Solutions to problems from 31st of May

1 Problem 1

1.1 Problem

Consider an SU(2) gauge theory with 2 Dirac flavours of fermions in the fundamental representation of the gauge group. Show that strong interaction will lead to the flavour symmetry breaking pattern SU(4) $\to$ Sp(4), resulting in 5 Goldstone bosons.

Recall that the defining properties of Sp(2N) are

$$M^\dagger M = 1 \tag{1}$$

$$M\Omega M^T = \Omega \tag{2}$$

where Ω is a $2N \times 2N$ symplectic matrix, typically chosen to be:

$$\Omega = \begin{pmatrix} 0_{N\times N} & 1_{N\times N} \\ -1_{N\times N} & 0_{N\times N} \end{pmatrix} \tag{3}$$

1.2 Solution

We have 4 independent fermion fields: u_L, d_L, u_R and d_R. As explained in the appendix A below, we can massage the right-handed fields in such a way that they transform like left handed fields. The corresponding left-handed fields are $(i\sigma_s^2)_{\alpha\beta}u_R^{*\beta}$ and similarly for d, where the s subscript on the Pauli matrix indicates that it acts on the spinor space.

Now that we have four fields all transforming as left-handed spinors under the Lorentz group, we consider their gauge transormation properties. u_L and d_L transform under the fundamental respresentation, while the other two, due to complex conjugation, under antifundamental representation. In the case of QCD this is as far as we can go, because there is no way of relating 3 to $\bar{3}$ of SU(3). In the case of SU(2) we can do better, because as explained in the appendix B, we can relate $\bar{2}$ to 2 by acting with the $(i\sigma^2)$ matrix, this time in the colour space.

The resulting fields have the same transformation properties and can be combined into a single multiplet:

$$Q_\alpha^{ai} = \begin{pmatrix} (u_L)_\alpha^i \\ (d_L)_\alpha^i \\ (i\sigma_c^2)^{ij}(i\sigma_s^2)_{\alpha\beta}(u_R^*)_i^\beta \\ (i\sigma_c^2)^{ij}(i\sigma_s^2)_{\alpha\beta}(d_R^*)_i^\beta \end{pmatrix} \tag{4}$$

The symmetry will be broken by the fields Q forming a condensate. This condensate must be a Lorentz scalar and a singlet under the gauge group. It can then be written as:

$$\Sigma^{ab} = \langle Q_\alpha^{ai}(i\sigma_c^2)^{ij}(i\sigma_s^2)^{\alpha\beta}Q_\beta^{bj}\rangle \tag{5}$$

Because both $(i\sigma^2)$ matrices are antisymmetric and the fields Q are fermions satisfying the anticommutation relation:

$$Q^{ai}_\alpha Q^{bj}_\beta = -Q^{bj}_\beta Q^{ai}_\alpha \tag{6}$$

we conclude that the condensate must be antisymmetric in the flavour indices

$$\Sigma^{ab} = -\Sigma^{ba} \tag{7}$$

We also note that under flavour rotations, Σ transforms as

$$\Sigma \to U\Sigma U^T \tag{8}$$

From this we can see that the SU(4) flavour group will be broken to a subgroup, which leaves Σ invariant, which is the Sp(4) group. This can be seen most easily by choosing $\Sigma = \Omega$.

For more details and applications see e.g. arxiv/1402.0233

A Revision - Lorentz transformation of spinor fields

Recall that the generators of the Lorentz group are the 3 rotation generators J_i and 3 boost generators K_i. Any wavefunction has to transform under the unitary representation of the Lorentz group, which implies that the generators are Hermitian $J_i = J_i^\dagger$, $K_i = K_i^\dagger$.

On the other hand, it is convenient to define the following combinations of the operators J and K:

$$A_i = J_i + iK_i \tag{9}$$

$$B_i = J_i - iK_i \tag{10}$$

The operators A and B satisfy the commutation relations

$$[A_i, A_j] = i\epsilon_{ijk}A_k \tag{11}$$

$$[B_i, B_j] = i\epsilon_{ijk}B_k \tag{12}$$

$$[A_i, B_j] = 0 \tag{13}$$

which implies that A and B form 2 independent representations of SU(2). On the other hand, A and B are not hermitian, because from the definition:

$$A^\dagger = B \tag{14}$$

$$B^\dagger = A \tag{15}$$

so the resentations of (A,B) will not be unitary.

We define the left-handed spinor as transforming under (2,0) representation of (A,B). The hermitian conjugate of a left-handed field then transforms as

$$(L)^\dagger = \left(e^{iA\cdot\theta}L\right)^\dagger = L^\dagger e^{-iB\cdot\theta} \tag{16}$$

i.e. as $(0, \bar{2})$. Similarly, the conjugate of the right-handed field (0,2) transforms as $(\bar{2}$,0).

A convenient way of keeping track of the transformation properties of Weyl spinors is the dotted/undotted notation, by assigning lower/upper dotted/undotted indices to the spinors as follows:

(2,0)	L_α
(0,2)	$R^{\dot{\alpha}}$
$(0,\bar{2})$	$(L^*)_{\dot{\alpha}}$
$(\bar{2}, 0)$	$(R^*)^{\alpha}$

so that:

- Complex conjugation adds/removes a dot without changing the position of the index
- To construct Lorentz scalars we contract upper and lower indices of the same type

Additionally, as discussed below, we can also contract e.g. two left-handed spinors using the $(i\sigma^2)^{\alpha\beta}$

B Fundamental and antifundamental representations of SU(2)

The defining properties of SU(2) matrices are:

$$U^\dagger U = 1 \tag{17}$$

$$\text{det} U = 1 \tag{18}$$

Reacalling that the determinant of a 2x2 matrix can be calculated as

$$\epsilon_{ij} U_{ia} U_{jb} = (\text{det} U)\epsilon_{ab} \tag{19}$$

where ϵ is the 2D Levi-Civita symbol, which can be written in terms of Pauli matrices as $i\sigma^2$.

It then follows that SU(2) matrices U satisfy

$$(i\sigma^2)U = U^*(i\sigma^2) \tag{20}$$

Then, if a field $\bar{\phi}$ transforms under $\bar{2}$ representation then $i\sigma^2\bar{\phi}$ transforms as:

$$i\sigma^2\bar{\phi} \rightarrow i\sigma^2 U^* \bar{\phi} = U(i\sigma^2)\bar{\phi} \tag{21}$$

which means that $i\sigma^2\bar{\phi}$ transforms under the fundamental respresentation.