Hidden Conformal Symmetry in Tree-Level Graviton Scattering

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based on *arXiv: 1802.05999*
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Graviton Scattering

Recent years show explosion of insights and interest in graviton scattering, e.g.

- BCJ Double Copy
- CHY and ambitwistor string
- BMS symmetries
- Gravitational wave physics ...

What is the symmetry of the quantum gravity S-matrix?

Yang–Mills vs Gravity:

- The actions of Yang–Mills theory and Einstein gravity look rather different.
- Perturbative quantization through gluons and gravitons shows striking similarities in amplitudes.

Conformal Symmetry of Gluon Amplitudes

4d Yang–Mills action invariant under conformal transformations:

$$\mathcal{S}_{\mathsf{YM}} = \int \mathrm{d}^4x \; \mathrm{tr}\, F^{\mu\nu} F_{\mu\nu}$$

Color-stripped gluon tree amplitudes:

$$A_n^\Sigma(k_i, \epsilon_i, a_i) = \sum_{\text{noncyc. perm}} \mathrm{Tr}\left[T^{a_1} T^{a_2} \dots T^{a_n}\right] A_n(k_i, \epsilon_i)$$

p momenta
ϵ polarizations
a colors

Momenta/polarizations (k, ϵ) traded for spinors/helicities (λ, $\bar{\lambda}$, h):

$$A_n(k_i, \epsilon_i) \to \quad A_n(\lambda_i, \bar{\lambda}_i, h_i).$$

Spinors λ^α, $\bar{\lambda}^{\dot{\alpha}}$ solve on-shell constraints:

$$k^{\alpha\dot{\alpha}} = k^\mu (\sigma_\mu)^{\alpha\dot{\alpha}} = \lambda^\alpha \bar{\lambda}^{\dot{\alpha}} \quad \Rightarrow \quad k^\mu k_\mu = 0$$

Conformal Ward identities: $\mathrm{J}^a A_n(\lambda_i, \bar{\lambda}_i, h_i) = 0$ [Witten 2003]

$$\mathrm{J}^a \in \begin{cases} \mathrm{P}^{\alpha\dot{\beta}} = \lambda^a \bar{\lambda}^{\dot{\beta}}, & \mathrm{L}^\alpha{}_\beta = \lambda^\alpha \partial_\beta - \frac{1}{2}\delta^\alpha_\beta \lambda^\gamma \partial_\gamma, \\ \mathrm{D} = \frac{1}{2}\partial_\gamma \lambda^\gamma + \frac{1}{2}\bar{\lambda}^{\dot{\gamma}} \bar{\partial}_{\dot{\gamma}}, & \mathrm{K}_{\alpha\dot{\beta}} = \partial_\alpha \bar{\partial}_{\dot{\beta}}. \end{cases}$$

Conformal Symmetry in Graviton Scattering?

Hints on hidden symmetries in graviton scattering:

- 4d YM trees are conformal and

$$\mathrm{YM}^2 \simeq \mathrm{Gravity}$$ [Kawai, '86 Lewellen, Tye] [Bern, Carrasco, Johansson '08] [...]

  Imprint of conformal symmetry on graviton amplitudes?

- In curved space at tree level:

$$\text{Einstein gravity} \simeq \text{conformal gravity}$$ [Metsaev 2011] [Maldacena 2011] [Adamo Mason 2013]

- Graviton amplitudes as conformal correlators on the celestial sphere [Cachazo Strominger 2014] [...]

  What is the role of conformal symmetry for graviton scattering?

Einstein Gravity

Conformal symmetry of graviton amplitudes?:

$$\mathcal{S}_{\mathsf{EH}} = \frac{1}{\kappa_d^2}\int \mathrm{d}^d x \sqrt{-g}\, R, \qquad\qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa_d h_{\mu\nu}.$$

Naive answer is **no**:

- coupling constant κ_d^2 has mass dimension $2-d$.
- forbids naive conformal symmetry for $d \neq 2$.
- so if there is a conformal symmetry, it must be hidden.

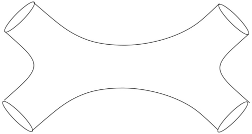
This talk:

- Motivation: Get inspiration from string soft limits of [Di Vecchia, Marotta Mojaza '15-'17].
- Method: Experiments with conformal generators and field theory graviton amplitudes.
- Conclusion: Tree-level graviton amplitudes have conformal symmetry in d dimensions.

Soft Dilatons and Graviton Scattering

Inspiration from Strings

In the $\alpha' \to 0$ limit, scattering of closed strings yields graviton field theory amplitude:



$$\xrightarrow{\alpha' \to 0}$$

Eventually we will be only interested in the field theory limit.

String Theory Soft Dilatons

Special case of string theory soft limit of [Di Vecchia, Marotta Mojaza '15-'17]: [see Paolo's talk]

Amplitude for n hard gravitons or dilatons and a soft dilaton ϕ with small momentum q:

$$\mathcal{M}^{\alpha'}_{n+\phi}(k_1,..,k_n,q) = \tfrac{\kappa_d}{\sqrt{d-2}}\Big[\delta(P)S_\delta + \big(q\cdot\partial_P\delta(P)\big)S_{\delta'} \\ + (S_W+S_V)\delta(P)\Big]\widetilde{M}^{\alpha'}_n + \mathcal{O}(q^2),$$

- $\mathcal{M}_n = \delta(P)\widetilde{M}^{\alpha'\to 0}_n$ is the field theory amplitude.
- Conformal generators enter:

$$S_\delta = 2 - \mathrm{D}_{\Delta=0} + q_\mu \mathrm{K}^\mu_{\Delta=0}, \qquad S_{\delta'} = 2 - \mathrm{D}_{\Delta=0}.$$

- as well as certain nonlocal operators (gauge transf. $W_i = k_i\cdot\partial_{\epsilon_i}$):

$$S_W = -\sum_{i=1}^n \frac{q\cdot\epsilon_i}{k_i\cdot q}\big(1+q\cdot\partial_{k_i}\big)W_i, \quad S_V = \sum_{i=1}^n \frac{q_\rho q^\sigma}{2k_i\cdot q}\big(S_i^{\rho\mu}S_{i,\mu\sigma} + d\,\epsilon_i^\rho\partial_{\epsilon_i,\sigma}\big).$$

Understand this formula in detail?

The Operator S_W

Contribution to the soft expansion:

$$\mathcal{M}^{\alpha'}_{n+\phi}(k_1,..,k_n,q) = \tfrac{\kappa_d}{\sqrt{d-2}}\Big[\cdots + S_W\,\delta(P)\widetilde{M}^{\alpha'}_n\Big] + \mathcal{O}(q^2),$$

$$S_W = -\sum_{i=1}^{n} \frac{q\cdot\epsilon_i}{k_i\cdot q}(1+q\cdot\partial_{k_i})\,W_i, \qquad\qquad W_i = k_i\cdot\partial_{\epsilon_i}.$$

- If one assumes that the amplitude is *manifestly* gauge invariant, the contribution from S_W could be dropped [Di Vecchia, Marotta Mojaza '16]
  But: manifestly gauge invariant form not known for $n > 4$

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- If one assumes that the amplitude is *manifestly* gauge invariant, the contribution from S_W could be dropped
  But: manifestly gauge invariant form not known for $n>4$
- Alternative argument: Write

$$-\sum_{i=1}^{n}\frac{q\cdot\epsilon_i}{k_i\cdot q}(k_i+q)\cdot\partial_{\epsilon_i}\mathcal{M}^{\alpha'}_n(k_1,\ldots,k_i+q,\ldots,k_n)+\mathcal{O}(q^2) = 0+\mathcal{O}(q^2).$$

  Note: requires reordering expansion in q!

Thus we assume that S_W can be dropped.

The Operator S_V

Contribution to the soft expansion:

$$\mathcal{M}^{\alpha'}_{n+\phi}(k_1,..,k_n,q) = \tfrac{\kappa_d}{\sqrt{d-2}}\left[\cdots + S_V \delta(P) \widetilde{M}^{\alpha'}_n\right] + \mathcal{O}(q^2),$$

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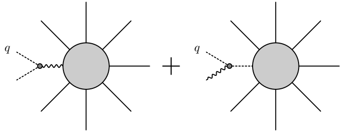
Use quadratic dependence of amplitude on polarizations:

$$\sum_{i=1}^{n} \frac{q_\rho q^\sigma}{2k_i \cdot q}\left(S_i^{\rho\mu} S_{i,\mu\sigma} + d\epsilon_i^\rho \partial_{\epsilon_i \sigma}\right)\delta(P)\left(\epsilon_i^\alpha \epsilon_i^\beta \widetilde{M}^{\alpha'}_{n,i,\alpha\beta}\right)$$

$$= \delta(P)\sum_{i=1}^{n}\left[\frac{q^\alpha q^\beta(\epsilon_i\cdot\epsilon_i)}{k_i\cdot q} + \frac{\eta^{\alpha\beta}(q\cdot\epsilon_i)^2}{k_i\cdot q}\right]\widetilde{M}^{\alpha'}_{n,i,\alpha\beta}.$$

- Interpretation of two terms: 

  since for the graviton $\epsilon_i \cdot \epsilon_i = 0$, while for the dilaton $\epsilon_i^\mu \epsilon_i^\nu \sim \eta^{\mu\nu}$.

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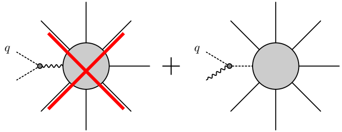

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What Remains?

Amplitude for n gravitons and a soft dilaton ϕ with small momentum q:

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$$+\tfrac{\kappa_d}{\sqrt{d-2}}\sum_{i=1}^{n}\frac{(q\cdot\epsilon_i)^2}{k_i\cdot q}\delta(P)M^{\alpha'}_n(k_1,\ldots,\phi(k_i),\ldots,k_n)+\mathcal{O}(q^2)\,,$$

Low engergy string action:

$$S_{\text{low-energy}}=\frac{1}{2\kappa_d^2}\int d^dx\sqrt{-g}\left[R-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\frac{1}{12}\mathrm{e}^{-\sqrt{\frac{8}{d-2}}\phi}\left(\partial_{[\mu}B_{\nu\rho]}\right)^2\right]\,.$$

- Dilaton and B-field couple only quadratically to the graviton.
- No virtual B-fields in graviton/dilaton amplitudes at tree level.

$\Rightarrow$ In the field theory limit $\alpha'\to 0$, tree-level amplitudes with n gravitons and an *odd* number of dilatons vanish!

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Conformal Symmetry?

Thus we are left with

$$0 = \Big[\delta(P)S_\delta + \big(q\cdot\partial_P\delta(P)\big)S_{\delta'}\Big]\widetilde{M}_n^{\alpha'\to 0} + \mathcal{O}(q^2).$$

with conformal generators entering as:

$$S_\delta = 2 - \mathrm{D}_{\Delta=0} + q_\mu \mathrm{K}^\mu_{\Delta=0}, \qquad\qquad S_{\delta'} = 2 - \mathrm{D}_{\Delta=0}.$$

If $\widetilde{M}_n^{\alpha'\to 0}$ were the field theory n-graviton amplitude, this would imply conformal invariance! But. . .

Amplitude vs Integral

...a priori $\widetilde{M}$ is not the amplitude:

$\widetilde{M}_n^{\alpha'\to 0}$ becomes the field theory amplitude if multiplied by $\delta(P)$.

To be more precise: The n-point bosonic string amplitude of massless closed states carrying momenta k_i and polarizations $\epsilon_i, \bar{\epsilon}_i$ is given by

$$\mathcal{M}_n^{\alpha'} = \delta(P)\widetilde{M}_n^{\alpha'},$$

with the integral

$$\begin{aligned}\widetilde{M}_n^{\alpha'} =& \frac{8\pi}{\alpha'}\left(\frac{\kappa_d}{2\pi}\right)^{n-2}\int \frac{\prod_{i=1}^n d^2 z_i}{dV_{abc}}\prod_{i<j}|z_i - z_j|^{\alpha' k_i k_j}\\ &\times \int\prod_{i=1}^n d\theta_i \exp\Bigg[\sum_{i<j}\frac{(\theta_i\epsilon_i)\cdot(\theta_j\epsilon_j)}{(z_i-z_j)^2} + \sqrt{\frac{\alpha'}{2}}\sum_{i\neq j}\frac{(\theta_i\epsilon_i)\cdot k_j}{z_i - z_j}\Bigg]\\ &\times \int\prod_{i=1}^n d\bar{\theta}_i \exp\Bigg[\sum_{i<j}\frac{(\bar{\theta}_i\bar{\epsilon}_i)\cdot(\bar{\theta}_j\bar{\epsilon}_j)}{(\bar{z}_i-\bar{z}_j)^2} + \sqrt{\frac{\alpha'}{2}}\sum_{i\neq j}\frac{(\bar{\theta}_i\bar{\epsilon}_i)\cdot k_j}{\bar{z}_i - \bar{z}_j}\Bigg],\end{aligned}$$

and $\qquad dV_{abc} = \frac{d^2 z_a\, d^2 z_b\, d^2 z_c}{|z_a - z_b|^2 |z_b - z_c|^2 |z_c - z_d|^2}, \qquad a, b, c \in \{1, \ldots, n\}$.

So what does this mean for field theory amplitudes?

Implications for Field Theory Amplitudes

Implications for Field Theory Amplitude

Consider soft-dilaton result order by order:

$$0 = \delta(P)(2-\mathrm{D}_{\Delta=0})\widetilde{M}_n^{\alpha'\to 0} + \left[\delta(P) q_\mu \mathrm{K}^\mu_{\Delta=0} + q\cdot\partial_P\delta(P)(2-\mathrm{D}_{\Delta=0})\right]\widetilde{M}_n^{\alpha'\to 0} + \mathcal{O}(q^2)\,.$$

Leading order:

$$\mathcal{O}(q^0): \qquad 0 = \mathrm{D}_{\Delta=\frac{d-2}{n}} \underbrace{\delta(P)M_n}_{\text{amplitude}}, \quad \text{with} \quad \mathrm{D}_{i,\Delta} = k_i\cdot\partial_{k_i} + \Delta$$

Next-to-leading order: Commute K^μ through $\delta(P)$:

$$\mathcal{O}(q^1): \qquad 0 = q_\mu \mathrm{K}^\mu_{\Delta=\frac{d-2}{n}} \underbrace{\delta(P)M_n}_{\text{amplitude}} - \frac{d-2}{n}\delta(P)\sum_{i=1}^n q\cdot\partial_{k_i}\widetilde{M}_n^{\alpha'\to 0},$$

with the special conformal generator:

$$\mathrm{K}^\mu_{i,\Delta} = \tfrac{1}{2}k_i^\mu\partial_{k_i}^2 - (k_i\cdot\partial_{k_i})\partial_{k_i}^\mu - \Delta\partial_{k_i}^\mu - iS_i^{\mu\nu}\partial_{k_i,\nu}, \quad S_j^{\mu\nu} = i\big(\epsilon_j^\mu\partial_{\epsilon_j}^\nu - \epsilon_j^\nu\partial_{\epsilon_j}^\mu\big)$$

$\Rightarrow$ Additional term drops out for $d=2$ and $\Delta=0$.

How about $d\neq 2$? Try!

Field (Theory) Work: Act on Tree Amplitudes

Special conformal generator:

$$\mathrm{K}^\mu_\Delta = \tfrac{1}{2} k^\mu \partial_k^2 - (k\cdot\partial_k)\partial_k^\mu - \Delta\partial_k^\mu - iS^{\mu\nu}\partial_{k,\nu}$$

Constraints:

- Momenta and polarization vectors obey

$$k_i \cdot k_i = 0, \qquad\qquad k_i\cdot\epsilon_i = 0\,.$$

- and amplitudes are gauge invariant:

$$\mathrm{W}_i \mathcal{A}_n = 0\,, \qquad\qquad \mathrm{W}_i = k_i\cdot\partial_{\epsilon_i}\,.$$

Special conformal generator does not commute with constraints:

$$\begin{aligned}
[\mathrm{K}^\mu_\Delta, k_i^2] &= (d-2-2\Delta)k_i^\mu + 2\epsilon_i^\mu \mathrm{W}_i,\\
[\mathrm{K}^\mu_\Delta, k_i\cdot\epsilon_i] &= \epsilon_i^\mu[(d-1-\Delta)+\epsilon_i\cdot\partial_{\epsilon_i}].\\
[\mathrm{K}^\mu_\Delta, \mathrm{W}_i] &= -\partial^\mu_{k_i} W_i + (1-\Delta)\partial^\mu_{\epsilon_i} - iS_i^{\mu\nu}\partial_{\epsilon_i,\nu}\,,\\
[\mathrm{K}^\mu_\Delta, \delta(P)] &= \frac{\partial\delta(P)}{\partial P^\nu}\big[(d-\mathrm{D}_\Delta)\eta^{\mu\nu} + \mathrm{J}^{\mu\nu}\big]
\end{aligned}$$

$\Rightarrow$ K^μ_Δ takes us off the kinematical constraint surface.

Back to Yang–Mills Amplitudes

Act on d-dimensional tree amplitudes in $\{k_i, \epsilon_i\}$ by [Bourjaily][Mafra Schlotterer].

Poincaré symmetry (modulo gauge transformations) dictates:

δ-stripped amplitude

$$\mathrm{K}^\mu_\Delta\, A_n = \sum_{i=1}^{n} \epsilon_i^\mu F_i + \sum_{i=1}^{n} k_i^\mu G_i,$$

We explicitly resolve on-shell constraints and momentum conservation:

$$k_a^\mu = -\sum_{i\neq a}^{n} k_i^\mu\,, \quad k_a^2 = \sum_{i\neq j\neq a}^{n} k_i \cdot k_j = 0\,, \quad \epsilon_a \cdot k_a = -\epsilon_a \cdot \sum_{i\neq a}^{n} k_i = 0\,,$$

and set the scaling dimension to $\Delta = \frac{d-2}{2}$ in four dimensions:

$$\mathrm{K}^\mu_\Delta\, A_n \overset{\Delta=1}{=} \sum_{i=1}^{n} \epsilon_i^\mu F_i \neq 0.$$

$\Rightarrow$ Even Yang–Mills amplitudes in $4d$ are not invariant in (k, ϵ)-space.

Yang–Mills Amplitudes in (k, ϵ)-Space

Note: Cyclic and reversal symmetry obeyed by YM amplitudes *only* modulo on-shell constraints:

$$A_n(1,2,\ldots,n) = A_n(2,\ldots,n,1),$$
$$A_n(1,2,\ldots,n) = (-1)^n A_n(n,\ldots,2,1).$$

Make symmetries manifest:

$$\mathcal{C}_n[A_n] := \tfrac{1}{2n} \sum_{\mathsf{Cyc}(1,2,\ldots,n)} \left[A_n(1,2,\ldots,n) + (-1)^n A_n(n,\ldots,2,1)\right] \; .$$

On shell we have $\mathcal{C}_n[A_n] = A_n$ but now (checked for $n \leq 6$):

$$\mathrm{K}^\mu_\Delta \mathcal{C}_n[A_n] = 0, \qquad \text{for} \quad \Delta = 1.$$

Observation: Manifest symmetries restore conformal properties!

(Pathological) Three-Point Example

The three-point delta-stripped YM amplitude takes the form

$$A_3 = (\epsilon_1 \cdot k_2)e_{23} - (\epsilon_2 \cdot k_1)e_{31} - (\epsilon_3 \cdot k_2)e_{12}\,, \qquad e_{ij} = \epsilon_i \cdot \epsilon_j,$$

with explicit implementation of momentum conservation using $k_3 = -k_1 - k_2$ and $(\epsilon_3 \cdot k_1) = -(\epsilon_2 \cdot k_2)$.

One finds

$$\mathrm{K}^\mu_{\Delta=1} A_3 = e_{12}\epsilon_3^\mu - e_{23}\epsilon_1^\mu + e_{31}\epsilon_2^\mu \neq 0,$$

Cyclic and reversal symmetrized form of amplitude

$$\mathcal{C}_3[A_3] = \tfrac{1}{2}\epsilon_1 \cdot (k_2 - k_3)\, e_{23} + \tfrac{1}{2}\epsilon_2 \cdot (k_3 - k_1)\, e_{31} + \tfrac{1}{2}\epsilon_3 \cdot (k_1 - k_2)\, e_{12}\,,$$

is annihilated:

$$\mathrm{K}^\mu_\Delta\, \mathcal{C}_3[A_3] = 0.$$

Works also for $n = 4, 5, 6$ for $\Delta = 1$ (for $n = 3$ Δ is actually arbitrary).

Gravity Tree-Amplitudes

Scattering data from Yang–Mills via KLT:

$$
\begin{aligned}
M_3(1,2,3) &= iA_3(1,2,3)A_3(1,2,3)\,, \\
M_4(1,2,3,4) &= -is_{12}A_4(1,2,3,4)A_4(1,2,4,3)\,, \\
M_5(1,2,3,4,5) &= is_{12}s_{34}A_5(1,2,3,4,5)A_5(2,1,4,3,5) \\
&\quad + is_{13}s_{24}A_5(1,3,2,4,5)A_5(3,1,4,2,5)\,, \\
M_6(1,2,3,4,5,6) &= -is_{12}s_{45}A_6(1,2,3,4,5,6)\Big[s_{35}A_6(2,1,5,3,4,6) \\
&\quad + (s_{34}+s_{35})A_6(2,1,5,4,3,6)\Big] + \mathcal{P}(2,3,4)\,.
\end{aligned}
$$

Similar situation as for Yang–Mills amplitudes:

δ-stripped amplitude

canonical dimension in 2d

$$
\mathrm{K}^\mu_\Delta M_n = \sum_{i=1}^n \epsilon_i^\mu F_i + \sum_{i=1}^n k_i^\mu G_i \overset{\Delta=0}{=} \sum_{i=1}^n \epsilon_i^\mu F_i \neq 0.
$$

Gravity Tree-Amplitudes

Note: Full permutation symmetry obeyed by graviton amplitudes *only* modulo on-shell constraints:

$$M_n(1,2,\ldots,n) = M_n(\mathcal{P}(1,2,\ldots,n)).$$

Make symmetries manifest:

$$\mathcal{P}_n[M_n(1,\ldots,n)] = \frac{1}{n!}\sum_{\mathcal{P}(1,\ldots,n)} M_n(1,\ldots,n).$$

We have on shell that $\mathcal{P}_n[M_n] = M_n$ but now (checked for $n \leq 6$):

$\mathrm{K}^{\mu}_{\Delta}\mathcal{P}_n[M_n] = 0,$ for any value of Δ.

Conformal Symmetry of Graviton Amplitudes

Special conformal generator:

$$\mathrm{K}^\mu_\Delta = \tfrac{1}{2} k^\mu \partial_k^2 - (k \cdot \partial_k)\partial_k^\mu - i S^{\mu\nu}\partial_{k,\nu} - \Delta \partial_k^\mu .$$

Independence of Δ means that

$$\sum_{i=1}^{n} \partial^\mu_{k_i} \mathcal{P}_n[M_n] = 0.$$

Consistent with soft-dilaton statement:

$$0 = q_\mu \mathrm{K}^\mu_{\Delta = \frac{d-2}{n}} \delta(P) M_n - \tfrac{d-2}{n} \delta(P) \sum_{i=1}^{n} q \cdot \partial_{k_i} \widetilde{M}_n^{\alpha' \to 0},$$

Hence, the manifestly symmetric amplitude $\mathcal{P}_n[M_n]$ furnishes a representation such that

$$\mathrm{K}^\mu_{\Delta = \frac{d-2}{n}} \delta(P) \mathcal{P}_n[M_n] = 0, \qquad \mathrm{D}^\mu_{\Delta = \frac{d-2}{n}} \delta(P) \mathcal{P}_n[M_n] = 0.$$

Together with translations P^μ and Lorentz rotations $\mathrm{L}^{\mu\nu}$ this yields full conformal invariance.

Summary

Invariance of full amplitude $\mathcal{A}_n = \delta^{(d)}(P) A_n$ under dilatations D_Δ and vanishing commutator of K_Δ with $\delta^{(d)}(P)$ requires

$$\text{YM}: \Delta = \frac{d-4}{n} + 1, \qquad \text{Gravity}: \Delta = \frac{d-2}{n}.$$

Poincaré invariance implies $\mathrm{K}^\mu_\Delta A_n = \sum_{i=1}^n \epsilon_i^\mu F_i + \sum_{i=1}^n k_i^\mu G_i$.

- YM: $G_i = 0$ for $\Delta = 1$.
- Gravity: $G_i = 0$ for $\Delta = 0$.

Non-zero F_i related to incompatibility of K^μ_Δ with constraints.

Make the physical symmetries, i.e. cyclic/reversal or full permutation symmetry, manifest such that (checked for $n \leq 6$)

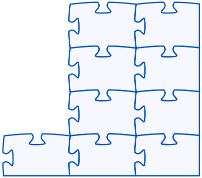
$$\left.\begin{array}{l} \text{YM: in } d=4 \text{ and for } \Delta = 1 \text{ we find } F_i = 0 \\ \text{Gravity: for any value of } d \text{ and } \Delta \text{ we find } F_i = 0 \end{array}\right\} \mathrm{K}^\mu_\Delta A_n = 0$$

Conclusion:

Graviton scattering shows conformal symmetry at tree level.

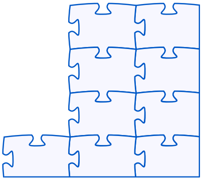
Puzzles for the Future

- Meaning of the d-dimensional conformal symmetry?
- Derivation from the Einstein–Hilbert action? Twistor methods?
- Relation to CHY, double copy, celestial sphere?
- Similar symmetry for d-dimensional Yang–Mills theory?
- Use the new symmetry?



Puzzles for the Future

- Meaning of the d-dimensional conformal symmetry?
- Derivation from the Einstein–Hilbert action? Twistor methods?
- Relation to CHY, double copy, celestial sphere?
- Similar symmetry for d-dimensional Yang–Mills theory?
- Use the new symmetry?



Thank you!