Conformal Higher Spin Theory
Scattering Amplitudes and Conserved Charges

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Linearised CHS

We will be interested in Conformal Higher Spin theory in four-dimensions and will only consider the bosonic theory [Fradkin & Tseytlin][Segal].
The starting point is the totally symmetric tensor

$$h_{a(s)} \equiv h_{a_1 \ldots a_s} = h_{(a_1 \ldots a_s)}$$

with gauge symmetries

$$\delta h_{a(s)} = \partial_{(a_1} \epsilon_{a_2 \ldots a_s)} + \eta_{(a_1 a_2} \alpha_{a_3 \ldots a_s)}$$

which correspond to generalised diffeomorphism and generalised Weyl invariance.

- Understanding this symmetry, and its non-linear generalisation, is in a sense the ultimate goal.

Quadratic Action

The equation of motion for the quadratic theory can be written as

$$P^{a(s)b(s)} h_{b(s)c(s)} = 0$$

where $P^{a(s)b(s)} \sim \partial^{2s}$ and is symmetric, traceless, and transverse.

$$P^{a(s)b(s)} = P^{b(s)a(s)} \ , \quad P_c^{\ ca(s-1)b(s)} = 0 \ , \quad P^{ca(s-1)b(s)} \partial_c = 0$$

which implies there are $s(s+1)$ degrees of freedom for each spin.

- For each spin, two of these are those of massless higher spin theory e.g. spin-2 gives the usual two states of Einstein graviton.
- The spectrum contains ghost states which render the theory non-unitary.
- However classically it can be extended to a fully consistent interacting theory of higher-spin fields.

The theory arises naturally in the study of free massless fields e.g. [Tseyltin], [Segal]

$$S = -\int d^d x \; \bar{\phi}\partial^2\phi$$

by coupling the infinite conserved currents

$$J(q) = \bar{\phi}(x+q/2)\phi(x-q/2) = \sum_s \frac{1}{s!} J_{a(s)} q^{a(s)}$$

i.e. $J_{a(s)} \sim \bar{\phi}\overleftrightarrow{\partial}_{a(s)}\phi$ to background fields

$$S_{int} = \frac{\ell^{s-2}}{s!}\int d^d x \; J_{a(s)} h^{a(s)}$$

Integrating out ϕ the resulting effective action is schematically

$$\mathcal{W}_\Lambda[h] = \mathcal{W}_{\rm fin}[h] + \log\Lambda \; \mathcal{W}_{\rm log}[h] + \sum_{n\geq -d/2} \Lambda^{2n+d}\mathcal{W}_n[h]$$

In four-dimensions $\mathcal{W}_{\rm log}[h]$ is the conformal higher spin action and is a practical tool for computing the action e.g. [Bekaert et al]

Higher Spin Curvatures

We can introduce generalised Weyl curvatures $C_{a_1 b_1 \ldots a_s b_s}$ which are trace free and have the symmetries

$$C_{a(s)b(s)} = C_{[a_1 b_1][a_2 b_2]\ldots[a_s b_s]} = C_{a_2 b_2 a_1 b_1 \ldots a_s b_s}$$

satisfying the usual Bianchi identities

$$C_{[a_1 b_1 a_2] b_2 \ldots b_s} = 0 = \partial_{[c} C_{a_1 b_1] a_2 \ldots b_s}$$

such that we can decompose them (using the usual notational identification of vector and bispinor indices $a \leftrightarrow AA'$.)

$$C_{a_1 b_1 a_2 b_2 \ldots a_s b_s} = \Psi_{A_1 B_1 \ldots A_s B_s} \epsilon_{A_1' B_1'} \cdots \epsilon_{A_s' B_s'} + \tilde{\Psi}_{A_1' B_1' \ldots A_s' B_s'} \epsilon_{A_1 B_1} \cdots \epsilon_{A_s B_s}$$

in terms of the (anti)-self dual Weyl spinors which are totally symmetric and traceless

$$\Psi_{A_1 B_1 \ldots A_s B_s} = \Psi_{(A_1 B_1 \ldots A_s B_s)} \ , \quad \Psi^{A}{}_{AB\ldots} = 0 \ .$$

and satisfy the wave equation

$$\partial^{AA'} \Psi_{AB\ldots} = 0$$

Higher Spin Actions

For spin-1, Maxwell theory $F_{ab} = \Psi_{AB}\epsilon_{A'B'} + \bar{\Psi}\epsilon_{AB}$ we write the action

$$S \sim \int d^4x \; F^{ab}F_{ab}$$

and similarly for spin-2, conformal gravity, the full *non-linear*, action is

$$S = \frac{1}{2\epsilon^2}\int d^4x \; C^{a_1b_1a_2b_2}C_{a_1b_1a_2b_2} = \frac{1}{2\epsilon}\int d^4x \; (\Psi^2 + \tilde{\Psi}^2)$$

For higher-spins we can write the quadratic action in terms of linearized curvatures

$$S \sim \frac{1}{2\epsilon^2}\int d^4x \; C^{a(s)b(s)}C_{a(s)b(s)} = \frac{1}{2\epsilon^2}\int d^4x \; (\Psi_s^2 + \tilde{\Psi}_s^2)$$

There is a non-linear action due to Segal which is somewhat implicit.

Higher Spin Twistor Actions

Alternatively we can try to follow the route of [Witten, Berkovits, Boels, Mason, Skinner, Adamo, ..]

Adding a total derivative term to the action we can rewrite it as

$$S = \int d^4x \, (G^{A(2s)} \Psi_{A(2s)} - \frac{\epsilon^2}{2} G^{A(s)} G_{A(s)})$$

This is the analogue of the Chalmers-Siegel action. In this imit $\epsilon \to 0$ this gives a self-dual higher-spin theory

$$\Psi_{A_1 A_2 \dots A_{2s}} = 0 \,, \quad \partial^{A_1 A'_1} \dots \partial^{A_s A'_s} G_{A_1 \dots A_s B_1 \dots B_s} = 0$$

describing an anti-self dual spin-s particle in a self-dual background.
This motivates an action in twistor space of the form

$$S \sim \int D^3 Z \; g_{\alpha(s-1)} (\bar{\partial} f^{\alpha(s-1)} + f^{\beta(s-1)} \wedge \partial_{\beta(s-1)} f^{\alpha(s-1)})$$
$$+ \text{interaction terms between different spins} + S_{\mathrm{MHV}}$$

Scattering Amplitudes

For our purposes we will define scattering amplitudes as the multi-field action evaluated on solutions of the linearized equations and in fact will focus on tree-level three-point amplitudes. Essentially this is a way of rewriting the cubic couplings.

Linearized solutions Starting with conformal gravity linearised around Minkowski space we can the solution (in transverse, traceless gauge $\partial^a h_{ab} = h^a{}_a = 0$)

$$h_{ab} = (A_{ab} + B_{ab} n \cdot x) e^{ik \cdot x}$$

where A_{ab} and B_{ab} are symmetric traceless, n is a time-like vector and there is the constraint $ik^a A_{ab} + n^a Bab = 0$.

Here we see that there are usual plane-wave solutions as well as solutions with polynomial growth in x.

Solutions in terms of h_{ab} are gauge dependent and it is useful to work directly with the Weyl curvatures which satisfy the linearised Bach equations

$$\partial^{AA'}\partial^{BB'}\Psi_{ABCD} = 0$$

Can rewrite this in terms of a twistor tensor $\Gamma^{\alpha}{}_{BCD} = \begin{pmatrix} \Psi^{A}{}_{BCD} \\ \gamma_{A'BCD} \end{pmatrix}$ with $\Psi^{A}{}_{ABC} = 0$ as

$$\partial^{BB'}\gamma_{A'BCD} = 0 \ , \quad \partial^{BB'}\Psi^{A}{}_{BCD} = i\gamma^{B'A}{}_{CD}$$

This can be written in terms of the *local twistor connection*, $\mathcal{D}^{BB'}$ which acts on twistor indices e.g.

$$\mathcal{D}^{BB'}\Gamma^{\alpha} = \nabla^{BB'}\Gamma^{\alpha} + i\begin{pmatrix} 0 & \epsilon_{B}{}^{A}\epsilon_{B'}{}^{C'} \\ P_{ACA'B'} & 0 \end{pmatrix}\begin{pmatrix} \Psi^{C} \\ \gamma_{C'} \end{pmatrix} = 0$$

with $P_{ab} = \Phi_{ab} - \Lambda\epsilon_{A'B'}\epsilon_{AB}\Lambda = 0$. Specifically one finds solutions

On-shell States

Solutions can be written as

$$\Gamma^{\alpha}{}_{BCD} = B^{\alpha}\phi_{BCD}$$

where $\partial^{BB'}\phi_{BCD} = 0$ e.g. $\phi_{BCD} = \lambda_B\lambda_C\lambda_D e^{ik\cdot x}$, $k_a = \lambda_A\tilde{\lambda}_{A'}$. The twistors B^{α} can be found perhaps easiestly by looking at the corresponding solutions in twistor space. The result is:

$$\Psi_{ABCD} = \begin{cases} \kappa\lambda_A\lambda_B\lambda_C\lambda_D e^{ik\cdot x}, & \text{neg. hel. graviton} \\ \tilde{\xi}_{(A}\lambda_B\lambda_C\lambda_{D)} e^{ik\cdot x}, & \text{neg. hel. vector} \\ \lambda_{(A}\lambda_B\lambda_C x_{D)}^{D'}\beta_{D'} e^{ik\cdot x}, & \text{neg. hel. ghost graviton} \end{cases}$$

Can do the same for the positive helicity states by introducing

$$\tilde{\Gamma}^{\alpha}{}_{B'C'D'} = A^{\alpha}\tilde{\lambda}_B\tilde{\lambda}_C\tilde{\lambda}_D e^{ik\cdot x}\ .$$

Amplitudes are functions of the polarization twistors and the particle momenta.

Higher Spin On-shell States

This can be generalised straightforwardly to higher-spin states. We now have twistor tensors satisfying

$$\mathcal{D}^{BB'}\Gamma^{\alpha_1\ldots\alpha_{s-1}}{}_{B_1\ldots B_{s+1}} = 0\ , \quad \text{and} \quad \Gamma^{\mathrm{A}\alpha(\mathrm{s}-2)}{}_{\mathrm{AB(s)}} = 0$$

implies the linearized CHS equation

$$\partial^{B(s)B'(s)}\Psi_{A(s)B(s)} = 0$$

with solutions

$$\Gamma^{\alpha_1\ldots\alpha_{s-1}}{}_{B_1\ldots B_{s+1}} = B^{\alpha(s-1)}\lambda_{B(s+1)}e^{ik\cdot x}$$

and the conjugate tensors $\tilde{\Gamma}^{\beta(s-1)}{}_{A'(s+1)} = A^{\beta(s-1)}\tilde{\lambda}_{A'(s+1)}e^{ik\cdot x}$. In total this gives the $s(s+1)$ onshell states.

Scattering Amplitudes

To find the $\overline{\mathrm{MHV}}$ three-point functions the easiest way to plug the twistor wavefunctions into the cubic terms in the action

$$\sim g_{\alpha(s-1)} f^{\beta(s-1)} \partial_{\beta(s-1)} f^{\alpha(s-1)}$$

For conformal gravity in flat space the amplitudes for three-gravitons are zero however there are non-vanishing amplitudes e.g. [Adamo, Nakach, Tseytlin] two spin-1 states and a graviton

$$\mathcal{M}(1^{-1}, 2^{+1}, 3^{+2}) = \epsilon\kappa \frac{[23]^4}{[12]^2} \delta^{(4)}(P)$$

which is result of Einstein-Maxwell theory. Find similar results for higher spin theory where again the scattering of unitary spin-s is vanishing however it's vanishing in an interesting way.

AdS Scattering Amplitudes

Using our *definition* of scattering amplitudes we can easily compute analogues in AdS with cosmological constant Λ i.e. evaluate the action on linearised solutions.

The result gives

$$\mathcal{M}(1^{-s}, 2^s, 3^s) = \Lambda^{s-1}\left(\frac{[23]^3}{[12][31]}\right)^s (1-\Lambda\Box_P)^{s-1}\delta(P)$$

- In the flat limit we find zero times the unique answer consistent with Poincaré and helicity constraints.
- For gravity this is the previously known amplitude version of the Anderson/Maldacena argument relating Conformal gravity and Einstein gravity on AdS. For higher spin it suggests some all spin analogue.
- There is a chiral flat space higher spin theory [Metsaev; Ponomarev and Skvortsov] of massless fields.
- We know all $\overline{\mathrm{MHV}}$ three-point amplitudes. MHV we know three-point for unitary states and n-point with $n-2$ spin-two fields.

Spin-lowering

There is a nice construction of conserved charges in linearised gravity due to Penrose which works by relating higher-spin fields to Maxwell theory and then using the definition of electric charge.

Starting from a spin-2 field satisfying the wave equation

$$\nabla^{AA'}\phi_{ABCD} = 0$$

we can form a solution to the Maxwell equation by using a solution to $\left[{2 \atop 0}\right]$-twistor equation

$$\nabla^{B_1'(B_1}L^{B_2B_3)} = 0$$

as

$$\chi_{AB} = \phi_{ABCD}L^{CD} \quad \text{satisfies} \quad \nabla^{\mathrm{AA'}}\chi_{\mathrm{AB}} = 0$$

and for which we can define the field strength

$$F_{ab} = \chi_{AB}\epsilon_{A'B'} + \bar{\chi}_{A'B'}\epsilon_{AB}$$

Conserved Charges

Given a Maxwell field there is a natural definition of electric charge as a two-surface integral

$$q = -\frac{1}{8\pi} \oint_S *F$$

or including the definition of magnetic charge

$$\boxed{\mu + iq = \frac{1}{4\pi} \oint_S \chi_{AB}\epsilon_{A'B'} dx^{AA'} \wedge dx^{BB'}} \qquad (1)$$

Hence for every solution of $\left[{2 \atop 0}\right]$-equation we find a complex conserved charge. For gravity this gives 10 complex charges only half of which are non-vanishing for real potentials.

Source Integrals

Defining

$$Q^{ab} = iL^{AB}\epsilon^{A'B'} - i\bar{L}^{A'B'}\epsilon^{AB}$$

and recalling the defintion of the linearised curvature in a source free region $R_{abcd} = \phi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \bar{\phi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD}$ we can write a real conserved density two-form as

$$\Theta_{i_1 i_2} = 4\mathrm{Re}\ \chi_{\mathrm{I_1 I_2}}\epsilon_{\mathrm{I'_1 I'_2}} = \mathrm{R}^{*}_{\mathrm{i_1 i_2 ab}}\mathrm{Q}^{\mathrm{ab}}$$

when R satisfies the Bianchi identity

$$d\Theta = {}^{*}E_{ab}\xi^{b}d^{3}x$$

where $\xi_b = 1/3\nabla_b Q^{ab}$ satisfies the Killing vector equation and E_{ab} is the linearized Einstein tensor or using the equations of motion the linearised T_{ab}. Hence

$$\mu \sim \int_V d\Theta$$

gives the charge as three volume integral over sources.

Higher Spin Charges

Repeating this construction for higher spin fields is immediate: Given a spin-s field , $\nabla^{A_1' A_1}\phi_{A_1 A_2 \ldots A_{2s}}$, define

$$\chi_{A_1 A_2}\phi_{A_1 A_2 A_3 \ldots A_{2s}} L^{A_3 \ldots A_{2s}}$$

which will satisfy the Maxwell equation if

$$\nabla^{A_1'(A_1} L^{A_2 \ldots A_{2s-1})} = 0$$

i.e. it defines a symmetric $\left[\begin{smallmatrix} 2s-2 \\ 0 \end{smallmatrix}\right]$-twistor.

- There are $1/3s(4s^2-1)$ solutions to this equation of which $2(s^2-3s+4)$ are non-vanishing for real potentials.
- These quantities are related to the multipole moments of stationary gravitational fields of Geroch as studied by [Curtis].

CHS Charges

If we start with the example of conformal gravity

$$\mathcal{D}^{BB'}\Gamma^{\alpha}{}_{BCD} = 0 \quad \& \quad \Gamma^{A}{}_{ABC} = 0$$

we can again form a solution to the Maxwell equation

$$\chi_{AB} = \Gamma^{\alpha}_{ABC} W_{\alpha}{}^{C} \quad \text{if} \quad \nabla^{(\mathrm{B}}_{(\mathrm{B}'}\mathrm{W}^{\mathrm{A})}_{\mathrm{A}')} = 0$$

i.e. if W defines $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$-twistor

- In fact only the trace-free part of the twistor is relevant and so can take it to me trace free.
- We can take this twistor to be Hermitian and in this case it corresponds to a conformal Killing vector.
- Can generalise this to higher spins. Now we have a $\left[\begin{smallmatrix}s/2\\s/2\end{smallmatrix}\right]$ traceless Hermitian twistor. This gives $1/12s^2(s+1)^2(2s+1)^2$ conformal Killing tensors which matches with the calculation of Eastwood in four-dimensions.

Conclusions

- Conformal higher spin theory is a consistent interacting theory with generalised diffeomorphism invariance.
- Usual plane-wave states have vanishing three-point amplitudes it has a non-trivial S-matrix in flat space if one includes "non-unitary" states
- Can identify a unitary sub-sector, analogues of EG inside CG, which can be used to reproduce MHV and MHV-bar 3-pt amplitudes up powers of cosmological constant.
- Twistor formulation appears to be convenient for doing explicit calculations.
- Can extend twistor construction of conserved charges to higher spin theories and to conformal theories. One important feature is that this construction generalises trivially to an conformally flat geometry, (A)dS, FRW etc.

Open Questions

- Can we compute more interesting amplitudes? Partition functions? Again the fact that twistor methods work on general conformally flat geometries should be useful.
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- Can we construct interesting classical solutions? Higher-spin "black holes"? Instantons?
- Can we understand the higher-spin gauge symmetries? Soft limits?
- There is a non-trivial higher spin self-dual theory - can we generalise the notions of integrability? Is there a quantum anomaly? Do higher spin symmetries imply any thing for high energy behaviour?
- Construction of the symmetries for gravity can be extended to non-linear level, can we repeat this for higher spin charges? Asymptotic higher spin symmetries? Ward identities?