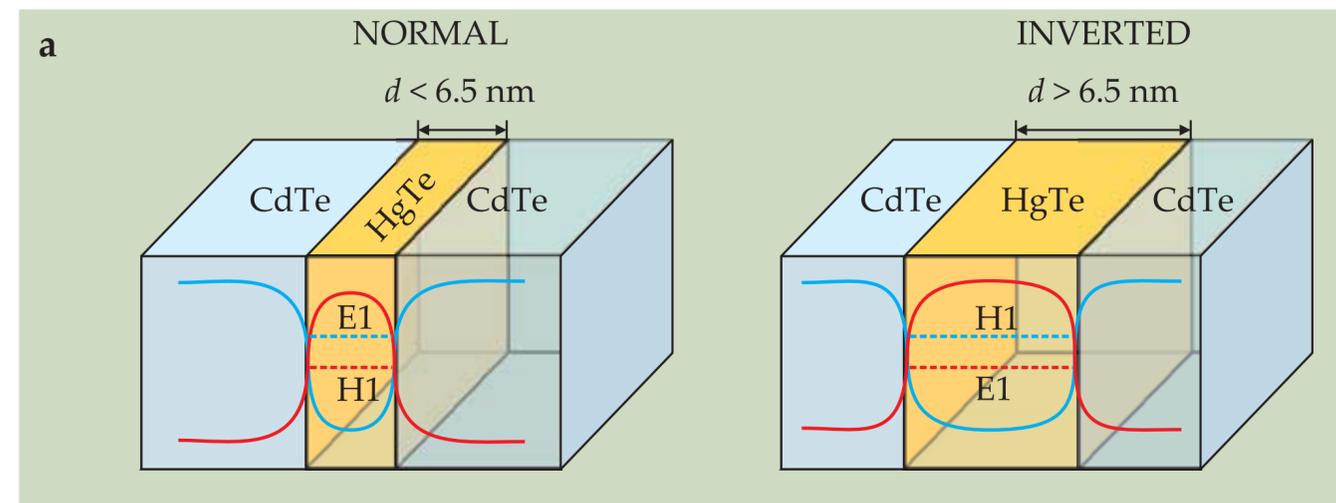
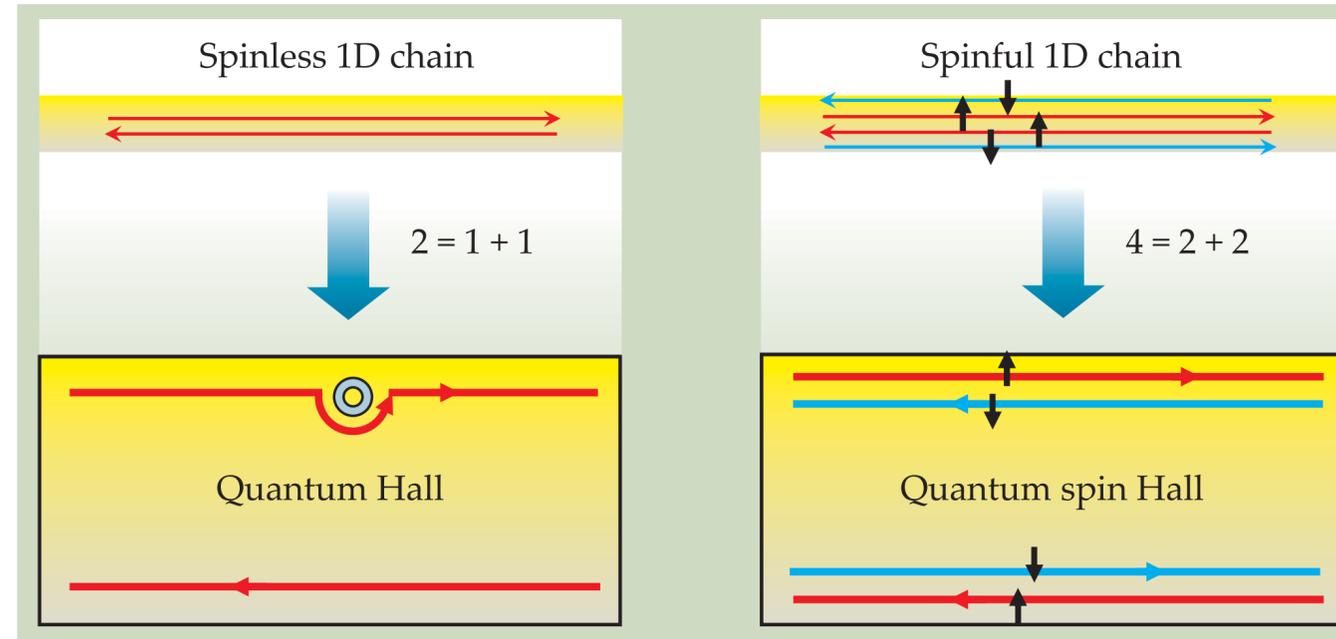


The next great era of awakening of human intellect may well produce a method of understanding the *qualitative* content of equations. Today we cannot. Today we cannot see that the water flow equations contain such things as the barber pole structure of turbulence that one sees between rotating cylinders. Today we cannot see whether Schrödinger's equation contains frogs, musical composers, or morality—or whether it does not.

Topology turns some complicated problems into simple ones

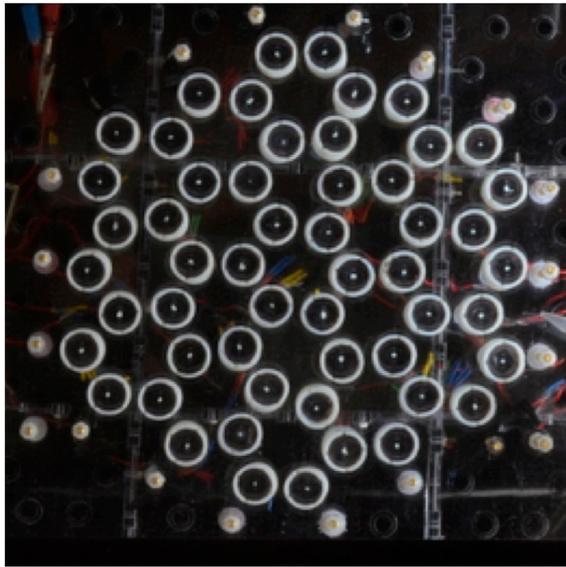
# Topological Insulators

(2019 Breakthrough Prize in Fundamental Physics)

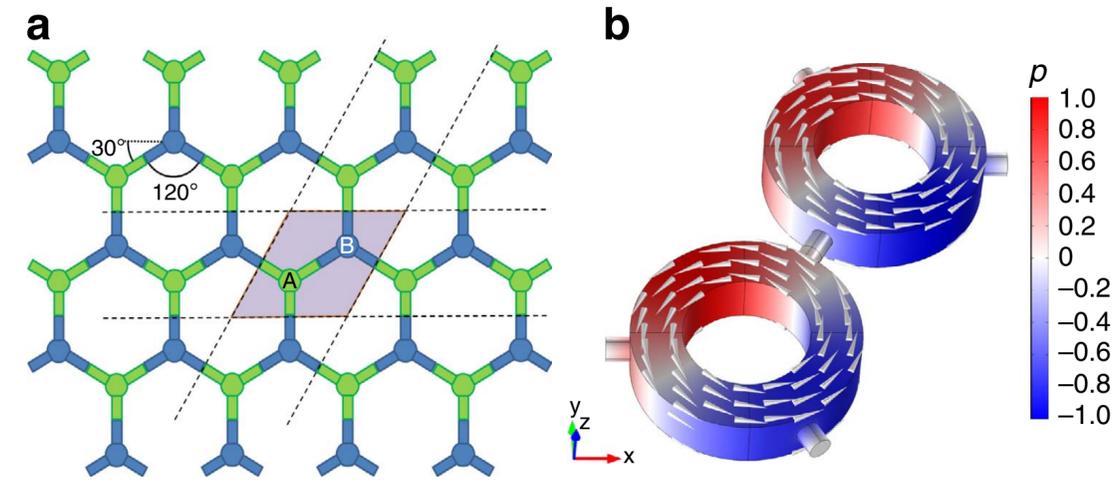


Qi and Zhang  
(2010).

# Waves of Topological Origin in Classical Systems

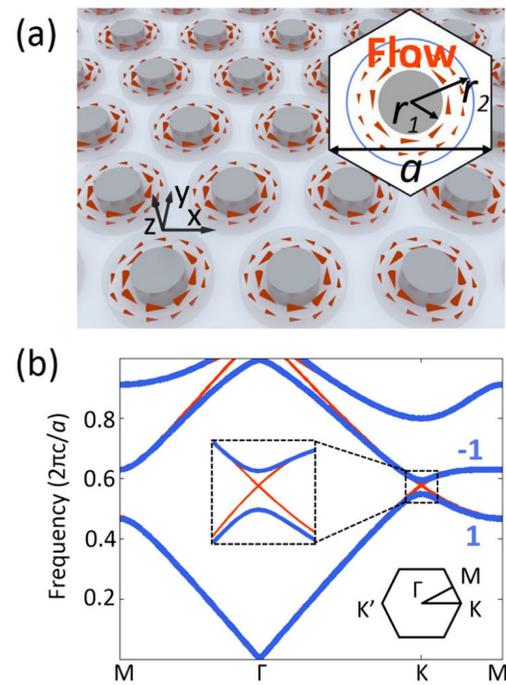


Nash *et al.* PNAS (2015)

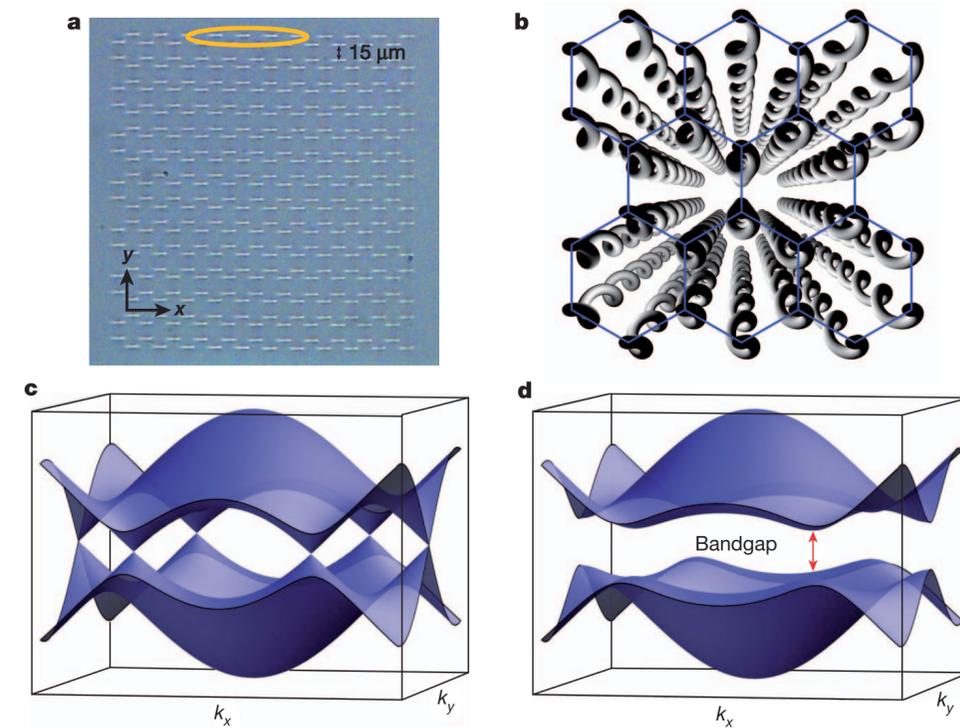


Khanikaev *et al.* Nature Communications (2015)

PRL 114, 114301 (2015) PHYSICAL

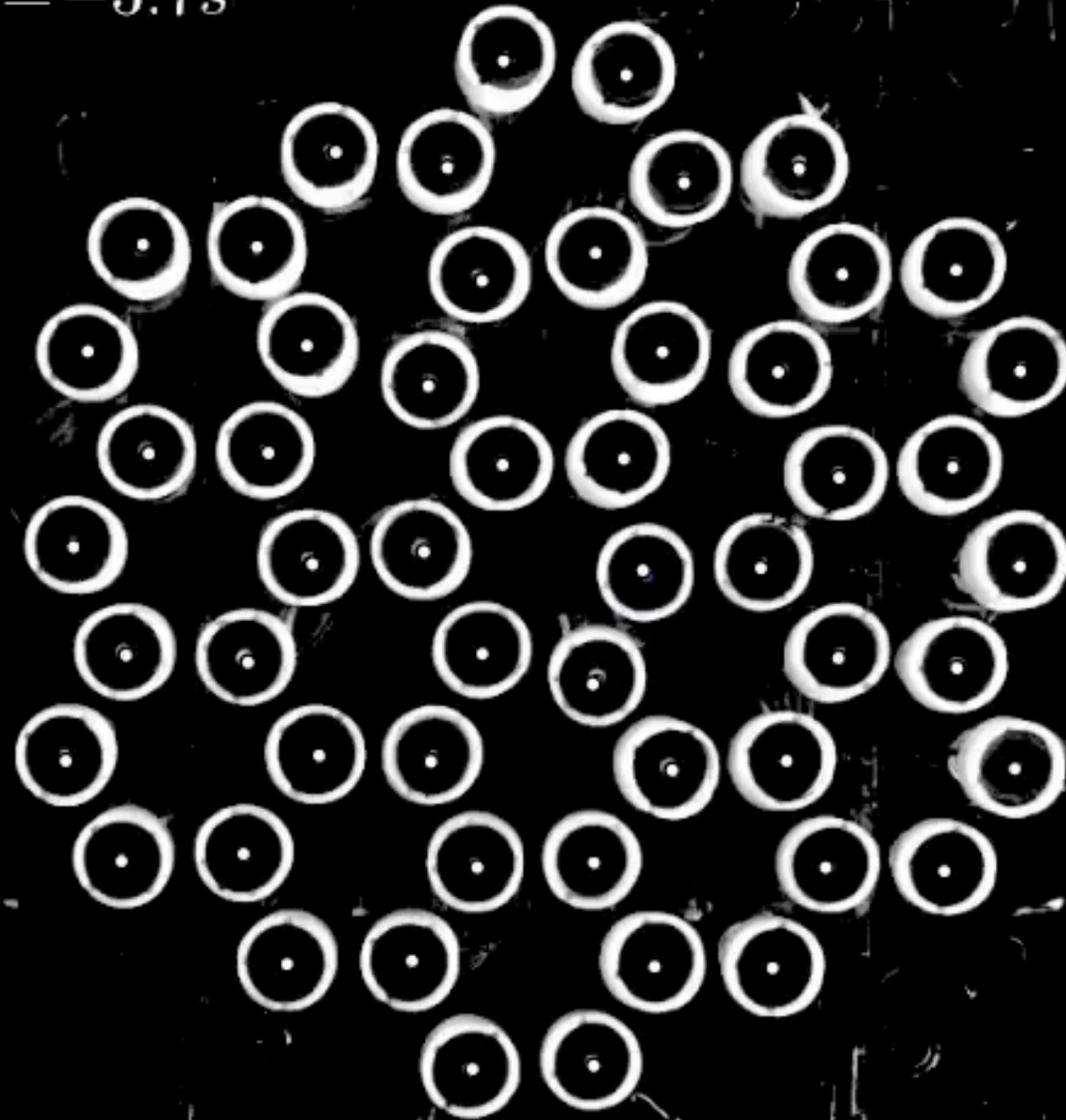


Yang *et al.* PRL (2015)



Rechtsman *et al.* Nature (2013)

$t = -3.7s$

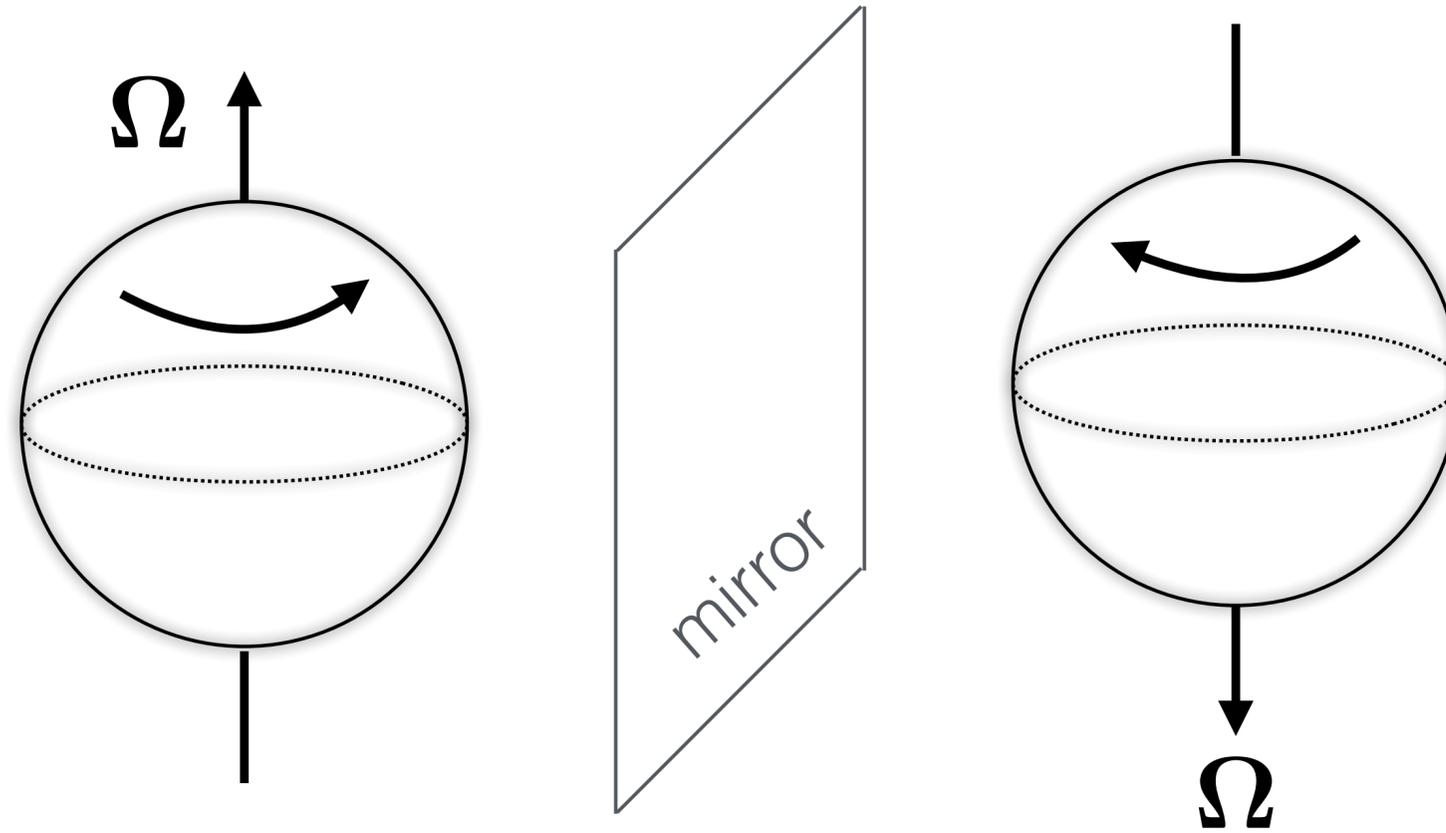


Nash *et al.* PNAS (2015)

# The (Continuous) Fluid Earth System?

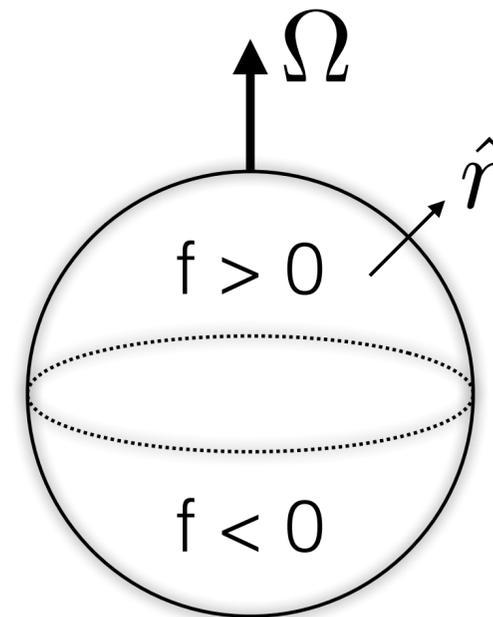


# CPT = I Theorem and Coriolis Force



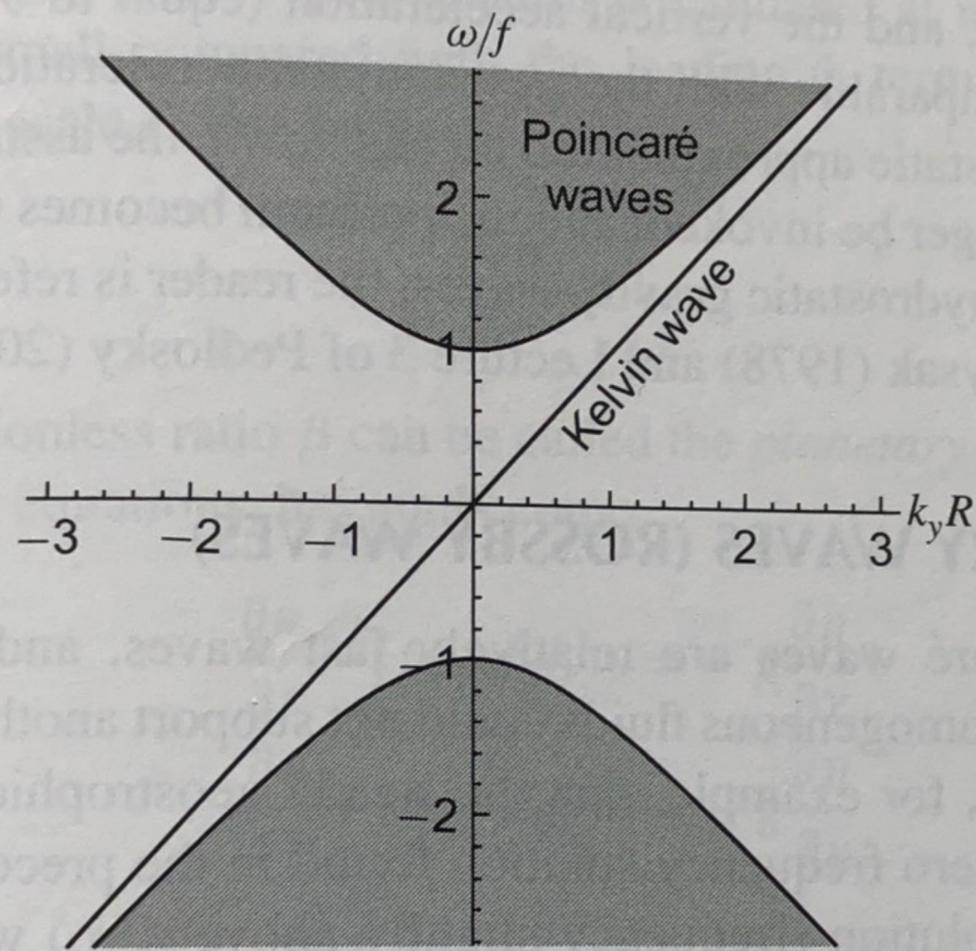
$$\mathbf{a}_{rot} = \frac{\mathbf{F}}{m} - 2\boldsymbol{\Omega} \times \mathbf{v}_{rot} - \boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \mathbf{r}$$

$$f \equiv 2\boldsymbol{\Omega} \cdot \hat{\mathbf{r}} = 2\Omega \cos(\theta)$$



# Objections Raised

- Too Simple (single component)
- No Lattice (no compact Brillouin zone)
- Bulk - interface correspondence?



**FIGURE 9.3** Recapitulation of the dispersion relation of Kelvin and Poincaré waves on the  $f$ -plane and on a flat bottom. Although Poincaré waves (gray shades) can travel in all directions and occupy therefore a continuous spectrum in terms of  $k_y$ , the Kelvin wave (diagonal line) propagates only along a boundary.

Venaille and Delplace, "Wave topology brought to the coast," PRR 3, 043002 (2021)

# Wave Tank Experiments by Satoshi Sakai (Kyoto University)

[http://gfd-dennou.org/library/gfd\\_exp/exp\\_e/exp/kw/1/res.htm](http://gfd-dennou.org/library/gfd_exp/exp_e/exp/kw/1/res.htm)



*Monday, 17th March 1879.*

Professor KELLAND, President, in the Chair.

The following Communications were read :—

1. On Gravitational Oscillations of Rotating Water.

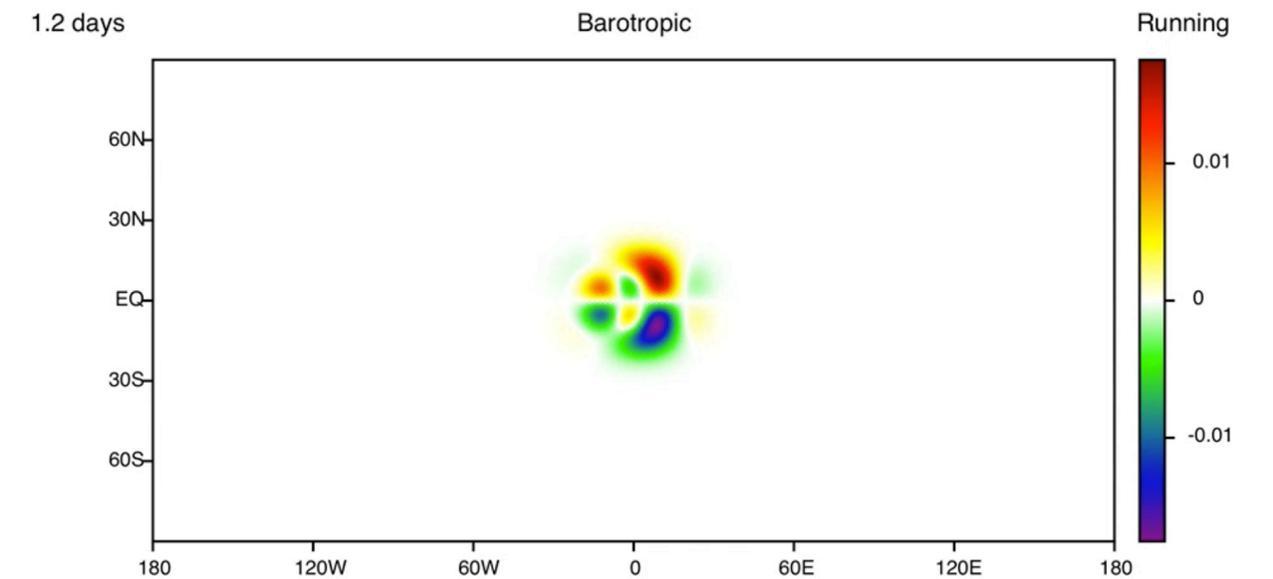
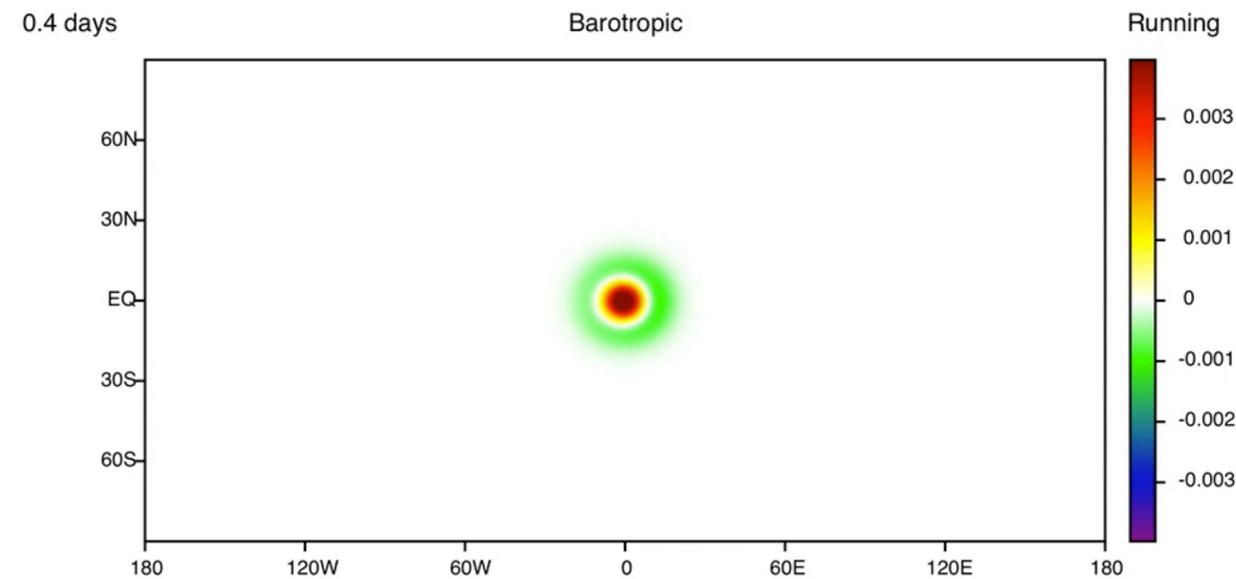
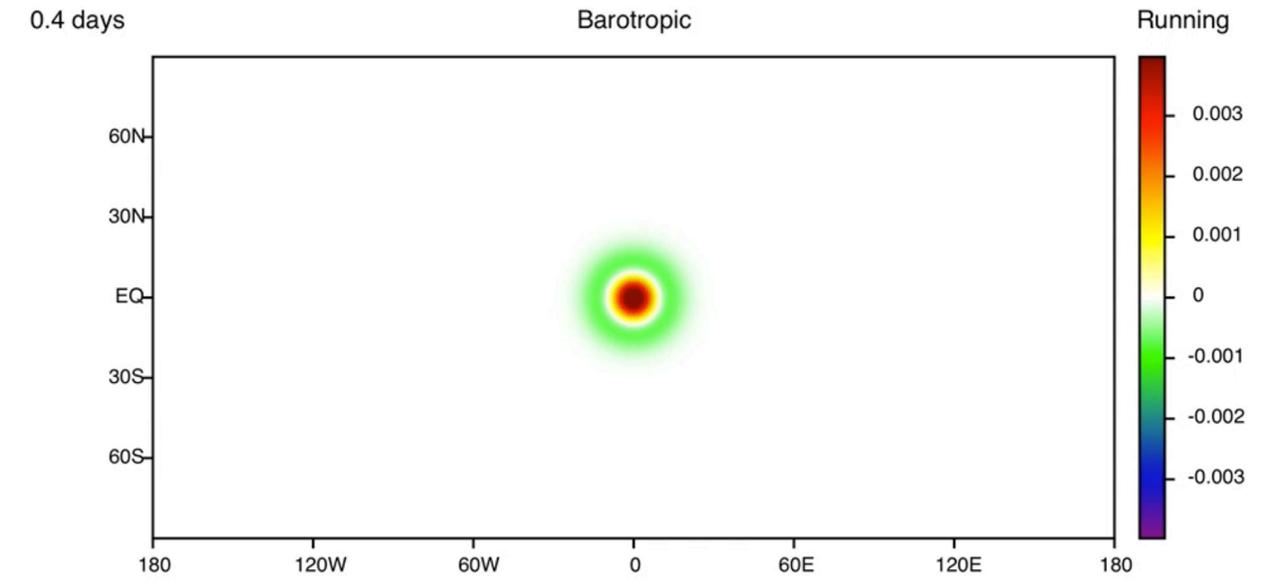
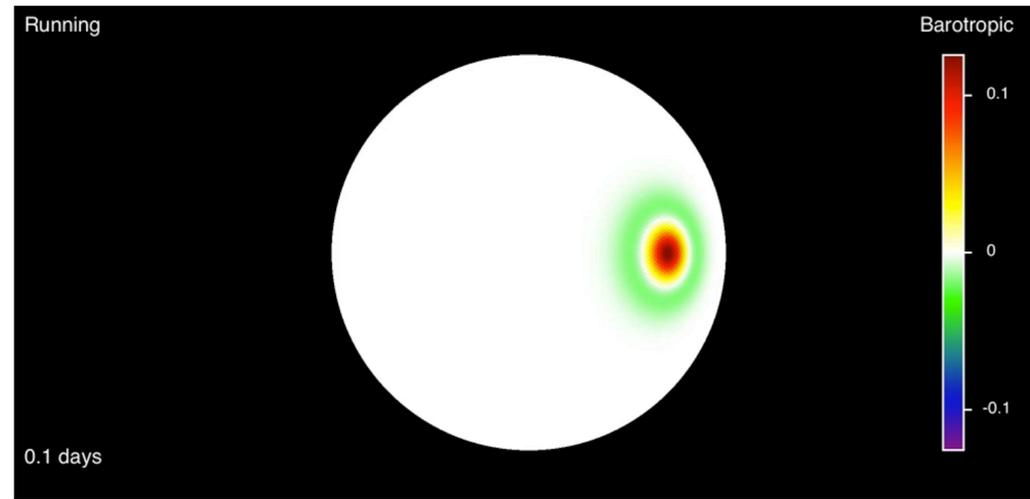
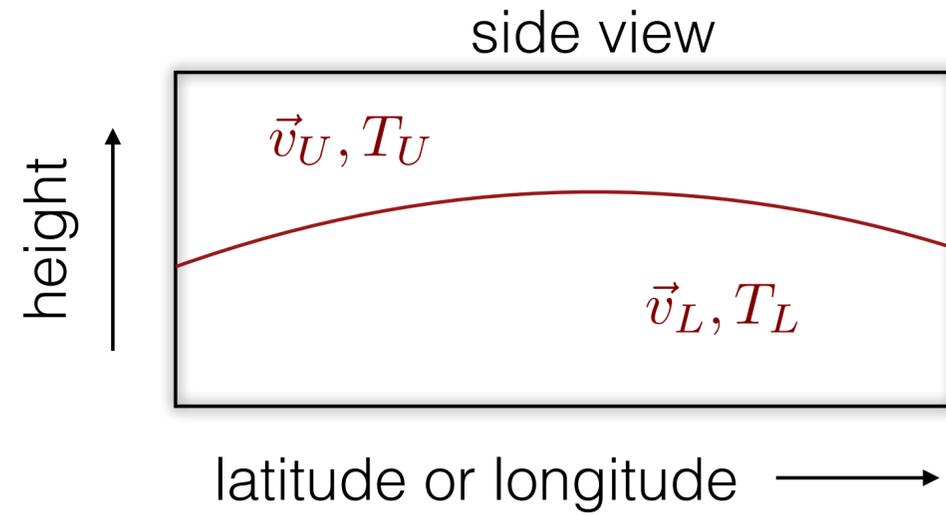
By Sir William Thomson.

*(Abstract.)*

This is really Laplace's subject in his Dynamical Theory of the Tides; where it is dealt with in its utmost generality except one important restriction,—the motion of each particle to be infinitely nearly horizontal, and the velocity to be always equal for all par-

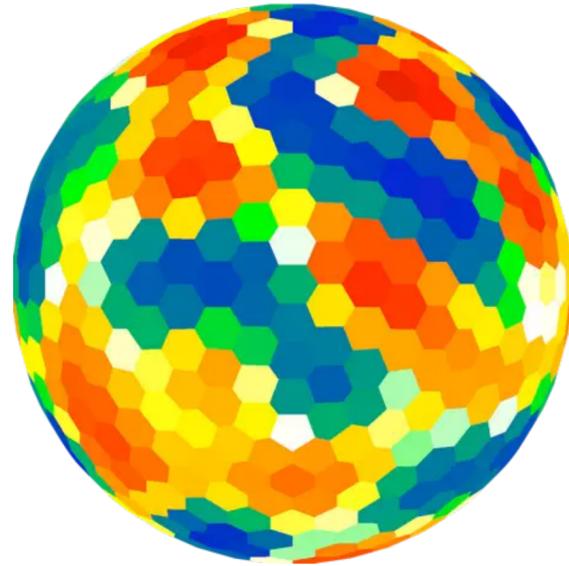
between the period of the oscillation  $\left(\frac{2\pi}{\sigma}\right)$ , the period of the rotation  $\left(\frac{2\pi}{\omega}\right)$ , and the time required to travel at the velocity  $\frac{\sigma}{m}$  across the canal. The more approximately nodal character of the tides on the north coast of the English Channel than on the south or French coast, and of the tides on the west or Irish side of the Irish Channel than on the east or English side, is probably to be accounted for on the principle represented by this factor, taken into account along with frictional resistance, in virtue of which the tides of the English Channel may be roughly represented by more powerful waves travelling from west to east, combined with less powerful waves travelling from east to west, and those of the

# 2 Layer Model



# Mac App Store Preview

Open the Mac App Store to buy and download apps.



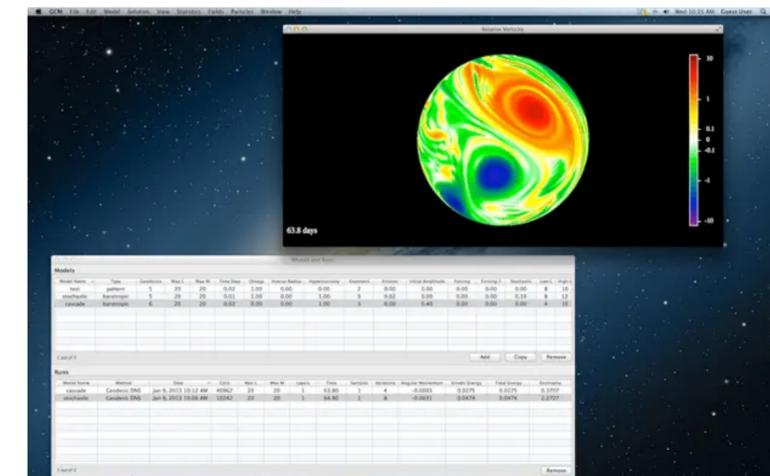
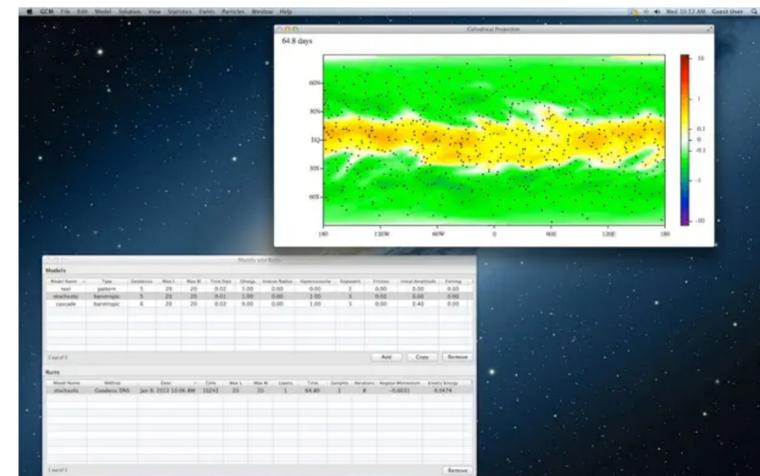
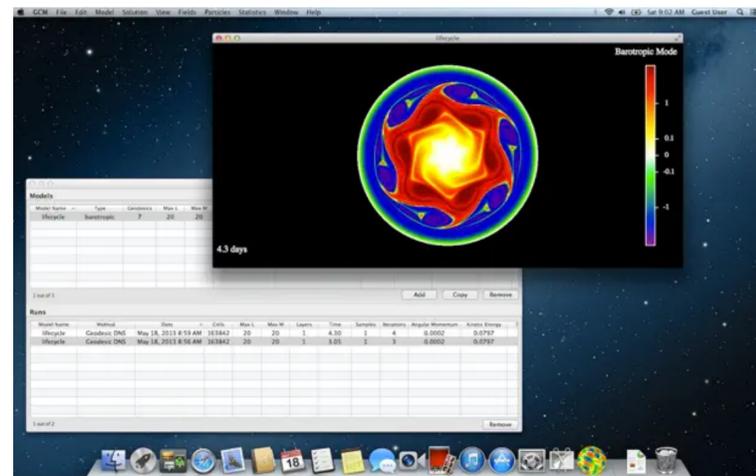
**GCM** 4+  
General Circulation Models  
Brad Marston

★★★★★ 4.8 • 5 Ratings

Free

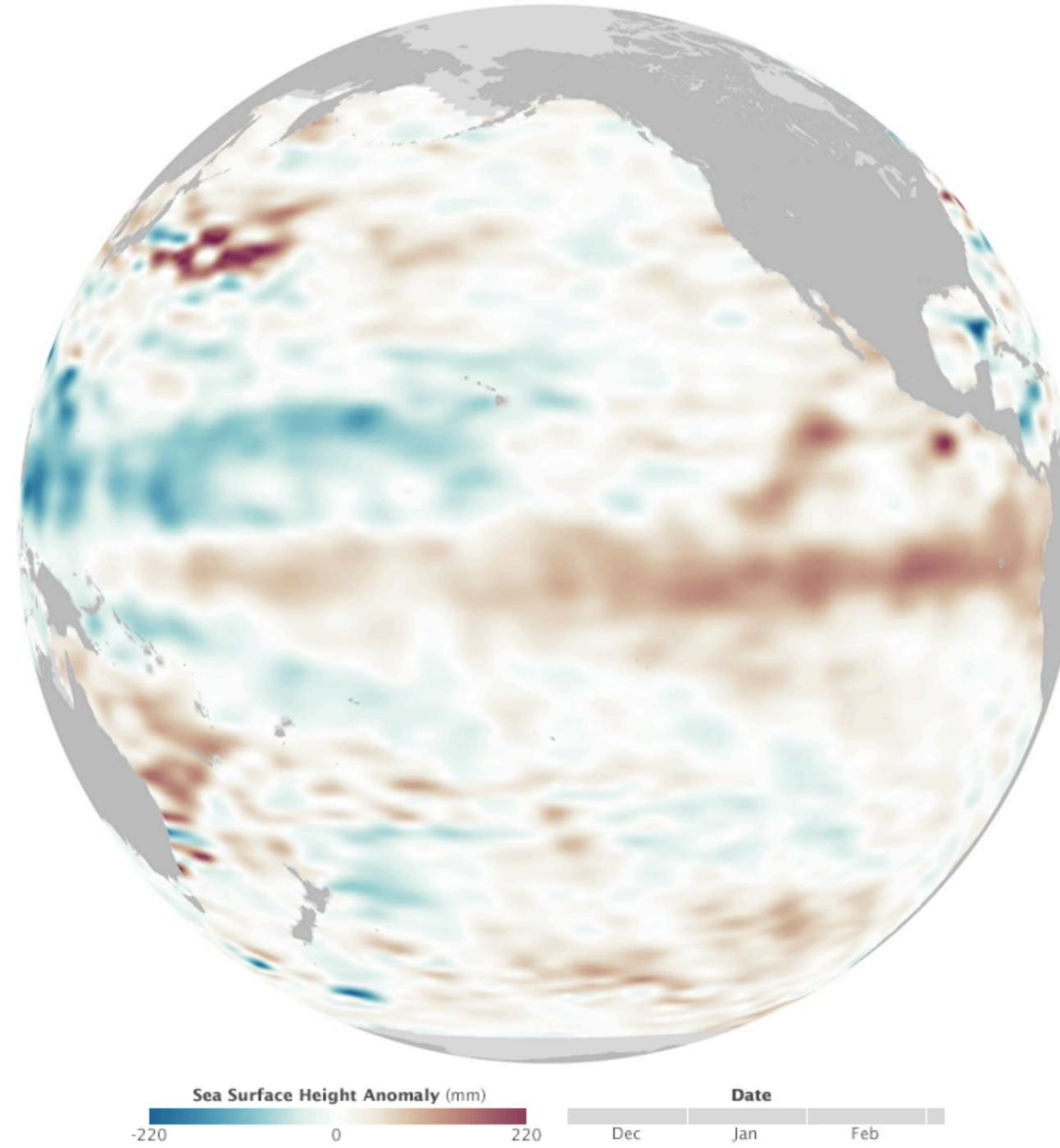
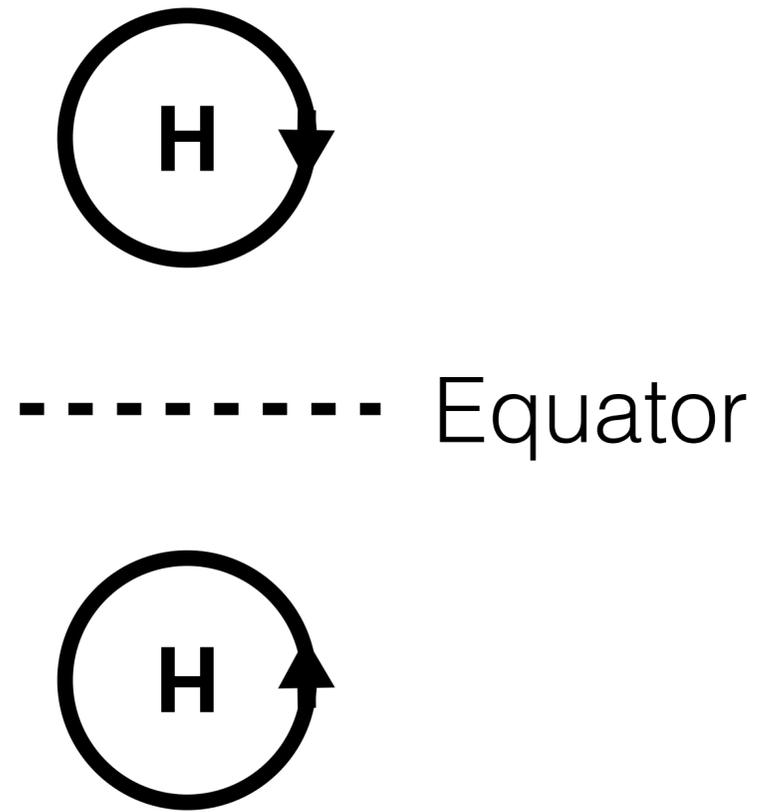
[View in Mac App Store ↗](#)

## Screenshots



Idealized General Circulation Models (GCMs) of planetary atmospheres, and a stellar tachocline, solved by a variety of methods. Now supports Dark Mode.

# El Niño 2009 - 2010



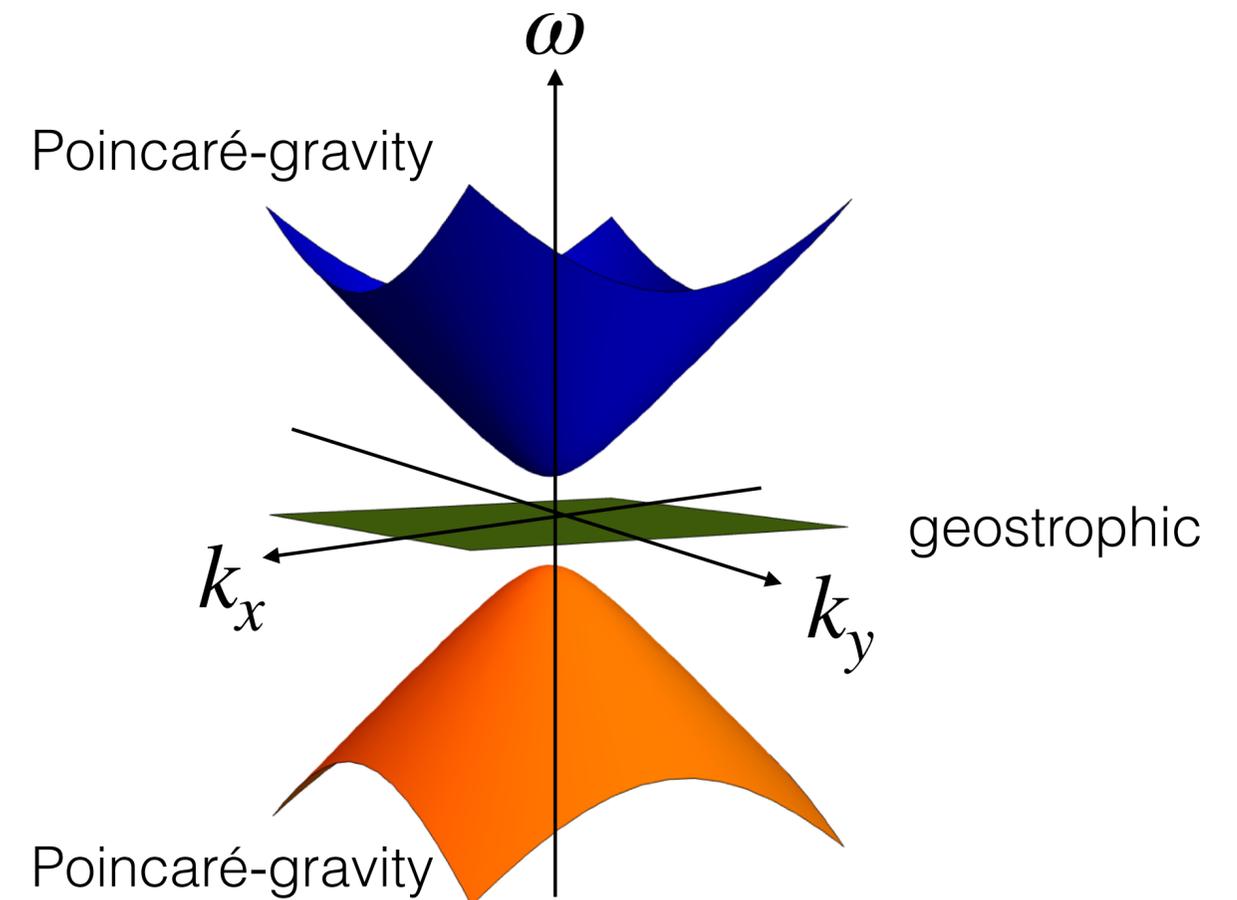
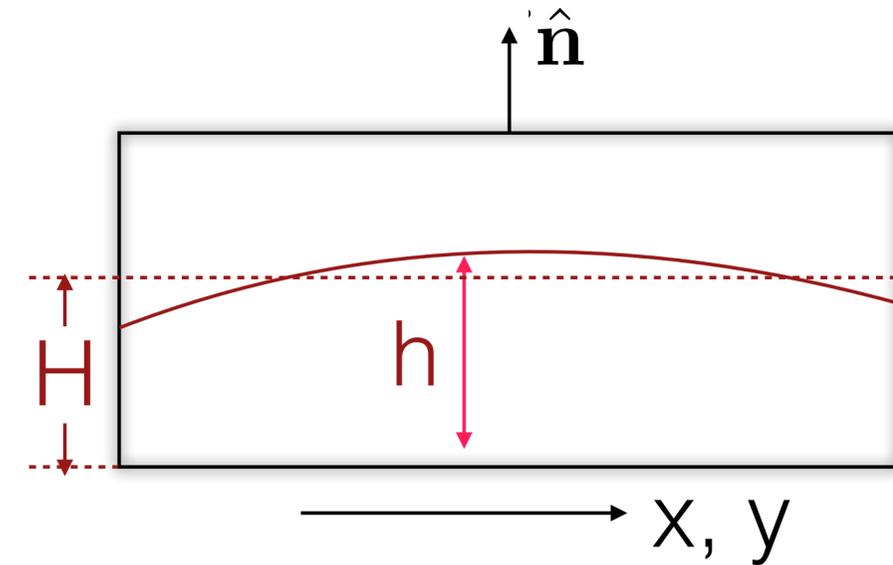
# Shallow Water Equations

$$\begin{aligned}\partial_t h + \nabla \cdot (h \mathbf{u}) &= 0, \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} &= -g \nabla h - f \hat{\mathbf{n}} \times \mathbf{u}.\end{aligned}$$

$$H = \begin{bmatrix} 0 & -if & gk_x \\ if & 0 & gk_y \\ Hk_x & Hk_y & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ h \end{bmatrix}$$

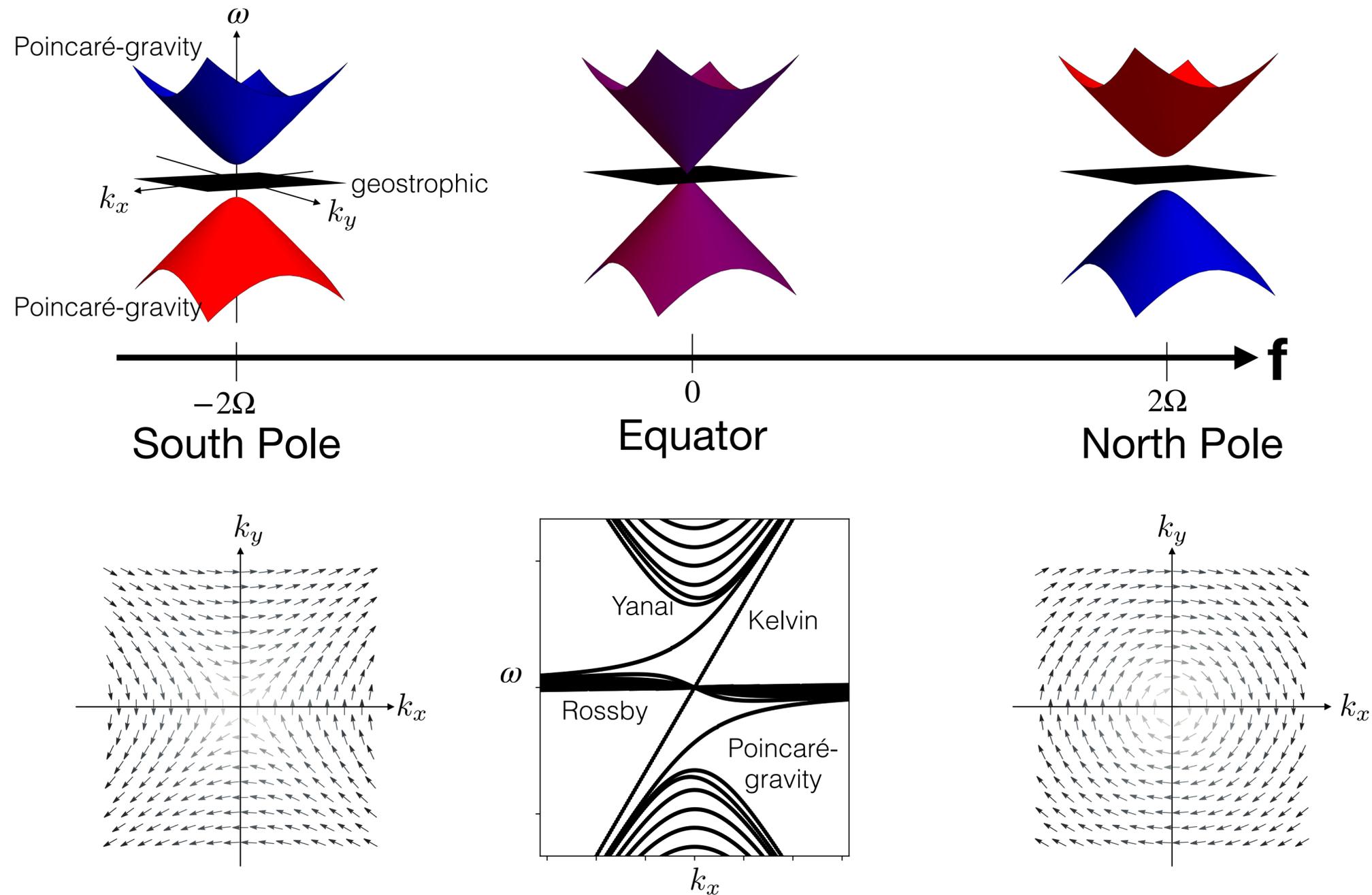
$$\omega_{\pm} = \pm \sqrt{gH(k_x^2 + k_y^2) + f^2}; \quad \omega_0 = 0$$

$$\psi_{\pm} = \begin{bmatrix} (k_x \pm ik_y) f \\ (k_y \mp ik_x) f \\ \pm H(k_x^2 + k_y^2) \end{bmatrix}$$



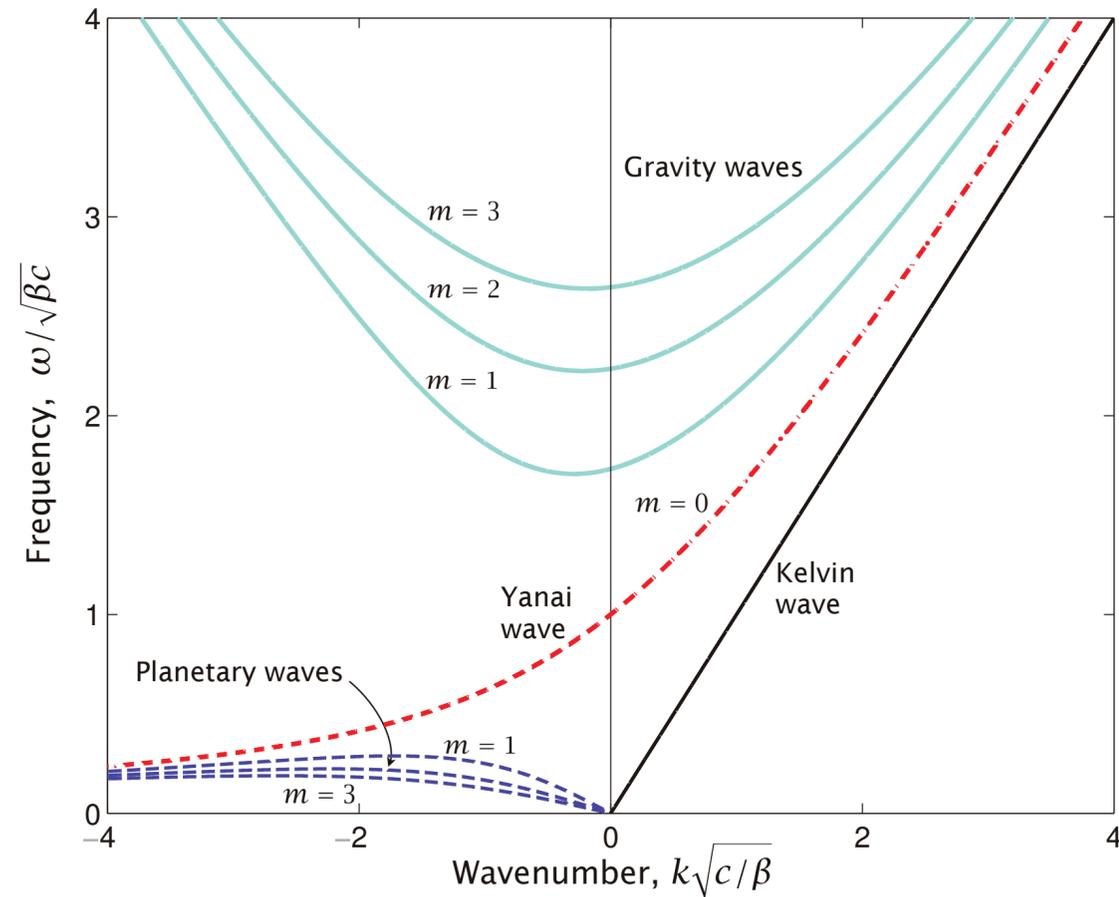
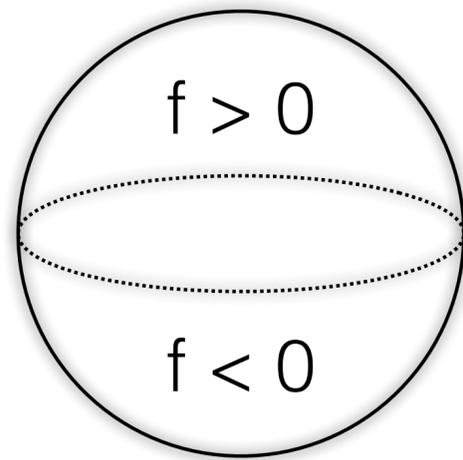
# Weyl Singularity & Gauge-Invariant Winding Number

Winding number changes by 2: 2 unidirectional waves



$$\Xi(k_x, k_y) = v(k_x, k_y) \cdot h^*(k_x, k_y)$$

# Shallow Water Waves Near Equator

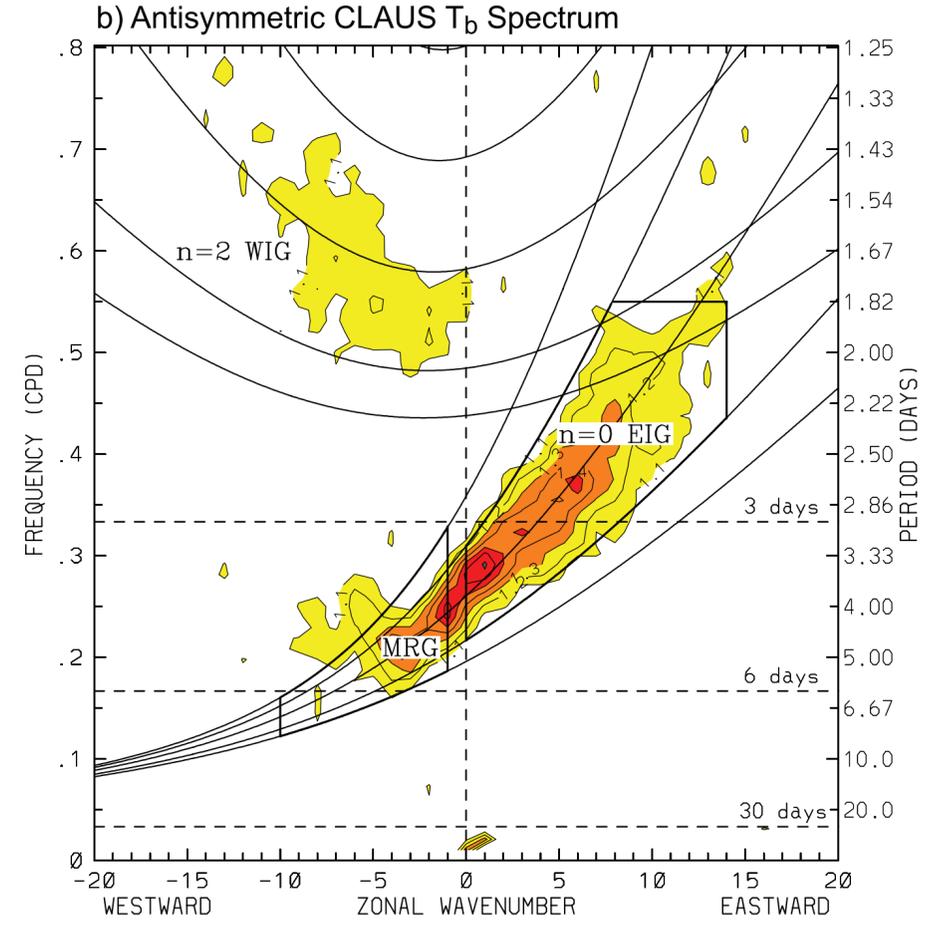
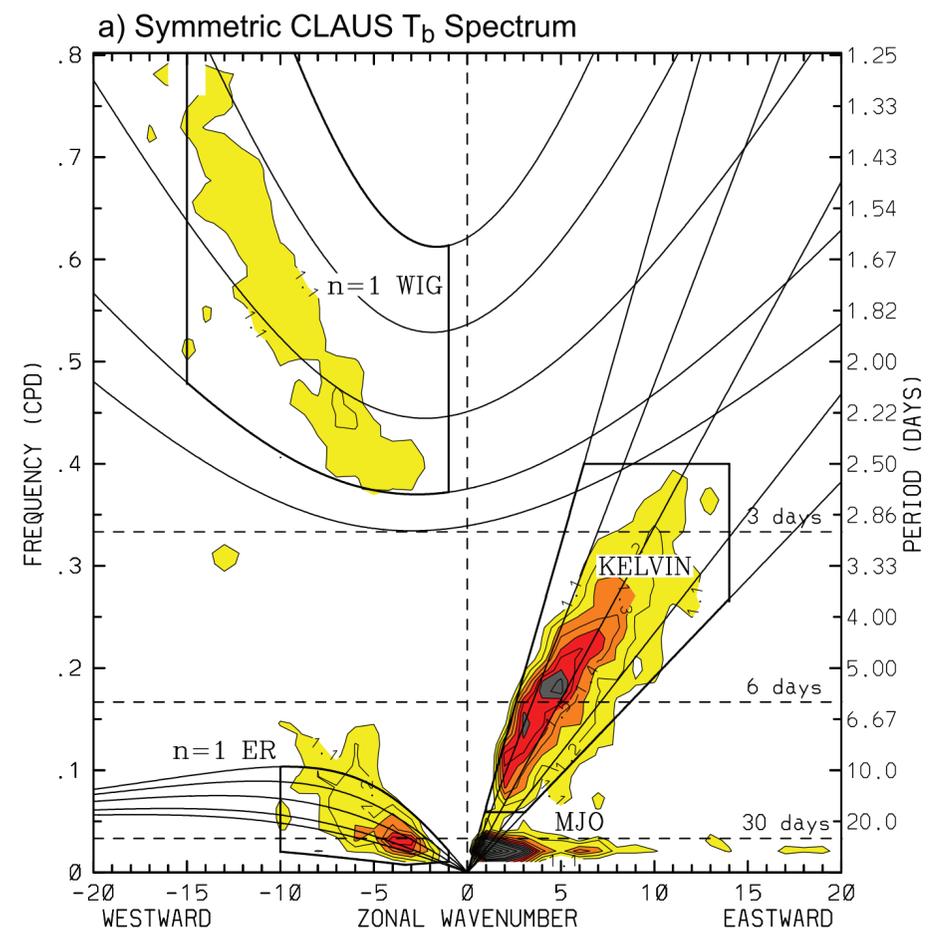
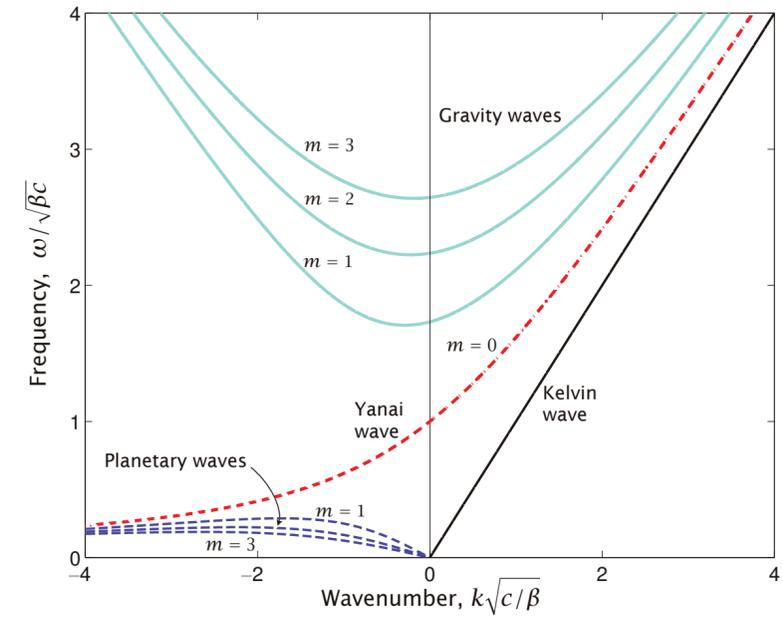
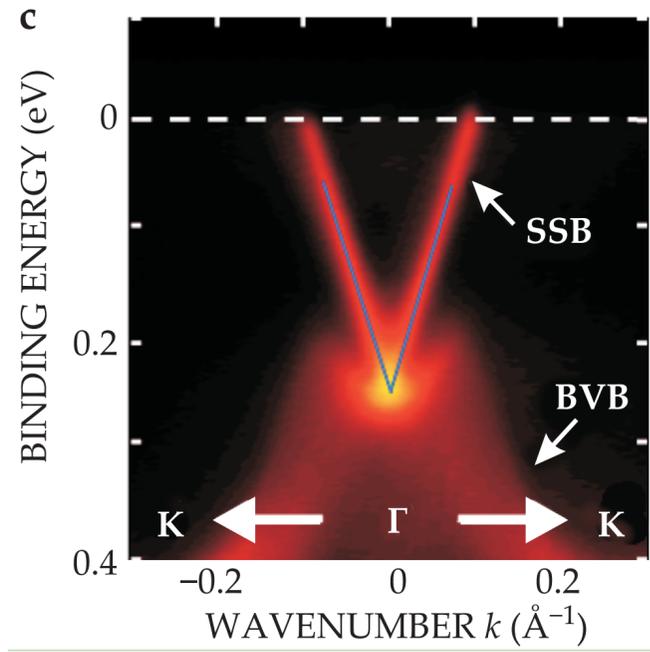


Matsuno 1966 and Longuet-Higgins 1968.

[Geoff Vallis, *Atmospheric and Oceanic Fluid Dynamics* (notes for a 2nd ed.)]

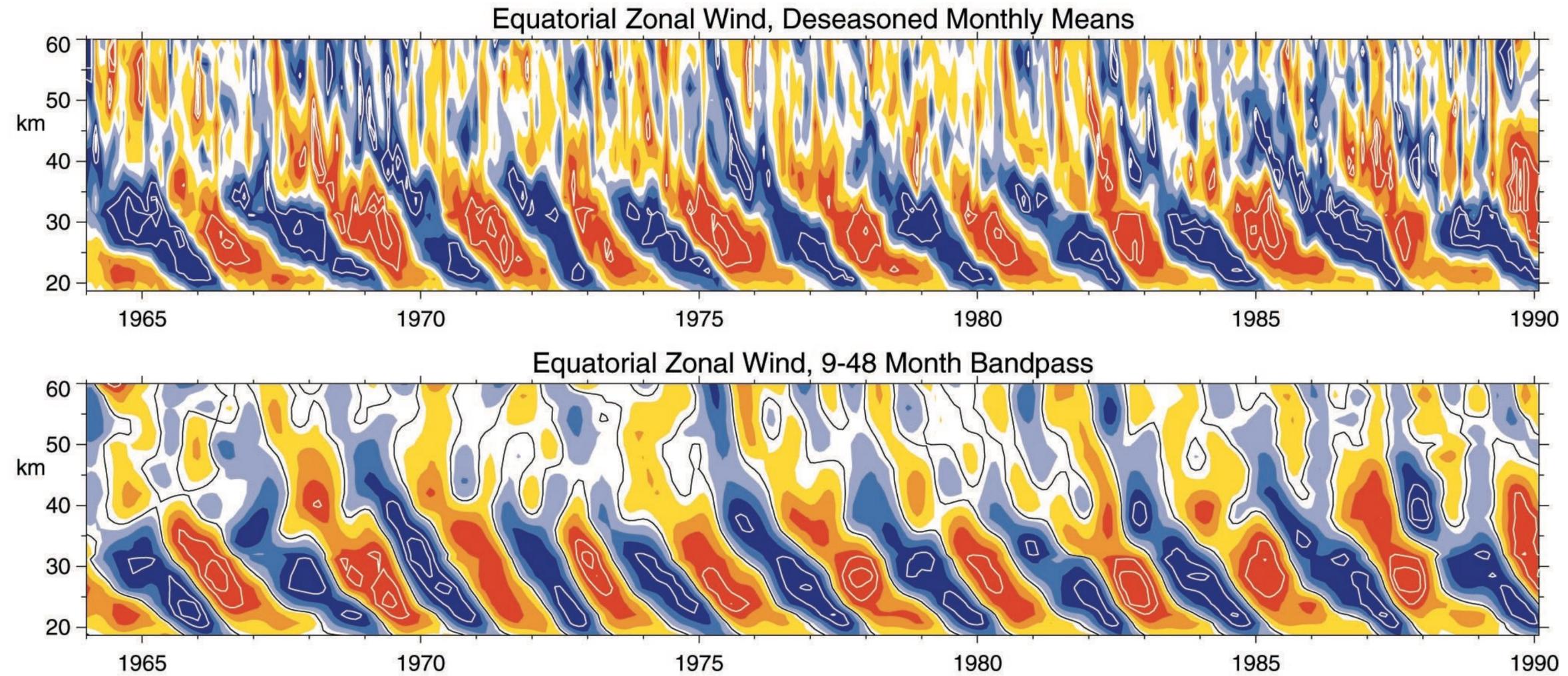
*The question of why the equator supports a uni-directional wave, the Kelvin wave, is interesting scientifically and important mathematically. I have asked the question myself and been unsatisfied with my own answer, as well as the answer of atmosphere / ocean scientists.*

Qi and Zhang  
(2010)



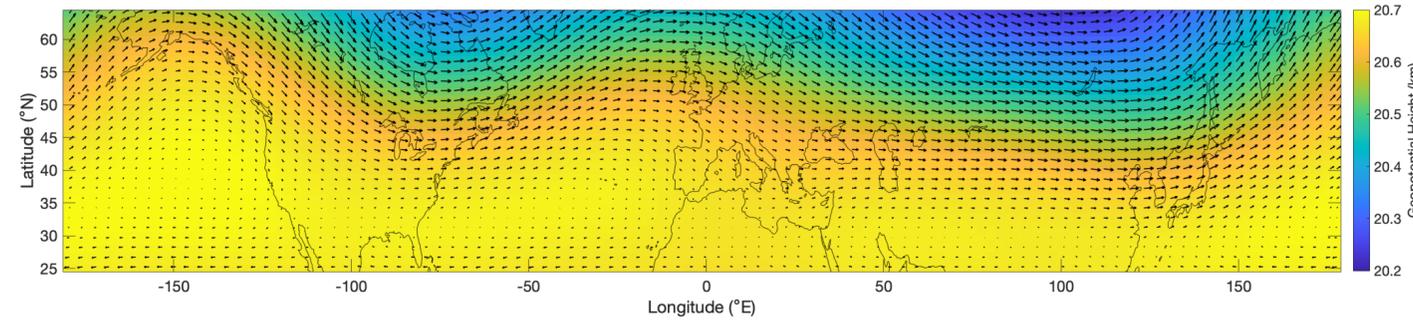
# Convectively Coupled Equatorial Waves: Kiladis & Wheeler (1999)

# Quasi-Biennial Oscillation (QBO) in the Stratosphere

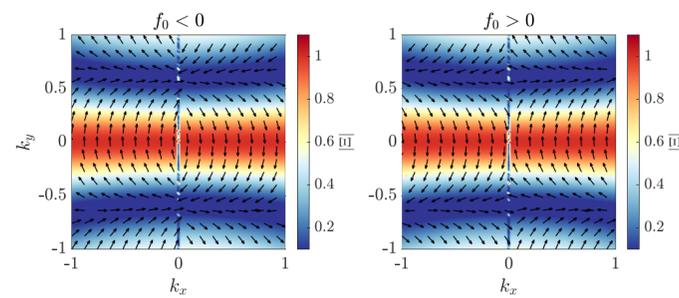


Baldwin (2001)

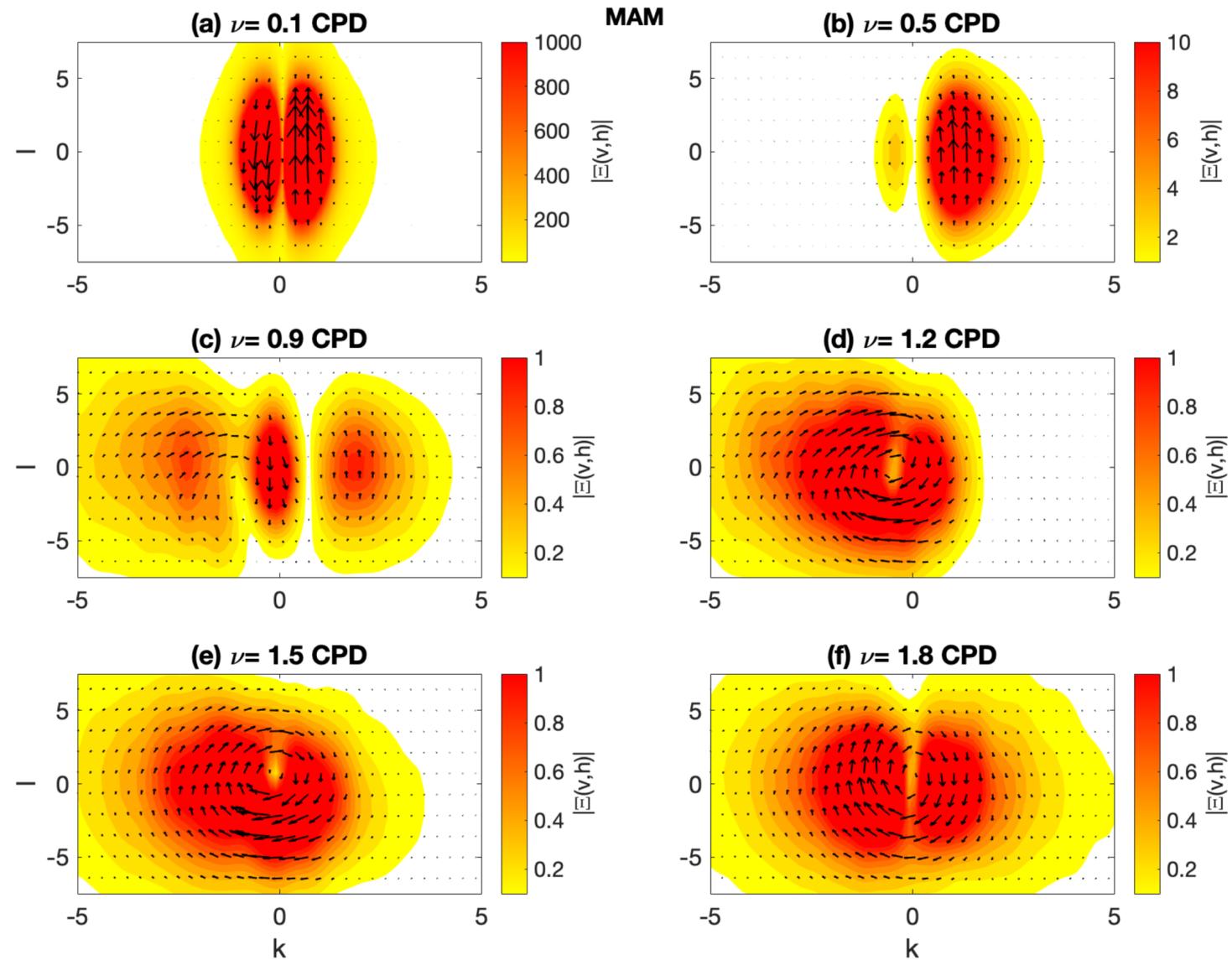
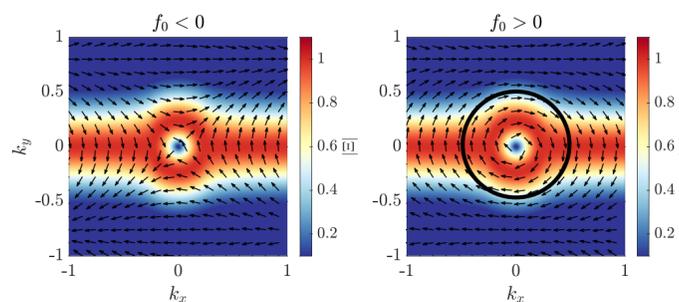
# ERA5 Reanalysis Data at 20km height from 25N to 65N



Rossby Waves

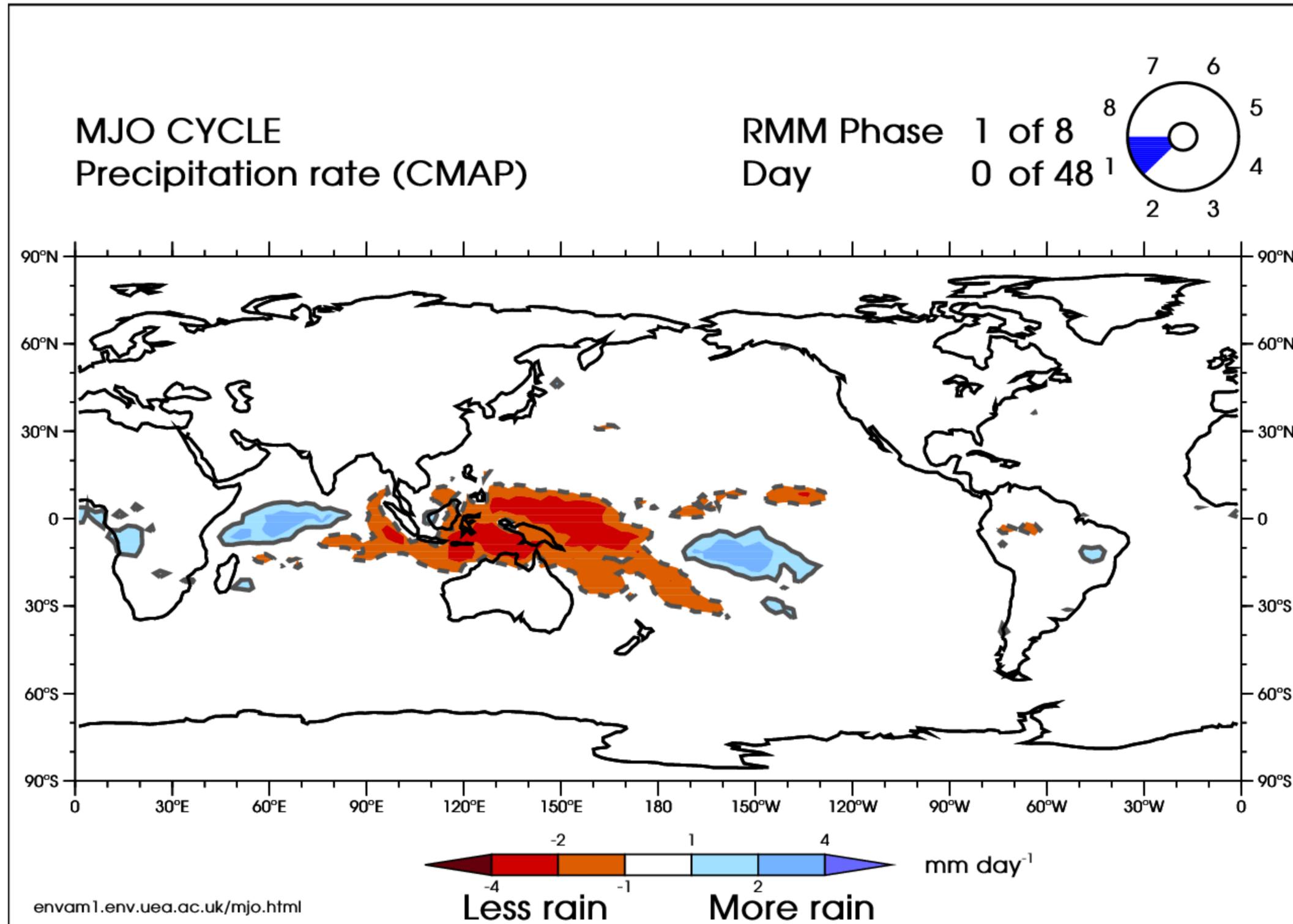


Poincaré-Gravity Waves



Non-trivial topology found!

# Madden-Julian Oscillation (MJO)



Matthews  
(2008)

# Robust wall states in rapidly rotating Rayleigh–Bénard convection

Benjamin Favier<sup>1,†</sup> and Edgar Knobloch<sup>2</sup>

*F. Zhang and J.-H. Xie*

*Robust wall states in rapidly rotating RBC*

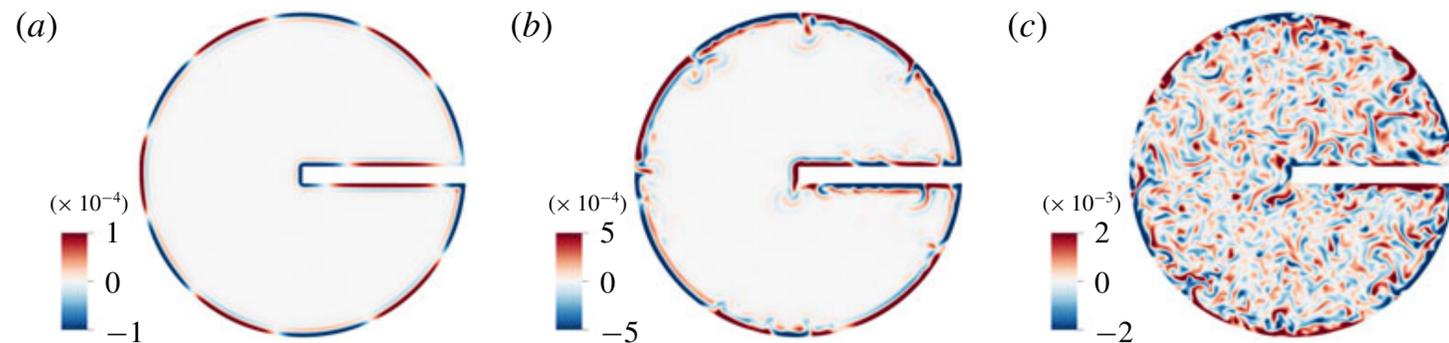


FIGURE 6. Vertical velocity in the mid-plane  $z = 0.5$  for a cylinder with a barrier. The Rayleigh number increases from (a) to (c): (a)  $Ra = 5 \times 10^7$ , (b)  $Ra = 5 \times 10^8$  and (c)  $Ra = 2 \times 10^9$ . Parameters are  $\Gamma = 1.5$ ,  $E = 10^{-6}$  and  $Pr = 1$ .

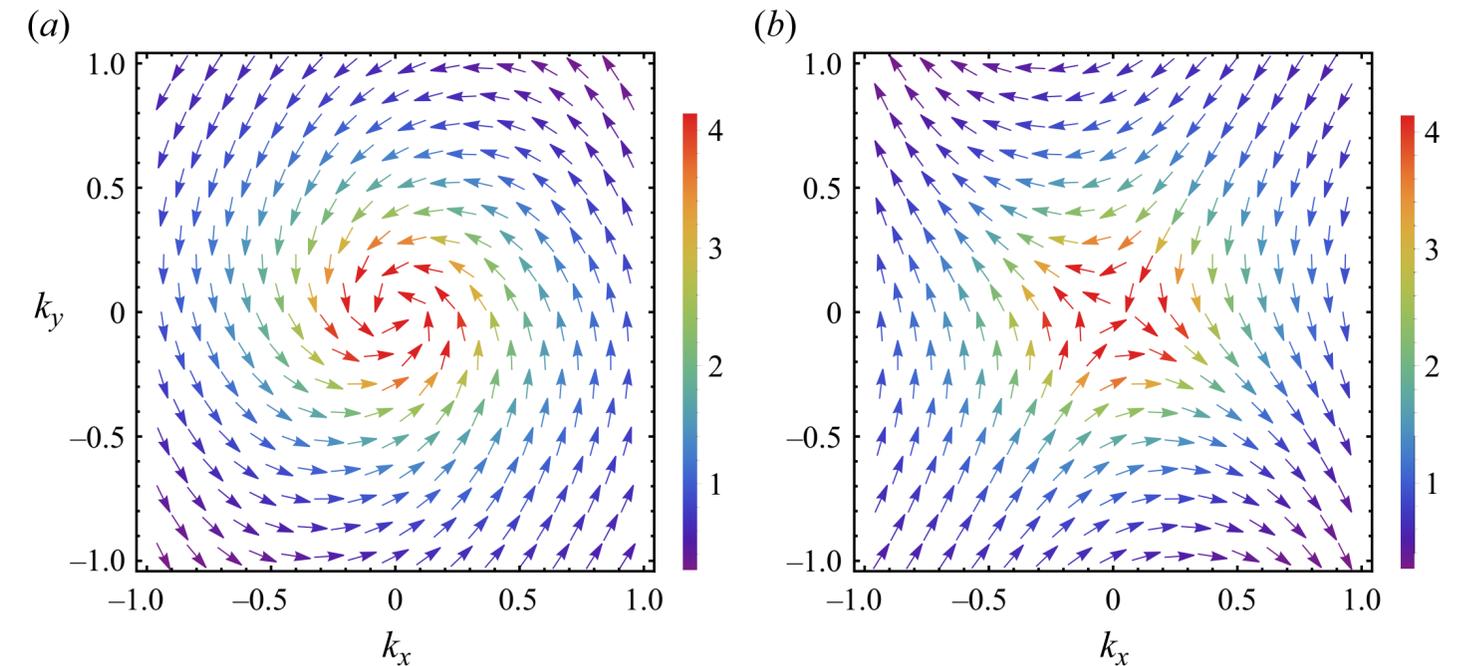
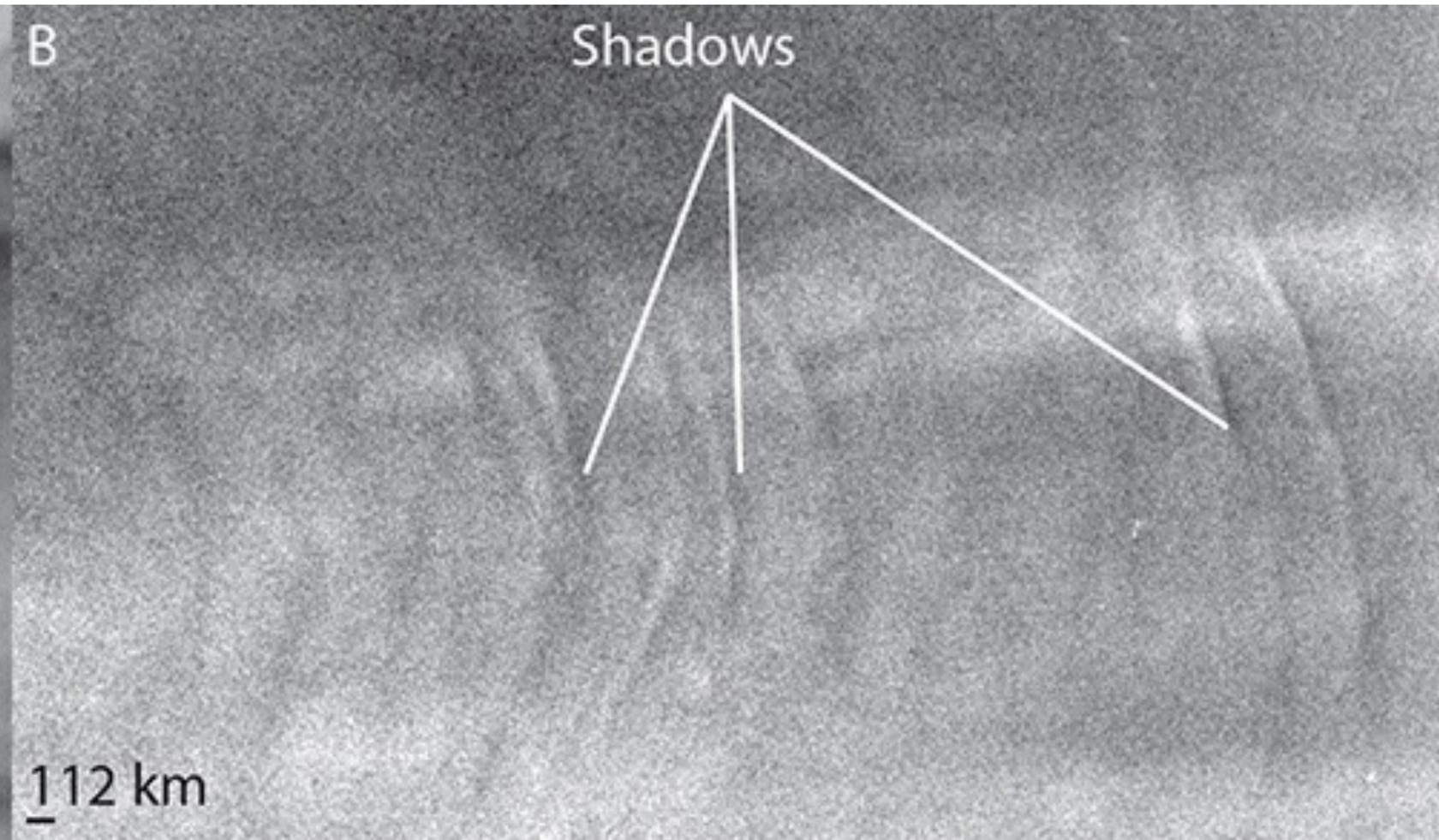
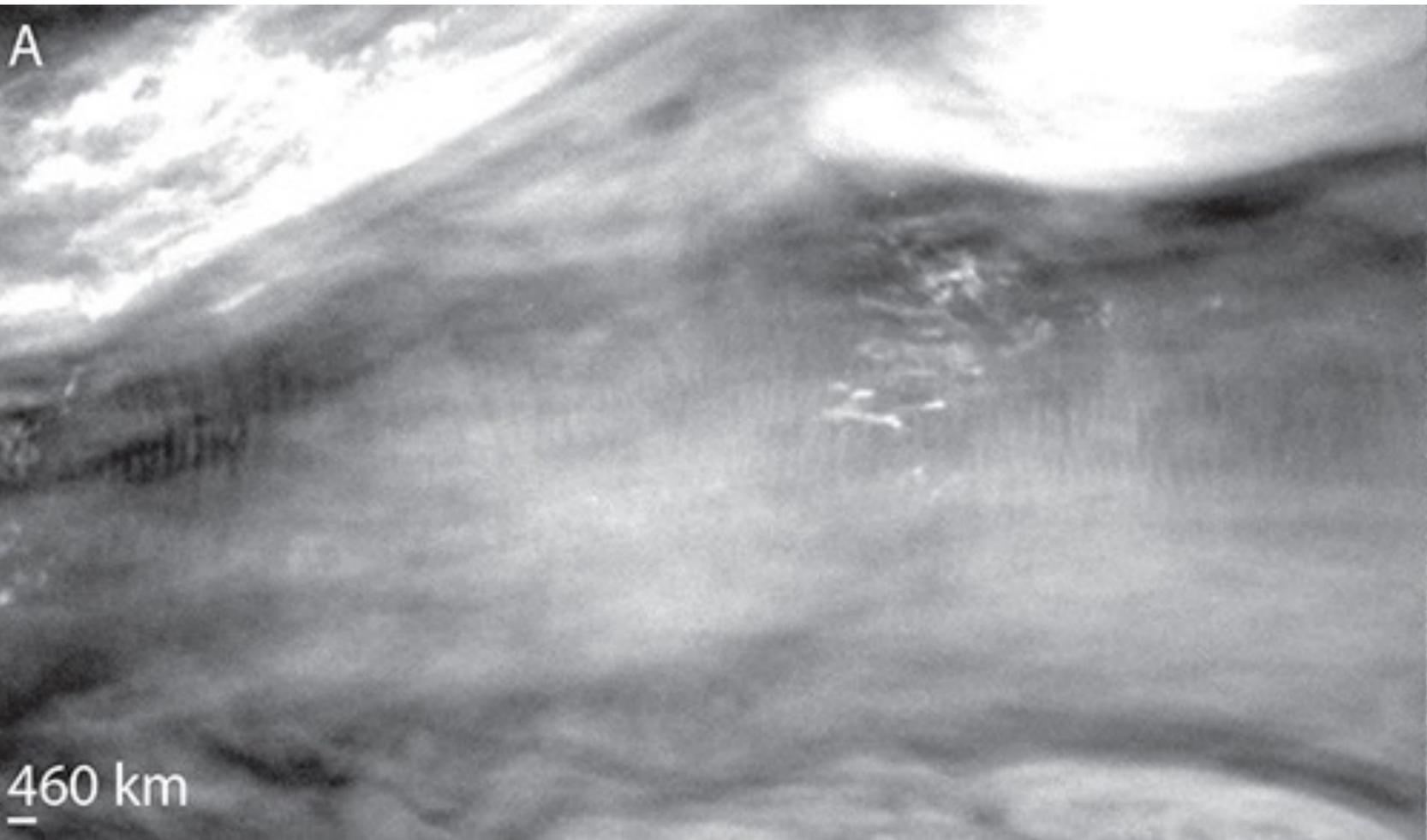
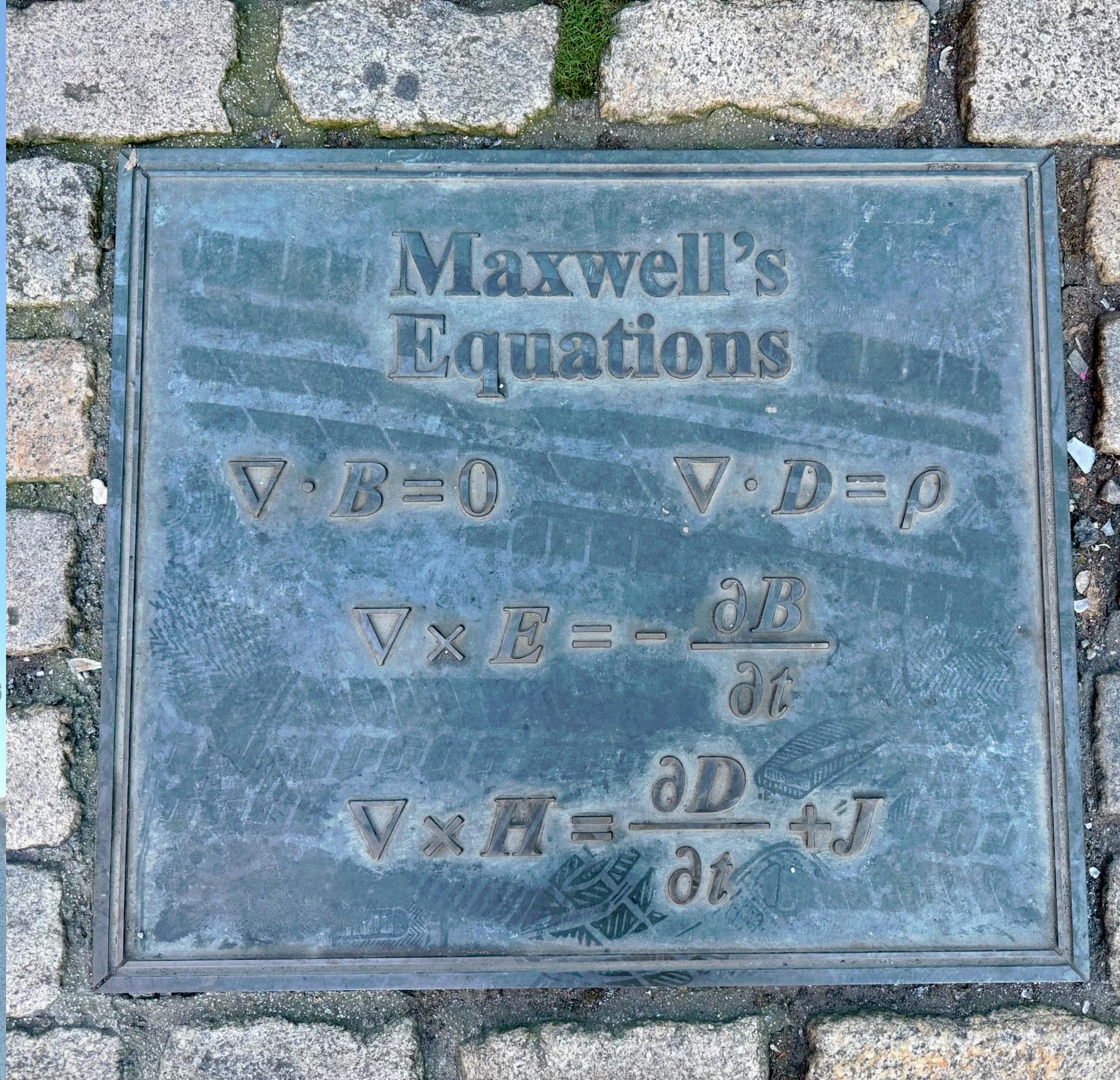


Figure 3. Arrows representing argument of  $\mathcal{E}(\mathbf{k}) = \hat{v}(\mathbf{k}) * \hat{w}^*(\mathbf{k})$  with  $\lambda = 1$  (a) and  $\lambda = -1$  (b). The  $x$  and  $y$  components of the arrow represent the real and imaginary parts of  $\mathcal{E}(\mathbf{k})$ . The length of the arrows is rescaled to be equal. Colours represent normalized magnitude  $|\mathcal{E}(\mathbf{k})|$  in arbitrary units. Other parameters are  $Pr = 1$ ,  $\alpha = 0.5$ ,  $k_z = 1 + 0.5i$ .

# Possible Equatorial Kelvin Waves in Jovian Atmosphere (seen by *New Horizons* and *Voyager 2*)





# Maxwell's Equations

$$\nabla \cdot B = 0 \qquad \nabla \cdot D = \rho$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times H = \frac{\partial D}{\partial t} + J$$

# Electromagnetic Waves in Magnetized Plasmas

$$m_e \frac{\partial \mathbf{v}_e}{\partial t} = -e(\mathbf{E} + \mathbf{v}_e \times \mathbf{B}_0)$$
$$\epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} - \mu_0 \mathbf{J} \quad \mathbf{J} = -n(r)e\mathbf{v}_e$$
$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$



Left to right:

Troy Carter (UCLA)

Ziyan (Zoe) Zhu (Stanford)

Brad Marston (Brown)

Jeff Parker (Google)

Pat Pribyl (UCLA)

Shreekrishna Tripathi (UCLA)

Not pictured:

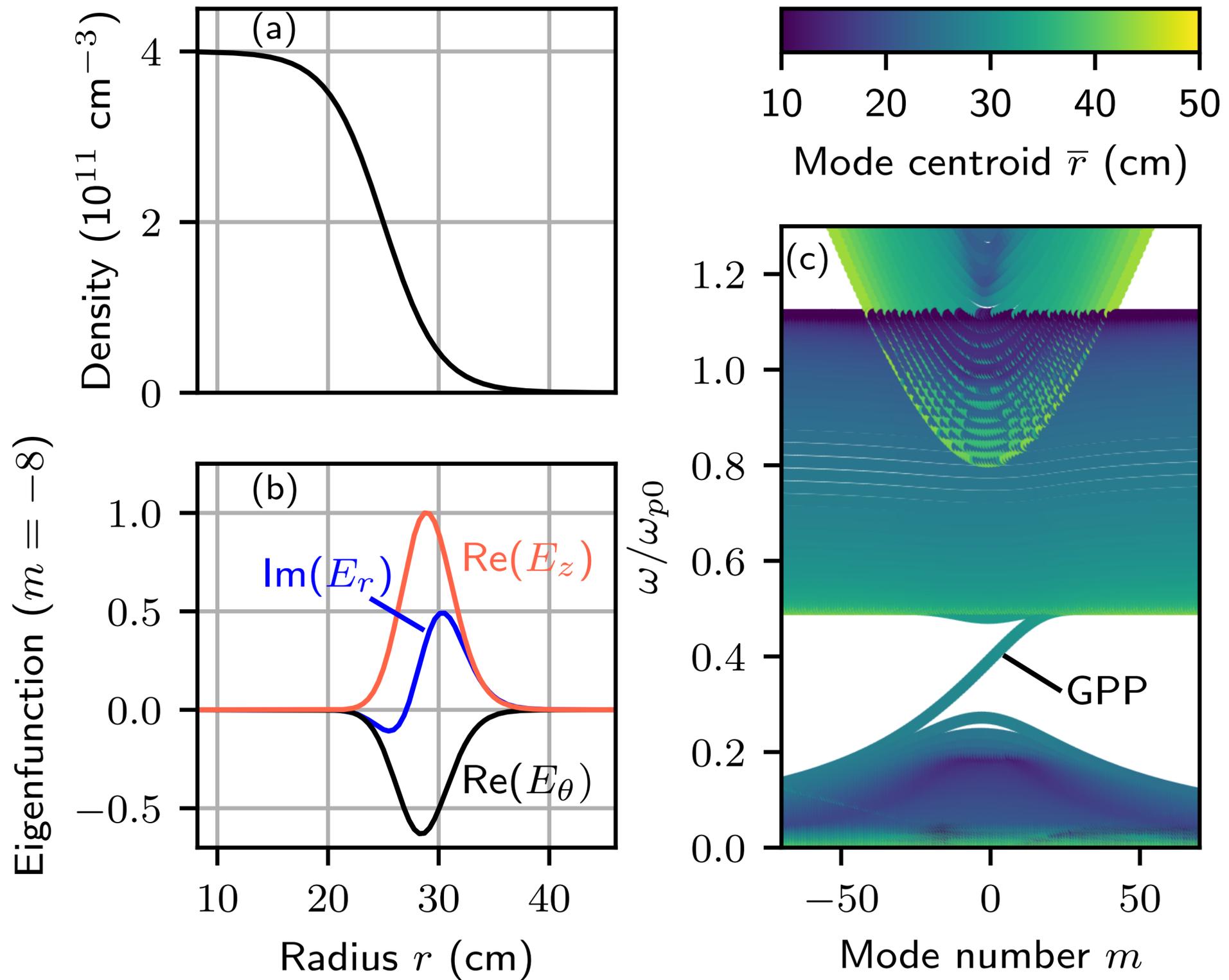
Joe Hall (MIT)

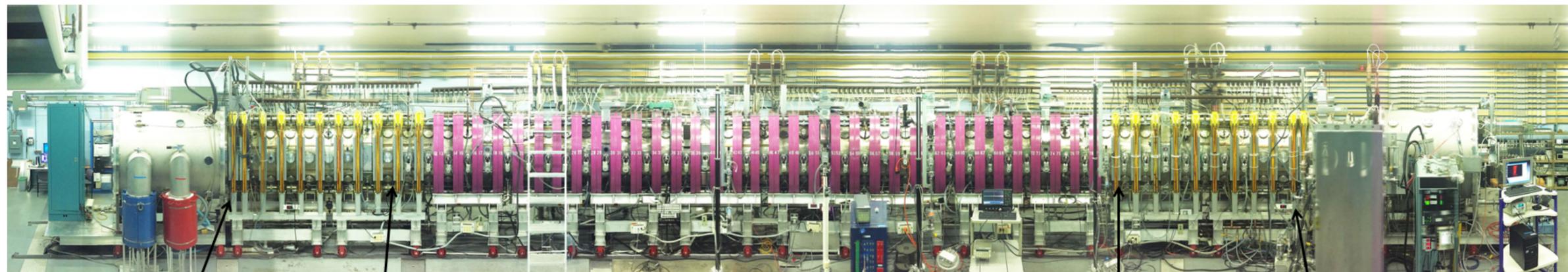
Walter Gekelman (UCLA)

Jia Han (UCLA)

Steve Tobias (Leeds)

# Gaseous Plasmon Polariton





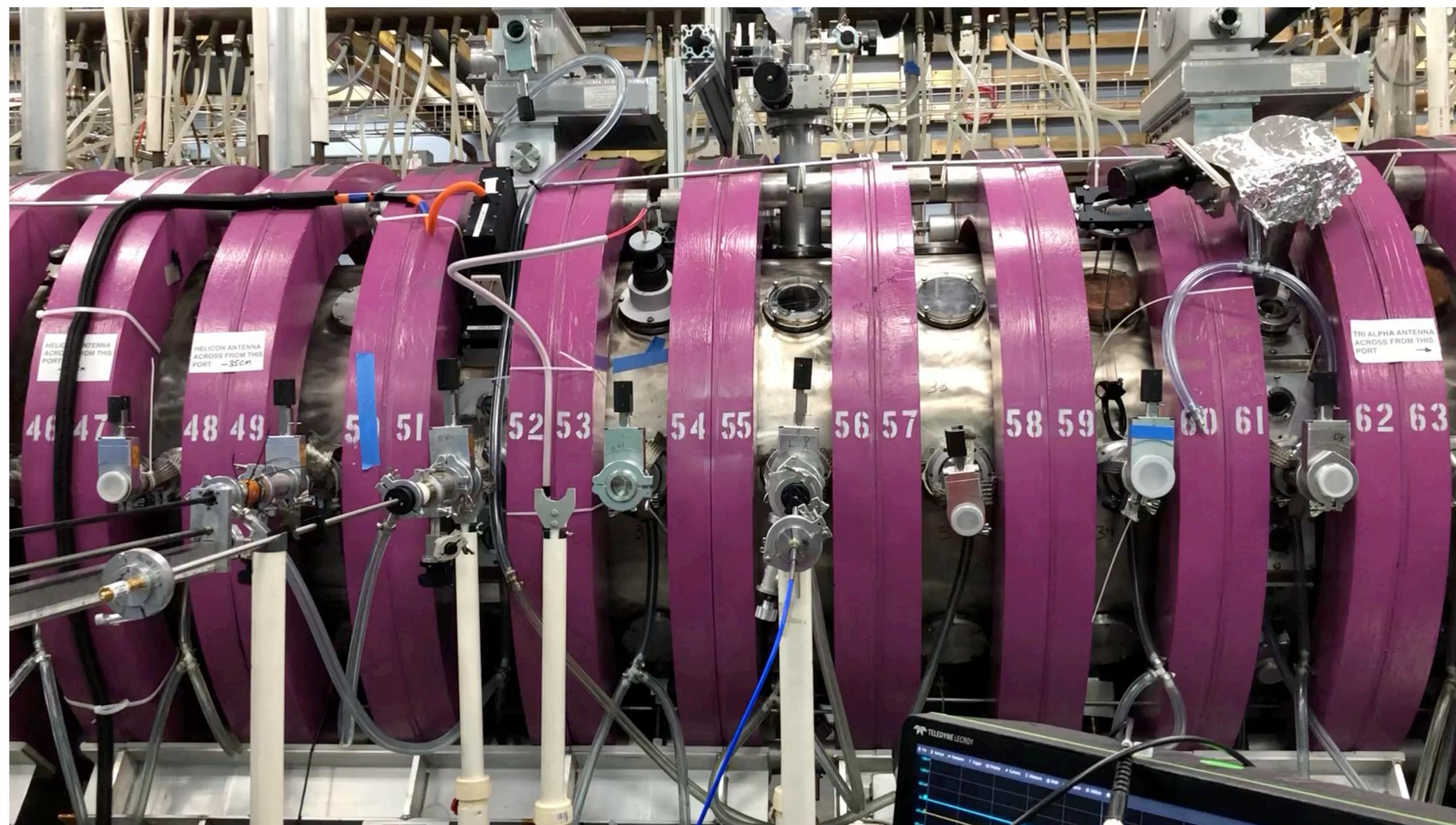
LaB6 Cathode

Anode

Measurement Region 18 m

Anode

LaB6 Cathode



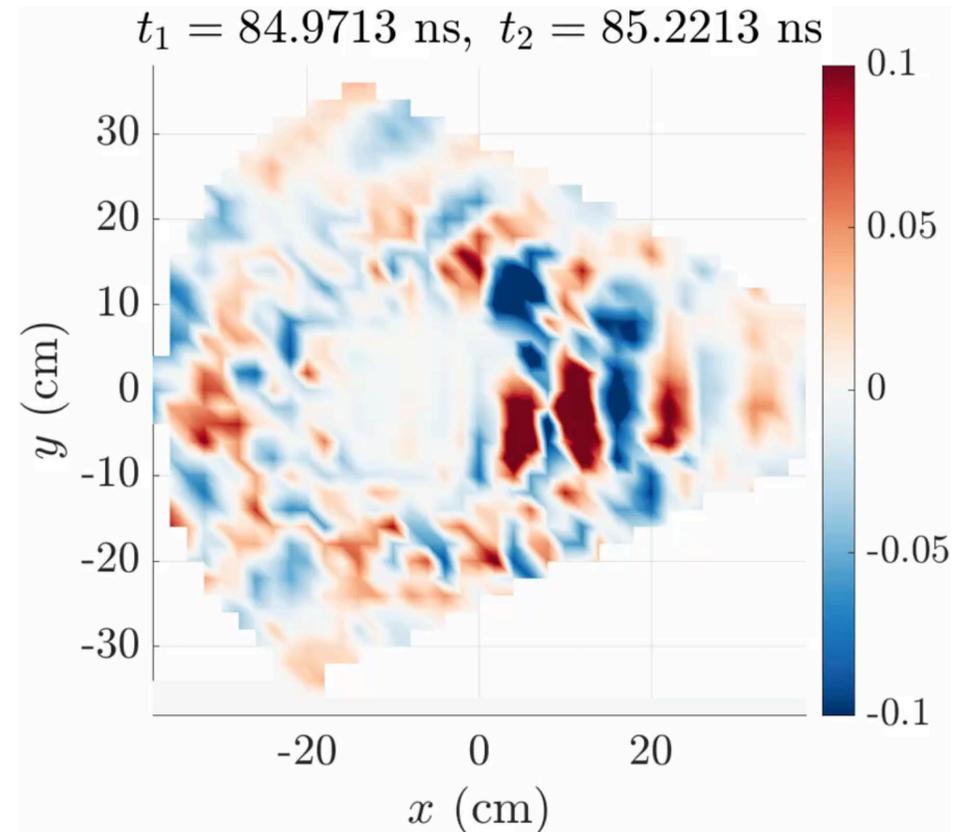
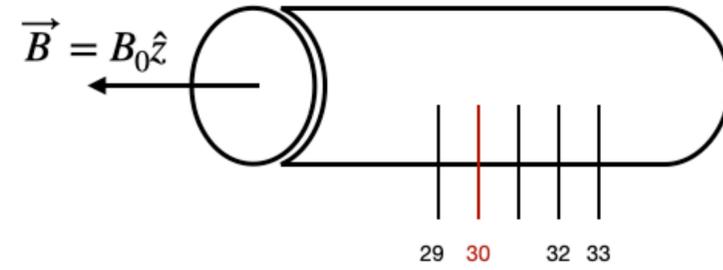
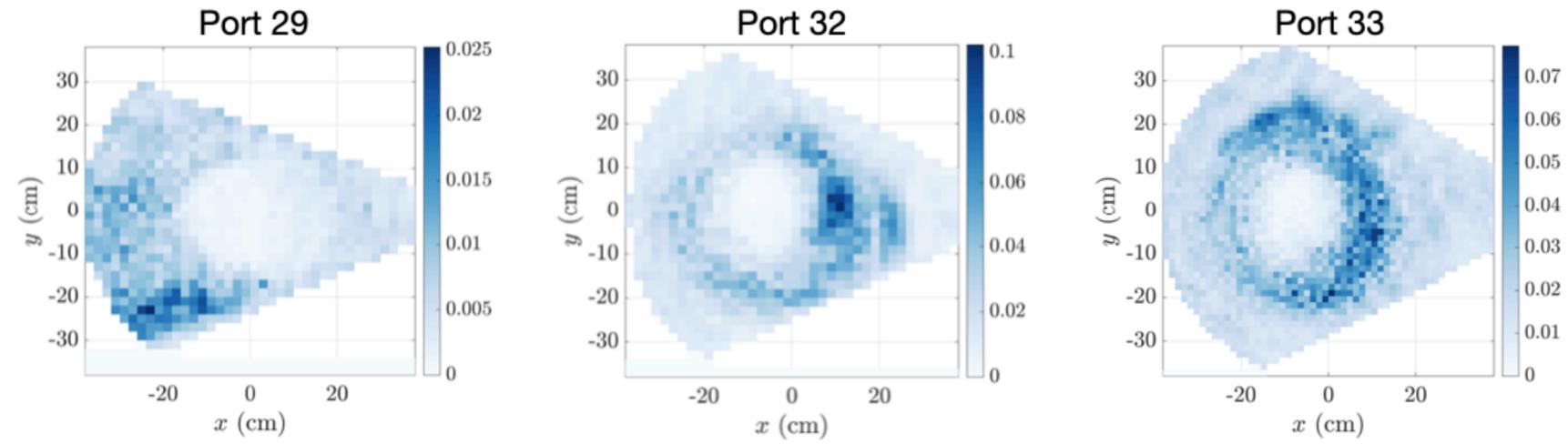
HELICON ANTENNA  
ACROSS FROM THIS  
PORT

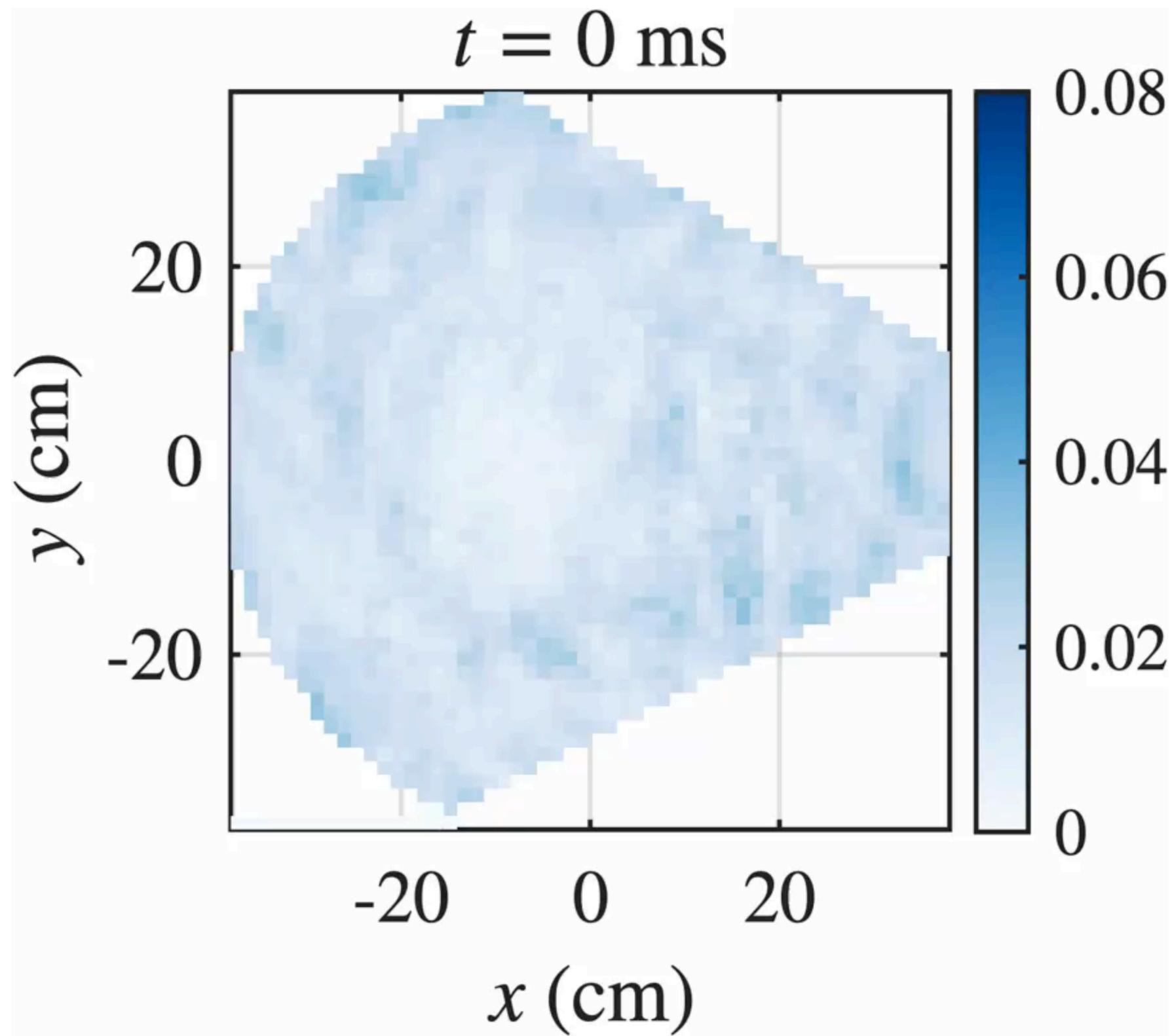
HELICON ANTENNA  
ACROSS FROM THIS  
PORT -35CM

TRI ALPHA ANTENNA  
ACROSS FROM THIS  
PORT

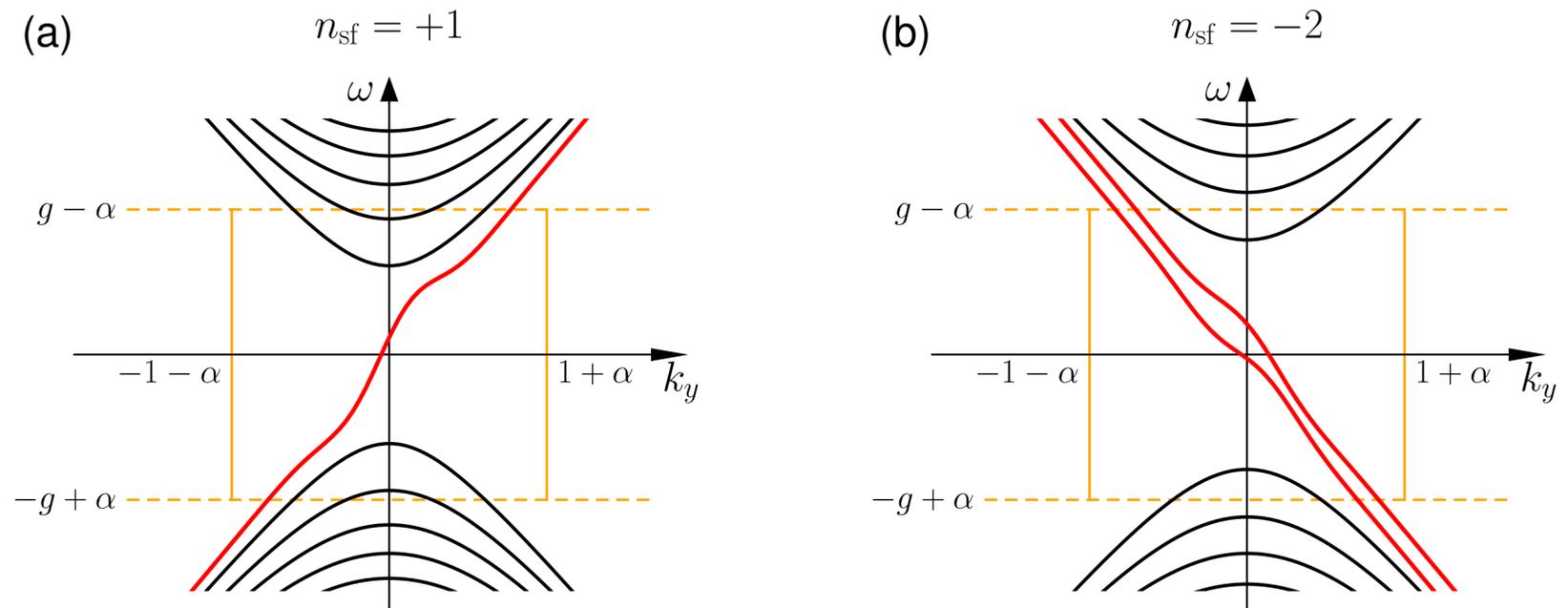
46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63

Comparison between three port positions (launcher at port 30)

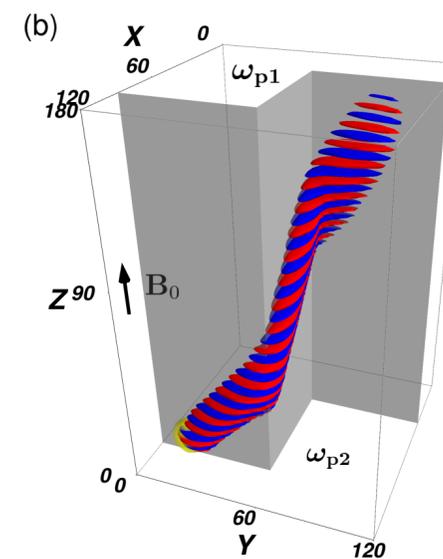
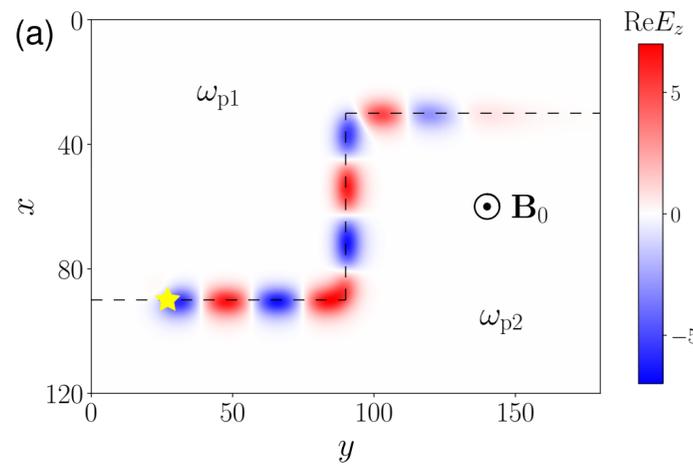
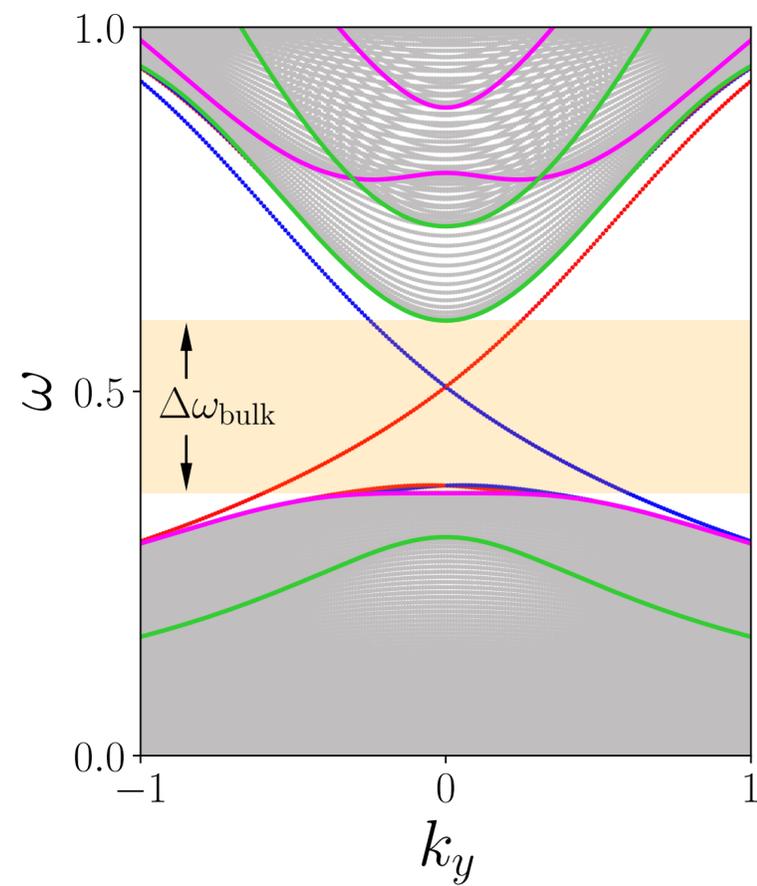




# Yichen Fu & Hong Qin



## Topological Langmuir–Cyclotron Wave





## Topological Modes in Stellar Oscillations

Armand Leclerc<sup>1</sup> , Guillaume Laibe<sup>1,2</sup> , Pierre Delplace<sup>3</sup>, Antoine Venaille<sup>3</sup>, and Nicolas Perez<sup>3</sup>

<sup>1</sup> Univ Lyon, Univ Lyon1, ENS de Lyon, CNRS, Centre de Recherche Astrophysique de Lyon, UMR5574, F-69230, Saint-Genis-Laval, France

[guillaume.laibe@ens-lyon.fr](mailto:guillaume.laibe@ens-lyon.fr)

<sup>2</sup> Institut Universitaire de France, France

<sup>3</sup> ENS de Lyon, CNRS, Laboratoire de Physique (UMR CNRS 5672), F-69342 Lyon, France

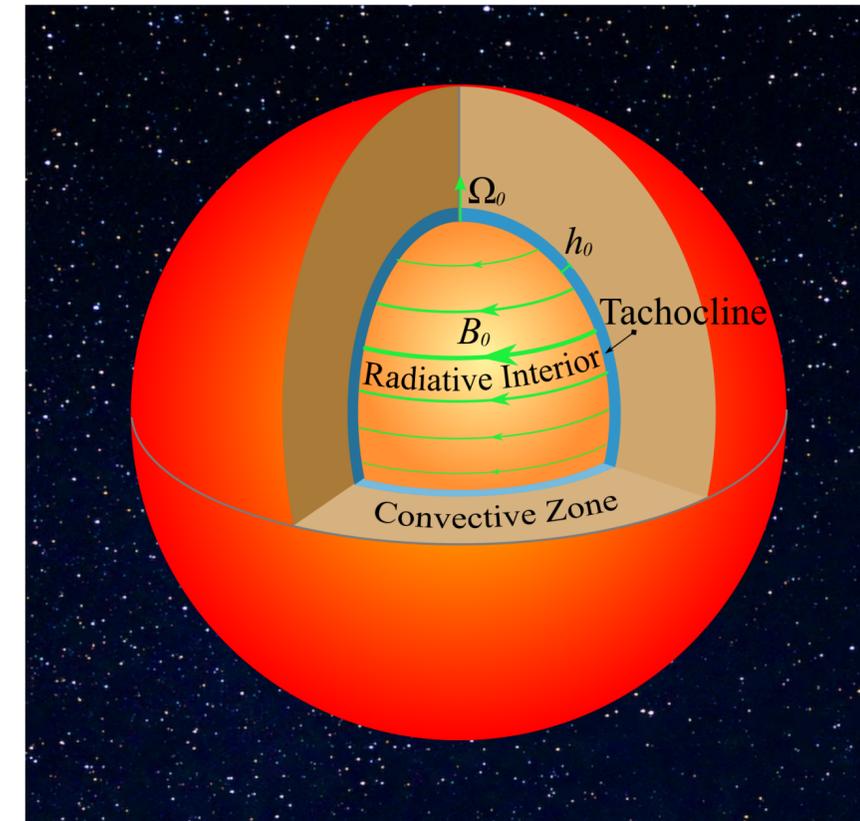
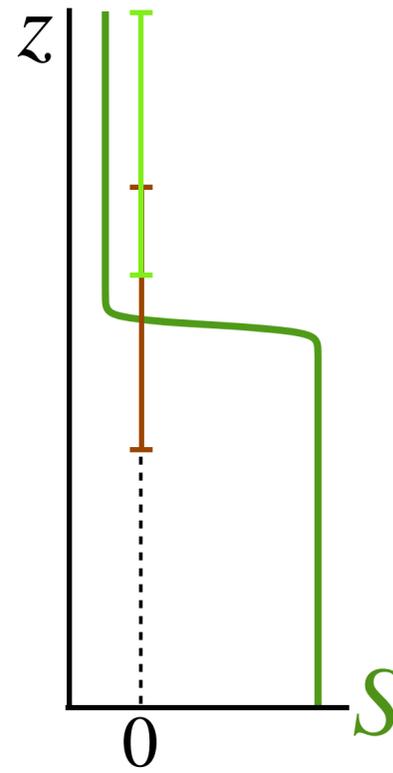
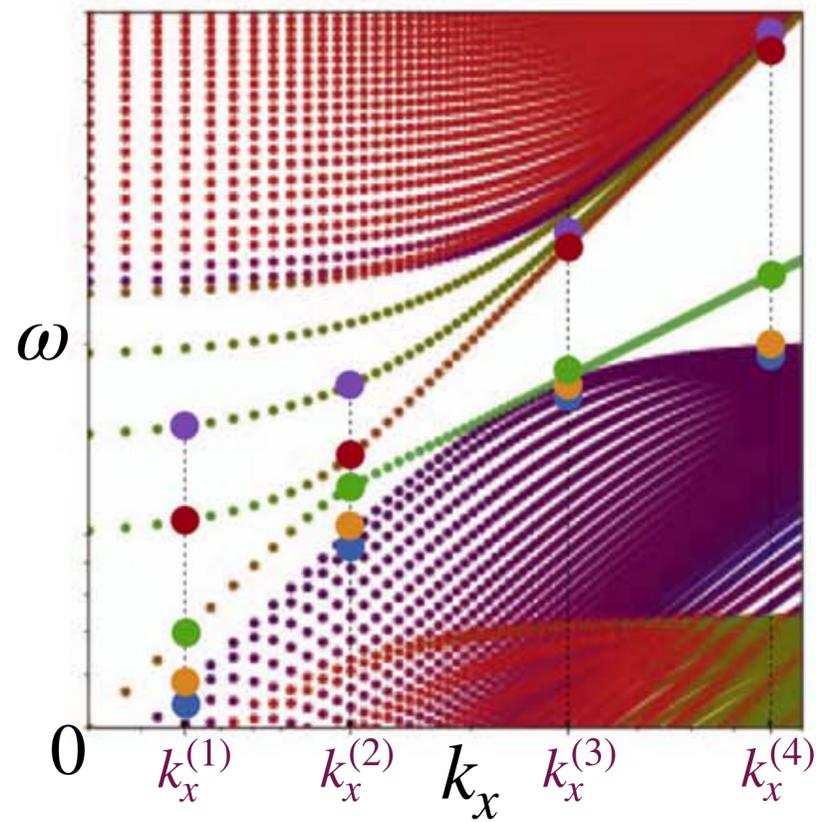
Received 2022 July 20; revised 2022 September 29; accepted 2022 October 3; published 2022 November 23

## Topological plasma oscillations in the solar tachocline

Ruben Lier,<sup>1,2,\*</sup> Richard Green,<sup>1,†</sup> Jan de Boer,<sup>1,2,‡</sup> and Jay Armas<sup>1,2,§</sup>

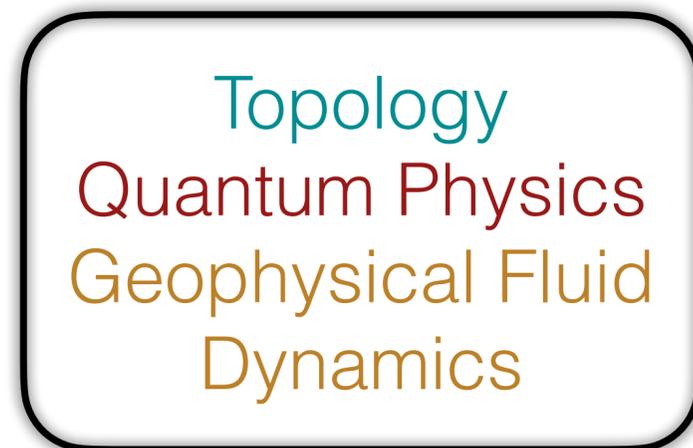
<sup>1</sup> Institute for Theoretical Physics, University of Amsterdam, 1090 GL Amsterdam, The Netherlands

<sup>2</sup> Dutch Institute for Emergent Phenomena (DIEP),  
University of Amsterdam, 1090 GL Amsterdam, The Netherlands

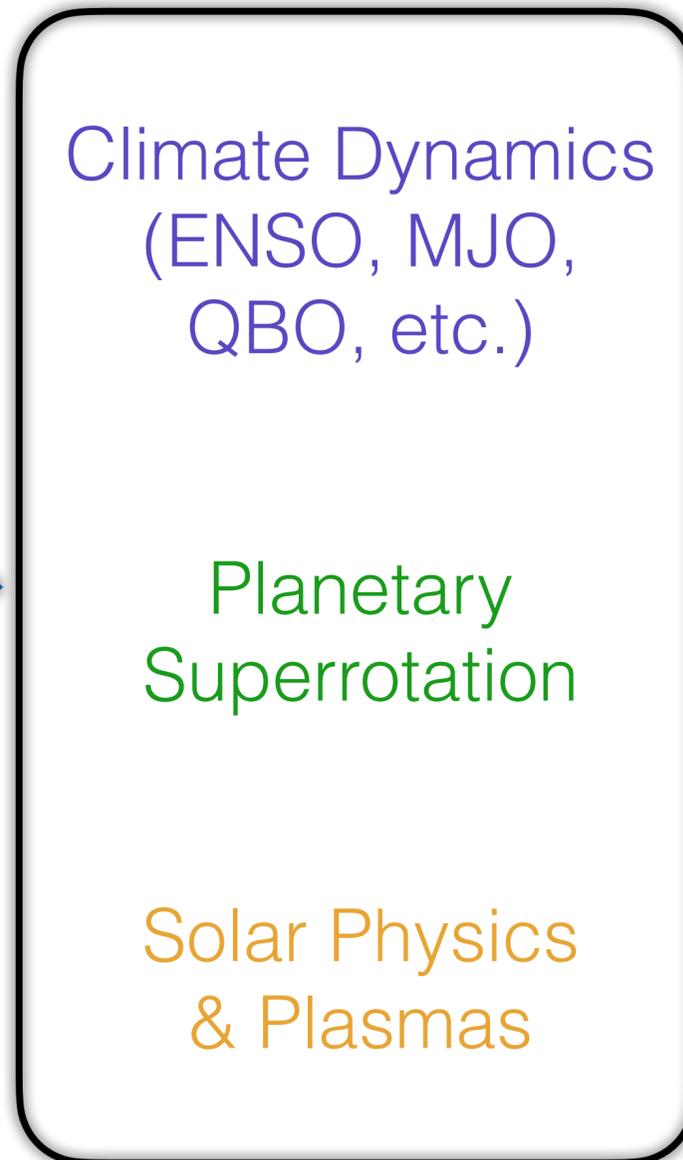


# Topological Geophysical / Astrophysical Fluid Dynamics

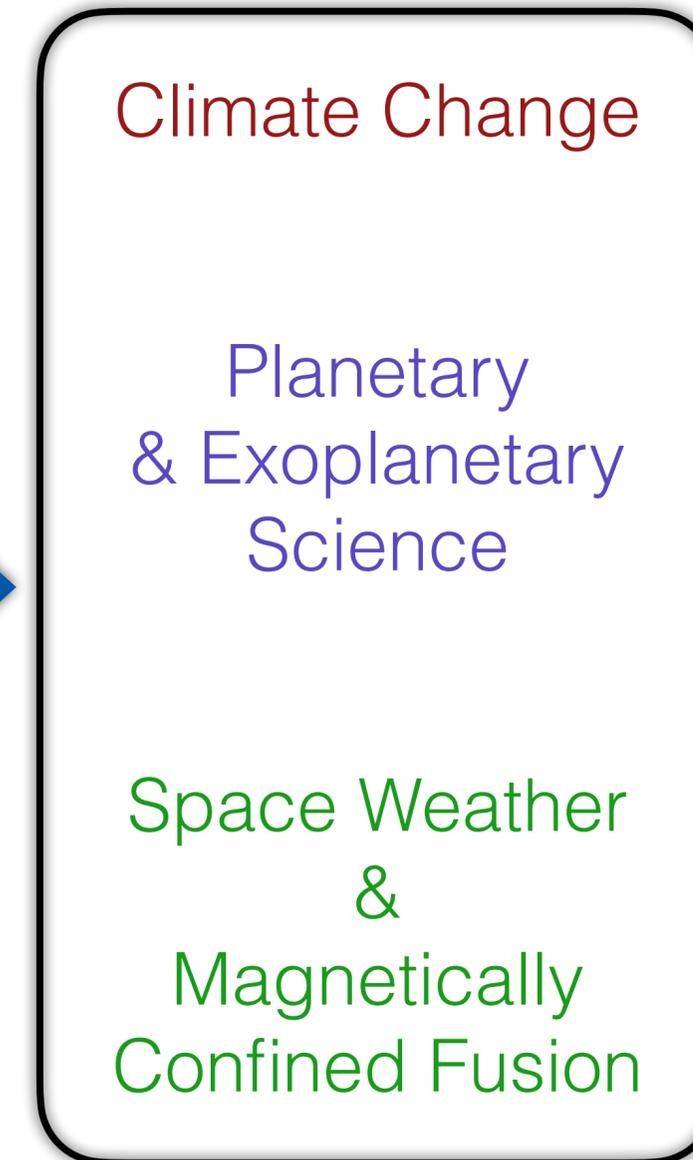
Foundations



Fundamental  
Science



Broad  
Impacts



What new waves of topological  
origin are waiting to be discovered?