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A new approach to quark masses using gradient flow

Standard Model parameters and observables from gradient flow – Edinburgh, Scotland

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Fabian Lange

in collaboration with Hiromasa Takaura and Robert Harlander

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Quark masses from the lattice

- Precise determinations of quark masses essential ingredient in precision tests of the Standard Model, entering e.g. Higgs physics and flavor observables
- On the lattice, determinations performed in intermediate renormalisation schemes followed by perturbative matching to $\overline{\text{MS}}$:
  - RI-(S)MOM schemes [Martinelli, Pittori, Sachrajda, Testa, Vladikas 1994; Aoki et al. 2007; Sturm, Aoki, Christ, Izubuchi, Sachrajda, Soni 2009]
  - Quark-current (or current-like) correlator method [HPQCD collaboration 2008]
  - Minimal renormalon-subtracted scheme [TUMQCD collaboration 2017]

⇒ Many results reported in FLAG [Flavour Lattice Averaging Group 2024]

- Use gradient flow? ⇒ also see talks by Oliver Witzel and Akhil Chauhan

Our suggestion

- Consider vacuum expectation values of flowed quark bilinears

$$S(t) = \langle \bar{\chi}(t,x)\chi(t,x)\rangle,$$
$$R(t) = \left\langle \bar{\chi}(t,x)\overleftrightarrow{\not{D}}\chi(t,x)\right\rangle$$

- Dimensional analysis implies

$$S(t) \sim \frac{m}{t} f_S(z,\alpha_\mathrm{s}),$$
$$R(t) \sim \frac{1}{t^2} f_R(z,\alpha_\mathrm{s}),$$

where $z = m^2 t$

- Wavefunction renormalisation cancels in suitable ratios [Lüscher 2013]

⇒ Compute, e.g.,

$$\frac{S_\mathrm{lat}(t)}{R_\mathrm{lat}(t)} = \frac{S_\mathrm{pert}(t)}{R_\mathrm{pert}(t)} = mt\frac{f_S(z,\alpha_\mathrm{s})}{f_R(z,\alpha_\mathrm{s})}$$

and extract mass

- Similar to definition of α_s through $\langle G_{\mu\nu}(t,x)G_{\mu\nu}(t,x)\rangle$ [Lüscher 2010]

Expected properties

$$S(t) = \langle \bar{\chi}(t,x)\chi(t,x)\rangle,$$
$$R(t) = \left\langle \bar{\chi}(t,x)\overleftrightarrow{\not{D}}\chi(t,x)\right\rangle$$

- One-point functions
- Ratios scheme independent
- Gauge invariant
- Straightforward to obtain $\overline{\mathrm{MS}}$ mass
- Non-perturbative corrections start at mass-dimensions three and four, respectively, but can be suppressed (more later)

⇒ Good for mass determination?

Status and mass range

- Perturbatively known only in small-mass limit $z = m^2 t \ll 1$:

$$S(t) = -\frac{N_c m}{8\pi^2 t}\left[s_0(z) + \frac{\alpha_s}{4\pi}s_1(z=0) + \left(\frac{\alpha_s}{4\pi}\right)^2 s_2(z=0) + \mathcal{O}(\alpha_s^3) + \mathcal{O}(z\alpha_s)\right],$$

$$R(t) = -\frac{2N_c}{(4\pi t)^2}\left[(1 + zr_{01}) + \frac{\alpha_s}{4\pi}(r_{10} + zr_{11}) + \left(\frac{\alpha_s}{4\pi}\right)^2 (r_{20} + zr_{21}) + \mathcal{O}(\alpha_s^3) + \mathcal{O}(z^2)\right]$$

[Lüscher 2013; Makino, Suzuki 2014; Artz, Harlander, **FL**, Neumann, Prausa 2019; **FL** PhD 2021]

- However, assuming typical window to ensure suppression of lattice artifacts and perturbative control:

$$a^2 \ll 8t \ll \Lambda_{\mathrm{QCD}}^{-2}$$

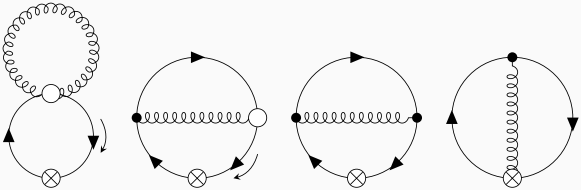
$$\Rightarrow \qquad 0.1 \ll 8m_c^2 t \ll 20 \qquad \text{and} \qquad 1.0 \ll 8m_b^2 t \ll 200$$

⇒ Need full mass-dependence:
  - Two loop: small- and large-mass expansions and numerically [Takaura, Harlander, **FL** 2025]

    ⇒ this talk
  - Three loop: numerically [Harlander, Mason 2025] ⇒ see Robert Mason's talk

Perturbative calculation

- Follow (first part of) standard perturbative approach for gradient flow [Lüscher, Weisz 2011; Artz, Harlander, **FL**, Neumann, Prausa 2019] $\Rightarrow$ see Robert Harlander's talk:
  - Generate all diagrams



  - Insert Feynman rules
  - Dirac and colour algebra
- Typical integral to be computed:

$$I(z) \sim \int_p \int_q \frac{e^{-2t(p^2+q^2)}}{(p^2+m^2)((p-q)^2)(q^2+m^2)},$$

Method: Laplace transform

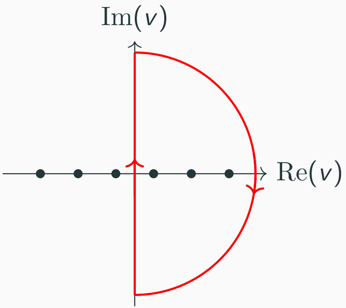
- Perform Laplace transform:

$$\tilde{I}(v) \equiv \int_0^\infty \mathrm{d}z\, z^{-v-1} I(z)$$

- Inverse transformation:

$$I(z) = \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}v\, \tilde{I}(v)\, z^v$$

- Manifests expansion terms as poles in the v-plane



- Use Cauchy's residue theorem:
  - For $z = m^2 t \ll 1$ close contour on the right:

$$I(z \ll 1) = -\sum_{v_{\mathrm{sing}}>0} \mathrm{Res}[\tilde{I}(v) z^v]|_{v=v_{\mathrm{sing}}}$$

  - For $z = m^2 t \gg 1$ close contour on the left:

$$I(z \gg 1) = \sum_{v_{\mathrm{sing}}<0} \mathrm{Res}[\tilde{I}(v) z^v]|_{v=v_{\mathrm{sing}}}$$

- Similar approach as in [Neubert 1994; Kitano, Takaura 2022]

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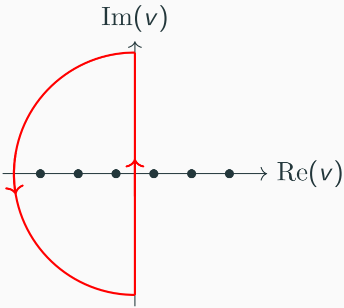

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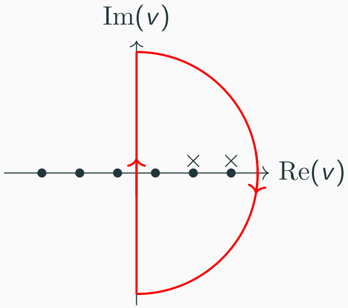
One-loop example

- Full contribution at one loop:

$$S|_{1-\text{loop}} = -4N_c \int_p \frac{m}{m^2+p^2} e^{-2tp^2} \equiv -4N_c\, t^{1/2-d/2}\, I(z)$$

- Laplace transform:

$$\tilde{I}(v) = t^{d/2} \int_p e^{-2tp^2} \int_0^\infty \mathrm{d}z\, \frac{z^{-v-1/2}}{z+tp^2} = \frac{\pi}{\cos(\pi v)} \frac{1}{(4\pi)^{2-\epsilon}} \frac{\Gamma(3/2-\epsilon-v)}{\Gamma(2-\epsilon)} 2^{v-3/2+\epsilon}$$



- For $z = m^2 t \ll 1$ $(v > 0)$ singularities at

$$v = n - \frac{1}{2} \qquad \text{and} \qquad v = n + \frac{1}{2} - \epsilon\,, \qquad n \in \mathbb{N}$$

Summing residues – small-mass expansion

- They give

$$-\mathrm{Res}[\tilde{I}(v)z^v] = \begin{cases} \dfrac{z^{1/2}}{32\pi^2} + \mathcal{O}(\epsilon)\,, & v = 1/2 \\ \dfrac{z^{3/2}}{16\pi^2}\left(\dfrac{1}{\epsilon} + 1 + 3\log 2 + \log\pi\right) + \mathcal{O}(\epsilon)\,, & v = 3/2 \\ -\dfrac{z^{3/2-\epsilon}}{16\pi^2}\left(\dfrac{1}{\epsilon} + 1 - \gamma_\mathrm{E} + \log(4\pi)\right) + \mathcal{O}(\epsilon)\,, & v = 3/2-\epsilon \\ \qquad\vdots & \end{cases}$$

- $\frac{1}{\epsilon}$-poles cancel and the full expansion reads

$$S|_{1-\mathrm{loop}} = -\frac{N_\mathrm{c}}{8\pi^2}\frac{m}{t}\left[1 + 2m^2t\,L_{mt} + 4(m^2t)^2(-1+L_{mt}) + \mathcal{O}((m^2t)^3)\right]$$

with $L_{mt} = \log(2m^2t) + \gamma_\mathrm{E}$

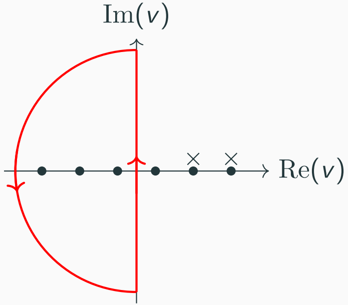
Summing residues – large-mass expansion

$$\tilde{I}(v) = t^{d/2} \int_p e^{-2tp^2} \int_0^\infty dz \, \frac{z^{-v-1/2}}{z + tp^2} = \frac{\pi}{\cos(\pi v)} \frac{1}{(4\pi)^{2-\epsilon}} \frac{\Gamma(3/2-\epsilon-v)}{\Gamma(2-\epsilon)} 2^{v-3/2+\epsilon}$$

- For $z = m^2 t \gg 1$ $(v < 0)$ singularities at

$$v = -n + \frac{1}{2}\,, \qquad n \in \mathbb{N}$$

- No $\frac{1}{\epsilon}$-poles



- Summing residues:

$$S|_{1-\text{loop}} = -\frac{N_c}{16\pi^2} \frac{1}{mt^2} \left[1 - \frac{1}{m^2 t} + \frac{3}{2} \frac{1}{(m^2 t)^2} + \mathcal{O}((m^2 t)^{-3}) \right]$$

Small-flow-time expansion

- Small-flow-time expansion for operators in terms of matching coefficients $\zeta_X(t)$ and unflowed operators [Lüscher, Weisz 2011; Lüscher 2013; Harlander, **FL**, Neumann 2020]:

$$\bar{\chi}(t,x)\chi(t,x) = \zeta_S^{(1)}(t)\frac{m}{t}\mathbb{1} + \zeta_S^{(3)}(t)m^3\mathbb{1} + \zeta_S(t)\bar{\psi}(x)\psi(x) + \mathcal{O}(t)$$

$$\bar{\chi}(t,x)\overleftrightarrow{\not{D}}\chi(t,x) = \zeta_2^{(0)}(t)\frac{1}{t^2}\mathbb{1} + \zeta_2^{(2)}(t)\frac{m^2}{t}\mathbb{1} + \zeta_{21}(t)\frac{1}{g^2}F^a_{\mu\nu}(x)F^a_{\mu\nu}(x) + \zeta_{22}(t)\bar{\psi}(x)\overleftrightarrow{\not{D}}\psi(x) + \zeta_{23}(t)m^4\mathbb{1} + \mathcal{O}(t)$$

- Take VEV to obtain small-mass expansion (briefly mentioned in [**FL** PhD 2021]):
  - $\zeta_X(t)$ known through NNLO [Makino, Suzuki 2014; Artz, Harlander, **FL**, Neumann, Prausa 2019; Borgulat, Harlander, Kohnen, **FL** 2023; Hieda, Suzuki 2016; Mereghetti, Monahan, Rizik, Shindler, Stoffer 2021; Borgulat, Harlander, Rizik, Shindler 2022; Harlander, **FL**, Neumann 2020; Harlander, Kluth, **FL** 2018]
  - VEVs known through NNLO for a long time [Broadhurst 1981; Spiridonov, Chetyrkin 1988; Harlander PhD 1998; Braaten, Narison, Pich 1992; Chetyrkin, Kühn 1994]

⇒ Can obtain $\mathcal{O}(\alpha_s m^3)$ for $S(t)$ and $\mathcal{O}(\alpha_s m^4)$ for $R(t)$ for free from literature results

- Some subtleties prevent immediate extension to NNLO

Results – small-mass expansion

$$S(t) = -\frac{N_{\rm c}}{8\pi^2}\frac{m}{t}\sum_{n,k=0}^{\infty} C_{n,k}^{S,\ll 1}(m^2,t,\mu^2)(m^2 t)^k \left(\frac{\alpha_{\rm s}(\mu^2)}{\pi}\right)^n$$

with

$$C_{0,0}^{S,\ll 1} = 1, \qquad C_{1,0}^{S,\ll 1} = C_{\rm F}(1+2\log 2),$$

$$C_{0,1}^{S,\ll 1} = 2(L_t - L_m), \quad C_{1,1}^{S,\ll 1} = C_{\rm F}\left(3 + 10\log 2 - 9\log 3 - 6\,\mathrm{Li}_2\left(\frac{1}{4}\right) - 4L_m - 3L_m^2 + 7L_t + 3L_m L_t\right),$$

$$\vdots \qquad\qquad \vdots$$

where

$$L_t \equiv \log(2\mu^2 t) + \gamma_{\rm E} \qquad \text{and} \qquad L_m \equiv \log(\mu^2/m^2)$$

- Computed up to $n=1$ and $k=5$
- Same for $R(t)$

Results – large-mass expansion

$$S(t) = -\frac{N_{\rm c}}{16\pi^2}\frac{1}{mt^2}\sum_{n,k=0}^{\infty} C_{n,k}^{S,\gg 1}(m^2,t,\mu^2)(m^2t)^{-k}\left(\frac{\alpha_{\rm s}(\mu^2)}{\pi}\right)^n$$

with

$$C_{0,0}^{S,\gg 1} = 1, \qquad C_{1,0}^{S,\gg 1} = -\frac{C_{\rm F}}{4}\Big(2 + 3L_m + 3L_t + 6\log 2 - 9\log 3\Big),$$

$$C_{0,1}^{S,\gg 1} = -1, \qquad C_{1,1}^{S,\gg 1} = \frac{C_{\rm F}}{16}\Big(22 + 27L_m + 21L_t + 42\log 2 - 51\log 3\Big),$$

$$\vdots \qquad \qquad \vdots$$

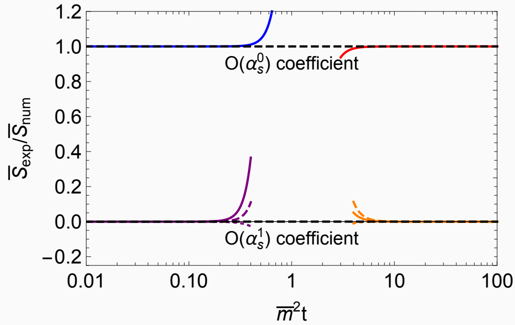
where

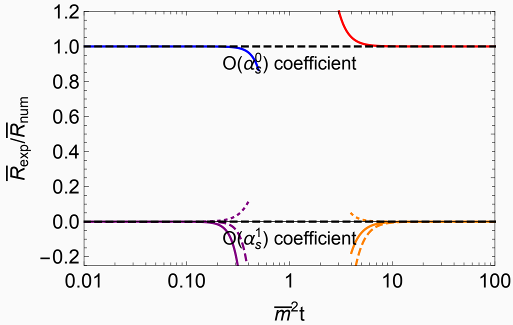
$$L_t \equiv \log(2\mu^2 t) + \gamma_{\rm E} \qquad \text{and} \qquad L_m \equiv \log(\mu^2/m^2)$$

- Computed up to $n = 1$ and $k = 5$
- Same for $R(t)$

Numerical results

- In addition to the expansions, compute integrals numerically as grid in $z = m^2 t$ with `ftint` [Harlander, Nellopoulos, Olsson, Wesle 2024] based on pySecDec [Heinrich, Jones, Kerner, Magerya, Olsson, Schlenk 2023]
- Perfect agreement in respective regions!





Lattice observables

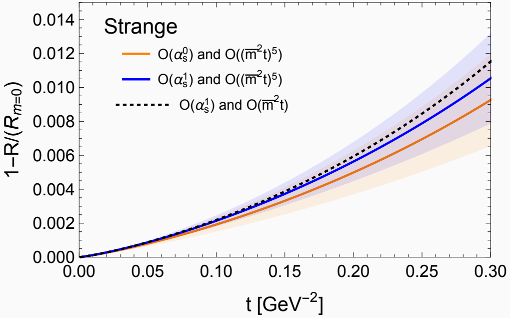
- Define observables

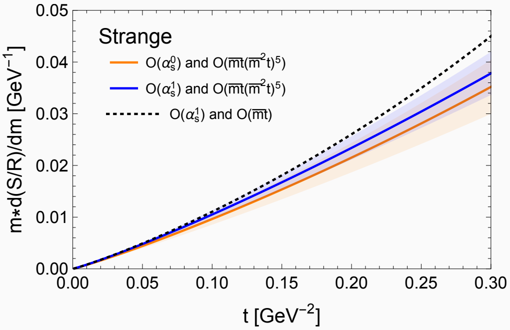
$$r_a(m) = \frac{S(t,m)}{R(t,m)}\,, \qquad r_b(m) = \frac{R(t,m)}{R(t,m=0)}\,, \qquad r_c(m) = m\frac{\mathrm{d}}{\mathrm{d}m}\left(\frac{S(t,m)}{R(t,m)}\right)$$

$\Rightarrow$ Finite and renormalisation group invariant

- Study impact of mass effects as well as sensitivity to mass and expected uncertainties in mass determination

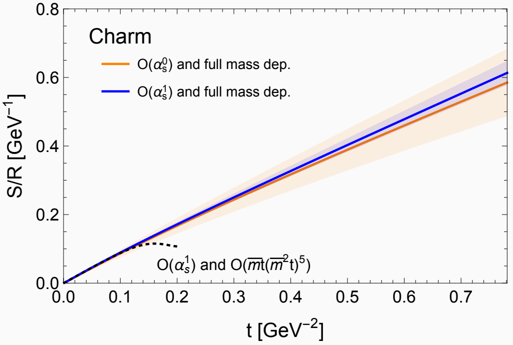
Results for the strange region

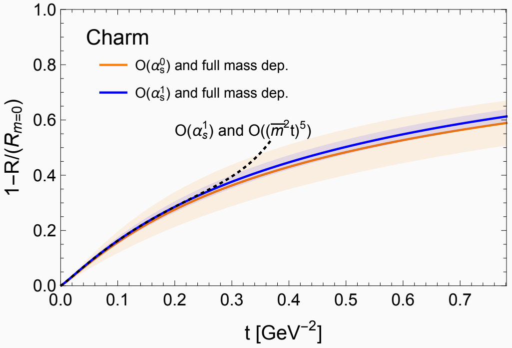




- Small-mass expansion sufficient
- Newly computed higher-order mass effects of similar importance as perturbative orders
- Sizeable perturbative uncertainty remaining

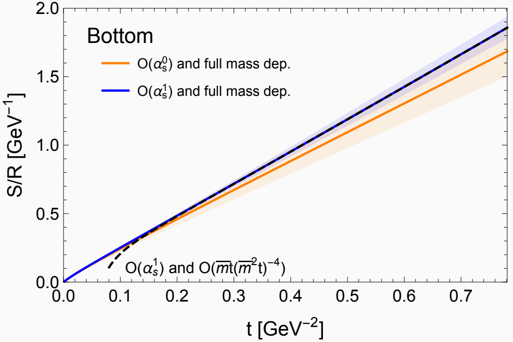
Results for the charm region

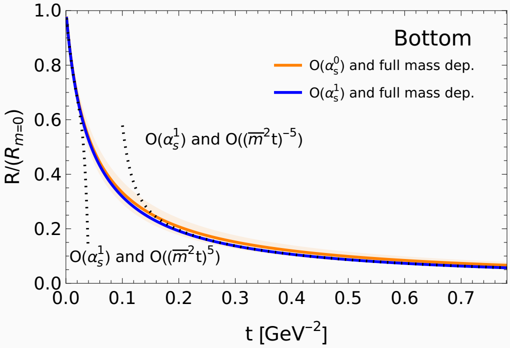




- Numerical result required, as neither asymptotic expansion provides adequate approximation over the full range
- Perturbative series shows good convergence, dependence on the renormalisation scale shrinks

Results for the bottom region





- Large-mass expansion provides excellent description over most of the relevant range
- Perturbative behaviour very stable, with small higher-order corrections and weak scale dependence

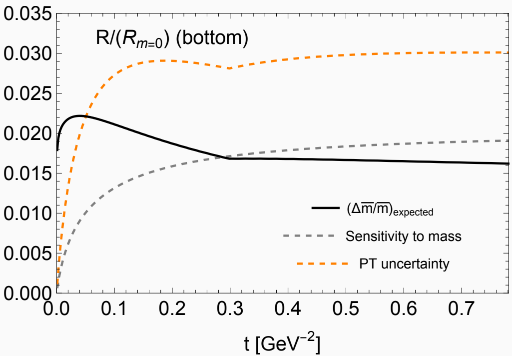
Sensitivity and precision

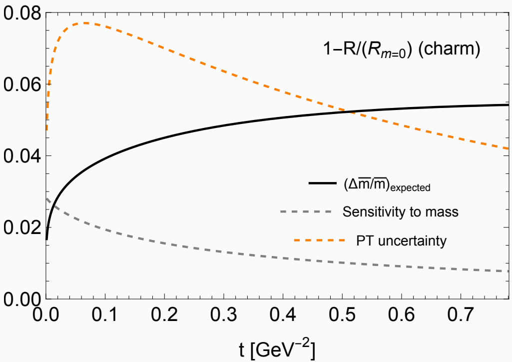
- Achievable precision with NLO:

$$\Delta m = \frac{\delta r}{\frac{\partial r}{\partial m}}$$

  - δr: perturbative uncertainty of observable
  - $\frac{\partial r}{\partial m}$: sensitivity of observable to mass

$\Rightarrow$ $1-2$ % achievable for bottom, $4-6$ % for charm, $\gtrsim 10$ % for strange (not shown)





A word on non-perturbative effects

- Estimate for $m^2 t \ll 1$ based on the small-flow-time expansion:

$$S = S_{\rm PT} + \zeta_{\rm S}(t)\left[\langle\bar\psi\psi\rangle - \langle\bar\psi\psi\rangle_{\rm PT}\right] + \mathcal{O}(t),$$

$$R = R_{\rm PT} + \zeta_{21}(t)\frac{1}{g^2}\left[\langle F^a_{\mu\nu}F^a_{\mu\nu}\rangle - \langle F^a_{\mu\nu}F^a_{\mu\nu}\rangle_{\rm PT}\right] + \zeta_{22}(t)\left[\left\langle\bar\psi\,\overleftrightarrow{\not{D}}\,\psi\right\rangle - \left\langle\bar\psi\,\overleftrightarrow{\not{D}}\,\psi\right\rangle_{\rm PT}\right] + \mathcal{O}(t)$$

- Differences between full result and perturbation theory single out non-perturbative effects, e.g.:

$$\langle\bar\psi\psi\rangle - \langle\bar\psi\psi\rangle_{\rm PT} = c_0\Lambda^3_{\rm QCD} + c_1 m\Lambda^2_{\rm QCD} + c_2 m^2\Lambda_{\rm QCD}$$

- For strange, $\frac{\Lambda^3_{\rm QCD}}{m/t} > 1$ possible
- Suppress non-perturbative effects through

$$1 - r_b(m) = 1 - \frac{R(t,m)}{R(t,m=0)}\,, \qquad r_c(m) = m\frac{\mathrm{d}}{\mathrm{d}m}\left(\frac{S(t,m)}{R(t,m)}\right)$$

- Not much known about non-perturbative effects for $m^2 t \gg 1$

Summary and outlook

- Gradient-flow observables provide gauge-invariant and scheme independent framework for quark mass determinations
- Computed relevant observables at NLO, including full dependence on the quark mass
- Method based on Laplace transform allows one to obtain both small- and large-mass expansions
- Promising precision estimated for heavy quarks: $1-2$ % for bottom, $4-6$ % for charm already with NLO
- NNLO available in the meantime [Harlander, Mason 2025] $\Rightarrow$ see Robert Mason's talk
- Future directions:
    - Proper estimate of non-perturbative effects
    - Lattice studies!