Solutions to problems from 28th of May

1 Problem 1

1.1 Problem

Show that in the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\alpha\partial_\mu A_\nu \partial^\mu A^\nu - \frac{1}{2}\beta\partial_\mu A_\nu \partial^\nu A^\mu + \frac{m^2}{2}A_\mu A^\mu + J_\mu A^\mu \tag{1}$$

the field A_μ describes a particle of spin 1 only, provided that $\alpha = -\beta$.

1.2 Solution

This follows Weinberg ch. 7.5 (page 320). (NB Weinberg uses “mostly positive” convention for the metric, so we get an extra minus for each pair of contracted indices).

The equation of motion can be obtained by applying the Euler-Lagrange equation

$$\partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\nu A_\mu} = \frac{\partial \mathcal{L}}{\partial A_\mu} \tag{2}$$

gives

$$\alpha\partial_\nu\partial^\nu A^\mu + \beta\partial^\mu\partial^\nu A_\nu + m^2 A^\mu + J^\mu = 0 \tag{3}$$

Taking a derivative ∂_μ gives:

$$(\alpha+\beta)\left(\left(\partial_\nu\partial^\nu + \frac{m^2}{\alpha+\beta}\right)(\partial_\mu A^\mu) + \frac{1}{\alpha+\beta}\partial_\mu J^\mu\right) = 0 \tag{4}$$

This is an equation for a scalar field $\partial_\mu A^\mu$ in the presence of external current – compare with the scalar Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{M}{2}\phi^2 - J\phi \tag{5}$$

This means that the original Lagrangian (1) contains a spin-0 degree of freedom unless $\alpha + \beta = 0$. In this case, eq. 4 becomes

$$\partial_\mu A^\mu = -\frac{1}{m^2}\partial_\mu J^\mu \tag{6}$$

which means that $\partial_\mu J^\mu$ no longer corresponds to a degree of freedom, because it can be solved as a function of the external current.

2 Problem 2

2.1 Problem

Starting from the propagator of a massive spin-1 particle, show that in $m \to 0$ limit the corresponding field A_μ must couple to the conserved current. Show that gauge invariance is recovered in this limit.

2.2 Solution

The propagator of the massive spin-1 particle is given by:

$$\frac{\eta^{\mu\nu} - \frac{p^\mu p^\nu}{m^2}}{p^2 - m^2 + i\epsilon} \tag{7}$$

we can see that the second term in the numerator is divergent in the $m^2 \to 0$ limit for any $p^\mu \neq 0$. This is an unphysical singularity and it must be removed. Since the field couples to the current (J_μ in the Lagrangian), which gives the Feynman rules for the vertices which the photon couples to, the simplest way of removing this term is by requiring that, in momentum space

$$p^\mu J_\mu(p) = 0 \tag{8}$$

or equivalently

$$\partial_\mu J^\mu(x) = 0 \tag{9}$$

The remaining Lagrangian is:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu \tag{10}$$

and it can be seen by inspection that it is invariant under $A^\mu \to A^\mu + \delta^\mu \alpha$ - $F^{\mu\nu}$ is invariant by itself, while the extra term coming from the current $\partial_\mu \alpha J^\mu$ can be integrated by parts and vanishes by current conservation.

BONUS:

One may wonder what happens to the longitudinal degree of freedom of the massive particle - after all massive spin-1 particle has 3 possible spin states while the massless one has only two. The current conservation condition also ensures that it decouples. To see this, separate the longitudinal degree of freedom from the transverse ones:

$$A^\mu = A^\mu_T + \partial^\mu \pi \tag{11}$$

where A^μ_T is the transverse component and π is the longitudinal one. We can immediately see that gauge invariance ensures that the longitudinal degree of freedom will disappear from the Lagrangian.

3 Problem 3

3.1 Problem

Choosing the units such that $c = 1$ but $\hbar \neq 1$, work out the dimensionalities of fields and couplings in the Lagrangian.

3.2 Solution

The action has the dimensions of $\hbar$, so the Lagrangian, which satisfies $S = \int d^4x\mathcal{L}$ has dimensions $\hbar/L^4$.

The dimension of the derivative is $[D] = 1/L$, by definition.

The dimensions of the fields can be determined from their respective kinetic terms. For example, the scalar field $\delta_\mu\phi\delta^\mu\phi$ must have the the same dimensions as the Lagrangian, so the dimensionality of the field must satisfy

$$[\phi]^2 * (1/L)^2 = \hbar/L^4 \tag{12}$$

$$[\phi] = \sqrt{\hbar}/L \tag{13}$$

The derivation for spin-1/2 and spin-1 fields proceeds along the same lines.

The dimensions of masses and couplings can then be determined using known field dimensionalities, e.g. $m^2\phi^2$:

$$[m^2] * [\phi]^2 = \hbar/L^4 \tag{14}$$

$$[m^2] = 1/L^2 \tag{15}$$