The many faces of (next-to) soft physics

Chris White, Queen Mary University of London

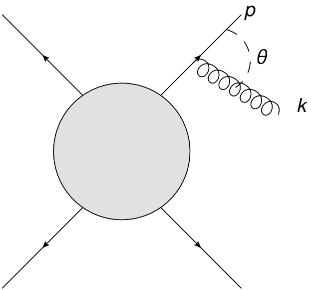
Symmetries of the S-matrix and Infrared Physics

Overview

- Brief introduction to (next-to-) soft divergences.
- Applications in Collider Physics (mainly QCD).
- Applications in high energy scattering (mainly gravity).
- Outlook.

Infrared divergences

- In scattering amplitudes, get singularities due to soft or collinear gauge bosons:



$$\frac{1}{p \cdot k} = \frac{1}{|\boldsymbol{p}||\boldsymbol{k}|(1 - \cos\theta)}.$$

- Formal divergences cancel upon combining real and virtual graphs (Block, Nordsieck).

- Both soft and collinear radiation is *universal*.
- Physics: it has an infinite wavelength, so cannot resolve the underlying amplitude.

- Universality of soft / collinear radiation is expressed in *factorisation formulae*.
- Example: consider a tree-level amplitude $\mathcal{A}_{n+1}(\{p_i\}, k)$ where momentum k becomes soft. We then get the *soft theorems*

$$\lim_{k^\mu \to 0} \mathcal{A}_{n+1}(\{p_i\}, k) = \mathcal{S}^{(0)}(\{p_i\}, k)\mathcal{A}_n(\{p_i\}),$$

where

$$\mathcal{S}^{(0)}_{\mathrm{QED}} = \sum_{i=1}^{n} \frac{\epsilon_\mu(k) p_i^\mu}{p_i \cdot k}, \quad \mathcal{S}^{(0)}_{\mathrm{grav.}} = \sum_{i=1}^{n} \frac{\epsilon_{\mu\nu}(k) p_i^\mu p_i^\nu}{p_i \cdot k}$$

(Yennie, Frautschi, Suura; Weinberg).
- All dependence on the soft momentum k is in the overall factor $\mathcal{S}$.

- It is also possible to write such formulae at one order higher in the k expansion (Cachazo, Strominger; Casali):

$$\mathcal{A}_{n+1}(\{p_i\}, k) = \left[\mathcal{S}^{(0)} + \mathcal{S}^{(1)}\right] \mathcal{A}_n(\{p_i\}),$$

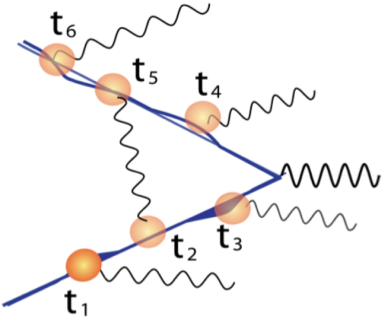
  with

$$\mathcal{S}^{(1)}_{QED} = \sum_{i=1}^{n} \frac{\epsilon_\mu k_\rho J^{(i)\mu\rho}}{p_i \cdot k}, \quad \mathcal{S}^{(1)}_{grav.} = \sum_{i=1}^{n} \frac{\epsilon_\mu k_\rho J^{(i)\mu\rho}}{p_i \cdot k},$$

  where $J^{(i)}_{\mu\nu}$ is the total angular momentum of (hard) particle i.
- Next-to-next-to-soft also possible for gravity.
- These and similar results have a surprisingly long history...

- Next-to-soft effects were first studied in gauge theory (QED) by Low (1958).
- He considered external scalars; generalised to fermions by Burnett and Kroll (1968).
- Both groups only considered massive particles (no collinear effects).
- Similar work in gravity by Gross, Jackiw (1968).
- Del Duca (1990) generalised the Low-Burnett-Kroll result to include collinear effects.

- Next-to-soft effects for massive particles considered using worldline methods by Laenen, Stavenga, White (2008).



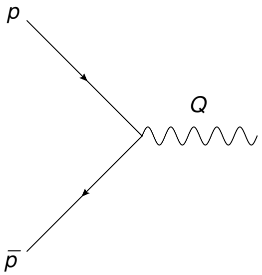
- Can replace propagators for external legs by quantum mechanics path integrals.
- Leading term in perturbative expansion is classical trajectory (soft limit).
- First-order wobbles give next-to-soft behaviour.

- Also works for gravity (White, 2011).

- The tree-level (next-to)-soft theorems can be obtained using Ward identities associated with asymptotic symmetries.
- This is the focus of much of this meeting!
- However, the history of next-to-soft physics suggests that there are many other applications of next-to-soft physics.
- Indeed, these have been reinvigorated by the recent work on next-to-soft theorems.
- The aim of this talk is to review some of these applications.

Key message: next-to-soft physics connects `hep-th`, `hep-ph`, `hep-ex` and `gr-qc`!

- A major application of (next-to) soft physics is to collider physics.
- We saw earlier that IR singularities cancel when real and virtual diagrams are combined.
- However, the cancellation can leave behind large contributions to perturbative quantities.
- Consider e.g. the production of a vector boson at a collider ("Drell-Yan production"):



- Let $z = Q^2/s$ be the fraction of (squared) energy s carried by the vector boson.
- At LO, $z = 1$, and thus the cross-section is

$$\frac{d\sigma^{(0)}}{dz} \propto \delta(1 - z).$$

- At next-to-leading order (NLO), radiation can carry energy, so that

$$0 \leq z \leq 1.$$

- The NLO cross-section then turns out to be

$$\frac{d\sigma^{(1)}_{q\bar{q}}}{dz} \sim \frac{\alpha_s}{2\pi}\left[4(1+z^2)\left(\frac{\ln(1-z)}{1-z}\right)_+ - 2\frac{1+z^2}{1-z}\ln(z) + \delta(1-z)\left(\frac{2\pi^2}{3}-8\right)\right].$$

- It contains highly divergent terms as $z \to 1$.
- Looks like perturbation theory is in trouble!
- Let's go one order higher and see what happens...

- At NNLO the problem is even worse! One has

$$\frac{d\sigma^{(2)}_{q\bar{q}}}{dz} \sim C_F^2 \left(\frac{\alpha_s}{2\pi}\right)^2 \left[128 \left(\frac{\ln^3(1-z)}{1-z}\right)_+ - 256 \left(\frac{\ln(1-z)}{1-z}\right)_+ + \ldots\right],$$

  where $\ldots$ denotes terms suppressed by $(1-z)$.
- Logs get higher at higher orders in perturbation theory...
- ... which indeed breaks down as $z \to 1$.
- Precisely the regime where the vector boson is produced near threshold, so that extra radiation is soft / collinear!
- The problem terms are echoes of IR singularities having been present.
- Thus, this problem affects many different scattering processes...

- For heavy particles produced near threshold, we can define a ξ, where $\xi \to 0$ at threshold (e.g. $\xi = (1 - z)$).
- Then the general structure of any such cross-section is:

$$\frac{d\sigma}{d\xi} = \sum_{n,m} \alpha^n \left[c_{nm}^{(0)} \left(\frac{\ln^m \xi}{\xi} \right)_+ + c_{nm}^{(1)} \ln^m \xi + \dots \right].$$

- First set of terms correspond to (leading) threshold logs: pure soft and / or collinear.
- Second set of terms is next-to-leading power (NLP) threshold logs: next-to-soft and / or collinear.
- For $\xi \to 0$, we need to rethink perturbation theory.

- The solution to this problem is to somehow work out what the large logs are to all orders in α_s.
- Then we can sum them up to get a function of α_s that is better behaved than any fixed order perturbation expansion.
- Toy example: consider the function

$$e^{-\alpha_s x} = \sum_{n=0}^{\infty} \frac{\alpha_s^n (-x)^n}{n!}.$$

- Each term diverges as $x \to \infty$, but the all-order result is well-behaved.

- Many approaches exist for resumming leading threshold logs.
- There are many (hundreds?) of observables at e.g. the LHC for which this is relevant.
- Original diagrammatic approaches by e.g. Sterman; Catani, Trentadue),
- Can also use Wilson lines (Korchemsky, Marchesini), or the renormalisation group (Forte, Ridolfi).
- A widely used approach is to treat soft and collinear gluons as separate fields in an effective theory: soft-collinear effective theory (SCET) (Becher, Neubert; Schwartz; Stewart).
- All approaches have the *factorisation* of soft / collinear physics at their heart.

Soft-collinear factorisation

- The general structure of an n-point amplitude is

$$\mathcal{A}_n = \mathcal{H}_n \times \mathcal{S} \times \frac{\prod_i J_i}{\prod_i \mathcal{J}_i}.$$

- This is the virtual generalisation of the soft theorem.
- Here $\mathcal{H}_n$ is the *hard function*, and is IR finite.
- The *soft* and *jet functions* $\mathcal{S}$ and J_i collect soft / collinear singularities respectively.
- The *eikonal jets* $\mathcal{J}_i$ remove any double counting.
- The soft and jet functions have universal definitions in terms of Wilson line operators.

- The soft-collinear factorisation formula leads directly to resummation of threshold effects.
- Related ideas in other approaches (e.g. SCET).
- Summing successive towers of threshold logs requires calculating the soft and jet functions to a given order in perturbation theory.
- State of the art is two loops (Sterman, Aybat, Dixon, Kidonakis, Mitov, Sung, Becher, Neubert, Beneke, Falgari, Schwinn, Ferroglia, Pecjak, Yang).
- Progress towards three-loops and beyond (Gardi, Laenen, Stavenga, Smillie, White, Almelid, Duhr, Korchemsky, Henn, Huber, Grozin, Marquard, Correa, Maldacena, Sever).

Next-to-leading power logs

- To date, much less has been known about NLP effects.
- Known for a while to be numerically significant e.g. in Higgs production (Kramer Laenen, Spira; Harlander, Kilgore; Catani, de Florian, Grazzini, Nason).
- This has been confirmed by recent N^{3}LO Higgs results (Anastasiou, Duhr, Dulat, Herzog, Mistlberger).
- There are three good reasons to study NLP logs:
  1. Resummation of them will improve precision.
  2. Even without resummation, NLP logs may provide good approximate N^{n}LO cross-sections.
  3. Can improve the stability of numerical codes.

- Next-to-soft effects in particular scattering processes classified to all orders by (Almasy, Moch, Presti, Soar, Vermaseren, Vogt).
- Can also be classified using the *method of regions* (Beneke, Smirnov, Pak, Jantzen) (see e.g. Bonocore, Laenen, Magnea, Vernazza, White).
- None of the previous approaches is fully general - but strong hints of an underlying structure.
- Can we predict NLP logs in an arbitrary process?
- Can they be written in terms of universal functions (like LP effects)?
- Encouraging recent progress...

- It is well-known that LP effects can be described using *Soft-Collinear Effective Theory* SCET (Stewart, Schwartz, Bauer, Fleming; Becher, Neubert).
- The same language can be extended to NLP level.
- Originally explored in B physics (Beneke, Campanario, Mannel, Pecjak).
- Recent study for scattering amplitudes (Larkoski, Neill, Stewart).
- Phenomenology explored by Feige, Kolodrubetz, Moult, Stewart, Rothen, Tackmann, Zhu; Boughezal, Liu, Petriello.
- Recent resummation of leading NLP log for some observables (Stewart et. al.).

Factorisation approach

- The soft-collinear factorisation formula can be generalised to next-to-leading power level (Bonocore, Laenen, Magnea, Melville, Vernazza, White).
- This provides a loop-level generalisation of the next-to-soft theorem.
- A new quantity appears at nex-to-soft level: the *jet emission function*.
- Has been calculated at one-loop level for quarks.
- Non-trivial check: reproduces all NLP terms up to NNLO in Drell-Yan.
- Observable loop-level corrections to the tree-level next-to-soft theorem!

Collider Physics - Summary

- Next-to-soft physics has a large number of applications in collider physics.
- Typically this involves summing up large terms in perturbative cross-sections...
- ... or finding approximate forms for fixed-order cross-sections.
- Such calculations improve the precision of theory predictions at the LHC.
- Current data demands this precision!

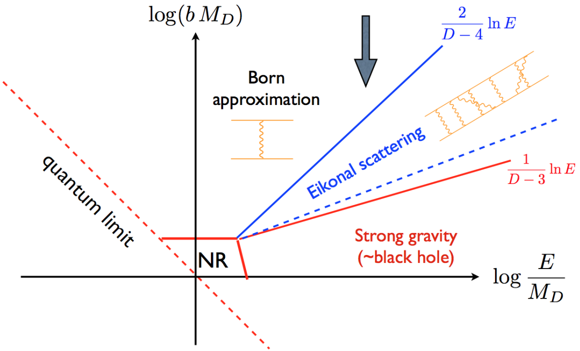
- Much of this conference focuses on relating (next-to) soft physics with asymptotic symmetries in gravity.
- However, (next)-to soft corrections have a different role to play in understanding the conceptual structure of quantum gravity...
- ...and may even have phenomenological consequences!
- More specifically, they are relevant to high energy scattering.
- Many papers from the 1990s onwards (Amati, Ciafaloni, Veneziano, Colferai, Falcioni; 't Hooft; Verlinde^2; Jackiw, Kabat, Ortiz).

- More specifically, we will focus on $2 \to 2$ scattering in the high energy or *Regge* limit

$$s \gg |t|,$$

where s is the squared centre of mass energy, and $|t|$ the momentum transfer.
- Corresponds to scattering above the Planck scale in gravity.
- Naïvely, we might think that non-renormalisability is a problem.
- However, in this limit infinite numbers of *soft* gravitons are exchanged, and the results are well-behaved!

- Can consider different regions in impact parameter b (conjugate to $|t|$), and energy $E \sim \sqrt{s}$:

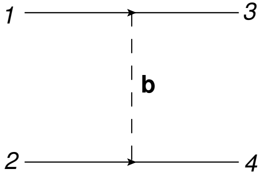


(see e.g. Giddings, Schmidt-Sommerfeld, Andersen).

- Next-to-soft corrections probe unknown parts of this diagram.

- Previous work on this topic focused on gravity only, including possible string theory corrections.
- Recent studies have used QCD methods to analyse gravity scattering (Akhoury, Saotome, Sterman; Melville, Naculich, Schnitzer, White).
- Idea is to develop a common language, that makes the structure of both theories clear.
- Let us look first at QCD...

- The Regge limit can be described by two Wilson lines separated by a transverse distance (Korchemsky, Korchemskaya).
- See also Balitsky; Caron-Huot.



- Take particles of mass m, such that

$$s \gg -t \gg m^2.$$

- $\boldsymbol{b}$ is the (2-d) impact parameter (distance of closest approach).

- In the asymptotic high energy limit, the incoming / outgoing particles follow classical straight line trajectories i.e. they do not recoil.
- The only quantum behaviour they are allowed is to experience a phase change.
- However, gauge-covariance of the amplitude restricts this phase to have the form (for each particle)

$$\mathcal{P} \exp \left[i g_s \int_{\mathcal{C}} dx^\mu \mathbf{A}_\mu(x) \right],$$

  where $\mathcal{C}$ is the spacetime contour of the particle.
- This is a Wilson line!

- The momentum space amplitude is then given by

$$\tilde{\mathcal{A}} = \int d^2\boldsymbol{b} e^{-i\boldsymbol{b}\cdot\boldsymbol{q}} \langle 0|\mathcal{W}(p_1, 0)\mathcal{W}(p_2, z)|0\rangle,$$

  where

$$\mathcal{W}(p, z) = \mathcal{P}\exp\left[ig_s p^\mu \int_{-\infty}^{\infty} ds \mathbf{A}_\mu(sp + z)\right].$$

- The momentum $\boldsymbol{q}$ is conjugate to the impact parameter, and satisfies $t \simeq -\boldsymbol{q}^2$.
- Can now calculate the position space amplitude at one-loop, using dimensional regularisation.

- The answer is (in $d = 4 - 2\epsilon$ dimensions)

$$\mathcal{A}^{(1)} = \frac{g_s^2 \Gamma(1-\epsilon)}{4\pi^{2-\epsilon}} \frac{(\mu^2 \boldsymbol{b}^2)^\epsilon}{2\epsilon} \left[i\pi \mathbf{T}_s^2 + \mathbf{T}_t^2 \log\left(\frac{s}{-t}\right) + \frac{1}{2}\left(\log\left(-\frac{t}{m^2}\right) - i\pi\right) \sum_{i=1}^{4} C_i \right] + \mathcal{O}(\epsilon^0),$$

  where

$$\mathbf{T}_s^2 = (\mathbf{T}_1 + \mathbf{T}_2)^2, \quad \mathbf{T}_t^2 = (\mathbf{T}_1 + \mathbf{T}_3)^2$$

  are quadratic colour operators for pure s- and t-channel exchanges; C_i the quadratic Casimir of particle i.
- From the known properties of Wilson lines, we can immediately exponentiate this!

- One then has

$$\mathcal{A} = \exp\left\{ K \left[i\pi \mathbf{T}_s^2 + \mathbf{T}_t^2 \log\left(\frac{s}{-t}\right)\right] + \ldots \right\}, \quad K = \frac{g_s^2 \Gamma(1-\epsilon)}{4\pi^{2-\epsilon}} \frac{(\mu^2 \boldsymbol{b}^2)^\epsilon}{2\epsilon}$$

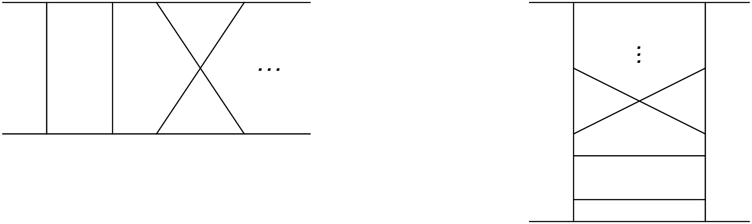
- There are two terms with non-trivial colour dependence:
  - (I) A t-channel term: $\propto \mathbf{T}_t^2 \log(\frac{s}{-t})$.
  - (II) A pure *eikonal phase*: $\propto i\pi \mathbf{T}_s^2$.
- The former is responsible for *Reggeisation* of t-channel exchanges:

$$-\frac{i\eta_{\mu\nu}}{q^2} -> -\frac{i\eta_{\mu\nu}}{q^2}\left(\frac{s}{-t}\right)^\alpha$$

- The latter describes a spectrum of bound states (e.g. positronium).

Eikonal phase and Regge trajectory

- The eikonal phase comes from horizontal (crossed) ladder diagrams, whereas the Regge trajectory comes from vertical ladders.



(a) *(b)*

- In QCD, the vertical ladders dominate.
- It is known that horizontal ladders dominate in gravity: the eikonal phase is enhanced by a factor $s/(-t)$ w.r.t. the Reggeisation term.
- The Wilson line approach gives an elegant view on this.

- First, we need to find appropriate Wilson lines for gravity.
- Here, we mean specifically the operator describing soft graviton emission.
- The relevant quantity has appeared in various places (Brandhuber, Heslop, Spence, Travalgini; Naculich, Schnitzer; White):

$$\exp\left[\frac{i\kappa}{2}\int_{\mathcal{C}} ds\,\dot{x}^{\mu}\,\dot{x}^{\nu}\,h_{\mu\nu}(x)\right].$$

- For straight line contours $x^{\mu} = x_0^{\mu} + p^{\mu}s$, this becomes

$$\exp\left[\frac{i\kappa}{2}\,p^{\mu}\,p^{\nu}\int_{\mathcal{C}} ds h_{\mu\nu}(x)\right].$$

- Closely related to its QCD counterpart!

- The Wilson line approach for the QCD Regge limit (Korchemsky, Korchemskaya) can be ported directly to gravity.
- The momentum space gravity amplitude is given by

$$\tilde{\mathcal{M}} = \int d^2\boldsymbol{b} e^{-i\boldsymbol{b}\cdot\boldsymbol{q}} \langle 0|\mathcal{W}_g(p_1, 0)\mathcal{W}_g(p_2, z)|0\rangle,$$

where

$$\mathcal{W}_g(p, z) = \exp\left[\frac{i\kappa}{2} p^\mu\, p^\nu \int_{-\infty}^{\infty} ds h_{\mu\nu}(sp + z)\right].$$

- Exponentiation of the one-loop calculation can be carried out as before.

- One finds

$$\mathcal{M} = \exp\left\{ -K_g (\mu^2 \boldsymbol{b}^2)^{\epsilon} \left[i\pi s + t \log\left(\frac{s}{-t} \right) \right] + \mathcal{O}(\epsilon^0) \right\},$$

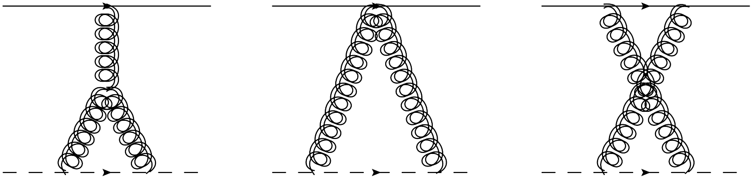
$$K_g = \left(\frac{\kappa}{2}\right)^2 \frac{\Gamma(1-\epsilon)}{8\pi^{2-\epsilon}}.$$

- The eikonal phase wins as $\frac{s}{-t} \to \infty$, in contrast to QCD.
- However, the structure of the result is basically the same, and can be obtained by the procedure

$$g_s \to \frac{\kappa}{2}; \quad \mathbf{T}^2_{s,t} \to s, t; \quad C_i \to 0.$$

- This is the BCJ double copy! (see also Akhoury, Saotome; Sabio Vera, Campillo, Vazquez-Mozo, Johansson).

- Diagrammatic study of Regge limit by Akhoury, Saotome, Sterman.



- Considered a light particle scattering on a black hole.
- Next-to-soft corrections lead to a modifed eikonal phase:

$$\chi \to \chi_{\mathrm{E}} + \chi_{\mathrm{NE}},$$

where $\chi_{\mathrm{NE}} \propto R_s$ (Schwarzschild radius of black hole).
- Correction corresponds to deflection angle of light particle (see also D'Appollonio, Di Vecchia, Russo, Veneziano; Bjerrum-Bohr, Donoghue, Holstein, Plante, Vanhove).

Wilson line approach

- Can also extend the Wilson line approach to next-to-soft level (Lanen, Stavenga, White).
- Has been applied to the Regge limit in both QCD and gravity (Luna, Melville, Naculich, White).
- General case of two massive particles.
- In QCD, get a power-suppressed correction to the Regge trajectory of the gluon.
- In gravity, the correction to the NE phase corresponds to two simultaneus deflection angles for the colliding particles (as conjectured by Andersen, Schmidt-Sommerfeld, Giddings).
- Previous results of Akhoury, Saotome, Sterman emerge as a special case.

- Next-to-soft corrections are relevant to transplanckian scattering in gravity.
- More generally, similar methods can be applied to understand radiation from scattering black holes.
- Full solutions for colliding shockwaves / black holes are not always known.
- The next-to-soft calculation allows us to build them up perturbatively i.e. order-by-order in the deflection angle.
- Methods exist for relating QCD and gravity results.

- (Next-to)-soft physics has a large number of applications, in different areas of physics.
- For hep-ph, hep-ex: increased precision for collider observables.
- For hep-th, gr-qc: transplanckian scattering in gravity, radiation in black hole scattering.
- Common languages for QCD and gravity (e.g. Wilson lines) make underlying structures / common behaviour clearer.

Open questions

- Can we resum next-to-leading power (NLP) threshold logs?
- Other applications in precision physics?
- Do next-to-soft methods help in calculating radiation from scattering black holes?
- What are gravitational Wilson lines useful for?
- What does anything in this talk have to do with BMS symmetry?

Thanks for listening!