Scattering amplitudes from superconformal symmetry

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Based on

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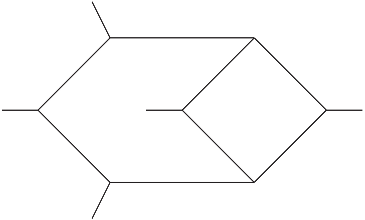
PRL **121** *(2018) 021602, D.C., J. M. Henn, E. Sokatchev*

and work in progress with Johannes Henn, Emery Sokatchev, Simone Zoia

Outline

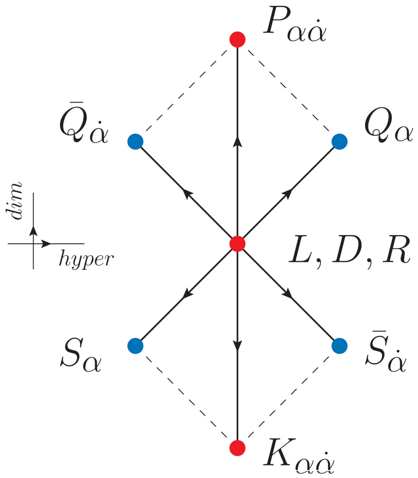
Q: How to profit from the underlying (super)conformal symmetry of the theory in calculations of nonplanar amplitudes/Feynman integrals?



IR and UV finite in 6D

A: Find the (super)conformal anomaly of the vertex functions; Use it as a seed calculating the anomaly; Solve the anomalous Ward identity

Superconformal symmetry



- Supersymmetric theories of massless particles in $D = 4$
- Symmetry of the classical Lagrangian
- Symmetry of Feynman integrals/integrands for scattering at high energies
  $\to$ masses are irrelevant
  $\to$ scale invariance
  $\to$ conformal symmetry

Superconformal symmetry

$\beta(g) = 0$ at all loop orders in $\mathcal{N} = 4$ SYM theory

Correlation functions of protected composite operators

$$\langle \mathcal{O}(x_1) \dots \mathcal{O}(x_n) \rangle$$

are exactly (super)conformal

Scattering amplitudes

$$\langle \Phi(p_1, \eta_1) \dots \Phi(p_n, \eta_n) \rangle$$

contain IR/collinear divergences
$\Longrightarrow$ breakdown of the superconformal symmetry @ loop level

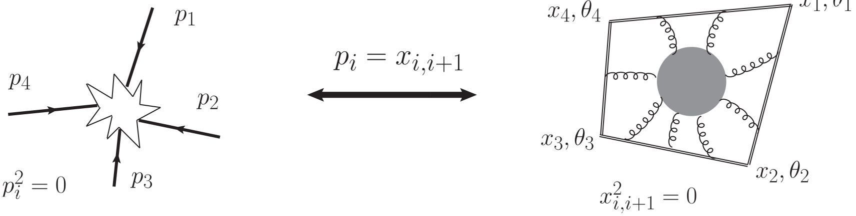
$\mathcal{N} = 4$ SYM in the planar limit

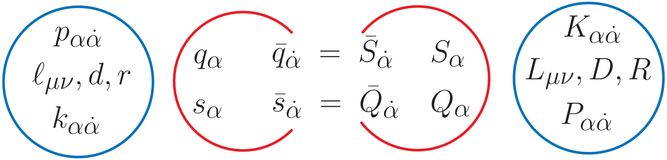
Ordinary superconformal symmetry

- Lagrangian symmetry
- Acts on the amplitude
- Chiral on-shell state $p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}, \eta$

Dual superconformal symmetry

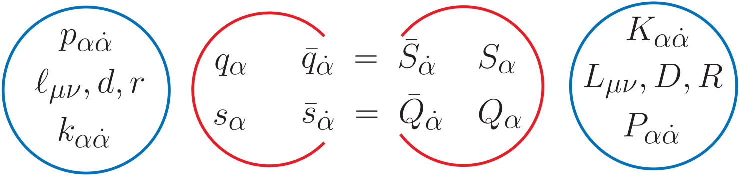
- Dynamical symmetry
- Acts on the super-Wilson Loop
- Chiral superspace $x_{\alpha\dot{\alpha}}, \theta_\alpha$





[Drummond, Henn, Korchemsky, Sokatchev '08]

$\mathcal{N} = 4$ SYM in the planar limit



Ordinary superconformal symmetry **Dual superconformal symmetry**

@ tree level both superconformal symmetries are exact (up to contact terms), but @ loop level

Exact: $q_\alpha,\ \bar q_\alpha,\ p_{\alpha\dot\alpha}, r, \ell_{\mu\nu}$

Broken: $k_{\alpha\dot\alpha},\ \bar s_{\dot\alpha},\ s_\alpha,\ d$

Exact: $Q_\alpha,\ \bar S_{\dot\alpha},\ P_{\alpha\dot\alpha},\ L_{\mu\nu}, R$

Broken: $\bar Q_{\dot\alpha},\ S_\alpha,\ K_{\alpha\dot\alpha},\ D$

$\mathcal{N}=4$ SYM in the planar limit

Breakdown of the dual superconformal symmetry is well understood

- Universal form of IR-divergences
- Ratio function and remainder function are finite

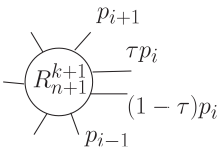
$$\mathcal{R}_{k,n} = \frac{\mathcal{A}_n^{N^k MHV}}{\mathcal{A}_n^{MHV}} \qquad , \qquad R_{k,n} = \frac{\mathcal{A}_n^{N^k MHV}}{\mathcal{A}_n^{BDS}}$$

- Exactly conformal @ all loop orders

$$K_{\alpha\dot{\alpha}} R_{k,n} = 0$$

- but $\bar{Q}_{\dot{\alpha}}$ is anomalous. Anomalous Ward identity (for the Wilson Loop)

$$\bar{Q}_{\dot{\alpha}} R_{k,n} = \gamma_{\rm cusp}(g) \sum_{i=1}^{n} \tilde{\lambda}_{i,\dot{\alpha}} \int \text{"} d\tau d^3\eta \text{"}$$



[Caron-Huot '11; Caron-Huot, He '11] [Bullimore, Skinner '11]

What about ordinary superconformal symmetry at loop level?

$$k_{\alpha\dot{\alpha}} = \frac{\partial^2}{\partial\lambda_\alpha \partial\tilde{\lambda}_{\dot{\alpha}}} \quad , \quad \bar{s}_{\dot{\alpha}} = \eta \frac{\partial}{\partial\tilde{\lambda}_{\dot{\alpha}}} \quad , \quad s_\alpha = \frac{\partial^2}{\partial\eta\partial\lambda_\alpha}$$

[Witten '03]

$\bar{s}_{\dot{\alpha}}$, s_α, $k_{\alpha\dot{\alpha}}$ are anomalous even for the finite remainder function

$$\bar{s}_{\dot{\alpha}} R_{k,n} \neq 0 \quad , \quad s_\alpha R_{k,n} \neq 0 \quad , \quad k_{\alpha\dot{\alpha}} R_{k,n} \neq 0$$

Anomaly is not a problem if it is under control

Exact symmetry $\longleftrightarrow$ Homogeneous DE

Anomalous symmetry $\longleftrightarrow$ Inhomogeneous DE

e.g. $\bar{s}_{\dot{\alpha}} = \bar{Q}_{\dot{\alpha}}$ and all-loop $\bar{Q}$-equation in the planar limit of $\mathcal{N} = 4$ SYM theory

Applications of superconformal symmetry for amplitudes rely on their duality with (super)-Wilson Loop, i.e.

- $\mathcal{N} = 4$ SYM theory
- Planar limit $N_c \to \infty$

But

- the ordinary superconformal symmetry does NOT need planarity!
- what are the implications for amplitudes in superconformal theories with less supersymmetry?

Our goal:

- Implications of the ordinary (super)conformal symmetry directly for amplitudes
- Anomalous Ward identity for s_α, $\bar{s}_{\dot{\alpha}}$, $k_{\alpha\dot{\alpha}}$ in the nonplanar sector

Holomorphic anomaly

Tree-level MHV amplitude of n gluons

$$\mathcal{A}^{\rm MHV}_{n;{\rm tree}} = \frac{\langle 12\rangle^3\,\delta^{(4)}\left(\sum_{i=1}^n \lambda_i\,\tilde\lambda_i\right)}{\langle 23\rangle\langle 34\rangle\ldots\langle n1\rangle}\,, \qquad \langle ij\rangle = \lambda_i^\alpha \epsilon_{\alpha\beta}\lambda_j^\beta$$

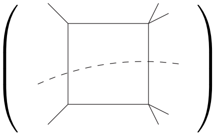
Singular denominators generate holomorphic anomaly [Cachazo, Svrcek, Witten '04]

$$\frac{\partial}{\partial\tilde\lambda^{\dot\alpha}}\frac{1}{\langle\lambda\chi\rangle} = 2\pi\tilde\chi_{\dot\alpha}\delta(\langle\lambda\chi\rangle)\delta([\tilde\lambda\tilde\chi]) \qquad \Longleftarrow \qquad \frac{\partial}{\partial\bar z}\frac{1}{z} = \pi\delta^2(z)$$

The anomaly of tree amplitudes is localized on collinear configurations of scattered particles (contact terms) [Bargheer, Beisert, Loebbert, McLoughlin, Galleas '09]

$$p_2\sim p_3 \quad , \quad p_3\sim p_4 \quad , \quad \ldots \quad , \quad p_{n-1}\sim p_n \quad , \quad p_n\sim p_1$$

One-loop $\bar s_{\dot\alpha}$ anomaly of ${\rm Disc}_{s_{1\ldots j}}\mathcal{A}_n^{\rm MHV}$ [Korchemsky, Sokatchev '09]

$\bar s_{\dot\alpha}$  $\neq 0$

- Anomaly of the IR-finite object because of collinear regions of loop integration
- How to lift this formula from the cut?

Conformal symmetry

We consider amplitude Feynman integrals instead of the scattering amplitudes

$$\mathcal{I}(p_1,\ldots,p_n)=\int d^Dk_1\ldots d^Dk_\ell\,\frac{\mathcal{N}(\{k_i\};\{p_i\})}{D_1\ldots D_N}$$

- Scattering of massless particles $p_i^2=0,\ i=1,\ldots,n$ and $p_1+\ldots+p_n=0$
- Massless propagators $D_i=(\pm k_{j_1}+\ldots\pm k_{j_a}\pm p_{l_1}+\ldots\pm p_{l_b})^2$
- Conformal interactions, e.g. Yukawa vertices and ϕ^4 in $D=4$; ϕ^3 in $D=6$
- Lagrangian is classically conformal
- UV- and IR-finite sector of the full theory, i.e. finite Feynman integrals
- Feynman integrals are naively conformal

$$\left(\sum_{i=1}^{n-1}\mathbb{K}_{i,\mu}\right)\mathcal{I}(p_1,\ldots,p_n)\stackrel{?}{=}0\,,\qquad \mu=0,1,\ldots,D-1$$

on-shell conformal generator $\mathbb{K}_\mu$ (2nd order diff operator)

Conformal boost generators in momentum space

Off-shell conformal boost K_μ with conformal dimension Δ,

$$K^\mu_\Delta = -q^\mu \Box_q + 2q^\nu \partial_{q^\nu} \partial_{q_\mu} + 2(D-\Delta)\partial_{q_\mu}$$

On-shell conformal boost $\mathbb{K}_\mu$

- $D=4$ realization is well-known, [Witten '03]
  Spinor-helicity parametrization of light-like momenta by $SL(2)$ spinors

$$\sigma^\mu_{\alpha\dot\alpha} p_\mu = \lambda_\alpha \tilde\lambda_{\dot\alpha} \qquad , \qquad \mathbb{K}_\mu = 2\,\tilde\sigma_\mu^{\dot\alpha\alpha} \frac{\partial^2}{\partial\lambda^\alpha \partial\tilde\lambda^{\dot\alpha}}$$

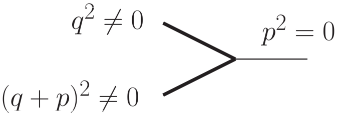
  $\lambda_\alpha, \tilde\lambda_{\dot\alpha}$ are defined up to phase.

- In $D=6$ we use chiral $SL(4)$ spinors λ^{Aa}, $A=1,\ldots,4$; [Cheung, O'Connell '09]
  helicity (little group) index $a=1,2$

$$p^\mu \tilde\sigma_\mu^{AB} = \lambda^{Aa}\lambda^B_a \qquad , \qquad \mathbb{K}_\mu = -\tilde\sigma_\mu^{AB} \frac{\partial^2}{\partial\lambda^{Aa}\partial\lambda^B_a}$$

$$\mathbb{K}^\mu = K^\mu_{\Delta=4} \quad \text{for the on-shell states} \quad \varphi = \varphi(\lambda^a \otimes \lambda_a)$$

6D vertex function ϕ^3

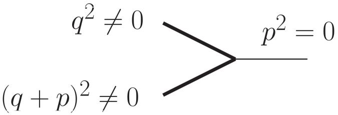


$$= \langle\, \phi(q)\, \phi(-q-p) \mid \phi(p)\, \rangle_g$$

$$\left(K^\mu_{\Delta=2} + \mathbb{K}^\mu\right)\ \frac{1}{(q^2+i0)((q+p)^2+i0)}$$

$$= \quad ???$$

6D vertex function ϕ^3



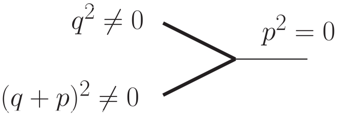
$$= \langle\, \phi(q)\, \phi(-q-p) \mid \phi(p)\, \rangle_g$$

$$(K^\mu_{\Delta=2} + \mathbb{K}^\mu)\ \frac{1}{(q^2+i0)((q+p)^2+i0)}$$

$$= 4i\pi^3\, p^\mu \int_0^1 d\xi\, \xi(1-\xi)\, \delta^{(6)}(q+\xi p)$$

Anomaly is contact and it lives on collinear configurations $q \sim p$ of momenta

6D vertex function ϕ^3



$$= \langle\, \phi(q)\, \phi(-q-p) \mid \phi(p)\, \rangle_g$$

$$(K^{\mu}_{\Delta=2} + \mathbb{K}^{\mu}) \;\; \frac{1}{(q^2 + i0)((q+p)^2 + i0)}$$

$$= 4 i \pi^3\, p^{\mu} \int_0^1 d\xi\, \xi (1-\xi)\, \delta^{(6)}(q + \xi p)$$

Anomaly is contact and it lives on collinear configurations $q \sim p$ of momenta

How to prove this distribution relation?

Anomaly of 6D vertex function ϕ^3

- Introduce Feynman parameter ξ and regulator ϵ,

$$\frac{1}{q^2(q+p)^2} = \int_0^1 d\xi \frac{1}{((q+\xi p)^2)^2} = \lim_{\epsilon\to 0} \int_0^1 d\xi \frac{1}{((q+\xi p)^2)^{2-\epsilon}}$$

- Act with conformal generators $-q^\mu \Box_q - p^\mu \Box_p + \dots$

$$(K^\mu_{\Delta=2} + \mathbb{K}^\mu) \int_0^1 d\xi \frac{1}{((q+\xi p)^2)^{2-\epsilon}} \sim p^\mu \int_0^1 d\xi \frac{\epsilon}{((q+\xi p)^2)^{3-\epsilon}}$$

- This distribution has a pole at $\epsilon \to 0$,

$$\frac{1}{((q+\xi p)^2)^{3-\epsilon}} = \frac{i\pi^2}{2\epsilon} \delta^{(6)}(q+\xi p) + \mathcal{O}(\epsilon^0)$$

- The limit $\epsilon \to 0$ produces the contact anomaly

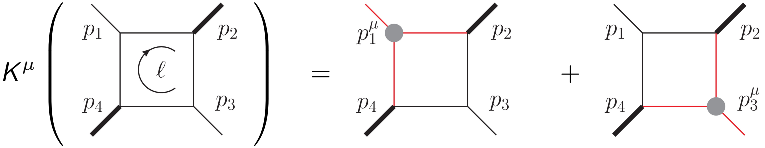
Conformal Ward identities for finite loop integrals

Consider 6D Box integrals: No UV or IR/collinear divergences

6D boxes = Finite parts of 4D boxes + 4D three-mass triangles

[Bern, Dixon, Kosower '93]

Two-mass-easy box in 6D is NOT invariant under $K^\mu \equiv \mathbb{K}_1^\mu + K_2^\mu + \mathbb{K}_3^\mu + K_4^\mu$ because of the 6D vertex anomalies



2nd order inhomogeneous DE for ℓ-loop $\mathcal{I}_{(\ell)}$ integrals with $(\ell-1)$-loop RHS

$$K^\mu \, \mathcal{I}_{(\ell)} = \int_0^1 d\xi \; A^\mu_{(\ell-1)}(\xi)$$

Bootstrap of multi-loop naively conformal integrals

2nd order DE are difficult to solve, but they are efficient for the bootstrap

Example. Planar pentabox integral in 6D

Five-particle massless scattering $\longrightarrow$ 26/31-letter alphabet of pentagon functions

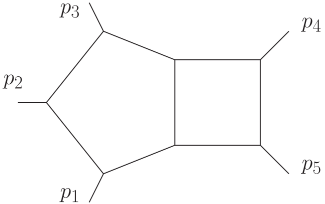
[Gehrmann, Henn, Lo Presti '15][D.C., Henn, Mitev '17]

Symbol ansatz of parity odd weight-5 integrable symbols

$$\mathcal{S}(\mathcal{I}_5) = \frac{1}{\sqrt{\Delta}} \sum_{i_1,\ldots,i_5} c_{i_1\ldots i_5} (W_{i_1} \otimes \ldots \otimes W_{i_5}) \,, \qquad \Delta = \det(p_i \cdot p_j)$$

161 free coefficients in the ansatz. They are uniquely fixed by just one projection

$$(n \cdot K)\, \mathcal{S}(\mathcal{I}_5) = (n \cdot p_1) A_1 + (n \cdot p_3) A_3 \,, \qquad (n \cdot p_i) = 0 \;\text{ at }\; i = 2, 4, 5$$



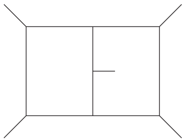
[see Zoia's talk at Loops&Legs 2018]

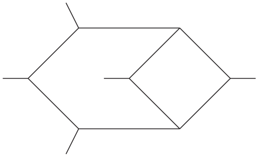
Conformal symmetry

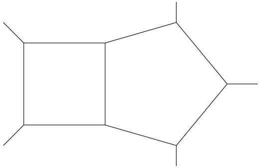
Anomalous Ward identities for K_μ are 2nd order DE, which are hard to solve, but knowing the alphabet and leading singularities we can bootstrap Feynman integrals.

Bootstrap of the symbol $\mathcal{S}(\mathcal{I})$ $\longrightarrow$ bootstrap of the hyperlogarithms

Relevant applications for the five-particle massless scattering







Superconformal symmetry

Anomalous Ward identity for S_α and $\bar{S}_{\dot{\alpha}}$ are 1st order DE.

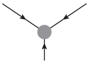
They can be integrated directly! No assumptions about alphabet!

$\mathcal{N} = 1$ Superconformal symmetry

$\mathcal{N} = 1$ matter supergraphs with on-shell states

Wess-Zumino model in 4D; chiral and antichiral off-shell superfields

$$\Phi(x, \theta) = \phi(x) + \theta^\alpha \psi_\alpha(x) + (\theta)^2 F(x)$$




$$S_{WZ} = \int d^4x\, d^2\theta d^2\bar\theta\, \bar\Phi \Phi + \frac{g}{3!} \int d^4x\, d^2\theta\, \Phi^3 + \frac{g}{3!} \int d^4x\, d^2\bar\theta\, \bar\Phi^3$$

$\cong$ kin terms +  +  + 

The action is classically superconformal $\mathfrak{su}(2,2|1)$

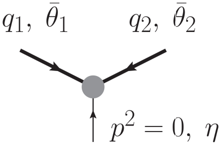
state	$\lvert\bar\psi\rangle$	$\lvert\bar\phi\rangle$	$\lvert\phi\rangle$	$\lvert\psi\rangle$
helicity	$-\frac{1}{2}$	0	0	$\frac{1}{2}$

$$\Psi(p, \eta) = |\psi\rangle + \eta|\psi\rangle$$
$$\bar\Phi(p, \eta) = |\bar\phi\rangle + \eta|\bar\psi\rangle$$

Two superstates with $\eta \equiv \theta^\alpha \lambda_\alpha$ (NOT one CPT superstate like in $\mathcal{N} = 4$ SYM)

Anomalies of vertex functions

$$\langle\, \bar\Phi(q_1,\bar\theta_1)\, \bar\Phi(q_1,\bar\theta_1)\,|\, \bar\Phi(p,\eta)\,\rangle_g =$$

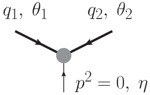


$$= \delta^{(4)}(P)\delta^{(2)}(Q)\frac{g}{q_1^2 q_2^2}$$

Off-shell S_α, $\bar S_{\dot\alpha}$ superconformal generators and their on-shell counterparts

$$\mathbb{S}_\alpha = \frac{\partial^2}{\partial\eta\partial\lambda^\alpha} \quad , \quad \bar{\mathbb{S}}_{\dot\alpha} = \eta\frac{\partial}{\partial\tilde\lambda^{\dot\alpha}}$$

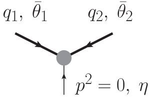
$$\left(\mathbb{S}_\alpha + S_\alpha^{(q_1)} + S_\alpha^{(q_2)}\right)$$



$$= 2i\pi^2\int_0^1 d\xi\, Q_\alpha\delta^{(4)}(q_1+\xi p)\delta^{(4)}(q_1+(1-\xi)p)$$

where $Q_\alpha \sim \lambda_\alpha$ on the collinear configuration $q_1 \sim q_2 \sim p = \lambda\otimes\tilde\lambda$

$$\left(\bar{\mathbb{S}}_{\dot\alpha} + \bar S_{\dot\alpha}^{(q_1)} + \bar S_{\dot\alpha}^{(q_2)}\right)$$

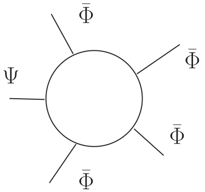


$$= 0$$

Five-particle NMHV amplitude supergraphs

We consider amplitude supergraphs, which are finite and naively superconformal

$\mathcal{A}_5^{\mathrm{NMHV}} =$



$$= \delta^{(4)}(P)\, \underbrace{\delta^{(2)}(Q) \cdot \Xi}_{\text{R-charge}\,=\,3} \cdot \mathcal{I}(\{\lambda, \tilde{\lambda}\})$$

Supercharges

$$Q_\alpha = \sum_i \eta_i \lambda_{i,\alpha}\,, \qquad \bar{Q}_{\dot\alpha} = \sum_i \tilde{\lambda}_{i,\dot\alpha} \frac{\partial}{\partial \eta_i}$$

The unique superinvariant @ 5 points: $Q\,\Xi = \bar{Q}\,\Xi = 0$

$$\Xi_{ijk} = \eta_i [jk] + \eta_j [ki] + \eta_k [ij]\,, \qquad [ij] := \tilde{\lambda}_{\dot\alpha} \epsilon^{\dot\alpha\dot\beta} \tilde{\lambda}_{\dot\beta}$$

Twistor collinearity operator arises from SUSY [Witten '03]

$$\mathbb{S}_\alpha\, \Xi_{ijk} = (F_{ijk})_\alpha \equiv [jk] \frac{\partial}{\partial \lambda_i^\alpha} + [ki] \frac{\partial}{\partial \lambda_j^\alpha} + [ij] \frac{\partial}{\partial \lambda_k^\alpha}$$

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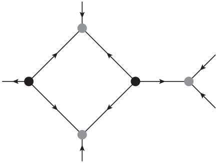

$$= \delta^{(4)}(P)\, \underbrace{\delta^{(2)}(Q) \cdot \Xi}_{\text{R-charge} = 3} \cdot \mathcal{I}(\{\lambda, \tilde{\lambda}\})$$

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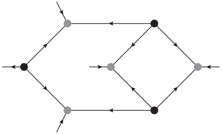
$\mathcal{A}_5^{\rm NMHV} =$  $= \delta^{(4)}(P)\, \underbrace{\delta^{(2)}(Q)\cdot \Xi}_{\text{R-charge}=3} \cdot \mathcal{I}(\{\lambda,\tilde\lambda\})$

Supercharges

$$Q_\alpha = \sum_i \eta_i \lambda_{i,\alpha}\,, \qquad \bar{Q}_{\dot\alpha} = \sum_i \tilde\lambda_{i,\dot\alpha} \frac{\partial}{\partial \eta_i}$$

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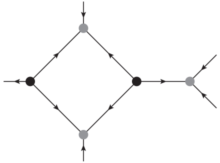
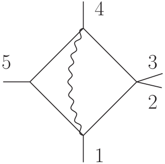
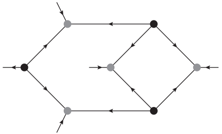
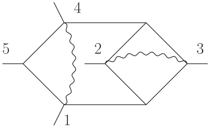
Ward identities for five-point integrals

Integrals with 'magic' numerators [Arkani-Hamed, Bourjaily, Cachazo, Trnka '10]

$\mathcal{A}_5^{\mathrm{NMHV}} =$  $\Longrightarrow \quad \mathcal{I}_5^{(1)}(\{\lambda, \tilde{\lambda}\}) =$ 

$\mathcal{A}_5^{\mathrm{NMHV}} =$  $\Longrightarrow \quad \mathcal{I}_5^{(2)}(\{\lambda, \tilde{\lambda}\}) =$ 

S_α-variation of $\mathcal{A}_5$ is anomalous $\Longrightarrow$ inhomogeneous DE for ℓ-loop Feynman integral $\mathcal{I}_5^{(\ell)}(\{\lambda, \tilde{\lambda}\})$ with collinearity operator F^{α}_{ijk},

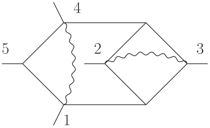
$$F^{\alpha}_{ijk}\, \mathcal{I}_5^{(\ell)}(\{\lambda, \tilde{\lambda}\}) = \sum_{r=1,2,3,4} \lambda_r^{\alpha} \int_0^1 d\xi\, A_r^{(-1+\ell)}(\xi, \{\lambda, \tilde{\lambda}\})$$

Solving DE for the nonplanar hexabox

Four dimensionless variables describe the five-point kinematics

$$x_1 = -1 - \frac{s_{14}}{s_{15}}\,, \quad x_2 = -1 - \frac{s_{14}}{s_{45}}\,, \quad x_3 = \frac{[12][34]}{[23][41]}\,, \quad x_4 = \frac{[23][45]}{[34][52]}$$

Integral $\mathcal{I} = \mathcal{I}(x_1, x_2, x_3, x_4)$ is a pure function, i.e. unit leading singularity

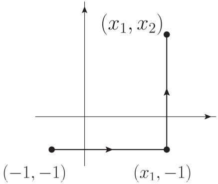
$\mathcal{I} =$ 

$$\tilde{d}\mathcal{I}(x_1, x_2, x_3, x_4) = a_1\, \tilde{d} \log x_1 + a_4\, \tilde{d} \log x_2$$
$$+ a_2\, \tilde{d} \log \frac{1 - x_1 x_2}{(1+x_2)(x_3-1)x_4 + (1+x_1)(x_3 x_4 - 1)}$$
$$+ a_3\, \tilde{d} \log \frac{1 - x_1 x_2}{(1+x_2)x_3 x_4 + (1+x_1)(x_3 x_4 - 1)}$$

where $\tilde{d} = dx_1\, \partial_{x_1} + dx_2\, \partial_{x_2}$; a_k – anomaly of k-th leg, weight-3 pure functions

Boundary conditions for DE:

- $\mathcal{I}(x_1 = -1, x_2 = -1) = 0$, i.e. at $s_{14} = 0$
- OR absence of nonphysical branch cuts



Summary

- Superconformal symmetry of amplitudes in $\mathcal{N} = 4$ SYM

- Conformal symmetry (2nd order DE)
    - ϕ^3 in 6D; Yukawa and ϕ^4 in 4D
    - Conformal anomalies of vertex functions
    - Anomalous conformal Ward identity for amplitude Feynman diagrams
    - Bootstrap of 5-particle Feynman integrals

- Superconformal symmetry (1st order DE)
    - 4D Wess-Zumino model of $\mathcal{N} = 1$ matter
    - Superconformal anomalies of vertex functions
    - Anomalous superconformal Ward identity for amplitude superdiagrams
    - Ward identity in the form of canonical DE