Scattering Strings on flat and curved space-times

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What will this talk be about?

We will describe a set of tools to compute String Theory scattering amplitudes on flat and curved space-times.

Why scattering amplitudes?

- Scattering amplitudes allow to test the predictions of our theory.
- They can teach us much about its structures/symmetries.

Why string theory?

- The most promising theory of quantum gravity but still poorly understood.
  - Computations are super hard :(
  - Essentially no progress on curved-backgrounds :(

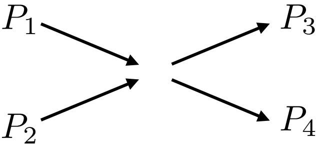
A fundamental question, essential for applications of string theory to quantum gravity, cosmology, black holes, ...

Surprising connections to number theory make progress possible :)

Scattering amplitudes

Scattering Amplitudes

Probability that two particles colliding (with momenta p_1, p_2) result into two other particles (with momenta p_3, p_4).



$$= \quad A(g, s, t, u)$$

- $A(g, s, t, u)$ depends on many things:
  - Which particles you are scattering (their masses, charges, etc)
  - The parameters/coupling constants of your model g.
  - The momenta of the particles being scattered:

$$s = -(p_1 + p_2)^2, \quad t = -(p_1 - p_3)^2, \quad u = -(p_1 - p_4)^2$$

$$s + t + u = 0 \quad \text{for massless scattering}$$

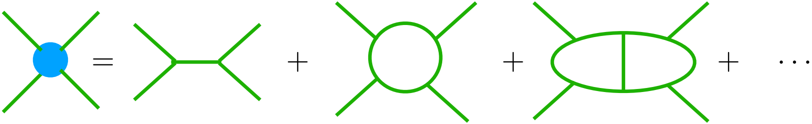
Scattering amplitudes via Feynman diagrams

No general methods to compute amplitudes for finite coupling g

Feynman recipe

- Amplitudes around $g = 0 \rightarrow$ Feynman diagrams

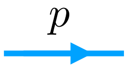
$$A(g) \;=\; g^2 A_0 \;+\; g^4 A_1 \;+\; g^6 A_2 \;+\; \cdots$$



- Lagrangian: fundamental particles + interactions among them $\rightarrow$ which diagrams we should draw.
- To each diagram corresponds an integral (Feynman integral) which we should do.

General Relativity

- Fundamental particles: Gravitons

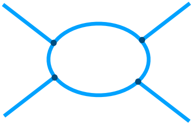


Massless particles, moving at the speed of light.

- Interactions among them: Einstein Hilbert Lagrangian

$$\mathcal{L}_{EH} = (\partial h)^2 + \sqrt{G_N} h(\partial h)^2 + \ldots$$

- The standard recipe leads to divergent integrals!



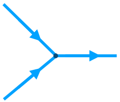
$\sim \quad \int_0^\infty d^4p \frac{p^6}{(p^2)^4} \rightarrow$ divergent! (for large p/UV)

Reason: The gravitons get too close to each other!

String theory and UV divergences

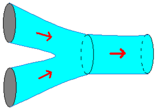
- Replace point gravitons by closed strings of finite length $\sqrt{\alpha'}$

Point particles



VS

Strings



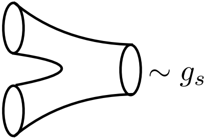
- At low energies $p^2 \ll 1/\alpha'$ and we recover GR

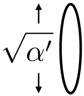
GR as an effective field theory

$$\mathcal{L} = (\partial h)^2 + \sqrt{G_N}\left(h(\partial h)^2 + \underbrace{\alpha'^{1/2} h^2 (\partial h)^2 + \alpha' h (\partial h)^3 + \cdots}_{\text{stringy corrections}} \right)$$

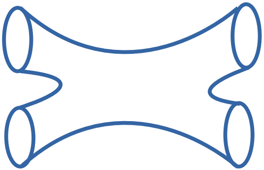
- At high energies stringy corrections start kicking-in, and make the amplitudes finite.
- We say string theory gives a UV completion of GR.

- In string theory we have two parameters



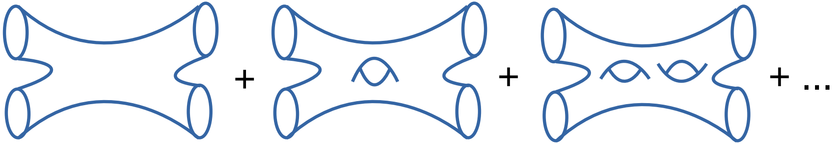


- We would like to compute scattering amplitudes



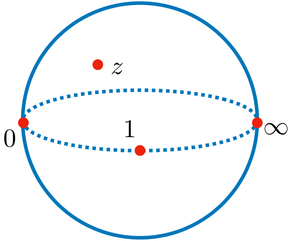
$$= \quad A(g_s, \alpha', s, t)$$

- The computation organises in a genus expansion



$$\underbrace{A^{(genus\ 0)}(\alpha', s, t)}_{\text{We focus on this}} + g_s^2 A^{(genus\ 1)}(\alpha', s, t) + g_s^4 A^{(genus\ 2)}(\alpha', s, t) + \cdots$$

- In flat space $A^{(genus\ 0)}(\alpha', s, t)$ follows from World-sheet theory



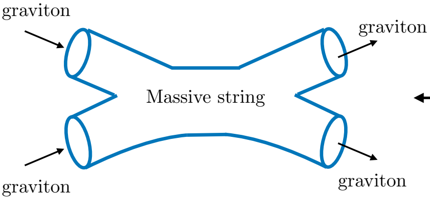
$$= \int_{\mathbb{C}} \frac{|z|^{-2\alpha' s}}{|z|^2} \frac{|1-z|^{-2\alpha' t}}{|1-z|^2} d^2 z$$

Four-graviton amplitude - Flat space

Genus 0

$$A_{flat}(\alpha', s, t) = \alpha'^3 \frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)\Gamma(-\alpha' u)}{\Gamma(1+\alpha' s)\Gamma(1+\alpha' t)\Gamma(1+\alpha' u)}$$

- Crossing symmetric ($s+t+u=0$)
- Poles due to the exchange of particles of mass $m=\sqrt{n/\alpha'}$



$$\longleftrightarrow \quad A_{flat}(\alpha', s, t) \sim \frac{P_\ell(t)}{s-m^2}$$

- α' expansion (low energy expansion)

$$A_{flat}(\alpha', s, t) \sim \underbrace{\frac{1}{s\,t\,u}}_{\text{gravity}} + \underbrace{2\zeta(3)\alpha'^3 + 2\zeta(5)\alpha'^5(s^2+t^2+u^2) + \cdots}_{\text{stringy corrections}}$$

String amplitudes and number theory

Beautiful connection to number theory

- We see the appearance of zeta values $\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$.
- Even ζ-values can be written in terms of π:

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}, \quad \zeta(4) = \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}, \quad \cdots$$

- Odd ζ-values are independent

$$\zeta(3) = \sum_{k=1}^{\infty} \frac{1}{k^3}, \quad \zeta(5) = \sum_{k=1}^{\infty} \frac{1}{k^5}, \quad \cdots$$

The α' expansion of the string amplitude contains only odd zeta values.

Quite deep from a mathematical perspective!

Single-valued zetas

- Zeta values can be defined in terms of sums

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$$

- Or in terms of polylogarithms evaluated at $z = 1$

$$Li_n(z) = \sum_{k=1} \frac{z^k}{k^n} \to Li_n(1) = \zeta(n)$$

- While these series converge for $|z| < 1$, polylogarithms can be analytically continued to the whole complex plane:

$$Li_1(z) = -\log(1-z), \quad Li_n(z) = \int_0^z Li_{n-1}(t) \frac{dt}{t}$$

- However these functions are not single-valued!

Single-valued polylogarithms

- Single-valued version of polylogarithms

$$Li_n(z) \to \mathcal{L}_n(z,\bar{z})$$

When you evaluate polylogarithms at $z=1$ you get zeta values. What if you evaluate their single-valued version at $z=\bar{z}=1$?

$$Li_n(1) \to \zeta(2), \zeta(3), \zeta(4), \zeta(5), \cdots$$

$$\mathcal{L}_n(1) \to \cancel{\zeta(2)}, \zeta(3), \cancel{\zeta(4)}, \zeta(5), \cdots$$

Punchline

- $\zeta(3), \zeta(5), \cdots$ are special!
- The α' expansion of the graviton string amplitude in flat space contains only these special zeta values!

String amplitudes on AdS

Can we compute string amplitudes on curved space-time?

- One of the biggest challenges of string theory.
- No world-sheet theory to help us.

Idea

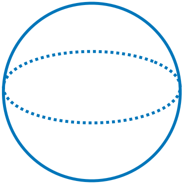
- Amplitudes in flat space were originally found from their properties (pre world-sheet theory) in what it was the birth of string theory!
- Can we do the same for curved backgrounds?

We will consider string theory on $AdS_5 \times S^5$ (sorry! we need $D = 10$)

Compute the (four-point) graviton amplitude for strings on $AdS_5 \times S^5$.

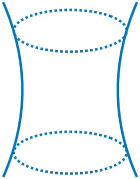
$AdS_5 \times S^5$

S^5



$$Y_0^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 + Y_5^2 = R^2$$

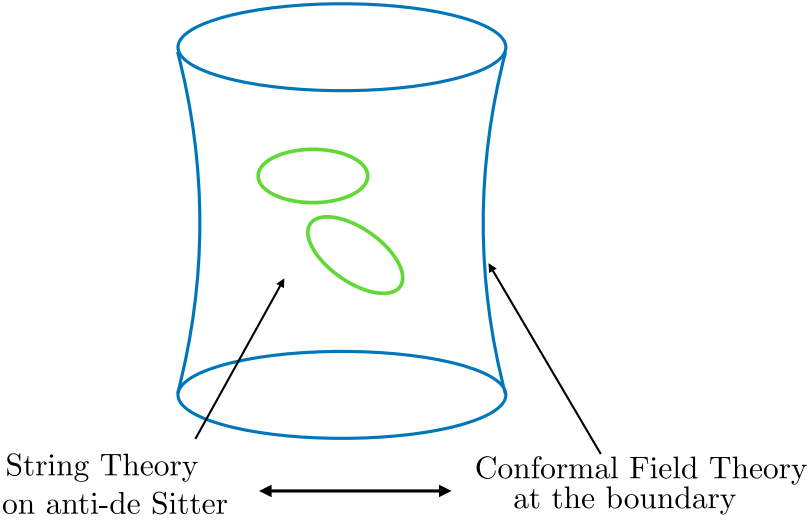
AdS_5



$$-X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 - X_5^2 = R^2$$

$$\text{As we take } R \to \infty \begin{cases} S^5 \to \text{Flat 5d space} \\ AdS_5 \to \text{Flat 5d space-time} \end{cases}$$

AdS/CFT



CFTs: very special theories with scale invariance.

Conformal Field Theories

- Special set of local operators

$$\mathcal{O}(\lambda x) = \lambda^{-\overset{\text{dimension}}{\Delta}}\mathcal{O}(x)$$

- Infinite such operators $\mathcal{O}_\Delta(x)$, each with their own dimension.

$$\Delta \leftrightarrow \text{energy levels in QM}$$

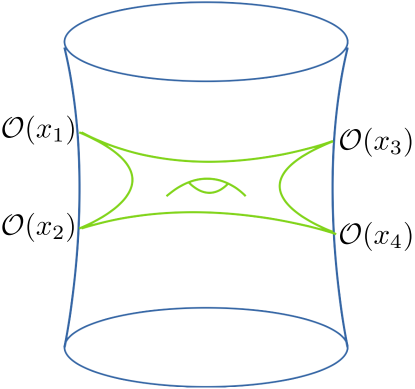
- Hard to compute and functions of the coupling of the theory $\Delta_i(g)$.
- Conformal symmetry severely restricts correlation functions

  - $\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \dfrac{1}{|x_1 - x_2|^{2\Delta}}$

  - $\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\rangle = \text{also fixed}$

  - $\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = G(U,V) \qquad U = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad V = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$

String amplitudes on $AdS_5 \times S^5$, radius R $\leftrightarrow$ Correlators in planar MSYM w/coupling g (large N QCD - like)

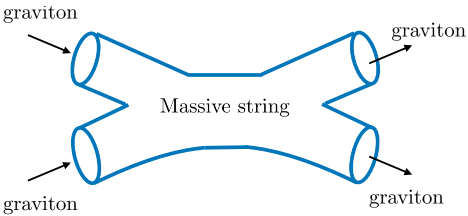


$$A_{AdS}(s,t) \xleftrightarrow{\text{integral transform}} \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle$$

$$\frac{R^2}{\alpha'} = g$$

$A_{AdS}(s,t)$ - properties

1. Crossing symmetric in s, t, u.
2. Poles corresponding to the exchange of intermediate strings



$$\longleftrightarrow \quad A_{AdS}(s,t) \sim \frac{Q_\ell(t)}{s-m^2}$$

3. Admits an expansion around flat space

$$A_{AdS}(s,t) = \underbrace{A_{flat}(s,t)}_{\text{Amplitude in flat space}} \underbrace{+ \frac{\alpha'}{R^2}A^{(1)}(s,t) + \frac{\alpha'^2}{R^4}A^{(2)}(s,t) + \cdots}_{\text{curvature corrections}}$$

Spectrum of strings in AdS?

$$A_{flat}(s,t) \sim \frac{P_\ell(t)}{s-m_{flat}^2} \quad \text{vs} \quad A_{AdS}(s,t) \sim \frac{Q_\ell(t)}{s-m_{AdS}^2}$$

- While the massive spectrum is very simple in flat space

$$m_{flat} = \sqrt{\frac{n}{\alpha'}}, \quad n = 1,2,3,\cdots$$

- In AdS it is currently unknown. However, thanks to AdS/CFT:

$$m_{AdS}(R) \quad \leftrightarrow \quad \Delta_i(g)$$

Spectrum of masses in AdS $\leftrightarrow$ Dimensions of ops in planar MSYM

Good news: much is known about $\Delta_i(g)$ from integrability!

From integrability results we can fix the location of the first few poles of $A_{AdS}(s,t)$. Very useful! but still not nearly enough...we need more...

String amplitude on AdS

$A_{AdS}(s,t)$

1. Crossing symmetry in s, t, u.
2. Exchanged strings lead to simple poles:

$$A_{AdS}(s,t) \sim \frac{Q_\ell(t)}{s-m^2}$$

3. Admits an expansion around flat space.

$$A_{AdS}(s,t) = A_{flat}(s,t) + \frac{\alpha'}{R^2}A^{(1)}(s,t) + \frac{\alpha'^2}{R^4}A^{(2)}(s,t) + \cdots$$

4. Extra condition: the low energy expansion of $A_{AdS}(s,t)$ contains only the special zeta values!

The last condition becomes incredibly powerful when combined with the other three! $\rightarrow$ It allows to compute $A_{AdS}(s,t)$ around flat space!

String amplitudes on AdS

World-sheet integral for amplitudes on AdS

- At present we don't have a World-sheet theory for strings on AdS...
- But! a World-sheet picture emerges from our results :)

Proposal order by order

$$A_{flat}(s,t) = \int d^2z |z|^{-2\alpha' s-2} |1-z|^{-2\alpha' t-2}$$

$$A^{(1)}(s,t) = \int d^2z |z|^{-2\alpha' s-2} |1-z|^{-2\alpha' t-2} \underbrace{\mathcal{L}_3(z,\bar{z})}_{\text{SV polylogs of weight 3}}$$

$$A^{(2)}(s,t) = \int d^2z |z|^{-2\alpha' s-2} |1-z|^{-2\alpha' t-2} \underbrace{\mathcal{L}_6(z,\bar{z})}_{\text{SV polylogs of weight 6}}$$

$$\vdots$$

Conclusions

- A beautiful world-sheet picture emerges for string amplitudes on curved backgrounds.
- Surprising connections to number theory.

Can we understand this from first principles?

This will pave the way to a deeper understanding of string theory on curved backgrounds, essential for connecting string theory to "real-world" physics, specially in the fields of quantum gravity, cosmology, black holes, ...