Solutions to problems from 29th of May

1 Problem

Starting from the 5-D action

$$S = \int d^5X \left(\sum_{i=1}^{2} \bar{\psi}_i(X) \left(i\Gamma^M D_M - M \right) \psi_i(X) \right) + \tilde{M}(\bar{\psi}_1\psi_2 + \bar{\psi}_2\psi_1) \tag{1}$$

where

$$X = (x^\mu, z) \tag{2}$$

$$\Gamma^M = (\gamma^\mu, -i\gamma^5) \tag{3}$$

and the fifth dimension (z) is finite of size L, derive the expression for the four-dimensional mass parameter of this theory.

2 Solution

To simplify the solution, I will ignore the mixing term between the fields ψ_1 and ψ_2, which will result in two separate equations of motion. The $\tilde{M}$ term will be put back later.

The equations of motion can be derived using the variational principle. Considering only one flavour for brevity:

$$\delta S = \int d^5X \delta\bar{\psi}(X) \left(i\Gamma^M D_M - M \right) \psi(X) \tag{4}$$

$$+ \bar{\psi}(X) \left(i\Gamma^M \overleftarrow{D_M} - M \right) \delta\psi(X) \tag{5}$$

$$+ \bar{\psi}\gamma^5\delta\psi\Big|_{z=0}^{L} + \delta\bar{\psi}\gamma^5\psi\Big|_{z=0}^{L} \tag{6}$$

The boundary terms mix the left and right-handed fields and so can be made to vanish by requiring at the boundary either:

$$\psi_L(x,0) = \psi_L(x,L) = 0 \tag{7}$$

or

$$\psi_R(x,0) = \psi_R(x,L) = 0 \tag{8}$$

The remaining equations of motion can be written in terms of Weyl spinors as

$$(i\not{D} - M \pm \partial_z)\psi_\pm(X) = 0 \tag{9}$$

where I relabelled $\psi_+ \equiv \psi_L$ and $\psi_- \equiv \psi_R$.

This partial differential equation can be solved by separation of variables

$$\psi_\pm(X) = \tilde{\psi}_\pm(x) f_\pm(z) \tag{10}$$

as it can be rewritten as

$$\frac{i\not{D}\tilde{\psi}_\pm(x)}{\tilde{\psi}_\pm(x)} = \frac{\mp\frac{d}{dz}f_\pm(z)}{f_\pm(z)} + M \tag{11}$$

Since the LHS is a function of x only, while the RHS is a function of z only, they must be both equal to a constant m. In the following we choose $m = 0$. The RHS is then the first order ordinary differential equation

$$\frac{d}{dz}f_\pm(z) = \pm M f_\pm(z) \tag{12}$$

with solution:

$$f_\pm(z) = c_\pm e^{\pm Mz}. \tag{13}$$

The integration constants $c_\pm$ can be obtained by requiring that the 4-D kinetic term is canonically normalised:

$$\left(\int d^4x\bar{\tilde{\psi}}_\pm(x)i\not{D}\tilde{\psi}_\pm(x)\right)\underbrace{\int_0^L dz f_\pm(z)^2}_{=1} \tag{14}$$

which can be solved to give

$$c_\pm = \sqrt{\frac{\pm 2M}{e^{\pm 2ML}-1}} \tag{15}$$

When ML is large this simplifies further to give

$$c_+ \approx \sqrt{2M}e^{-ML} \tag{16}$$

$$c_- \approx \sqrt{2M} \tag{17}$$

We now put back the mass term $\tilde{M}\bar{\psi}_1\psi_2$. Choosing the boundary conditions such that ψ_1 is left-handed and ψ_2 is right-handed gives the 4-D mass term:

$$2M\tilde{M}Le^{-ML}\int d^4x\bar{\psi}_1(x)\psi_2(x) + h.c. \tag{18}$$

and we can see that the mass term is exponentially suppressed.