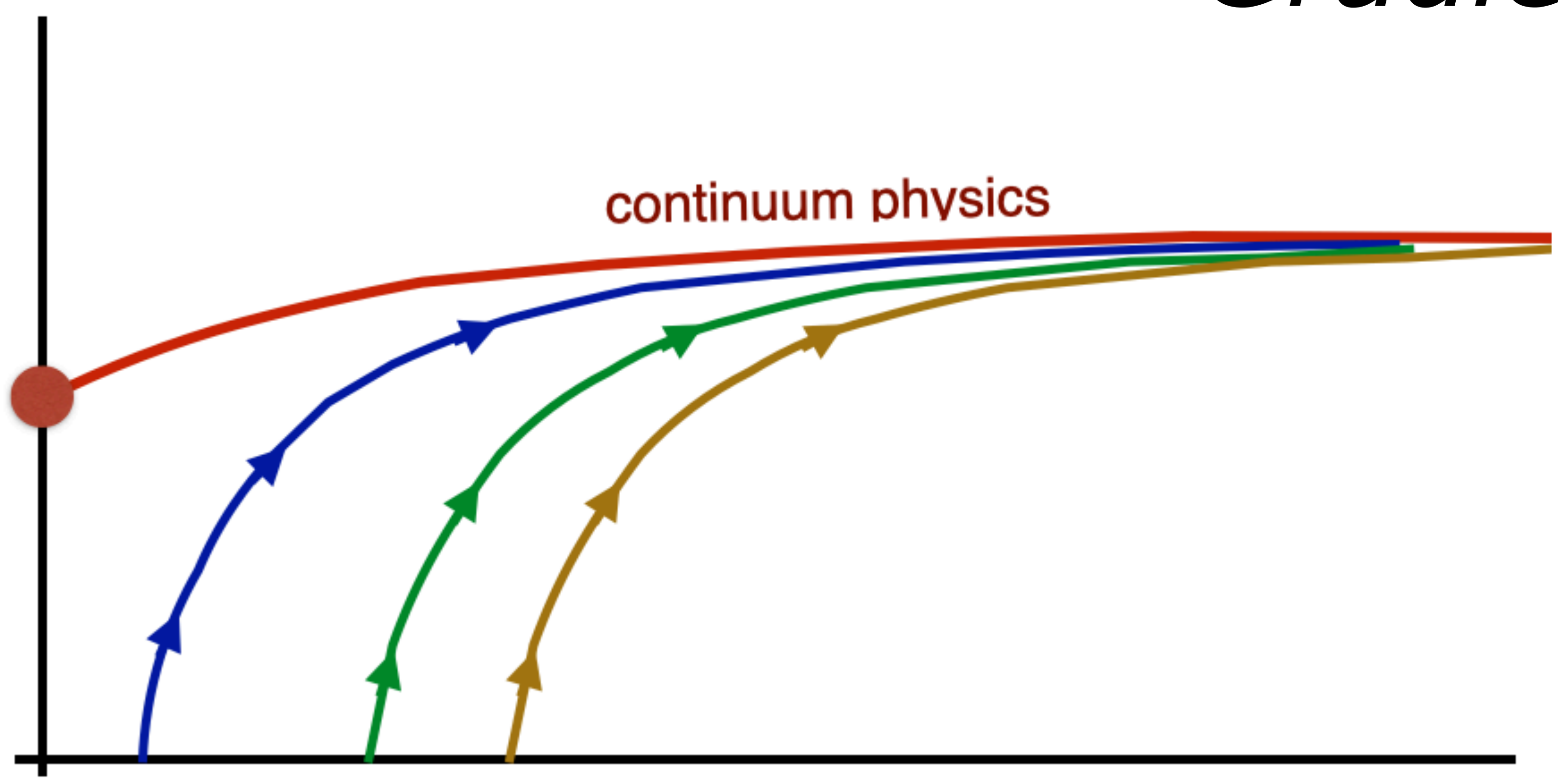
Lattice Gradient Flow

Anna Hasenfratz
University of Colorado Boulder

Gradient Flow Workshop, Edinburgh
May 13 2026

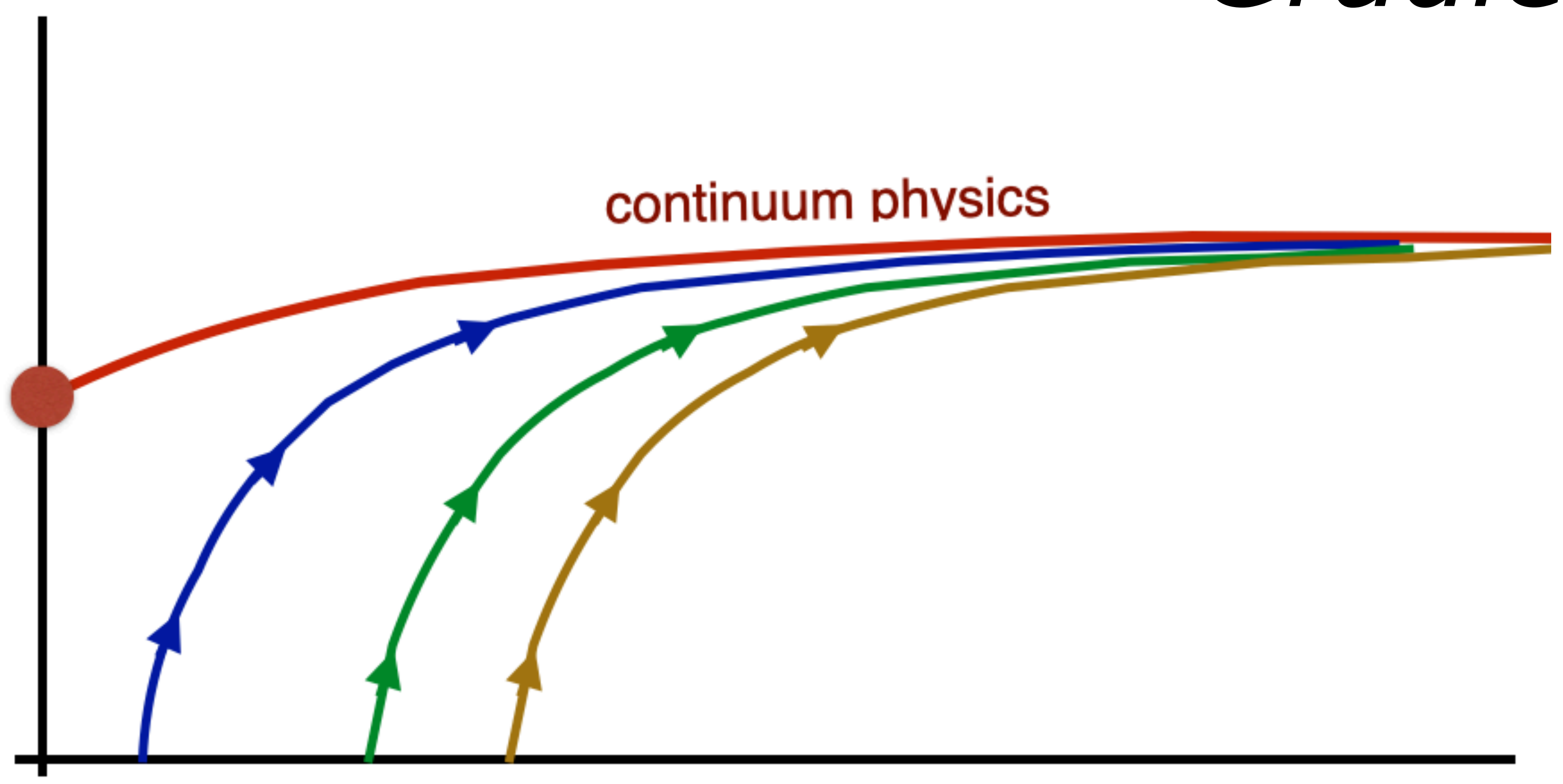


Lattice Gradient Flow

as a nonperturbative renormalization scheme

Anna Hasenfratz
University of Colorado Boulder

Gradient Flow Workshop, Edinburgh
May 13 2026



Gradient flow

0907.5491 M. Luscher: Trivializing maps, the Wilson flow and the HMC algorithm $\Longrightarrow$ algorithmic developments ($\longrightarrow$ normalizing flow)

1006.4518 M. Luscher: Properties and uses of the Wilson flow in lattice QCD $\Longrightarrow$ flowed gauge field is renormalized

1101.0963 M. Luscher, P. Weisz: Perturbative analysis of the gradient flow in non-abelian gauge theories $\Longrightarrow$ formal all-orders proof of renormalizability

1302.5246 M. Luscher: Chiral symmetry and the Yang--Mills gradient flow $\Longrightarrow$ GF for fermions

Gradient flow

0907.5491 M. Luscher: Trivializing maps, the Wilson flow and the HMC algorithm $\Longrightarrow$ algorithmic developments ($\longrightarrow$ normalizing flow)

- Gradient flow (GF): continuous, invertible smearing transformation (flow) for gauge fields

$$\partial_\tau B_\mu(\tau; x) = D_\nu G_{\nu\mu}(\tau; x), \quad B_\mu(0; x) = A_\mu(x)$$

Gradient flow

0907.5491 M. Luscher: Trivializing maps, the Wilson flow and the HMC algorithm $\Longrightarrow$ algorithmic developments ($\longrightarrow$ normalizing flow)

- Gradient flow (GF): continuous, invertible smearing transformation (flow) for gauge fields

$$\partial_\tau B_\mu(\tau; x) = D_\nu G_{\nu\mu}(\tau; x), \quad B_\mu(0; x) = A_\mu(x)$$

In this talk I will use τ to denote the gradient flow time, to distinguish it from the Euclidean time t

Gradient flow

1006.4518 M. Luscher: Properties and uses of the Wilson flow in lattice QCD $\Longrightarrow$ flowed gauge field is renormalized

- Gradient flow (GF): continuous, invertible smearing transformation (flow) for gauge fields

$$\partial_\tau B_\mu(\tau;x) = D_\nu G_{\nu\mu}(\tau;x), \quad B_\mu(0;x) = A_\mu(x)$$

- GF coupling

$$g^2_{GF}(\tau) = \mathcal{N}\tau^2\langle E\rangle_\tau$$

- Flowed system is renormalized at $\mu \sim 1/\sqrt{8\tau}$; g^2_{GF} is renormalized coupling

Gradient flow

1101.0963 M. Luscher, P. Weisz: Perturbative analysis of the gradient flow in non-abelian gauge theories

$\Longrightarrow$ formal all-orders proof of renormalizability

- Gradient flow (GF): continuous, invertible smearing transformation (flow) for gauge fields

  $\partial_\tau B_\mu(\tau;x) = D_\nu G_{\nu\mu}(\tau;x), \quad B_\mu(0;x) = A_\mu(x)$
- GF coupling

  $g^2_{GF}(\tau) = \mathcal{N}\tau^2\langle E\rangle_\tau$
- Flowed system is renormalized at $\mu \sim 1/\sqrt{8\tau}$; g^2_{GF} is renormalized coupling

- Scale setting: $g^2_{GF}(t_0) = 0.3\mathcal{N}$
- RG β function : two equivalent lattice approaches
  - Step scaling function
  - Continuous β function

Gradient flow

1302.5246 M. Luscher: Chiral symmetry and the Yang--Mills gradient flow $\Longrightarrow$ GF for fermions

- Gradient flow (GF): continuous, invertible smearing transformation (flow) for gauge fields

$$\partial_\tau B_\mu(\tau; x) = D_\nu G_{\nu\mu}(\tau; x), \quad B_\mu(0; x) = A_\mu(x)$$

- GF coupling

$$g_{GF}^2(\tau) = \mathcal{N}\tau^2\langle E\rangle_\tau$$

- Flowed system is renormalized at $\mu \sim 1/\sqrt{8\tau}$; g_{GF}^2 is renormalized coupling
- Fermion flow (on gauge background)

$$\partial_\tau \chi(\tau; x) = \Delta\,\chi(\tau; x)\,, \quad \partial_\tau \bar{\chi}(\tau; x) = \bar{\chi}(\tau; x)\overleftarrow{\Delta}$$

- Renormalization: only wave function renormalization needed

$$\chi_R(\tau; x) \;=\; Z_\chi^{1/2}(\tau)\,\chi(\tau; x)$$

$$O_{\rm GF}(\tau; x) = Z_\chi^{n/2}(\tau)\, O(\tau; x)$$

Gradient flow

1302.5246 M. Luscher: Chiral symmetry and the Yang--Mills gradient flow $\Longrightarrow$ GF for fermions

- Gradient flow (GF): continuous, invertible smearing transformation (flow) for gauge fields

$$\partial_\tau B_\mu(\tau;x) = D_\nu G_{\nu\mu}(\tau;x), \quad B_\mu(0;x) = A_\mu(x)$$

- GF coupling

$$g^2_{GF}(\tau) = \mathcal{N}\tau^2 \langle E \rangle_\tau$$

- Flowed system is renormalized at $\mu \sim 1/\sqrt{8\tau}$; g^2_{GF} is renormalized coupling
- Fermion flow (on gauge background)

$$\partial_\tau \chi(\tau;x) = \Delta\, \chi(\tau;x)\,, \quad \partial_\tau \bar{\chi}(\tau;x) = \bar{\chi}(\tau;x)\overleftarrow{\Delta}$$

- Renormalization: only wave function renormalization needed

$$\chi_R(\tau;x) \;=\; Z_\chi^{1/2}(\tau)\,\chi(\tau;x)$$
$$O_{\rm GF}(\tau;x) = Z_\chi^{n/2}(\tau)\, O(\tau;x)$$

- GF provides a complete nonperturbative regularization scheme
- Connect to $\overline{MS}$ with short flow time expansion (SFTX):

$$O_{\overline{\rm MS}}(\mu;x) = \lim_{\tau\to 0} \zeta_O^{-1}(\mu,\tau)\, O_{\rm GF}(\tau;x)$$

Gradient flow as real-space RG

A. Carosso, AH, E. Neil,
PRL 121,201601 (2018)

H.Sonoda, H.Suzuki,
PTEP,023B05 (2021)

GF is a continuous smearing transformation of radius $\rho \propto \sqrt{8\tau}$:

➡ defines "block spins" or "block links"

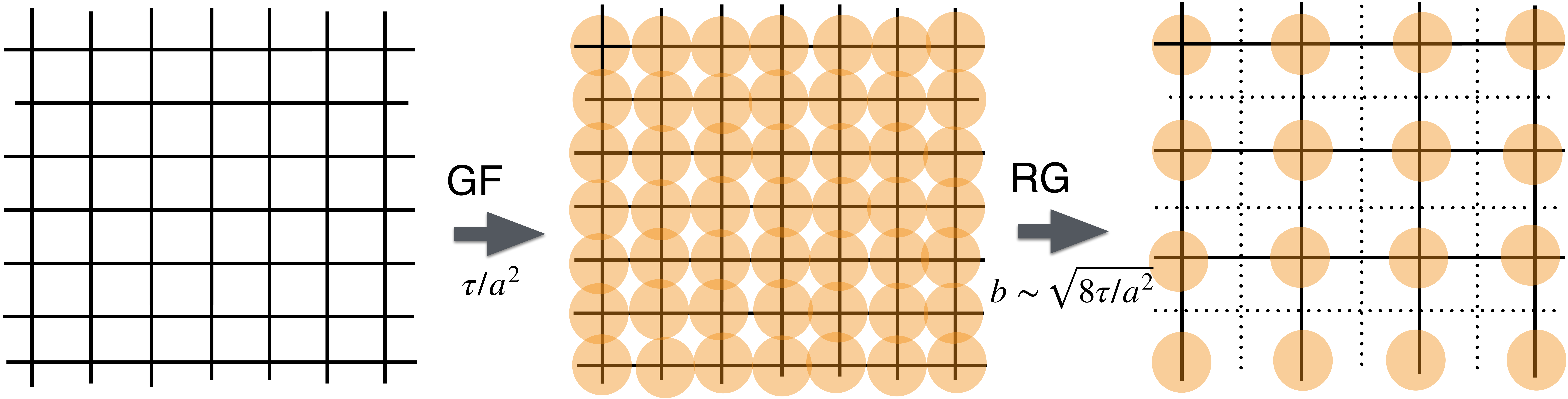
GF does not have coarse graining - not an RG transformation

BUT it can be *interpreted* as Wilsonian RG with $b \propto \sqrt{8\tau/a^2}$ scale change for *local* operators

Gradient flow as real-space RG

A. Carosso, AH, E. Neil, PRL 121,201601 (2018)
H.Sonoda, H.Suzuki, PTEP,023B05 (2021)

GF is a continuous smearing transformation of radius $\rho \propto \sqrt{8\tau}$:
➡ defines "block spins" or "block links"

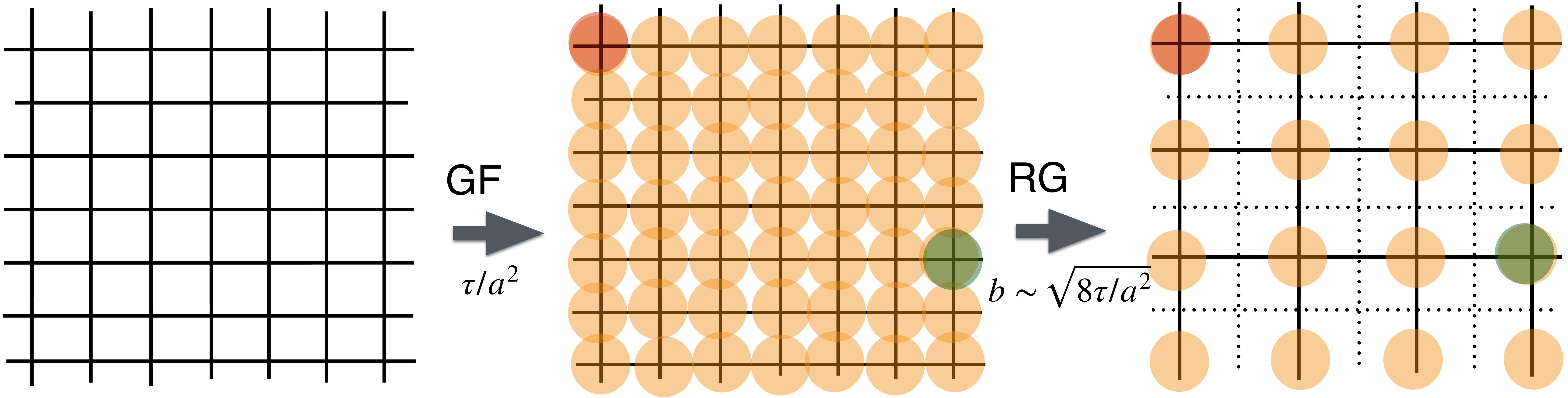


GF does not have coarse graining - not an RG transformation
BUT it can be *interpreted* as Wilsonian RG with $b \propto \sqrt{8\tau/a^2}$ scale change for *local* operators

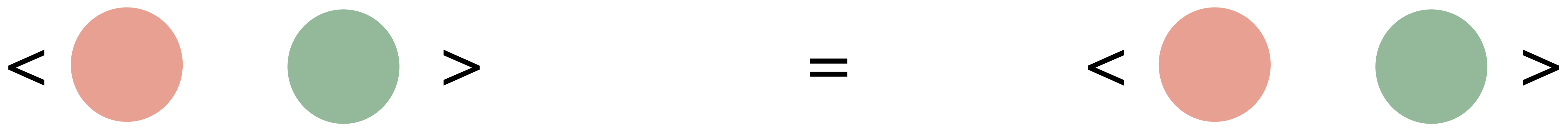
Gradient flow as real-space RG

A. Carosso, AH, E. Neil,
PRL 121,201601 (2018)
H.Sonoda, H.Suzuki,
PTEP,023B05 (2021)

GF is a continuous smearing transformation of radius $\rho \propto \sqrt{8\tau}$:
➡ defines "block spins" or "block links"



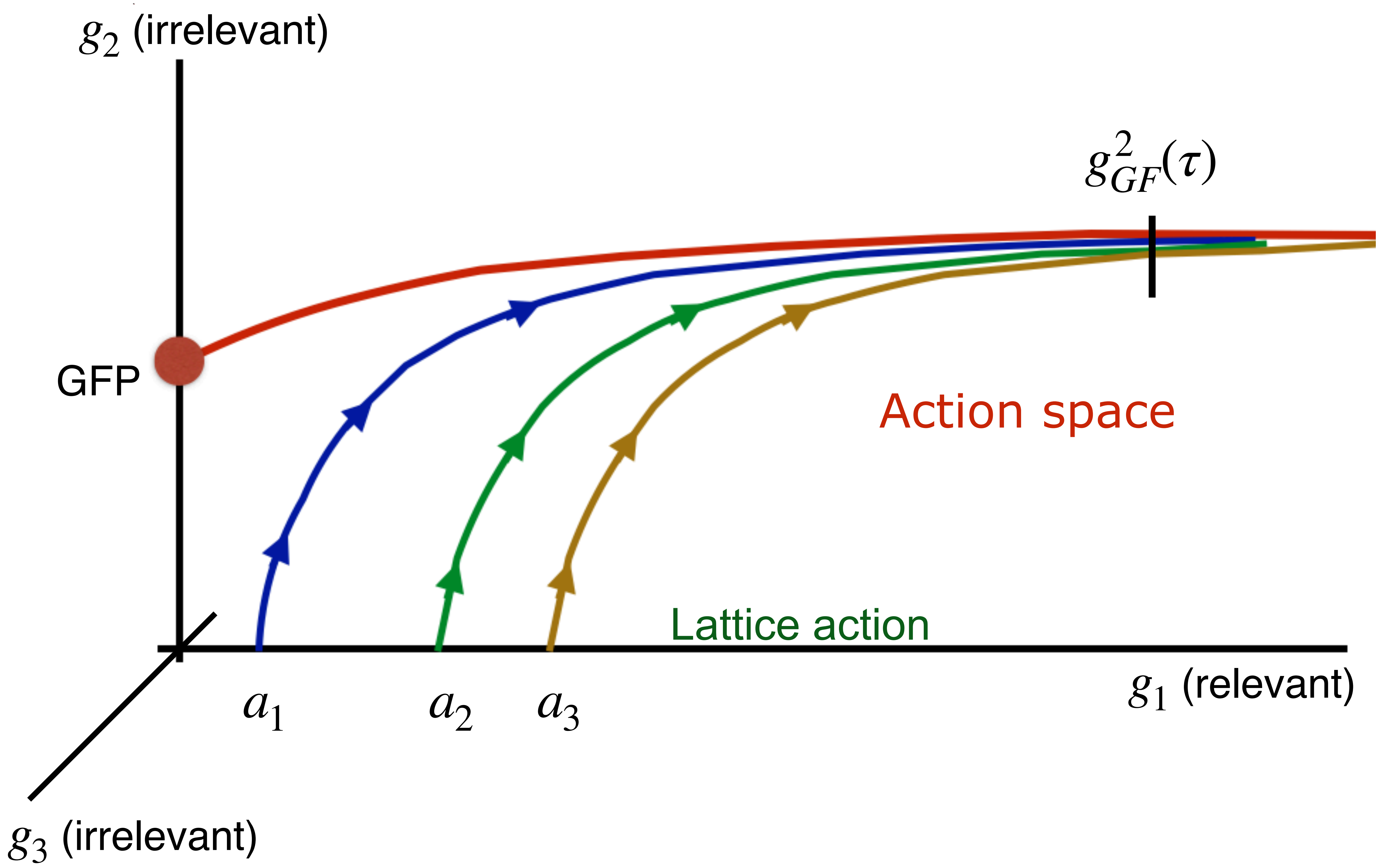
GF does not have coarse graining - not an RG transformation
BUT it can be *interpreted* as Wilsonian RG with $b \propto \sqrt{8\tau/a^2}$ scale change for *local* operators



GF/RG from lattice to continuum

Bare action parameter space:

GF/RG: $\tau = 0: \; S(g_i, m_i; a) \; :$ RG flow to $\tau: \; S(g_i(\tau), m_i(\tau))$

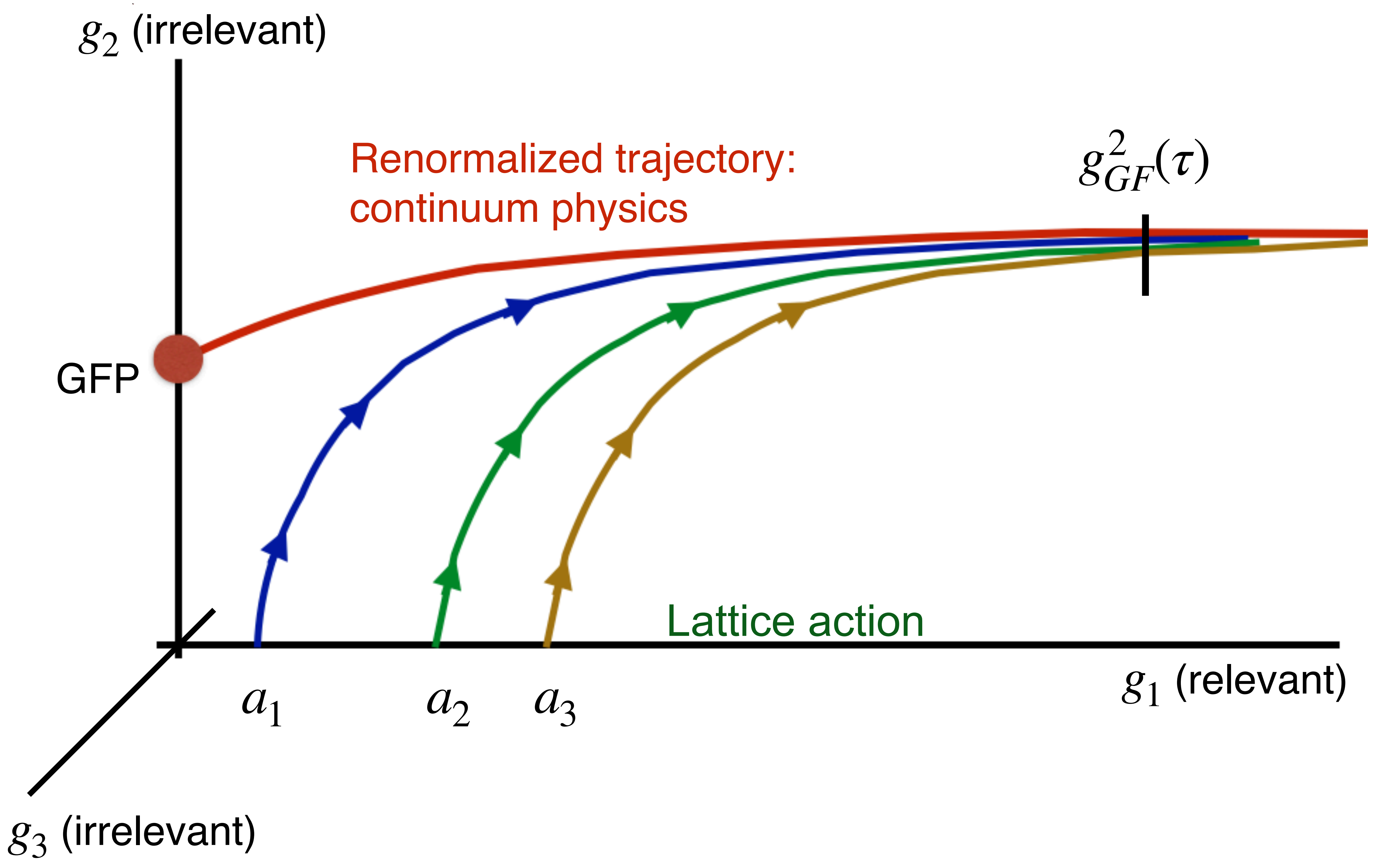


GF/RG from lattice to continuum

Bare action parameter space:

GF/RG: $\tau = 0 : S(g_i, m_i; a)$: RG flow to $\tau : S(g_i(\tau), m_i(\tau))$

- RG flow approaches the RT - RT describes continuum physics
- In the chiral limit RT is characterized by a single parameter $g^2_{GF}(\tau)$

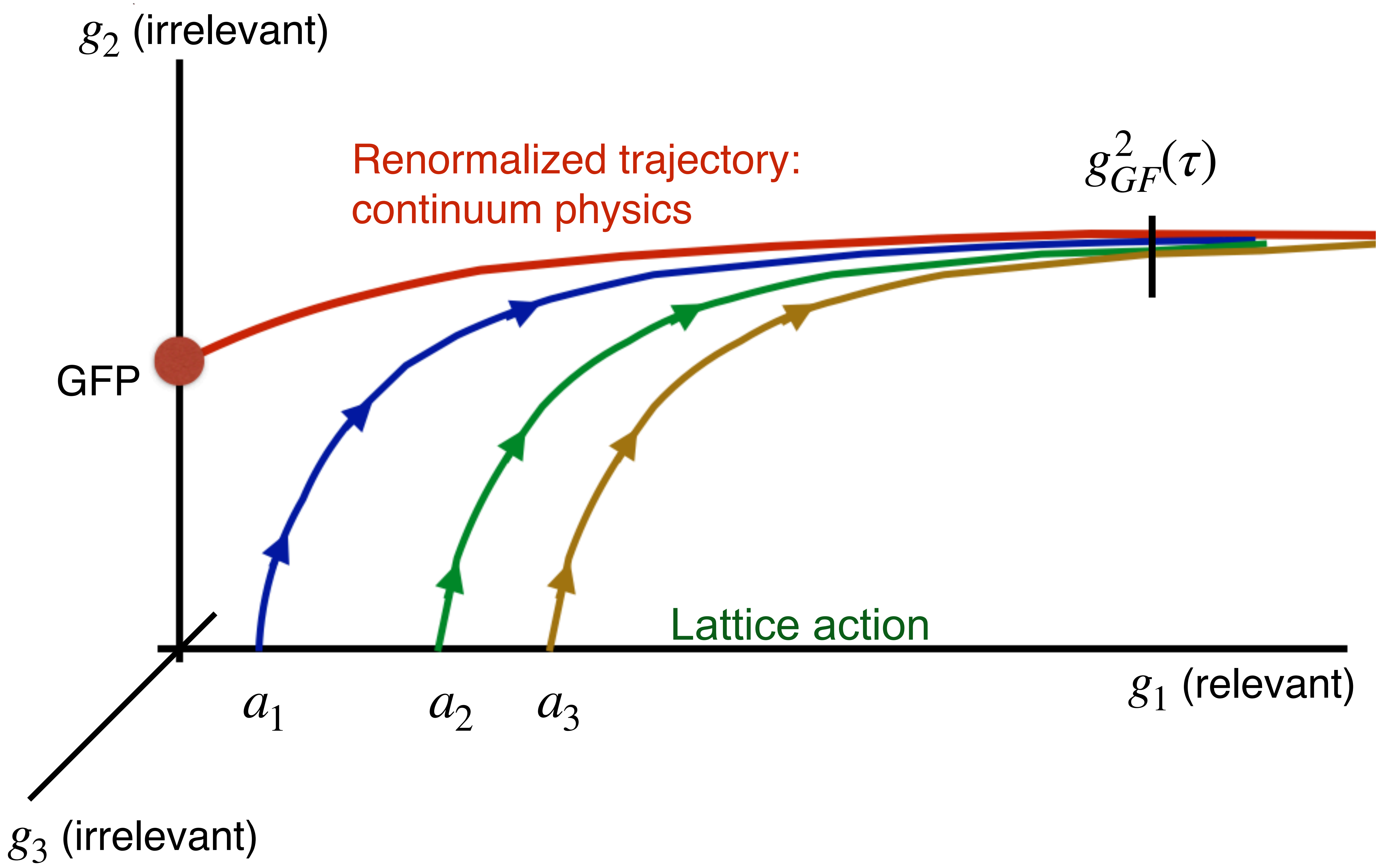


GF/RG from lattice to continuum

Bare action parameter space:

GF/RG: $\tau = 0: S(g_i, m_i; a)$: RG flow to $\tau: S(g_i(\tau), m_i(\tau))$

- RG flow approaches the RT - RT describes continuum physics
- In the chiral limit RT is characterized by a single parameter $g^2_{GF}(\tau)$



$g^2_{GF}(\tau)$ is renormalized coupling in GF scheme

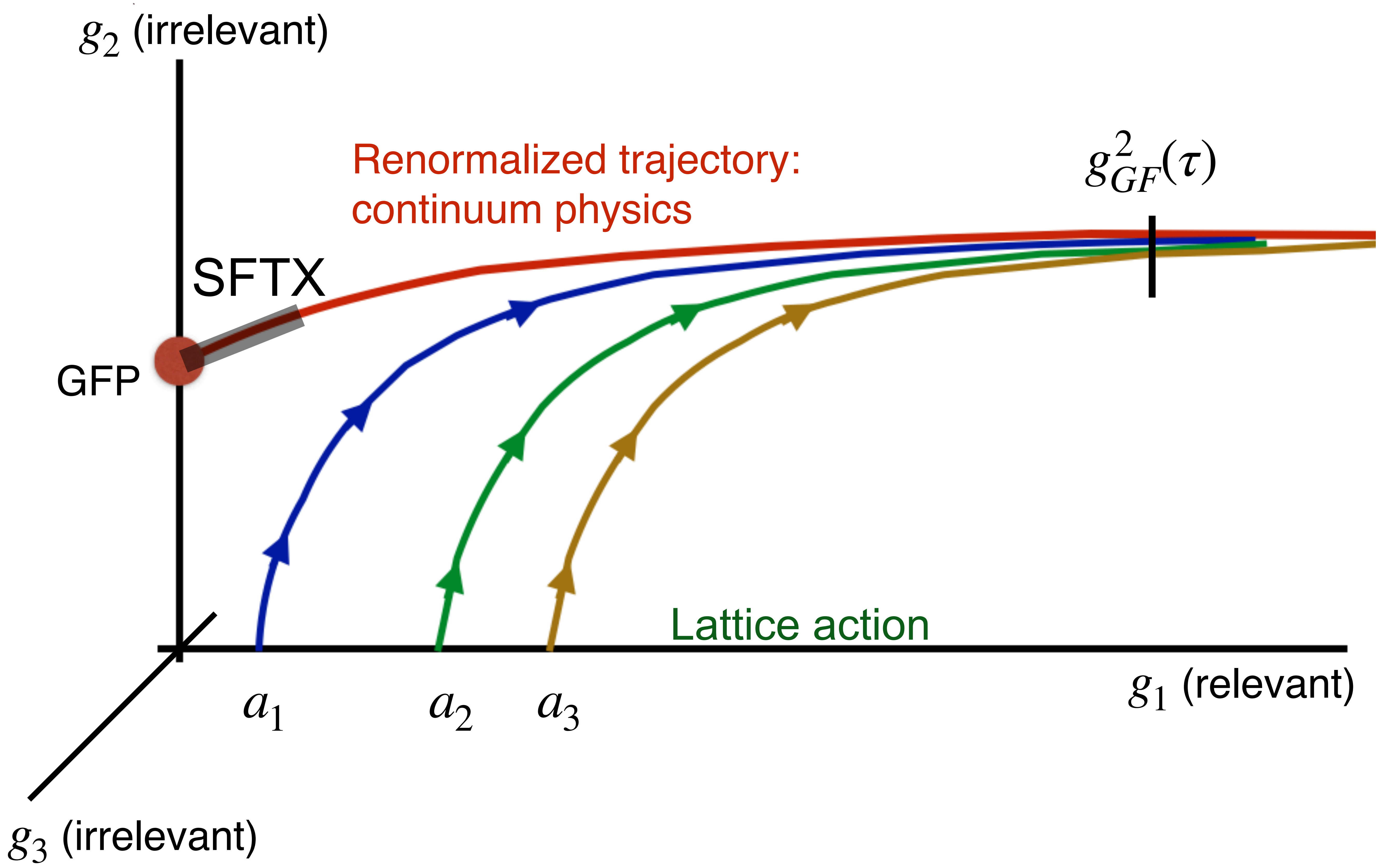
$$\beta_{GF}(g^2) = -\tau \frac{d, g^2_{GF}}{d\tau}$$

GF/RG from lattice to continuum

Bare action parameter space:

GF/RG: $\tau = 0: \; S(g_i, m_i; a)$: RG flow to $\tau: \; S(g_i(\tau), m_i(\tau))$

- RG flow approaches the RT - RT describes continuum physics
- In the chiral limit RT is characterized by a single parameter $g^2_{GF}(\tau)$



$g^2_{GF}(\tau)$ is renormalized coupling in GF scheme

$$\beta_{GF}(g^2) = -\tau \frac{d, g^2_{GF}}{d\tau}$$

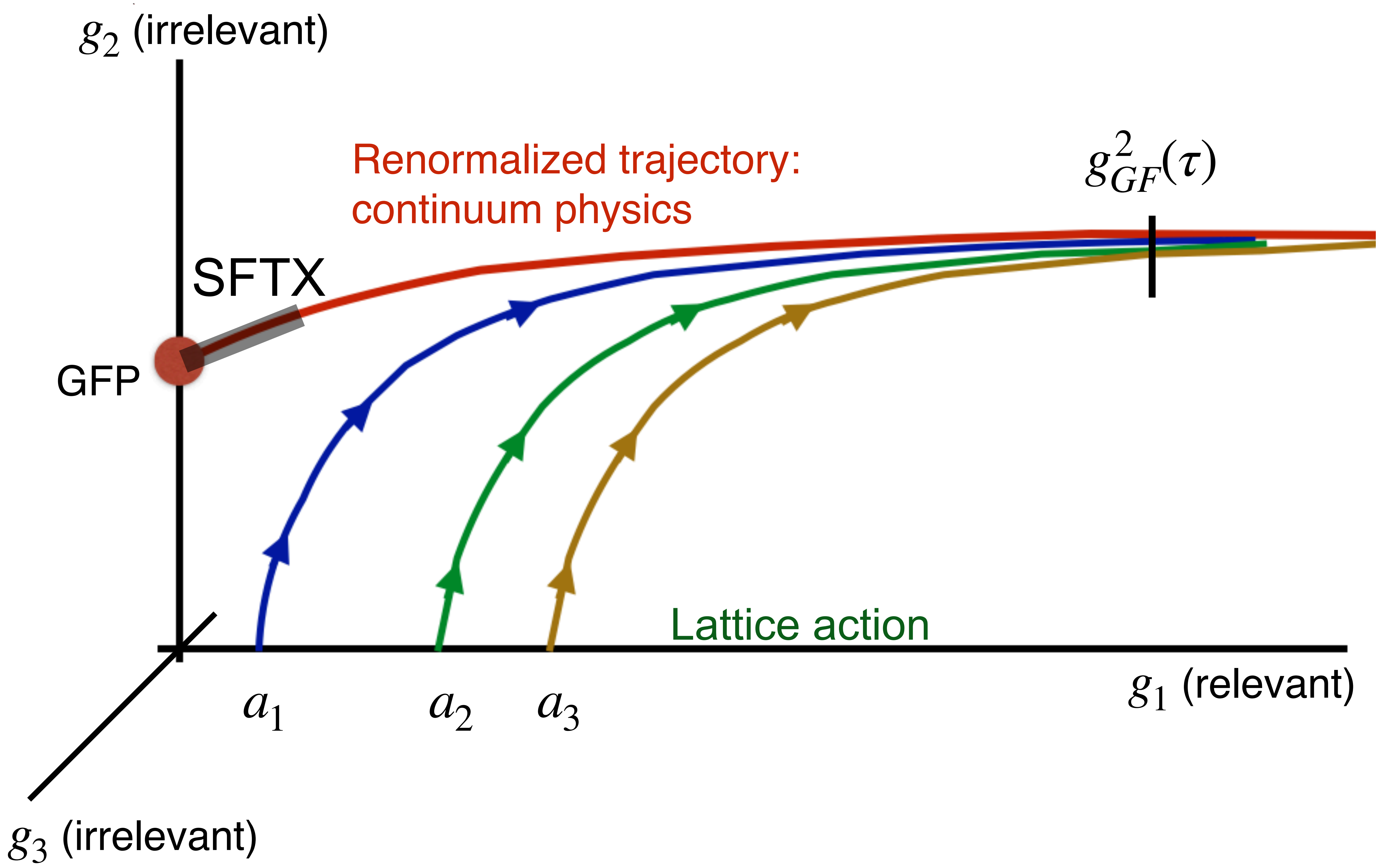
Connection to $\overline{MS}$ possible at high energies and $\tau \to 0$ limit : SFTX

GF/RG from lattice to continuum

Bare action parameter space:

GF/RG: $\tau = 0 : S(g_i, m_i; a)$: RG flow to $\tau : S(g_i(\tau), m_i(\tau))$

- RG flow approaches the RT - RT describes continuum physics
- In the chiral limit RT is characterized by a single parameter $g^2_{GF}(\tau)$



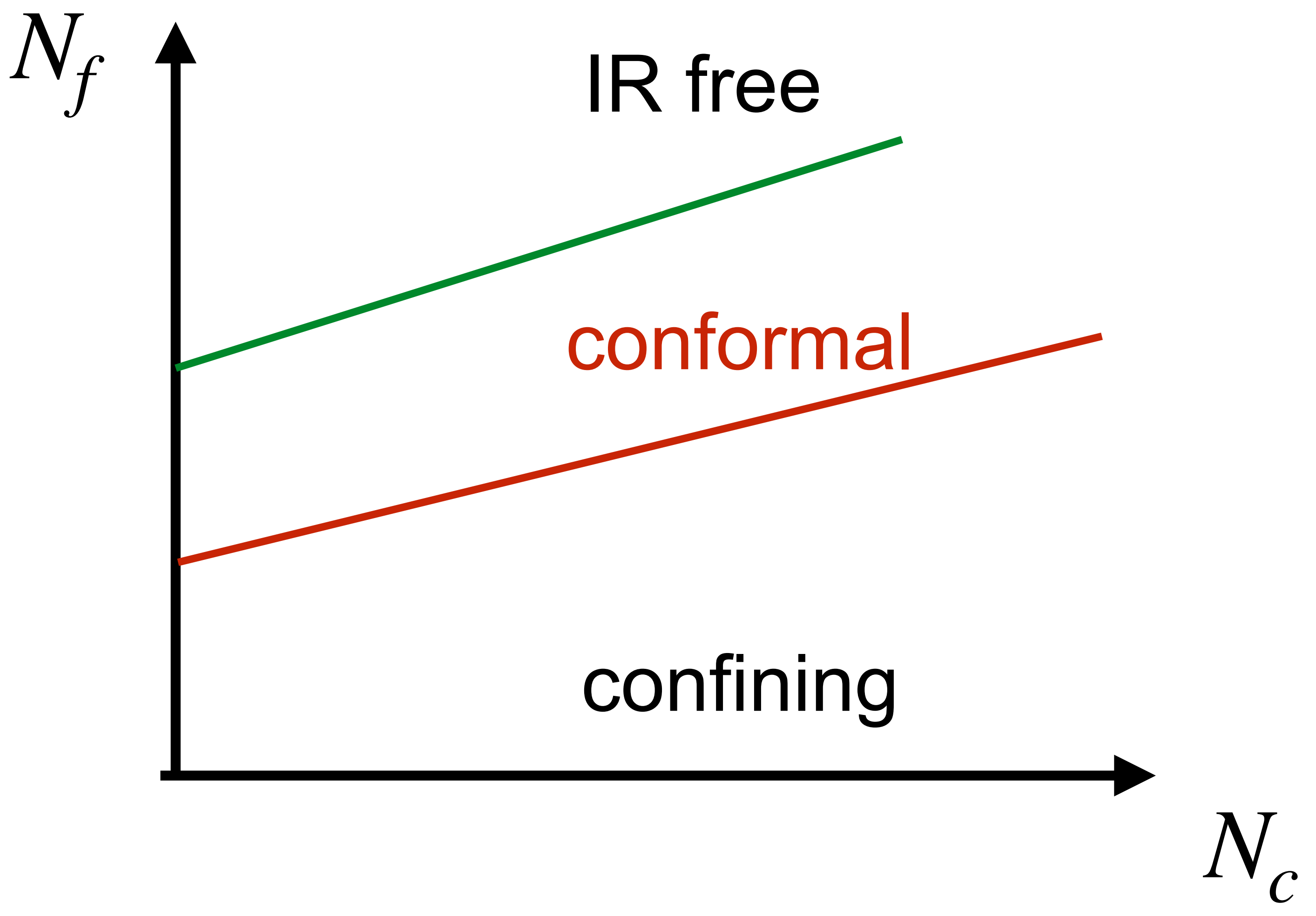
$g^2_{GF}(\tau)$ is renormalized coupling in GF scheme

$$\beta_{GF}(g^2) = -\tau \frac{d, g^2_{GF}}{d\tau}$$

Connection to $\overline{MS}$ possible at high energies and $\tau \to 0$ limit : SFTX

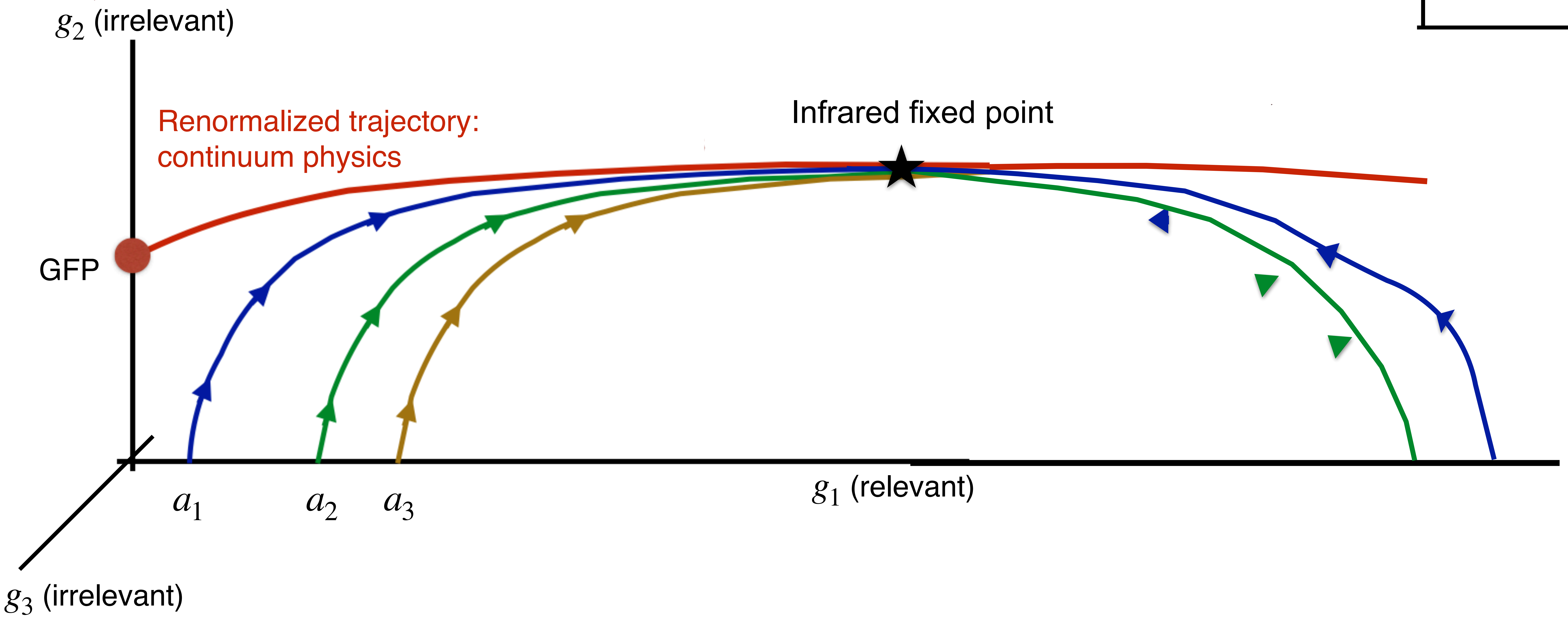
Wave function renormalization is equivalent to the η exponent of critical phenomena

GF/RG in the conformal window

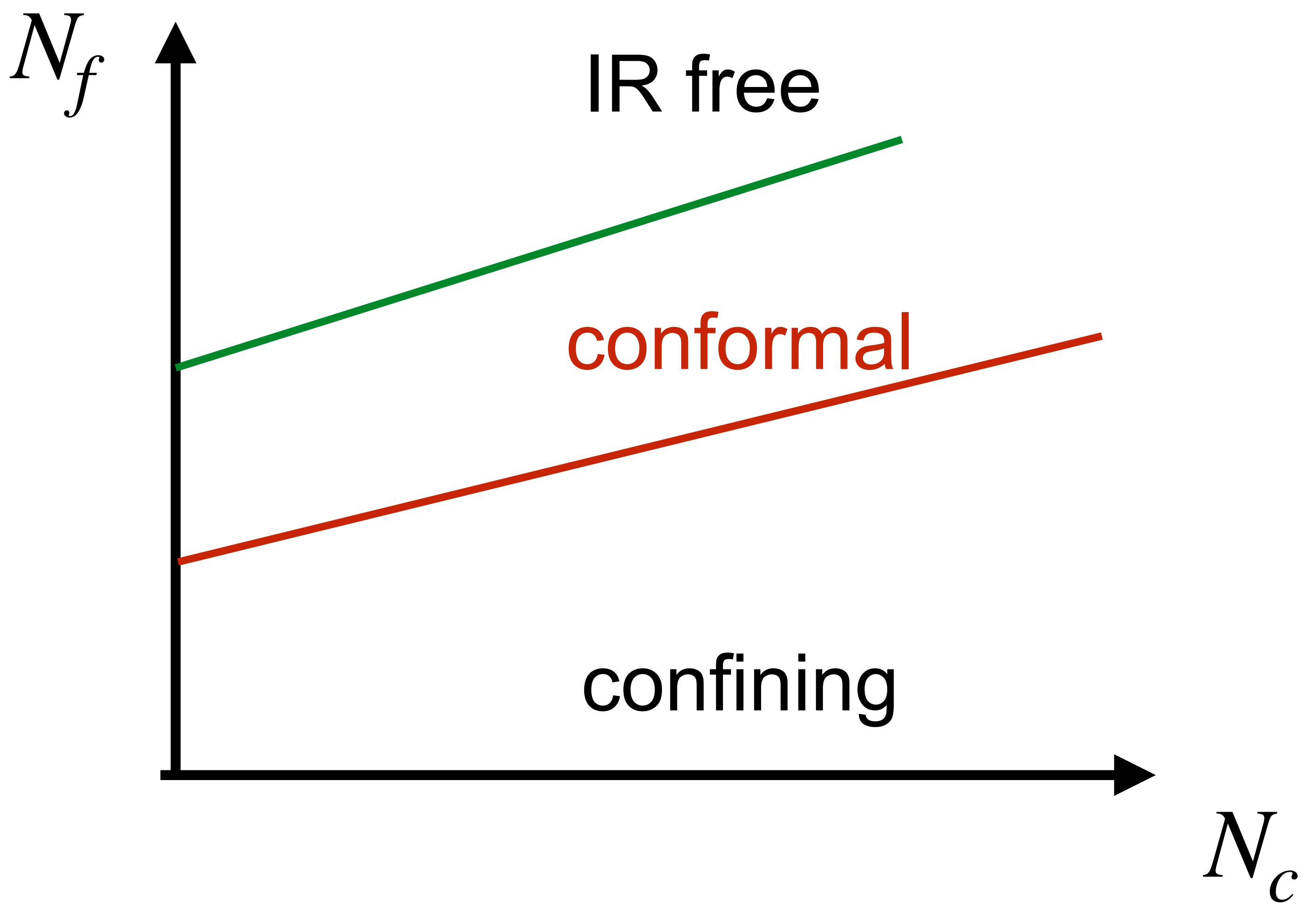


Bare action parameter space:

GF/RG: $\tau = 0 : S(g_i, m_i; a)$: RG flow to $\tau : S(g_i(\tau), m_i(\tau))$

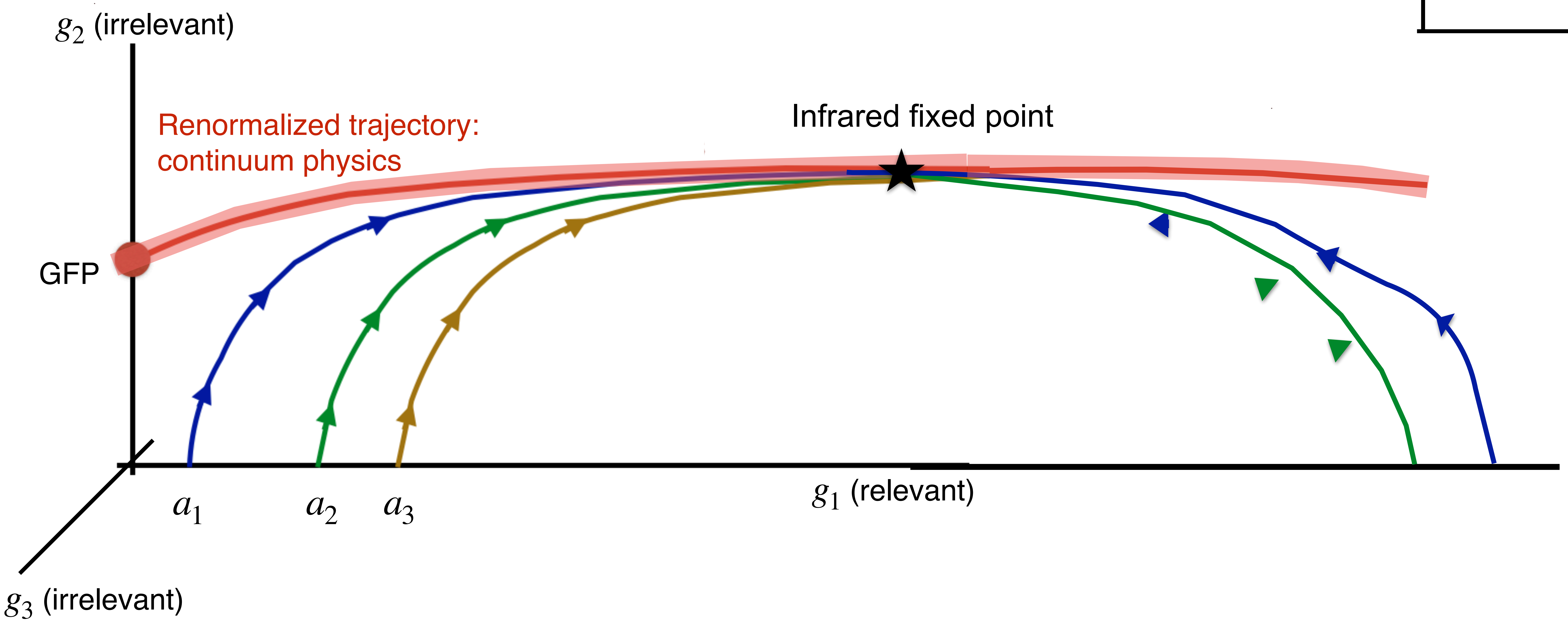


GF/RG in the conformal window

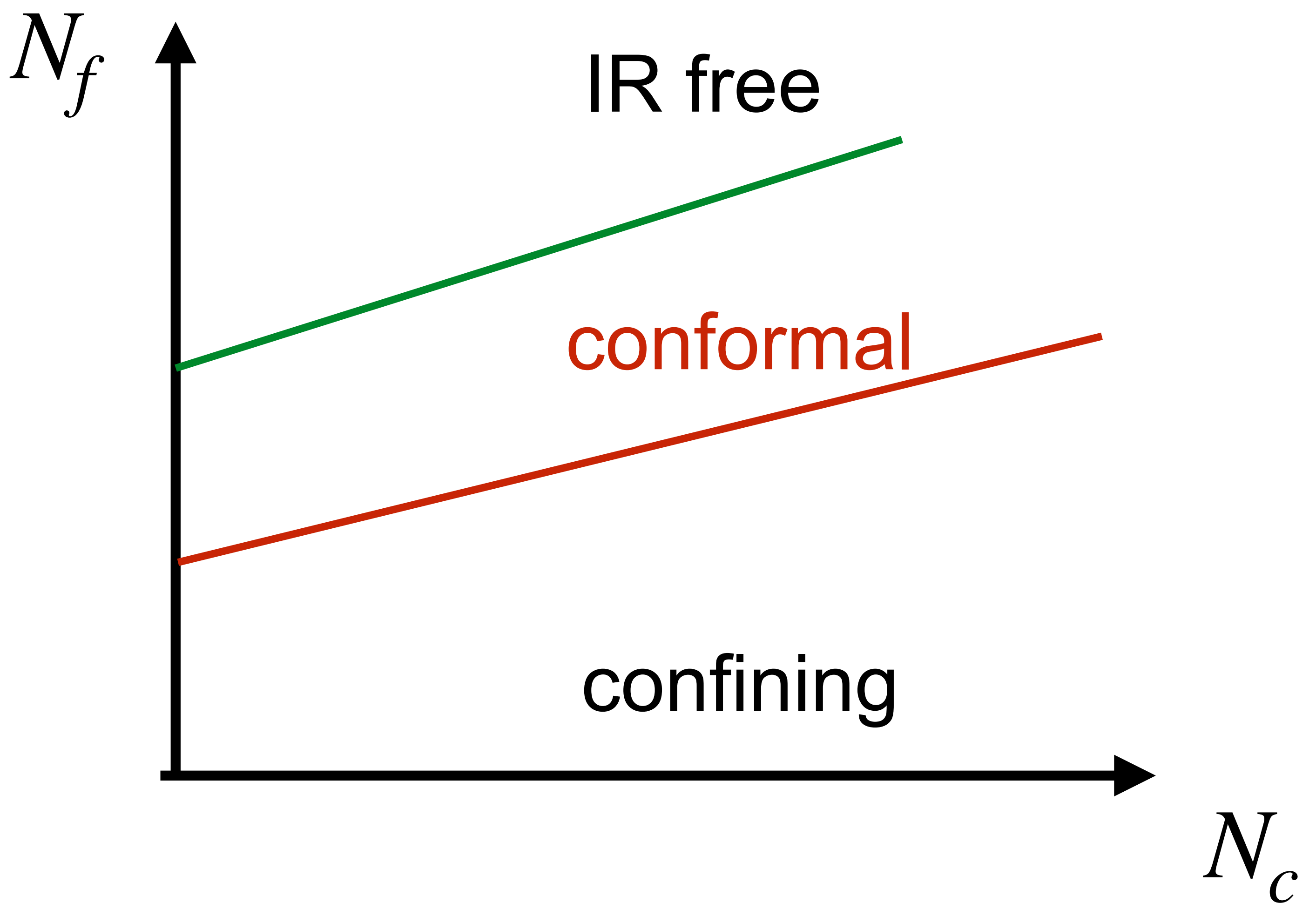


Bare action parameter space:

GF/RG: $\tau = 0 : S(g_i, m_i; a)$: RG flow to $\tau : S(g_i(\tau), m_i(\tau))$

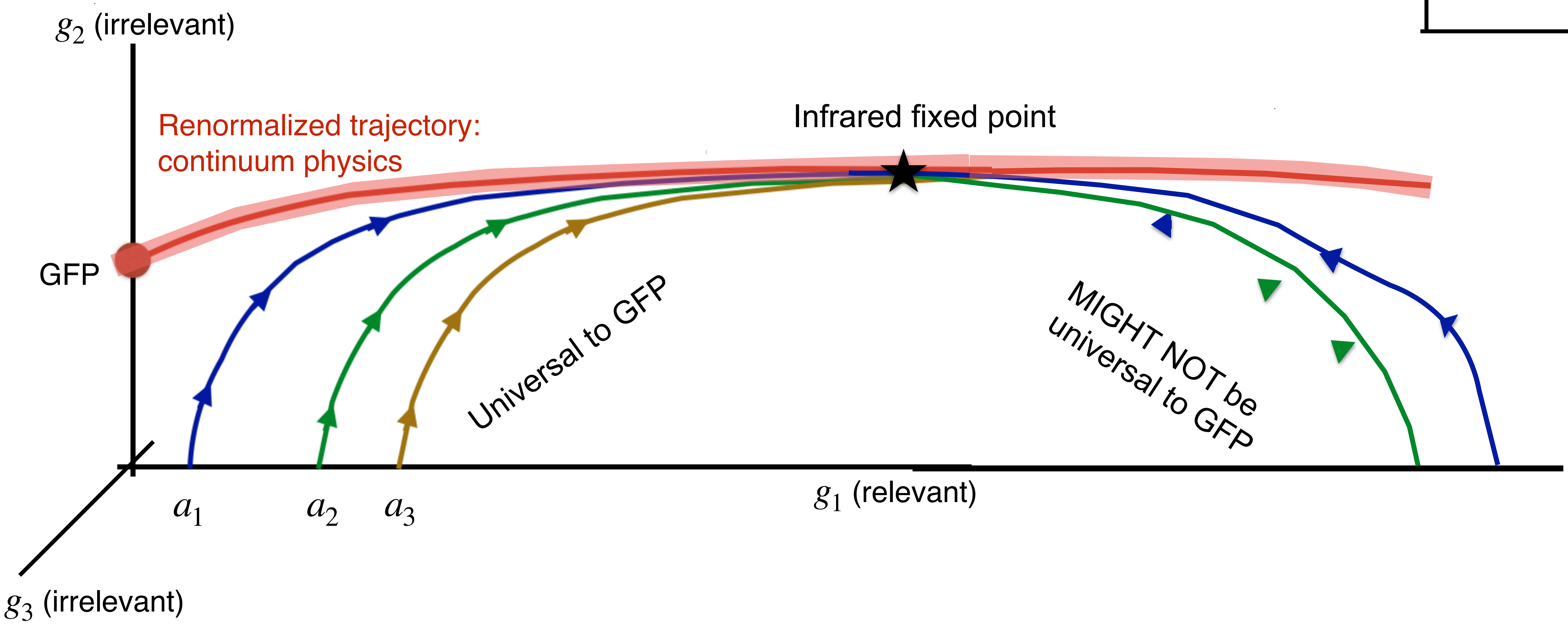


GF/RG in the conformal window



Bare action parameter space:

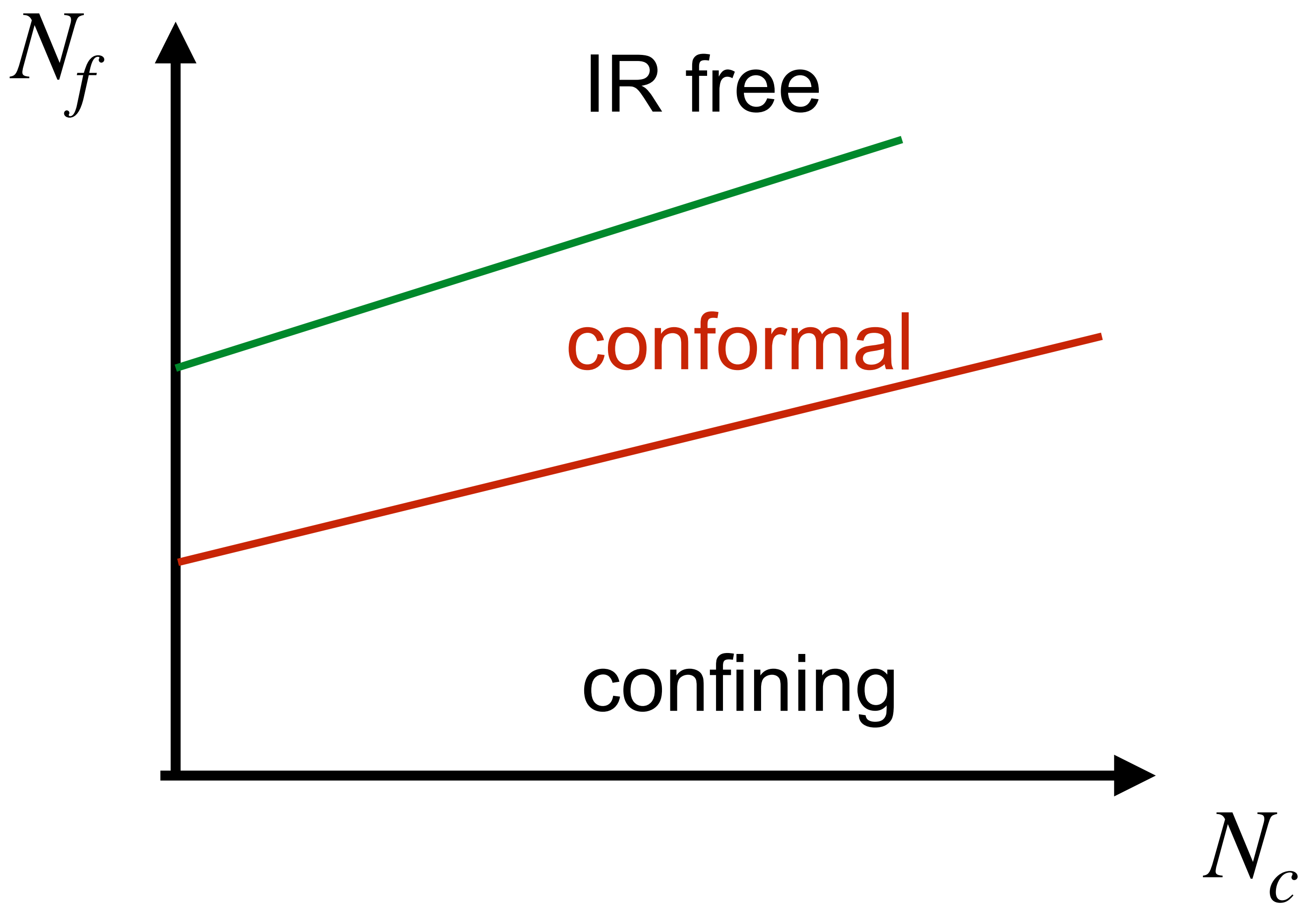
GF/RG: $\tau = 0 : S(g_i, m_i; a)$: RG flow to $\tau : S(g_i(\tau), m_i(\tau))$

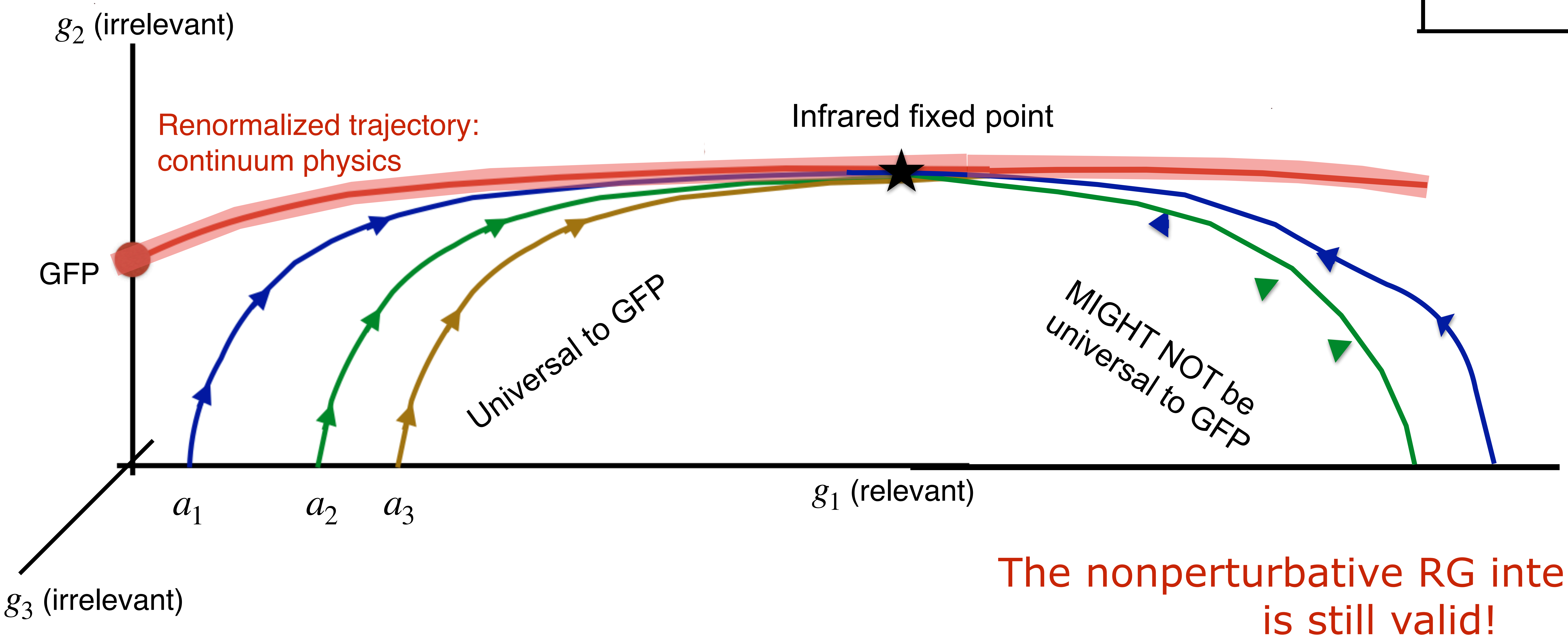


GF/RG in the conformal window

Bare action parameter space:

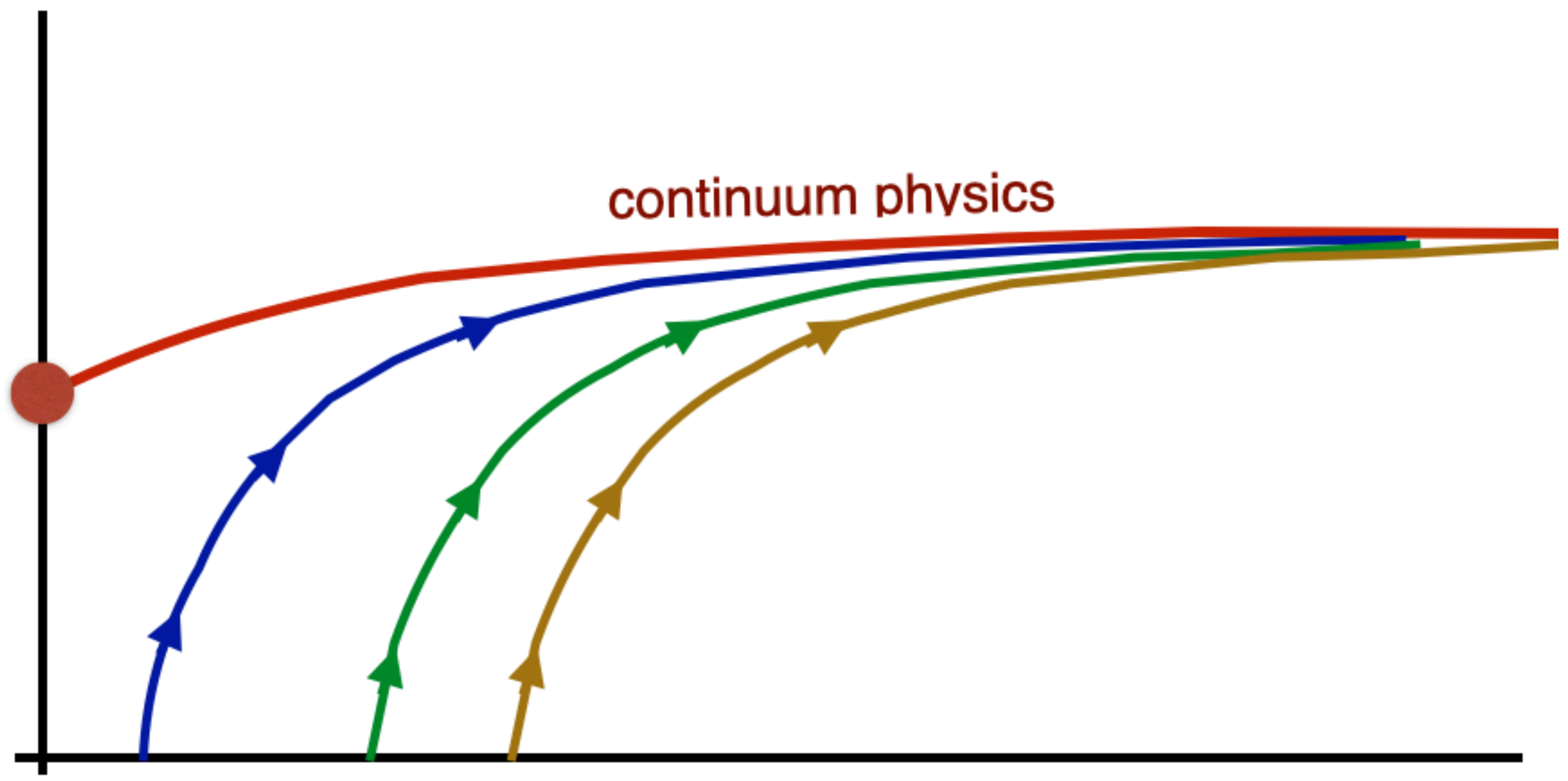
GF/RG: $\tau = 0: \; S(g_i, m_i; a) \;:$ RG flow to $\tau: \; S(g_i(\tau), m_i(\tau))$





The nonperturbative RG interpretation is still valid!

Limitations of GF



- The GF/RG picture is valid in infinite volume:
  - take the $1/L \to 0$ limit
    (at weak coupling $g^2(L=\infty) = g^2(L) + \frac{c}{L^4} + \ldots$)
- Step scaling uses finite volume ✓

- Need to cancel small-flow time cutoff effects
  - take the $a^2/\tau \to 0$ continuum limit (forces the bare coupling g_0^2 UVFP)

- Finite volume is OK at Gaussian FP and up to IRFP
  - not OK at an emerging UVFP or merged UV-IRFP

Yet there could be interesting strongly coupled physics!
(E.q. SU(2) with $N_f = 4$ or SU(3) with $N_f = 8$ flavors)

RG β function

The GF coupling : $g^2_{GF}(\tau, L; \beta) = \mathcal{N}\tau^2\langle E\rangle_\tau$

Luscher et al *Nucl.Phys.B* 413 (1994) 481
Fodor et al *JHEP* 11 (2012) 007

- Step scaling function: $\beta_{c,s}\left(g^2_{GF}(\tau, L; \beta_b)\right) = \dfrac{g^2_{GF}(s^2\tau, sL; \beta_b) - g^2_{GF}(\tau, L; \beta_b)}{\log(2s^2)}$

  L provides the length scale, $\tau = (cL)^2/8$
  Continuum limit: $L/a \to \infty$

- Continuous β function: $\beta(g^2_{GF}(\tau)) = -\tau \dfrac{d\, g^2_{GF}}{d\tau}$, $L/a \to \infty$

  Continuum limit : $a^2/\tau \to 0$

Fodor et al 1711.04833
AH, Witzel,PRD101 (2020) 3

- The two approaches are equivalent in the $s \to 0, c \to 0$ limits
- Require different analysis and have different numerical challenges

$\beta(g^2)$ can be determined in small volume (deconfined) simulations:
- $\beta_{GF}(g^2_{GF})$ is known for $g^2_{GF} \in (0+, 20+)$
- direct determination $g^2_{GF}(\tau)$ is possible only $g^2_{GF} \gtrsim 8$

RG β function - recent results:

N_f small : QCD like

Goal: determine the GF β function; predict Λ_{QCD}, α_s

(M. Luscher colloquium: "α_s demystified")

- Continuous β function studies:

  SU(3) gauge, no fermions — AH,Peterson,VanSickle,Witzel, arXiv: 2303.00704; Wong et al,2301.06611

  SU(3) gauge + $N_f = 3$ fundamental fermions; staggered (R. Wong's talk)

  SU(3) gauge + $N_f = 4$ fundamental fermions; staggered (Y. Mandlecha talk)

- Step scaling + decoupling

  Alpha collaboration

  Recent publication (A. Ramos' talk) — Dalla Brida et al, Nature volume 652, 328–334 (2026)

RG β function - recent results:

N_f small : QCD like

Goal: determine the GF β function; predict Λ_{QCD}, α_s
(M. Luscher colloquium: “α_s demystified”)

- Continuous β function studies:
  SU(3) gauge, no fermions — AH,Peterson,VanSickle,Witzel, arXiv: 2303.00704; Wong et al,2301.06611

  SU(3) gauge + $N_f = 3$ fundamental fermions; staggered (R. Wong’s talk)
  SU(3) gauge + $N_f = 4$ fundamental fermions; staggered (Y. Mandlecha talk)

- Step scaling + decoupling
  Alpha collaboration
  Recent publication (A. Ramos’ talk) — Dalla Brida et al, Nature volume 652, 328–334 (2026)

RG β function - recent results:

N_f small : QCD like

Goal: determine the GF β function; predict Λ_{QCD}, α_s
(M. Luscher colloquium: “α_s demystified”)

- Continuous β function studies:
  SU(3) gauge, no fermions

  AH,Peterson,VanSickle,Witzel, arXiv: 2303.00704
  Wong et al,2301.06611

  SU(3) gauge + $N_f = 3$ fundamental fermions; staggered (R. Wong’s talk)
  SU(3) gauge + $N_f = 4$ fundamental fermions; staggered (Y. Mandlecha talk)

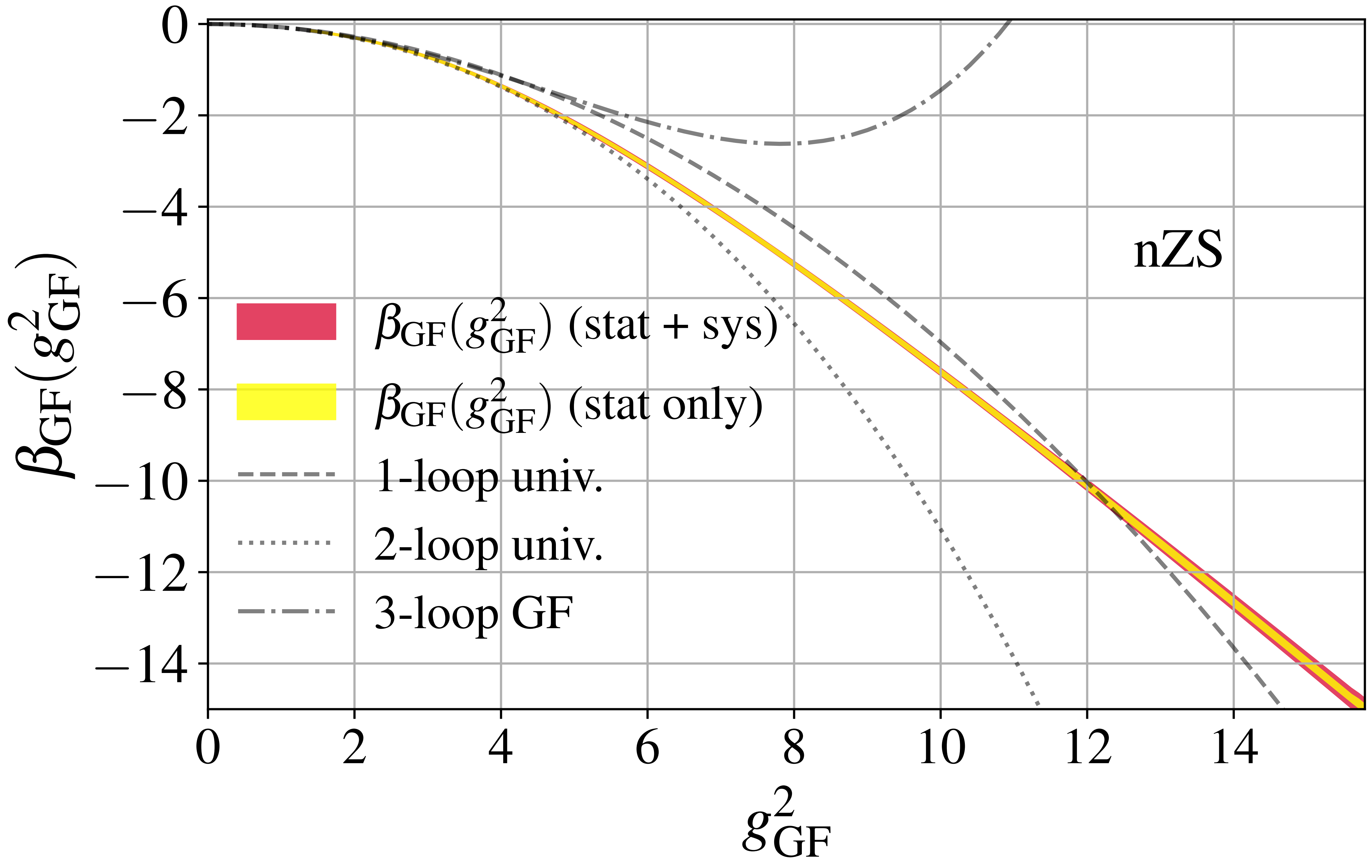
- Step scaling + decoupling
  Alpha collaboration
  Recent publication (A. Ramos’ talk)

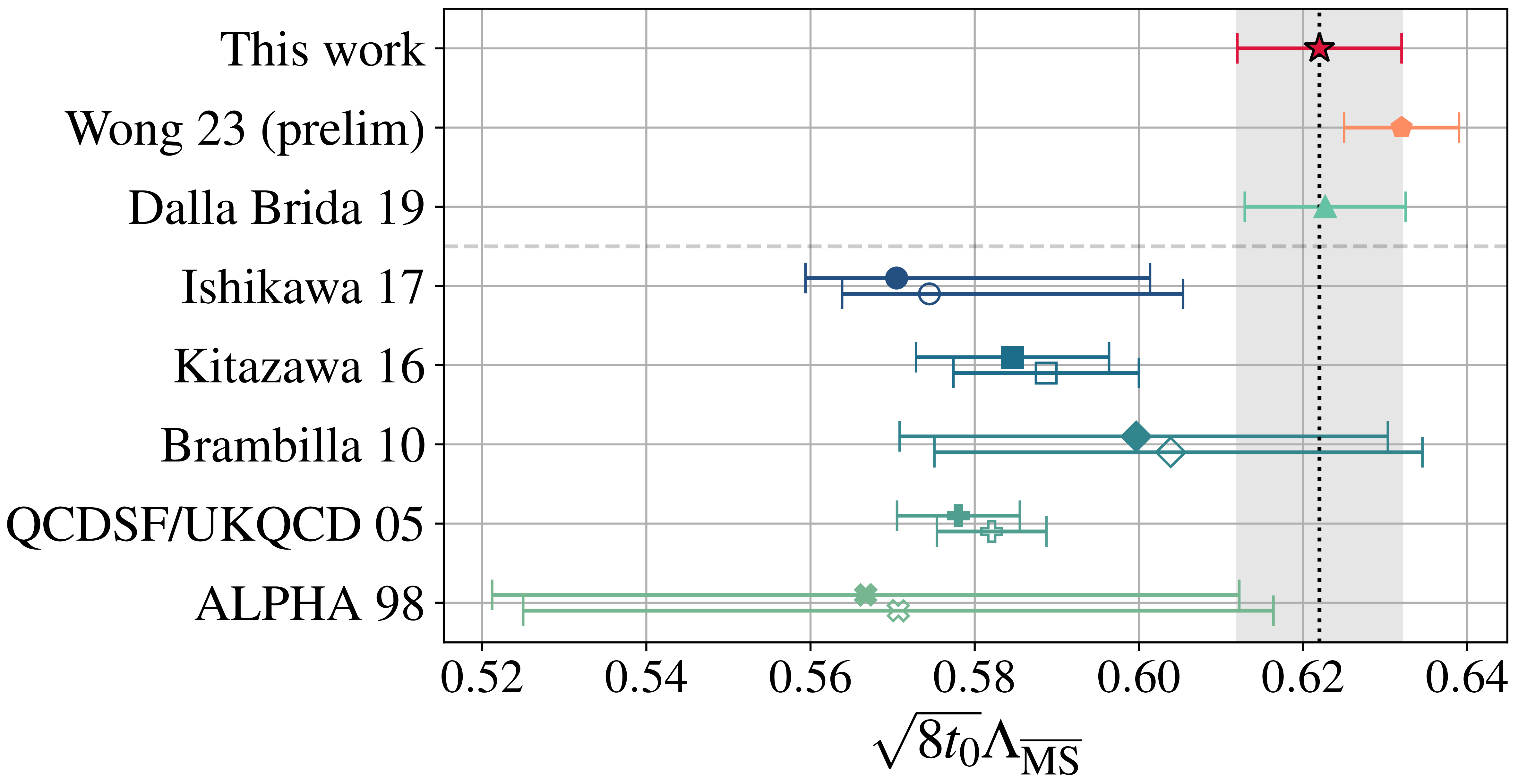
  Dalla Brida et al, Nature volume 652, 328–334 (2026)

RG β function - recent results:

AH,Peterson,VanSickle,Witzel, arXiv: 2303.00704

$N_f = 0$ GF β function, Λ_{QCD}



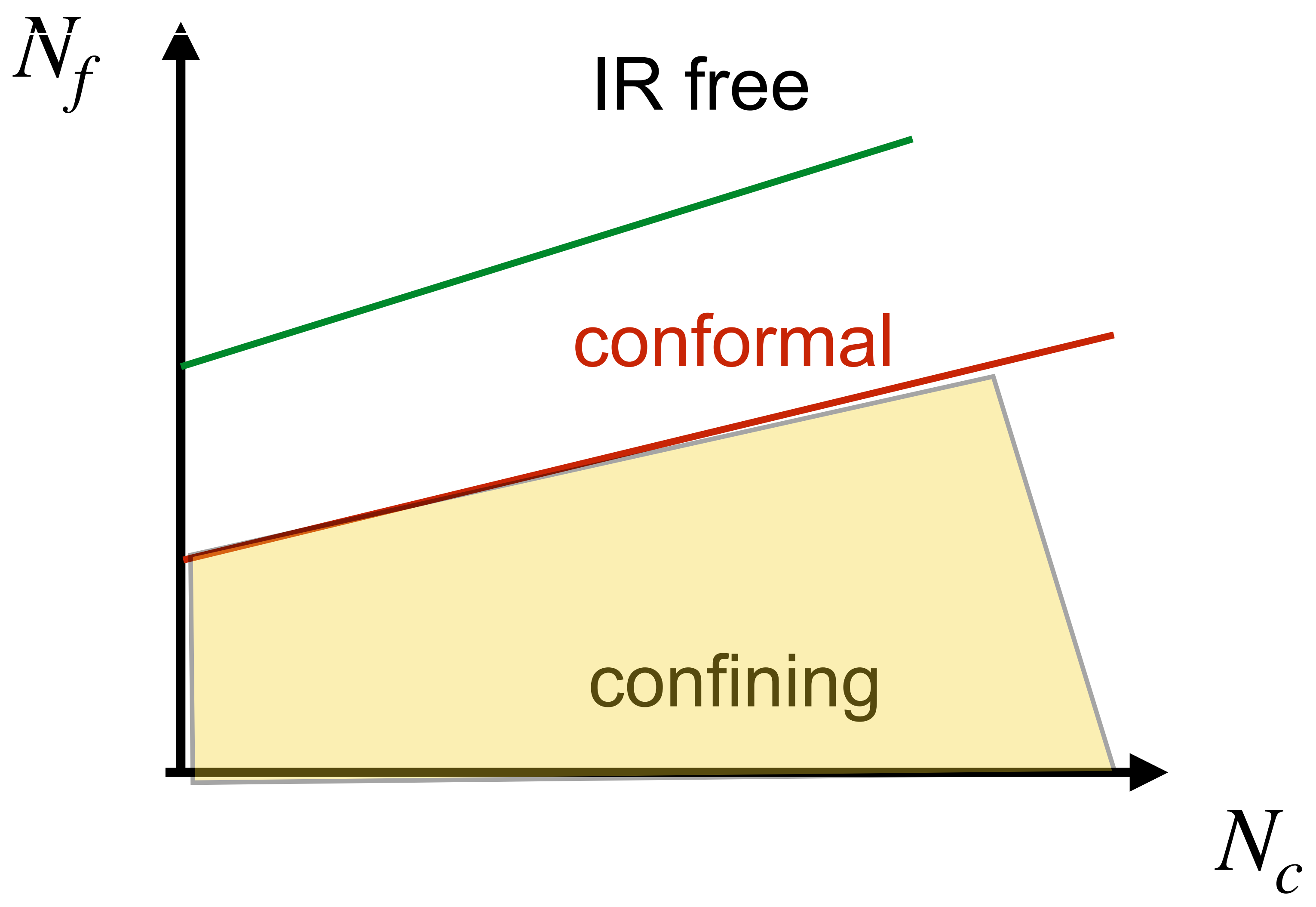


RG β function
- maps into perturbative curves
- almost linear in the strong coupling

$\Lambda_{\overline{MS}}$ consistent with other GF results
in tension with other (older) results

RG β function -conformal systems

$N_f > N_f^*$: within the conformal window

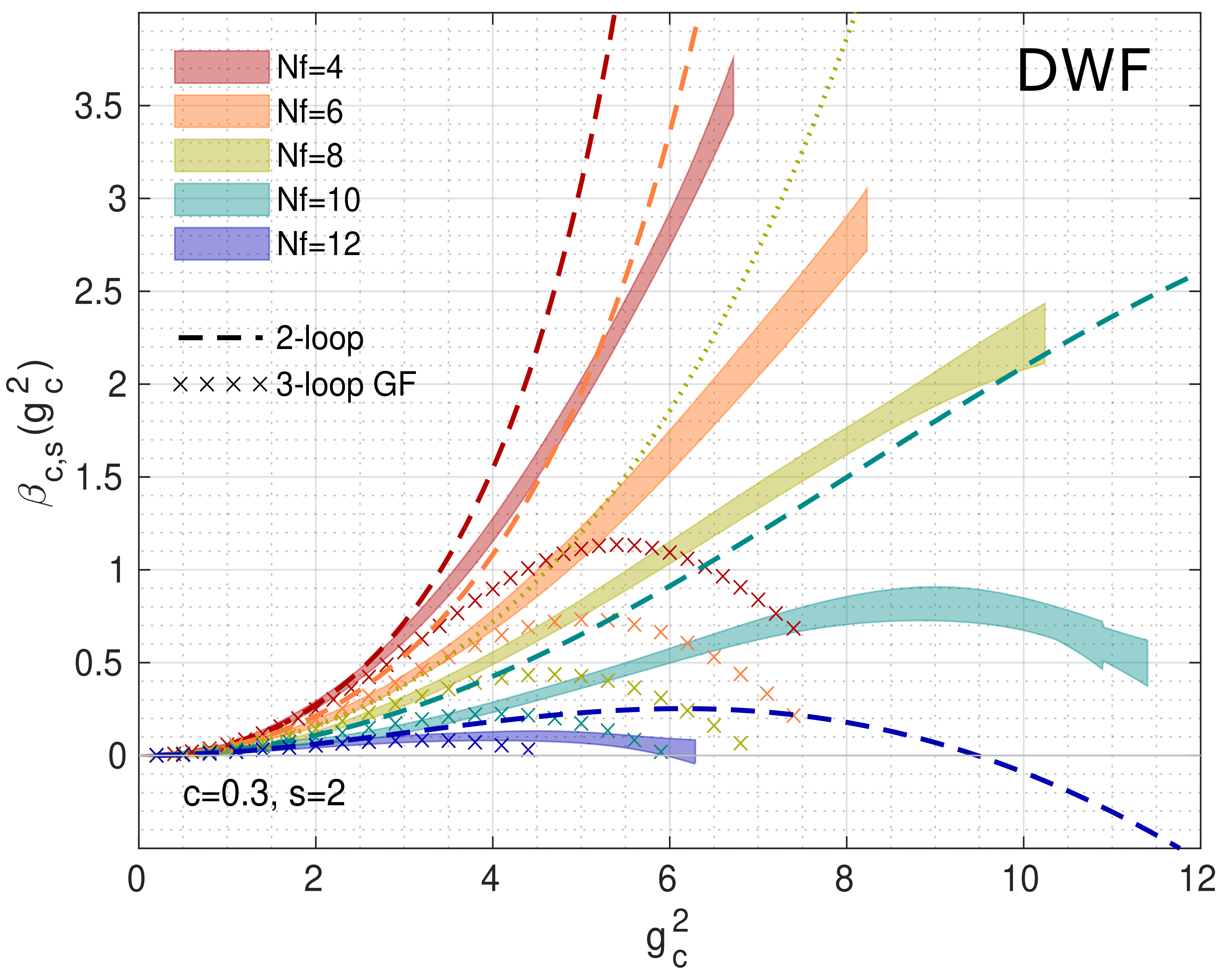


Goal:
- Identify the IRFP
- Find the opening of the conformal window
- Find universal critical exponents

- SU(3) gauge $N_f = 2 - 12$ domain wall
- SU(3) gauge + $N_f = 12$ staggered lattice fermions
- SU(3) gauge + $N_f = 10$ Wilson&staggered lattice fermion (N. Mackey's talk)
- SU(4) gauge + $N_f = 4 + 4$ fundamental+sextet fermions; Wilson lattice fermions
- SU(3) gauge + $N_f = 8$ staggered lattice fermions - conformal + new SMG phase!
- SU(2) gauge + $N_f = 4$ staggered- conformal + new SMG phase!

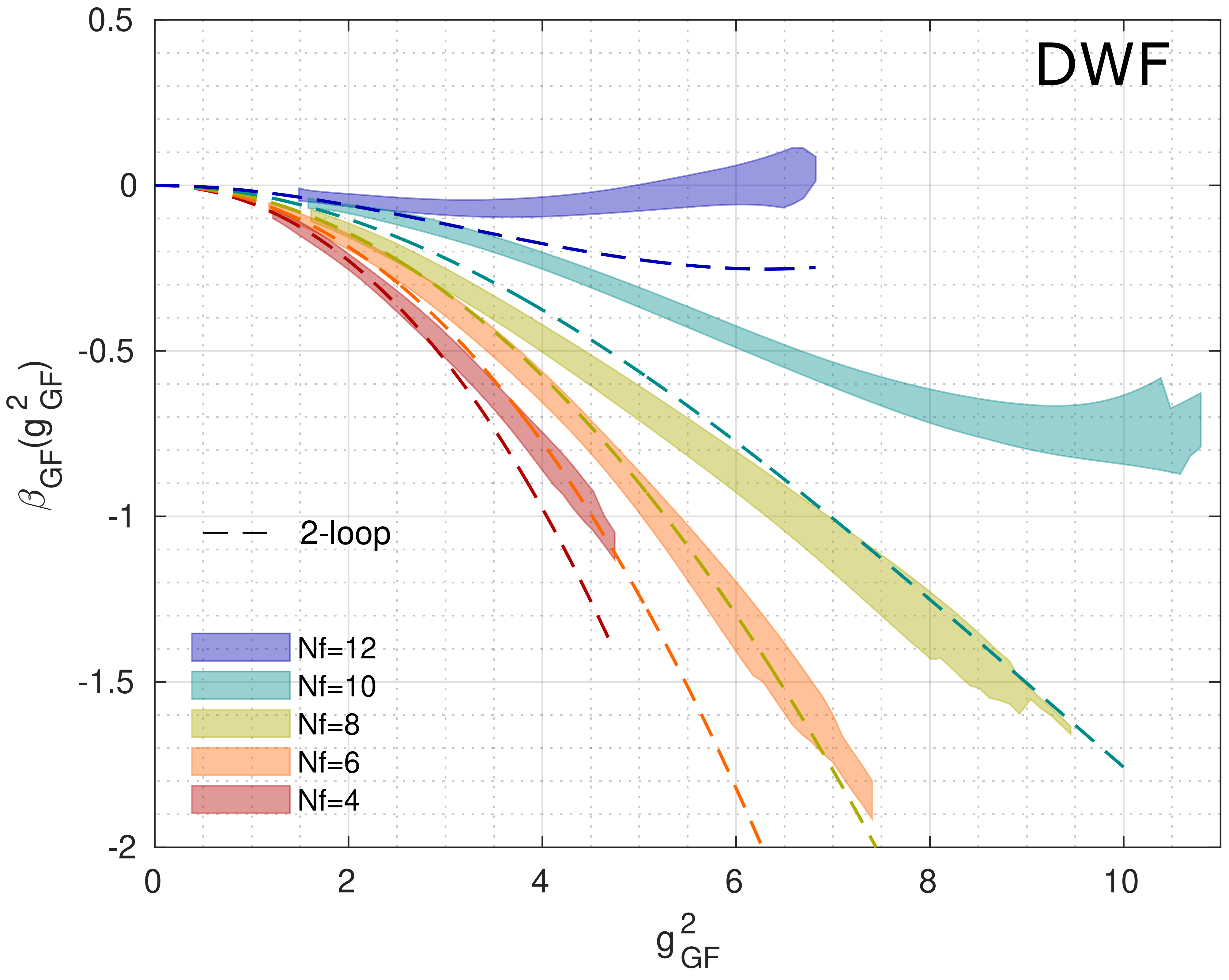
RG β function - compare $N_f = 2 - 12$

Step scaling



AH,Rebbi,Witzel, PRD107 (2022) 11

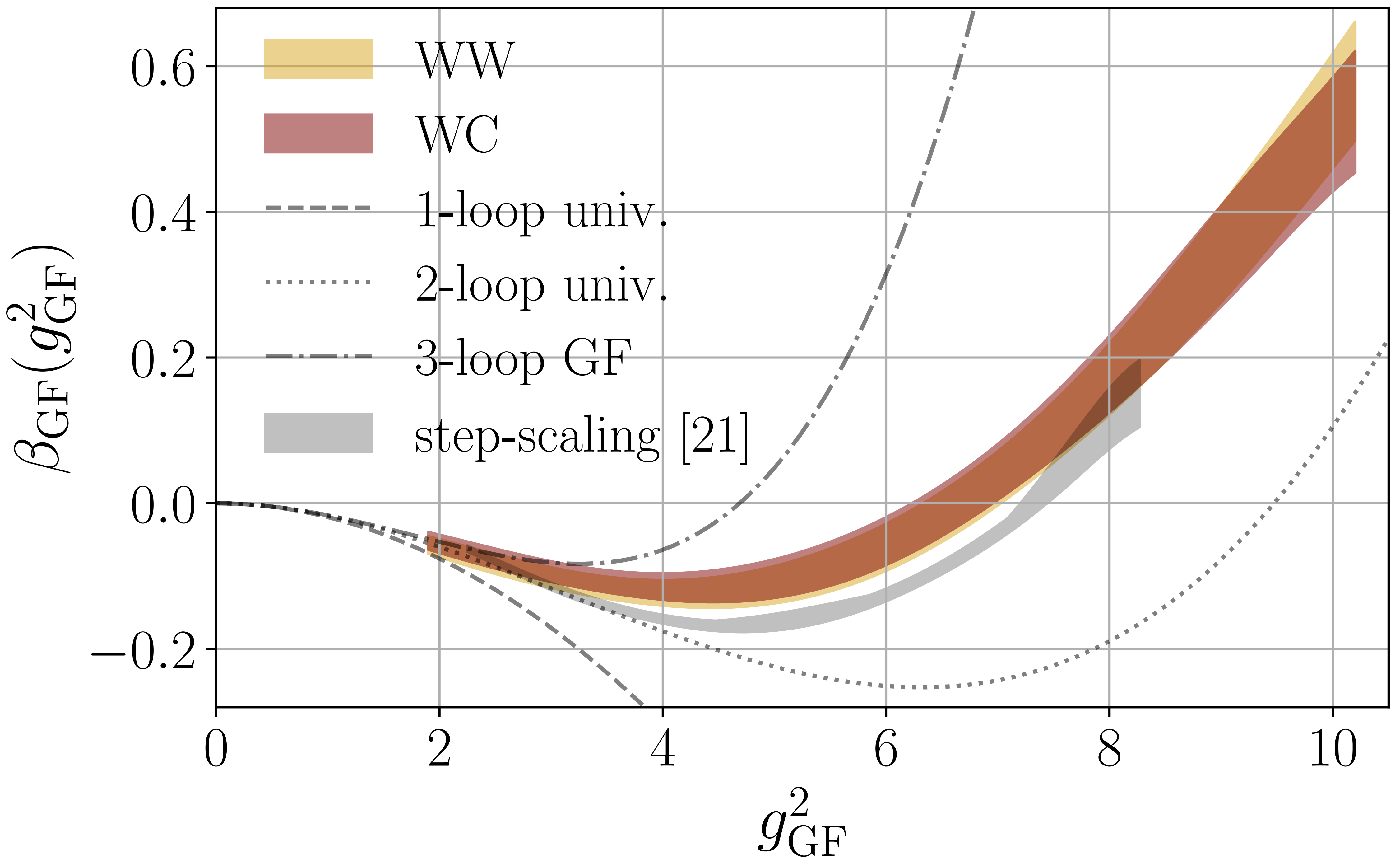
Continuous β function



AH,Witzel, Lattice 2023

RG β function - $N_f = 10, 12$

$N_f = 12$: step scaling & continuous β

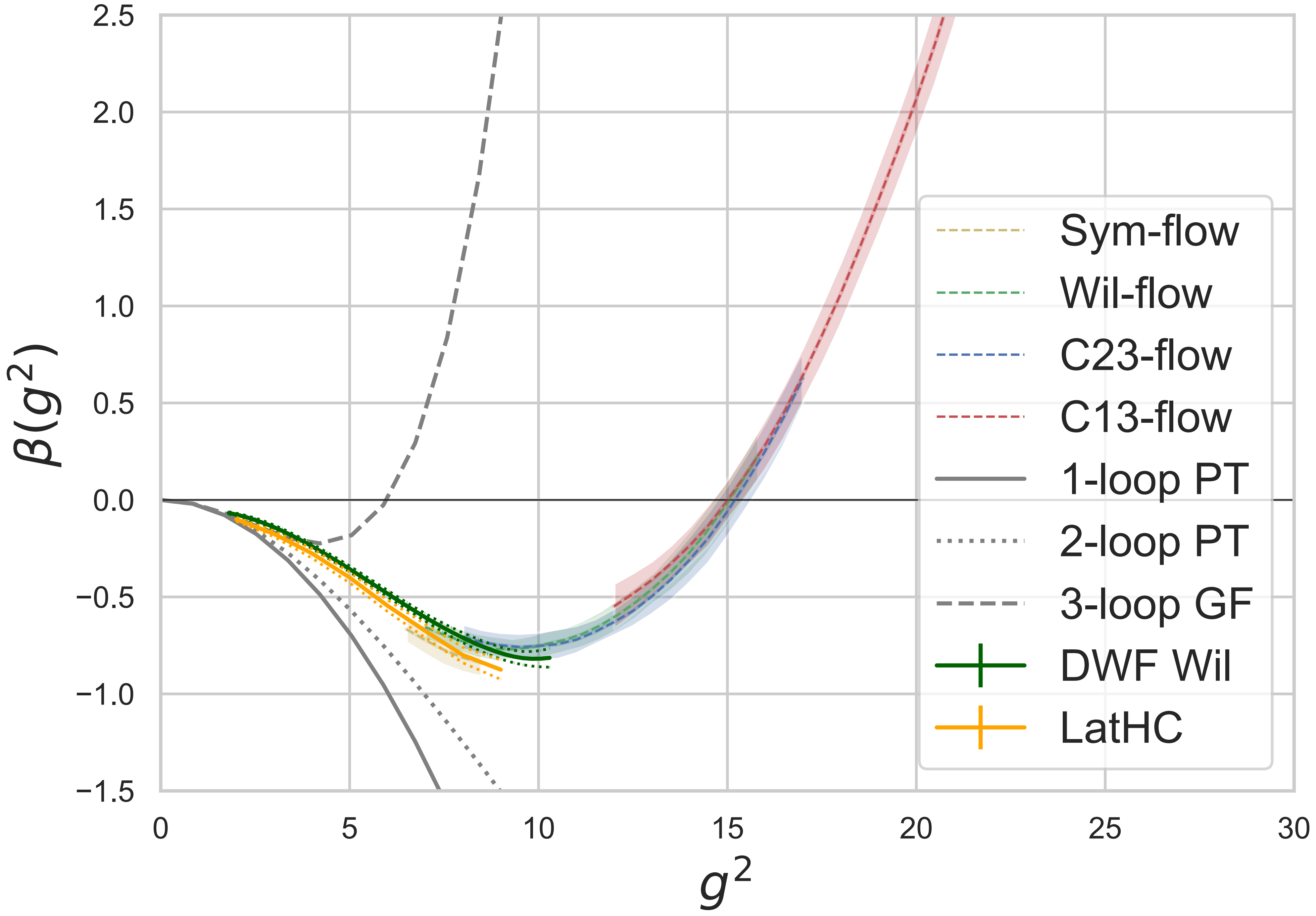


Peterson, Hasenfratz, PRD109 (2024) 11

[21] Hasenfratz, Schaich *JHEP* 02 (2018) 132
(step scaling - different scheme!)

$N_f = 10$: continuous β fn; Wilson fermions

See N. Mackey's talk for staggered



Hasenfratz, Neil, Shamir, Svetitsky, Witzel, PRD 108 (2023) 7

Gradient flow for fermions

- Fermion flow on gauge background

$$\partial_\tau \chi(\tau;x) = \Delta\, \chi(\tau;x)\,, \quad \partial_\tau \bar{\chi}(\tau;x) = \bar{\chi}(\tau;x)\overleftarrow{\Delta}$$

- Only wave function renormalization needed

$$\chi_R(\tau;x) \;=\; Z_\chi^{1/2}(\tau)\,\chi(\tau;x)$$
$$O_{\rm GF}(\tau;x) = Z_\chi^{n/2}(\tau)\, O(\tau;x)$$

- Connect to $\overline{MS}$ with short flow time expansion (SFTX):

$$O_{\overline{\rm MS}}(\mu;x) = \lim_{\tau\to 0} \zeta_O^{-1}(\mu,\tau)\, O_{\rm GF}(\tau;x)$$

Gradient flow for fermions

- Fermion flow on gauge background

$$\partial_\tau \chi(\tau;x) = \Delta\,\chi(\tau;x)\,, \quad \partial_\tau \bar{\chi}(\tau;x) = \bar{\chi}(\tau;x)\overleftarrow{\Delta}$$

- Only wave function renormalization needed

$$\chi_R(\tau;x) \;=\; Z_\chi^{1/2}(\tau)\,\chi(\tau;x)$$

$$O_{\mathrm{GF}}(\tau;x) = Z_\chi^{n/2}(\tau)\,O(\tau;x)$$

- Connect to $\overline{MS}$ with short flow time expansion (SFTX):

$$O_{\overline{\mathrm{MS}}}(\mu;x) = \lim_{\tau\to 0} \zeta_O^{-1}(\mu,\tau)\,O_{\mathrm{GF}}(\tau;x)$$

Correlators $\langle O(\tau;x)O_{probe}\rangle$ are straightforward to calculate

Renormalized correlators $\langle O_{GF}(\tau;x)O_{probe}\rangle = Z_\chi\langle O(\tau;x)O_{probe}\rangle$ require Z_χ

- If only ratios are considered, Z_χ is not needed:
  - Lifetime, bag parameter (A. Rago talk)
  - Quark masses (O. Witzel, A. Chauhan talk)
  - PDFs (A. Shindler)
- If absolute normalization is needed, we need Z_χ

GF renormalization schemes

Different Z_χ definitions lead to different renormalization schemes

Ringed scheme:

Makino, Suzuki, *PTEP* 2014 (2014) 063B02

$$\mathring{Z}_\chi(\tau) \propto \tau^2 \left\langle \bar{\chi}(\tau;x) \overleftrightarrow{\not{D}} \chi(\tau;x) \right\rangle^{-1/2}$$

$$O_{\overline{\mathrm{MS}}}(\mu) = \lim_{\tau \to 0} \mathring{\zeta}_O^{-1}(\mu,\tau)\, \mathring{Z}_\chi(\tau)\, O(\tau)$$

$\mathring{\zeta}_O(\mu,\tau)$ from SFTX

Borgulat et al, *JHEP* 05 (2024) 179

$\mathring{Z}_\chi$ is difficult to evaluate on the lattice;

$\langle \bar{\chi}(\tau;x) \overleftrightarrow{\not{D}} \chi(\tau;x) \rangle$ requires “backward” flow

GF renormalization schemes

Different Z_χ definitions lead to different renormalization schemes

Ringed scheme:

Makino, Suzuki, *PTEP* 2014 (2014) 063B02

$$\mathring{Z}_\chi(\tau) \propto \tau^2 \left\langle \bar{\chi}(\tau;x)\, \overleftrightarrow{\not{D}} \chi(\tau;x) \right\rangle^{-1/2}$$

$$O_{\overline{\mathrm{MS}}}(\mu) = \lim_{\tau\to 0} \mathring{\zeta}_O^{-1}(\mu,\tau)\, \mathring{Z}_\chi(\tau)\, O(\tau)$$

$\mathring{\zeta}_O(\mu,\tau)$ from SFTX

Borgulat et al, *JHEP* 05 (2024) 179

$\mathring{Z}_\chi$ is difficult to evaluate on the lattice;
$\langle \bar{\chi}(\tau;x)\, \overleftrightarrow{\not{D}} \chi(\tau;x) \rangle$ requires “backward” flow

Vector scheme:

The vector current is conserved :

- in $\overline{MS}$ scheme $V_{\overline{MS}}(\mu) = V_{\mathrm{bare}}$

Reproduce that in GF

- $\widetilde{Z}_\chi^{(V)}(\tau) \langle V(\tau) V_{\mathrm{probe}} \rangle = \langle V_{\mathrm{bare}} V_{\mathrm{probe}} \rangle$

$$O_{\overline{\mathrm{MS}}}(\mu) = \lim_{\tau\to 0} \widetilde{\zeta}_O^{-1}(\mu,\tau)\, \widetilde{Z}_\chi^{(V)}(\tau)\, O(\tau) \;;$$

$\widetilde{\zeta}_O(\mu,\tau)$ are new matching coefficients

GF renormalization schemes

Different Z_χ definitions lead to different renormalization schemes

Ringed scheme:

Makino, Suzuki, *PTEP* 2014 (2014) 063B02

$$\mathring{Z}_\chi(\tau) \propto \tau^2 \left\langle \bar{\chi}(\tau;x) \overleftrightarrow{\not{D}} \chi(\tau;x) \right\rangle^{-1/2}$$

Definition!

$$O_{\overline{\rm MS}}(\mu) = \lim_{\tau\to 0} \mathring{\zeta}_O^{-1}(\mu,\tau)\, \mathring{Z}_\chi(\tau)\, O(\tau)$$

$\mathring{\zeta}_O(\mu,\tau)$ from SFTX

Borgulat et al, *JHEP* 05 (2024) 179

$\mathring{Z}_\chi$ is difficult to evaluate on the lattice;

$\langle \bar{\chi}(\tau;x) \overleftrightarrow{\not{D}} \chi(\tau;x) \rangle$ requires "backward" flow

Vector scheme:

The vector current is conserved :

- in $\overline{MS}$ scheme $V_{\overline{MS}}(\mu) = V_{\rm bare}$

Reproduce that in GF

- $\widetilde{Z}_\chi^{(V)}(\tau)\langle V(\tau) V_{\rm probe}\rangle = \langle V_{\rm bare} V_{\rm probe}\rangle$

Definition!

$$O_{\overline{\rm MS}}(\mu) = \lim_{\tau\to 0} \widetilde{\zeta}_O^{-1}(\mu,\tau)\, \widetilde{Z}_\chi^{(V)}(\tau)\, O(\tau) \;;$$

$\widetilde{\zeta}_O(\mu,\tau)$ are new matching coefficients

GF renormalization schemes

Different Z_χ definitions lead to different renormalization schemes

Ringed scheme:

Makino, Suzuki, *PTEP* 2014 (2014) 063B02

$$\mathring{Z}_\chi(\tau) \propto \tau^2 \left\langle \bar{\chi}(\tau;x) \overleftrightarrow{\not{D}} \chi(\tau;x) \right\rangle^{-1/2}$$

Definition!

$$O_{\overline{\mathrm{MS}}}(\mu) = \lim_{\tau\to 0} \mathring{\zeta}_O^{-1}(\mu,\tau)\, \mathring{Z}_\chi(\tau)\, O(\tau)$$

$\mathring{\zeta}_O(\mu,\tau)$ from SFTX

Borgulat et al, *JHEP* 05 (2024) 179

$\mathring{Z}_\chi$ is difficult to evaluate on the lattice;
$\langle \bar{\chi}(\tau;x) \overleftrightarrow{\not{D}} \chi(\tau;x) \rangle$ requires "backward" flow

Vector scheme:

The vector current is conserved :

- in $\overline{MS}$ scheme $V_{\overline{MS}}(\mu) = V_{\mathrm{bare}}$

Reproduce that in GF

- $\widetilde{Z}_\chi^{(V)}(\tau) \langle V(\tau) V_{\mathrm{probe}} \rangle = \langle V_{\mathrm{bare}} V_{\mathrm{probe}} \rangle$

Definition!

$$O_{\overline{\mathrm{MS}}}(\mu) = \lim_{\tau\to 0} \widetilde{\zeta}_O^{-1}(\mu,\tau)\, \widetilde{Z}_\chi^{(V)}(\tau)\, O(\tau)\ ;$$

$\widetilde{\zeta}_O(\mu,\tau)$ are new matching coefficients

$\widetilde{\zeta}_O(\mu,\tau) = \mathring{\zeta}_O(\mu,\tau) \mathring{\zeta}_V^{-1}(\mu,\tau)$ from SFTX

$\widetilde{Z}_\chi^{(V)}(\tau)$ is straightforward to calculate on the lattice
Matches to $\overline{MS}$ with existing SFTX coefficients

GF renormalization schemes

Different Z_χ definitions lead to different renormalization schemes

Ringed scheme:

Makino, Suzuki, *PTEP* 2014 (2014) 063B02

$$\mathring{Z}_\chi(\tau) \propto \tau^2 \left\langle \bar{\chi}(\tau;x) \overleftrightarrow{\not{D}} \chi(\tau;x) \right\rangle^{-1/2}$$

Definition!

$$O_{\overline{\rm MS}}(\mu) = \lim_{\tau\to 0} \mathring{\zeta}_O^{-1}(\mu,\tau)\, \mathring{Z}_\chi(\tau)\, O(\tau)$$

$\mathring{\zeta}_O(\mu,\tau)$ from SFTX

Borgulat et al, *JHEP* 05 (2024) 179

$\mathring{Z}_\chi$ is difficult to evaluate on the lattice;

$\langle \bar{\chi}(\tau;x) \overleftrightarrow{\not{D}} \chi(\tau;x)\rangle$ requires "backward" flow

Vector scheme:

Axial scheme: $V \to A_0$

The vector current is conserved :

- in $\overline{MS}$ scheme $V_{\overline{MS}}(\mu) = V_{\rm bare}$

Reproduce that in GF

- $\widetilde{Z}_\chi^{(V)}(\tau)\langle V(\tau) V_{\rm probe}\rangle = \langle V_{\rm bare} V_{\rm probe}\rangle$

Definition!

$$O_{\overline{\rm MS}}(\mu) = \lim_{\tau\to 0} \widetilde{\zeta}_O^{-1}(\mu,\tau)\, \widetilde{Z}_\chi^{(V)}(\tau)\, O(\tau)\ ;$$

$\widetilde{\zeta}_O(\mu,\tau)$ are new matching coefficients

$\widetilde{\zeta}_O(\mu,\tau) = \mathring{\zeta}_O(\mu,\tau)\mathring{\zeta}_V^{-1}(\mu,\tau)$ from SFTX

$\widetilde{Z}_\chi^{(V)}(\tau)$ is straightforward to calculate on the lattice

Matches to $\overline{MS}$ with existing SFTX coefficients

GF A-scheme

GF renormalized operator :

$$\widetilde{O}^{(A)}_{GF}(\tau) = \widetilde{Z}^{(A)}_{\chi}(\tau) O(\tau) \;, \quad \widetilde{O}_{\overline{MS}}(\mu) = \lim_{\tau \to 0} (\mathring{\zeta}_O^{-1} \mathring{\zeta}_A)(\mu;\tau)\, \widetilde{Z}^{(A)}_{\chi}(\tau) O(\tau)$$

GF A-scheme

$$\widetilde{Z}_{\chi}^{(A)}(\tau)\langle A(\tau)A_{\text{probe}}\rangle = \langle A_{\text{bare}}A_{\text{probe}}\rangle$$

GF renormalized operator :

$$\widetilde{O}_{GF}^{(A)}(\tau) = \widetilde{Z}_{\chi}^{(A)}(\tau)O(\tau) \ , \quad \widetilde{O}_{\overline{MS}}(\mu) = \lim_{\tau\to 0}(\mathring{\zeta}_{O}^{-1}\mathring{\zeta}_{A})(\mu;\tau)\,\widetilde{Z}_{\chi}^{(A)}(\tau)O(\tau)$$

$$\widetilde{Z}_{\chi}^{(A)}(\tau) = \frac{Z_A\langle A_0(\tau=0;t)A_0(\tau=0;t=0)\rangle}{\langle A_0(\tau;t)A_0(\tau=0;t=0)\rangle} \equiv Z_A R_A(\tau)$$

if $A_{\text{bare}} \equiv A_0^{\text{conserved}} = Z_A A_0^{\text{lattice}}$, $A_{\text{probe}} = A_0(\tau=0)$

GF A-scheme

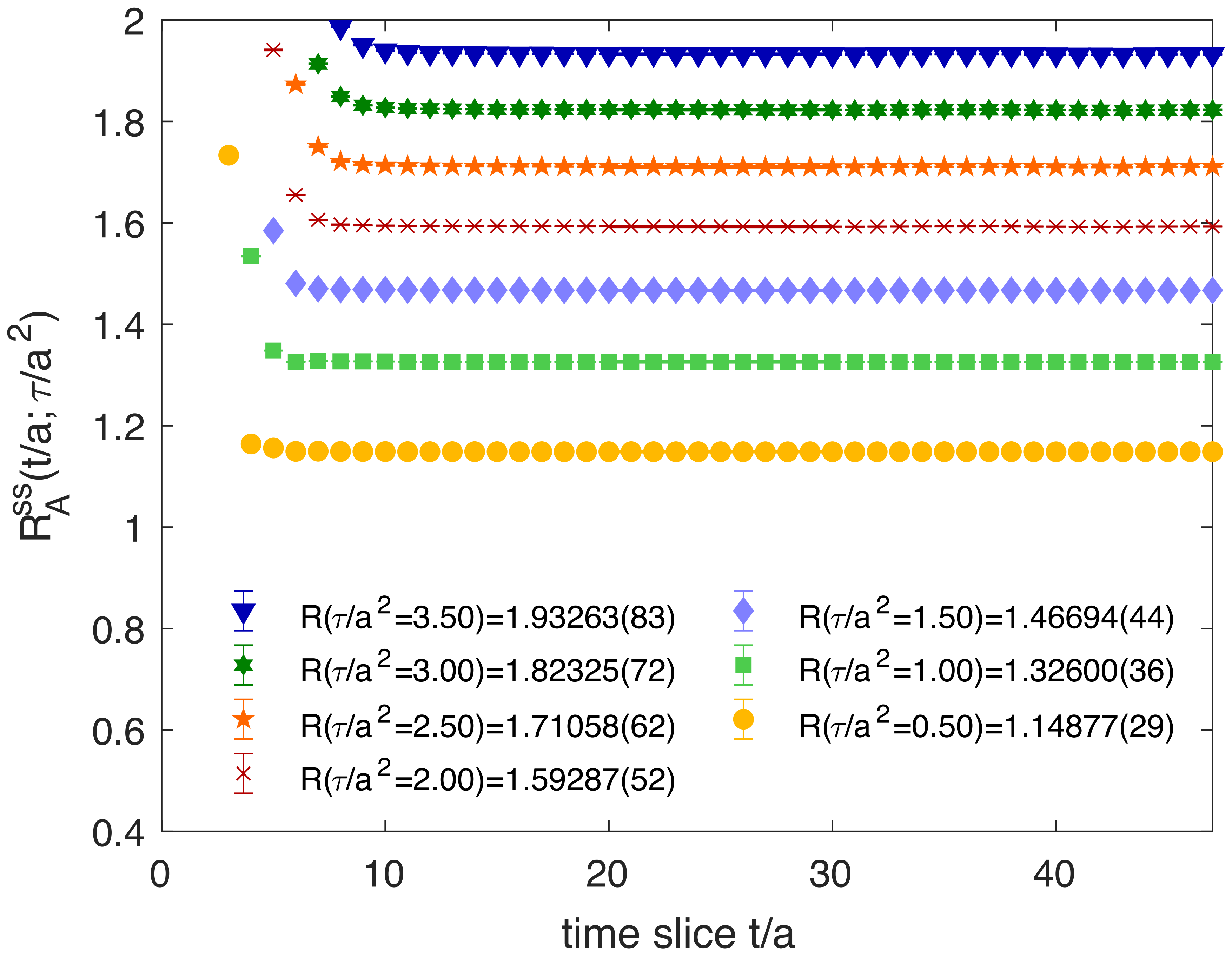
$$\widetilde{Z}^{(A)}_{\chi}(\tau)\langle A(\tau)A_{\rm probe}\rangle = \langle A_{\rm bare}A_{\rm probe}\rangle$$

GF renormalized operator :

$$\widetilde{O}^{(A)}_{GF}(\tau) = \widetilde{Z}^{(A)}_{\chi}(\tau)O(\tau) \;, \quad \widetilde{O}_{\overline{MS}}(\mu) = \lim_{\tau\to 0}(\mathring{\zeta}_O^{-1}\mathring{\zeta}_A)(\mu;\tau)\,\widetilde{Z}^{(A)}_{\chi}(\tau)O(\tau)$$

$$\widetilde{Z}^{(A)}_{\chi}(\tau) = \frac{Z_A\langle A_0(\tau=0;t)A_0(\tau=0;t=0)\rangle}{\langle A_0(\tau;t)A_0(\tau=0;t=0)\rangle} \equiv Z_A R_A(\tau)$$

if $A_{\rm bare} \equiv A_0^{\rm conserved} = Z_A A_0^{\rm lattice}$, $A_{\rm probe} = A_0(\tau=0)$



$R_A(\tau)$ is independent of Euclidean separation if $t \to \infty$ since both numerator and denominator couples to the same ground state

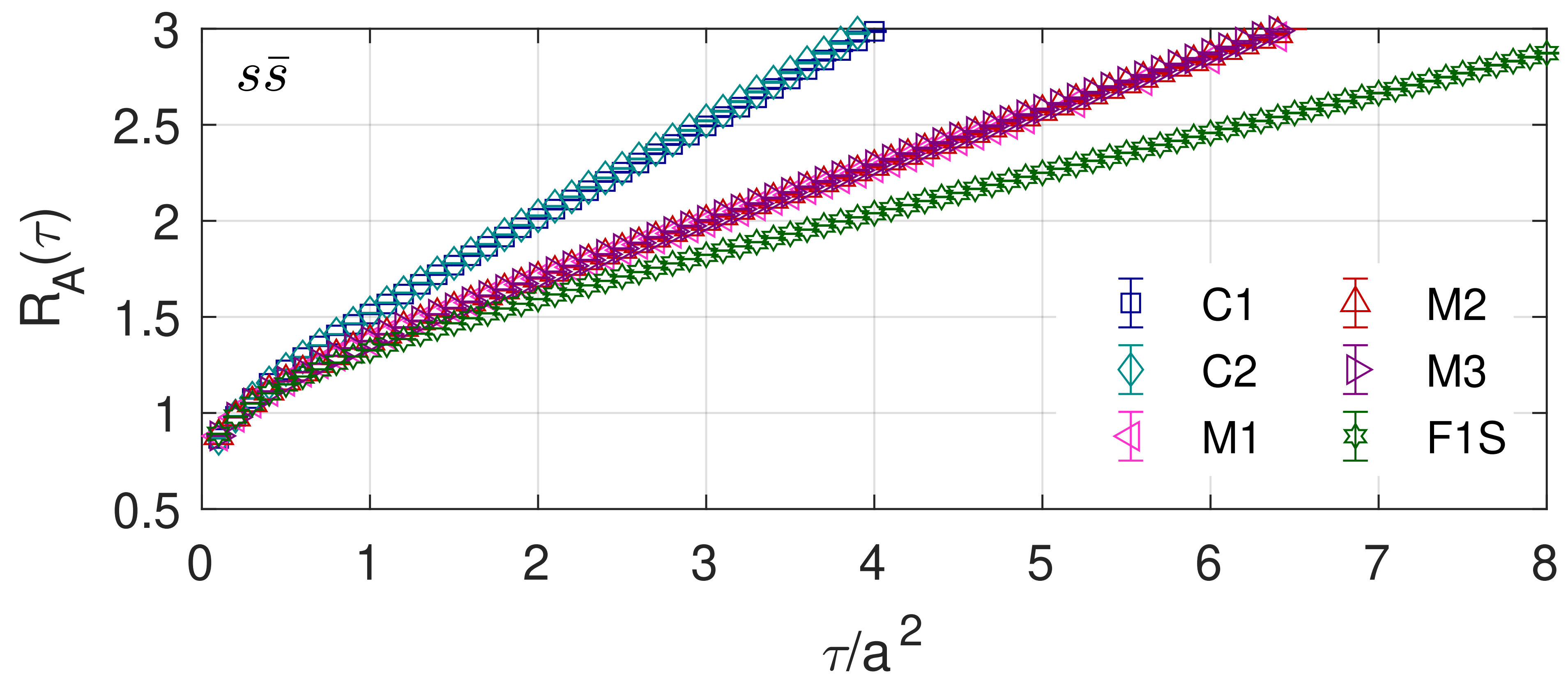
Numerical study: 2+1 flavor DWF fermions as
in Black et al 2506.16327, 2603.28516, 2603.28517
See O. Witzel talk

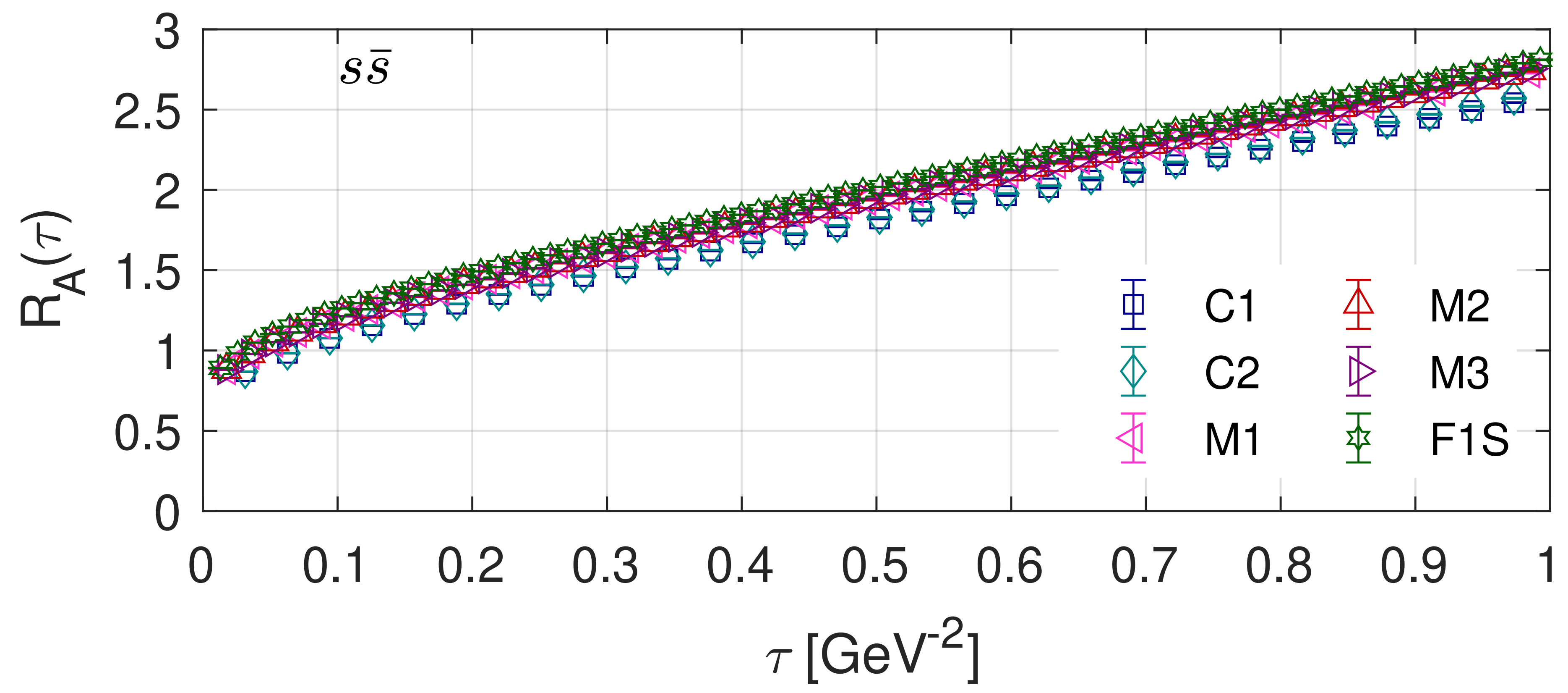
GF A-scheme

GF renormalized operator :

$$\widetilde{O}^{(A)}_{GF}(\tau) = \widetilde{Z}^{(A)}_{\chi}(\tau) O(\tau) \ , \quad \widetilde{O}_{\overline{MS}}(\mu) = \lim_{\tau \to 0} (\mathring{\zeta}_O^{-1} \mathring{\zeta}_A)(\mu; \tau) \, \widetilde{Z}^{(A)}_{\chi}(\tau) O(\tau) \ ,$$

$\widetilde{Z}^{(A)}_{\chi}(\tau) = Z_A R_A(\tau)$ is divergent, but only logarithmically



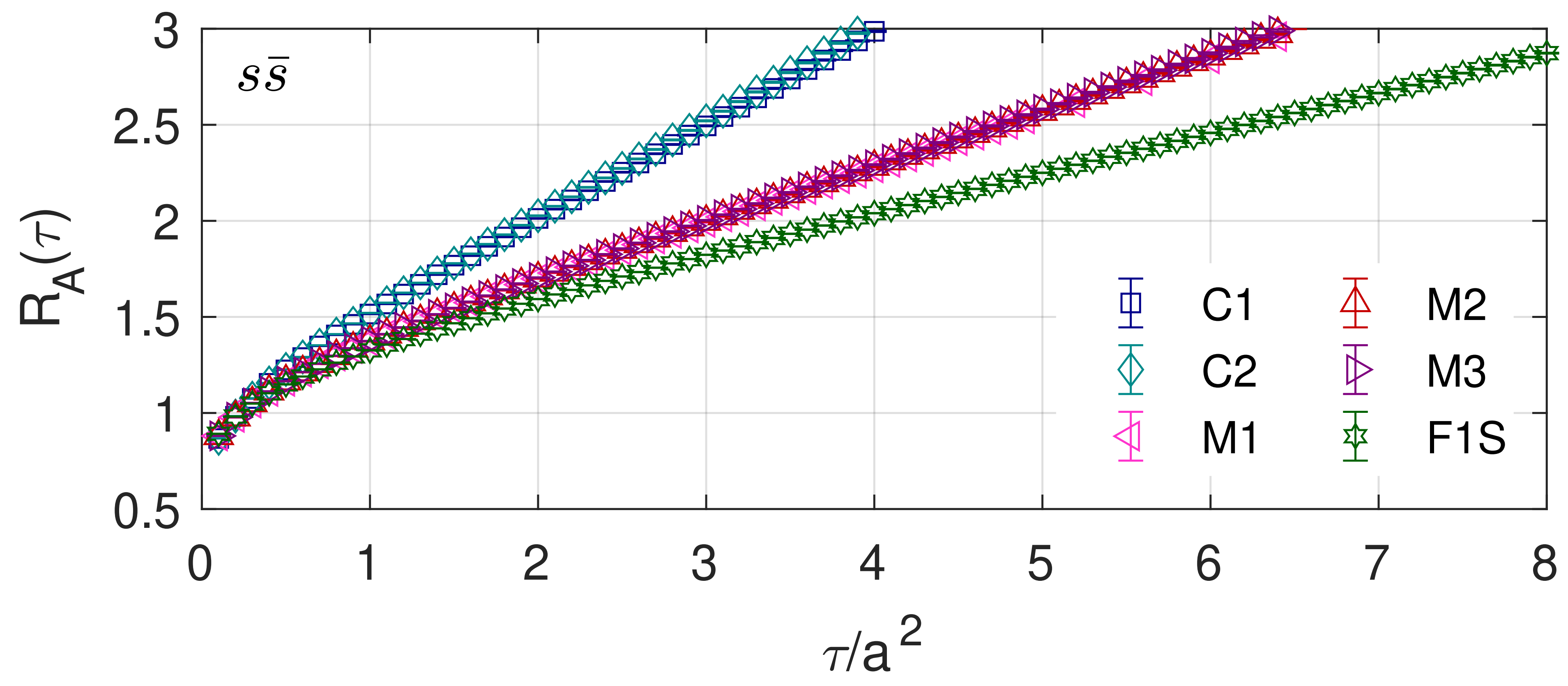


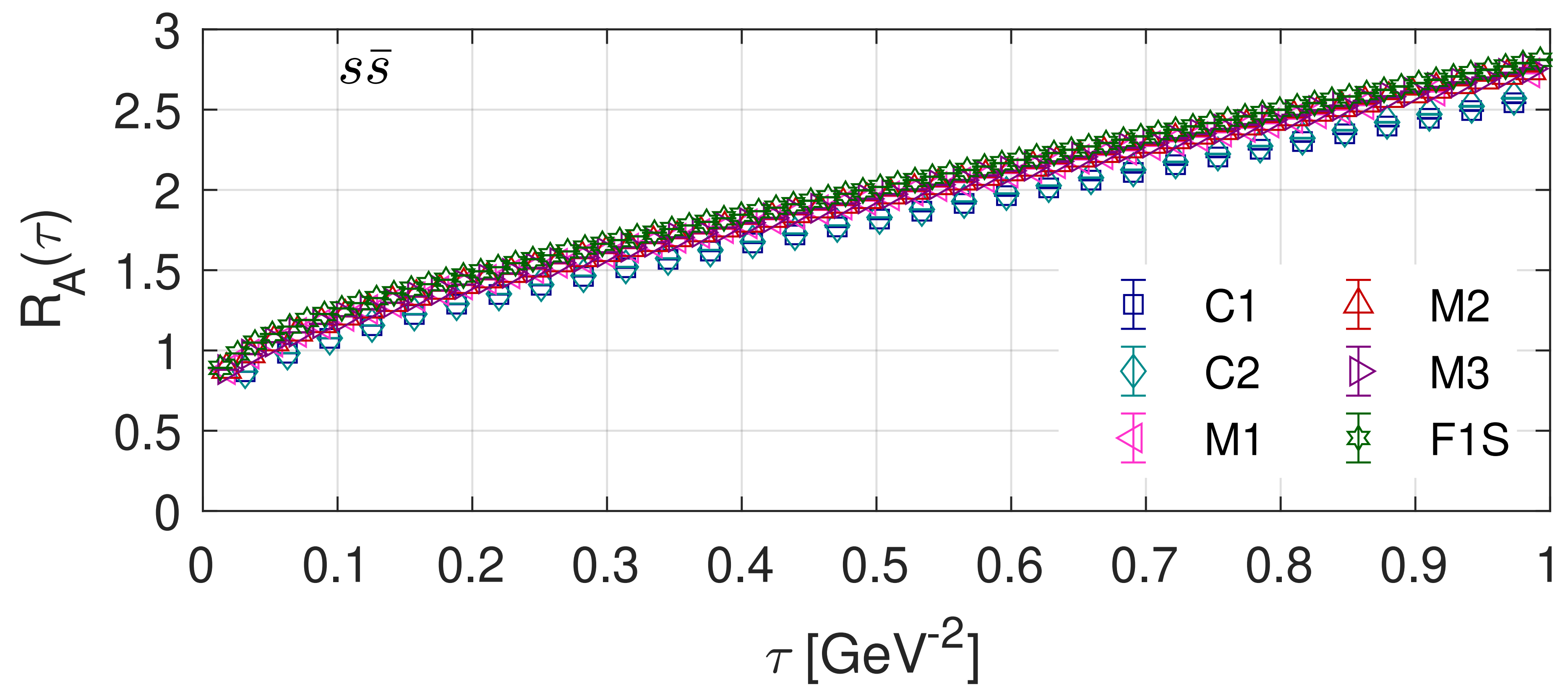
GF A-scheme

GF renormalized operator :

$$\widetilde{O}^{(A)}_{GF}(\tau) = \widetilde{Z}^{(A)}_{\chi}(\tau) O(\tau) \ , \quad \widetilde{O}_{\overline{MS}}(\mu) = \lim_{\tau\to 0} (\mathring{\zeta}_O^{-1} \mathring{\zeta}_A)(\mu;\tau)\, \widetilde{Z}^{(A)}_{\chi}(\tau) O(\tau) \ ,$$

$\widetilde{Z}^{(A)}_{\chi}(\tau) = Z_A R_A(\tau)$ is divergent, but only logarithmically





Numerical study: 2+1 flavor DWF fermions as
in Black et al 2506.16327, 2603.28516, 2603.28517
See O. Witzel talk

GF A-scheme anomalous dimension

GF renormalized operator :

$$\widetilde{O}^{(A)}_{GF}(\tau) = \widetilde{Z}^{A)}_{\chi}(\tau) O(\tau) \ , \quad \widetilde{O}_{\overline{MS}}(\mu) = \lim_{\tau \to 0} (\mathring{\zeta}_O^{-1} \mathring{\zeta}_V)(\mu; \tau) \, \widetilde{Z}^{A)}_{\chi}(\tau) O(\tau) \ , \quad Z^{A)}_{\chi}(\tau) = Z_A R_A(\tau)$$

GF A-scheme anomalous dimension

GF renormalized operator :

$$\widetilde{O}^{(A)}_{GF}(\tau) = \widetilde{Z}^{A)}_{\chi}(\tau) O(\tau) \ , \quad \widetilde{O}_{\overline{MS}}(\mu) = \lim_{\tau\to 0} (\mathring{\zeta}_O^{-1} \mathring{\zeta}_V)(\mu;\tau)\, \widetilde{Z}^{A)}_{\chi}(\tau) O(\tau) \ , \quad Z^{A)}_{\chi}(\tau) = Z_A R_A(\tau)$$

The ratio $\dfrac{\langle \widetilde{O}^{(A)}_{GF}(\tau) O(0)\rangle}{\langle O(0)O(0)\rangle} = \dfrac{Z_A R_A}{R_O}$ measures the overlap of $O^{(A)}_{GF}(\tau)$ with $O(0)$;

Its τ dependence is the anomalous dimension : $\gamma_{O/A}(\tau) = \tau \dfrac{d \log(\mathrm{R_A}(\tau)/\mathrm{R_O}(\tau))}{d\tau}$

Combine with the coupling $g^2_{GF}(\tau)$ to predict $\gamma_{O/A}(g^2_{GF})$

GF A-scheme anomalous dimension

GF renormalized operator :

$$\widetilde{O}^{(A)}_{GF}(\tau) = \widetilde{Z}^{A)}_{\chi}(\tau) O(\tau) \ , \quad \widetilde{O}_{\overline{MS}}(\mu) = \lim_{\tau\to 0} (\mathring{\zeta}_O^{-1} \mathring{\zeta}_V)(\mu;\tau)\, \widetilde{Z}^{A)}_{\chi}(\tau) O(\tau) \ , \quad Z^{A)}_{\chi}(\tau) = Z_A R_A(\tau)$$

The ratio $\dfrac{\langle \widetilde{O}^{(A)}_{GF}(\tau) O(0) \rangle}{\langle O(0) O(0) \rangle} = \dfrac{Z_A R_A}{R_O}$ measures the overlap of $O^{(A)}_{GF}(\tau)$ with $O(0)$;

Its τ dependence is the anomalous dimension : $\gamma_{O/A}(\tau) = \tau \dfrac{d \log(\mathrm{R_A}(\tau)/\mathrm{R_O}(\tau))}{d\tau}$

Combine with the coupling $g^2_{GF}(\tau)$ to predict $\gamma_{O/A}(g^2_{GF})$

$\gamma_{O/A}(g^2_{GF})$ can be determined in small volume (deconfined) simulations:

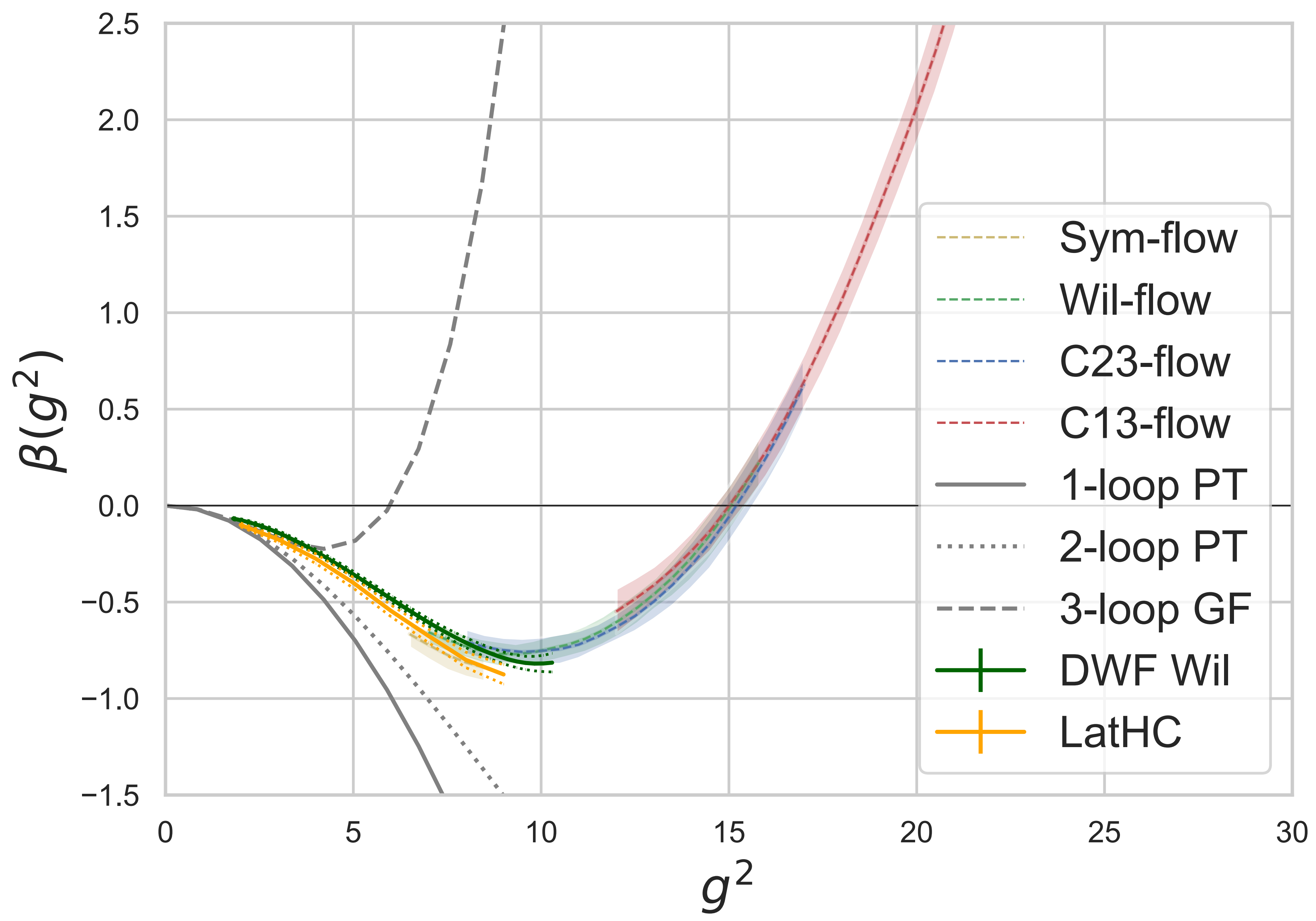
- direct determination $\gamma_{O/A}(\tau)$ is possible only for large g^2_{GF} (confining lattices)

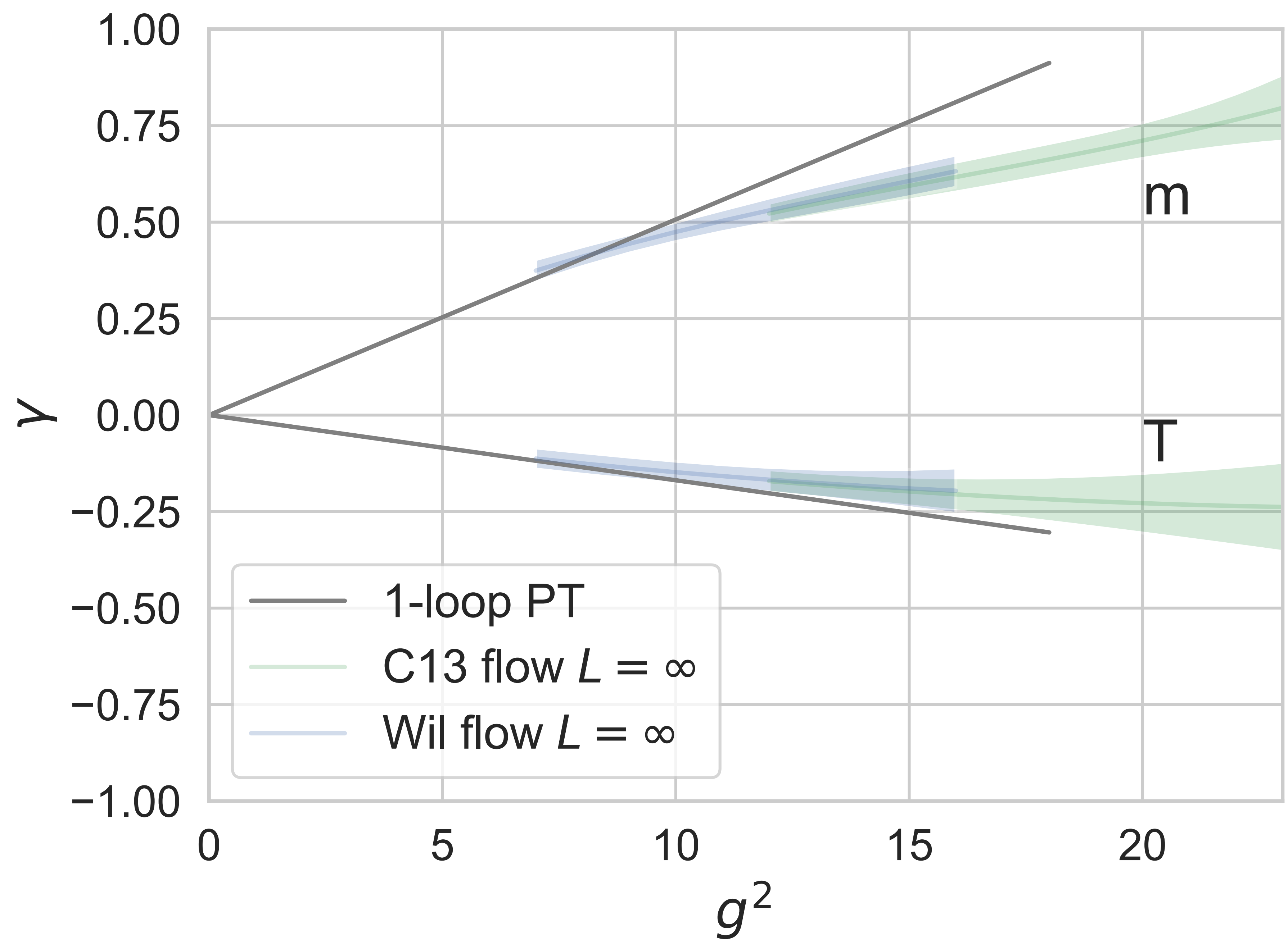
V-scheme is similar; just repace A_0 with V_k

Anomalous dimension - examples

Hasenfratz,Neil, Shamir, Svetitsky, Witzel,
Phys.Rev.D 108 (2023) 7

SU(3) with $N_f = 10$ massless flavors; Wilson fermions, V-scheme
Conformal system, the anomalous dimension is universal at the IRFP





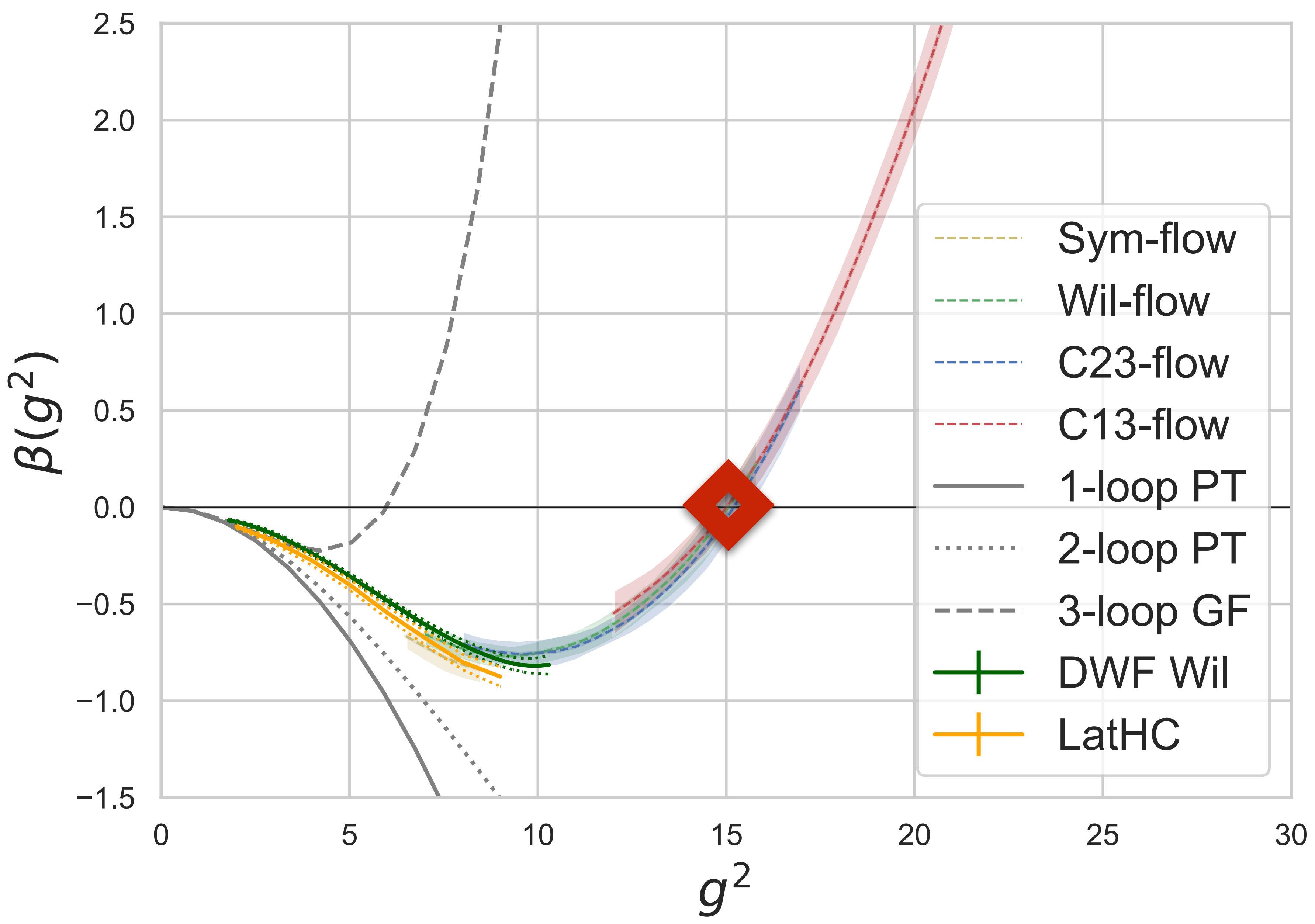
$\gamma(g^2)$ is not special at the IRFP; find g^2_{IRFP} from the beta function, then find γ_{IRFP}
(See N. Mackey's talk for staggered results)

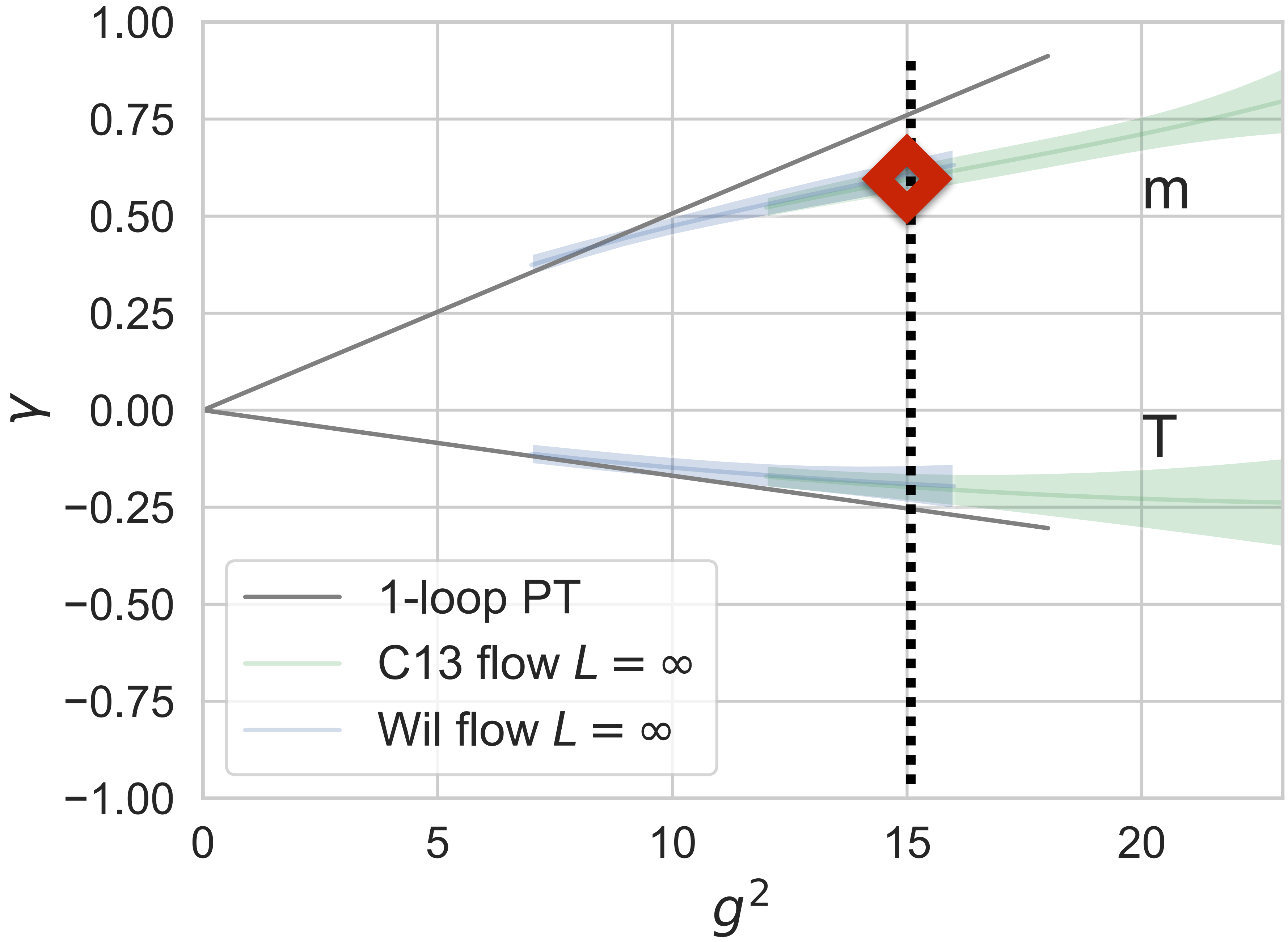
Similar ideas were explored in A. Carosso, AH, E. Neil, PRL 121,201601 (2018)
A.H.,C. Monahan, M. Rizik, A. Shindler and O. Witzel Lattice'21 (arXiv:2201:09740)

Anomalous dimension - examples

Hasenfratz,Neil, Shamir, Svetitsky, Witzel, *Phys.Rev.D* 108 (2023) 7

SU(3) with $N_f = 10$ massless flavors; Wilson fermions, V-scheme
Conformal system, the anomalous dimension is universal at the IRFP





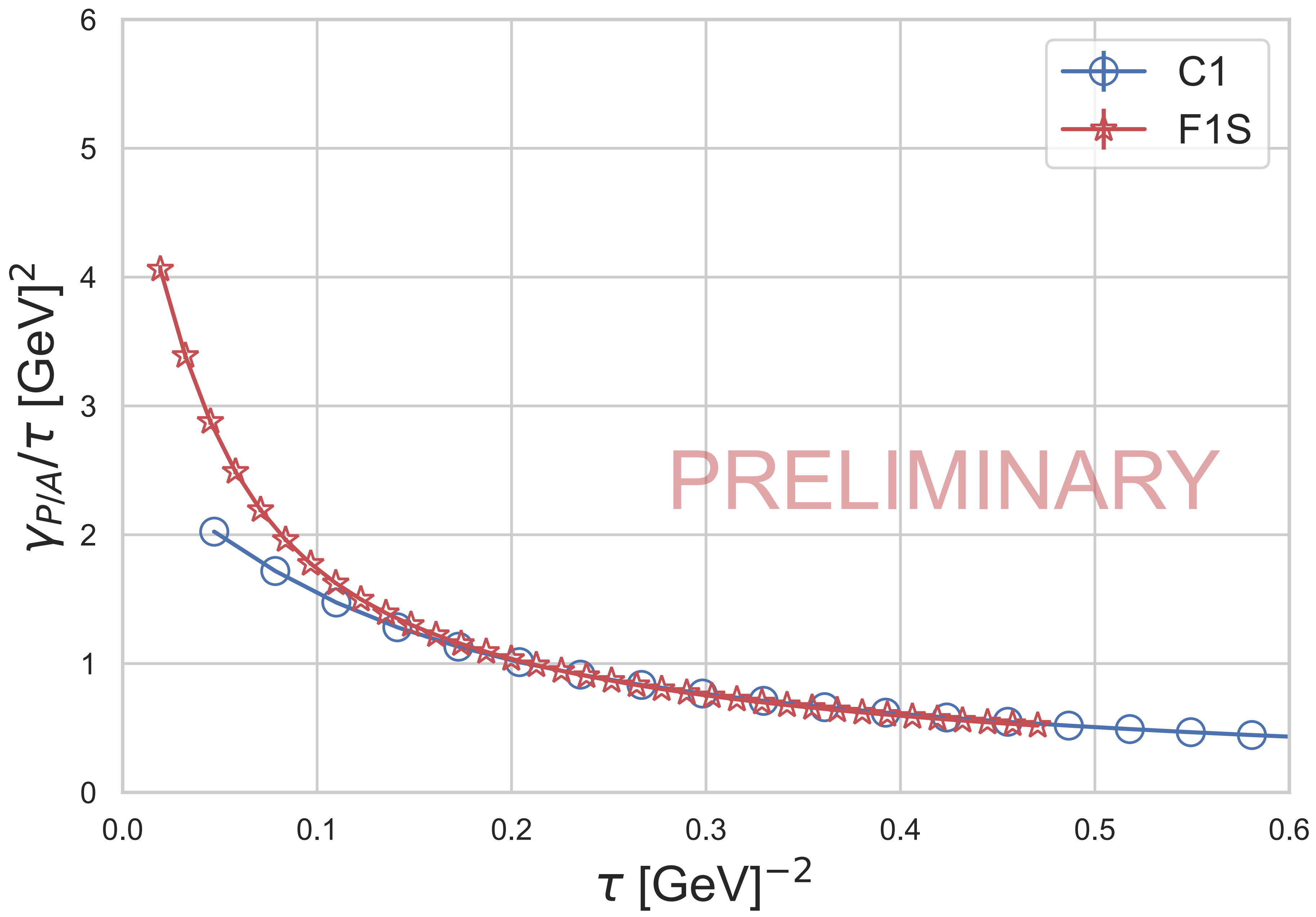
$\gamma(g^2)$ is not special at the IRFP; find g^2_{IRFP} from the beta function, then find γ_{IRFP}
(See N. Mackey's talk for staggered results)

Similar ideas were explored in A. Carosso, AH, E. Neil, PRL 121,201601 (2018)
A.H.,C. Monahan, M. Rizik, A. Shindler and O. Witzel Lattice'21 (arXiv:2201:09740)

Anomalous dimension - examples

2+1 flavor DWF fermions, $O \to P$ (mass anomalous dimension) Large volume simulations $\longrightarrow$

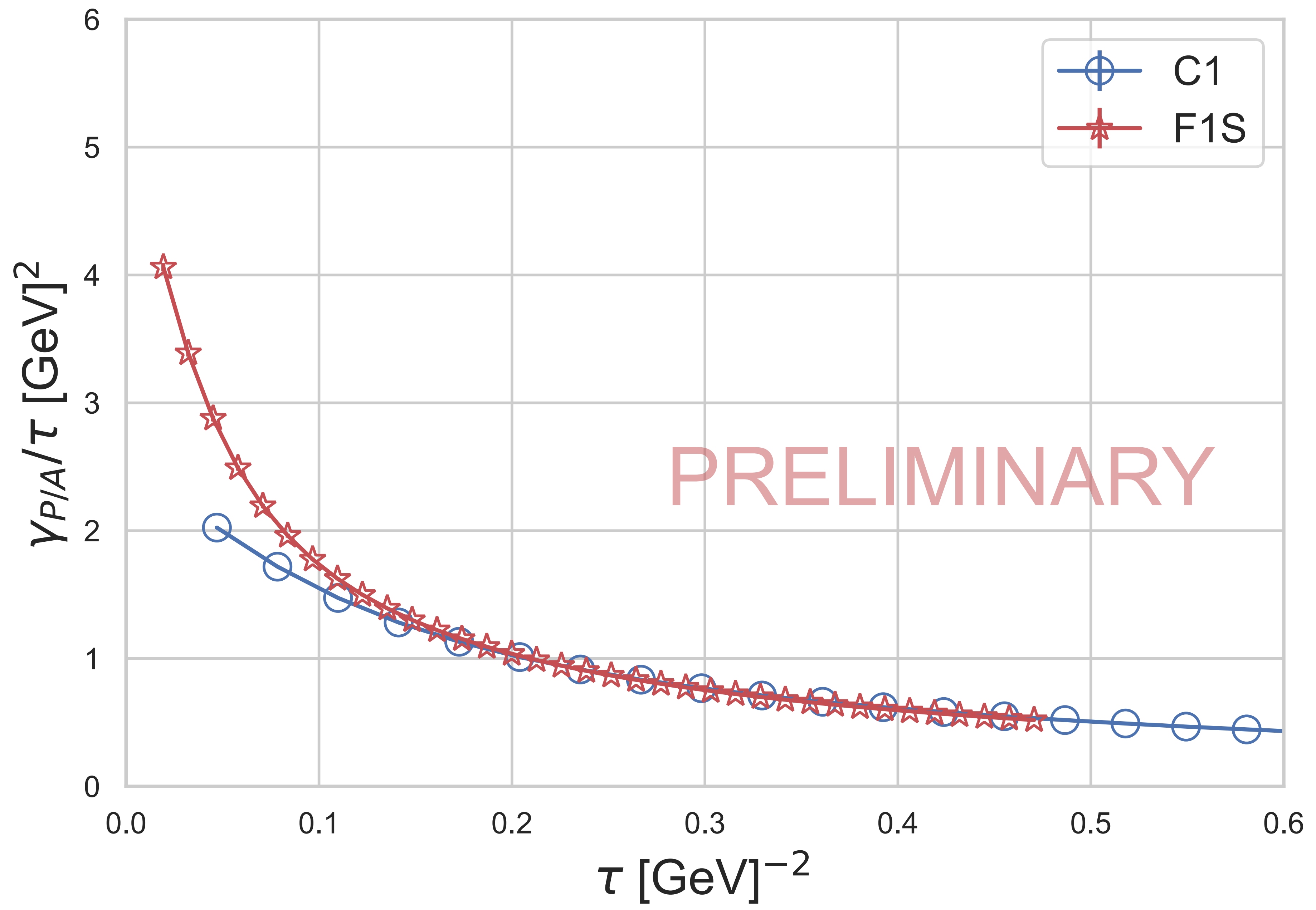
τ dependent anomalous dimension : $\gamma_{O/A} = \tau \dfrac{d \log(\mathrm{R_A}(\tau)/\mathrm{R_O}(\tau))}{d\tau}$



Anomalous dimension - examples

2+1 flavor DWF fermions, $O \rightarrow P$ (mass anomalous dimension) Large volume simulations $\longrightarrow$

τ dependent anomalous dimension : $\gamma_{O/A} = \tau \frac{d \log(\mathrm{R_A}(\tau)/\mathrm{R_O}(\tau))}{d\tau}$



Flow time evolution operator

$$C(\tau^*, \tau) = \exp\left(\int_{\tau}^{\tau^*} \frac{d\tau'}{\tau'} \gamma_{O/A}(\tau')\right), \quad C(\tau^*, \tau) = C(\tau^*, \tau_1) C(\tau_1, \tau)$$

connects renormalized correlators at τ^* and τ

$$\langle \widetilde{O}^{(A)}_{GF}(\tau^*) O(0) \rangle = C(\tau^*, \tau) \langle \widetilde{O}^{(A)}_{GF}(\tau) O(0) \rangle$$

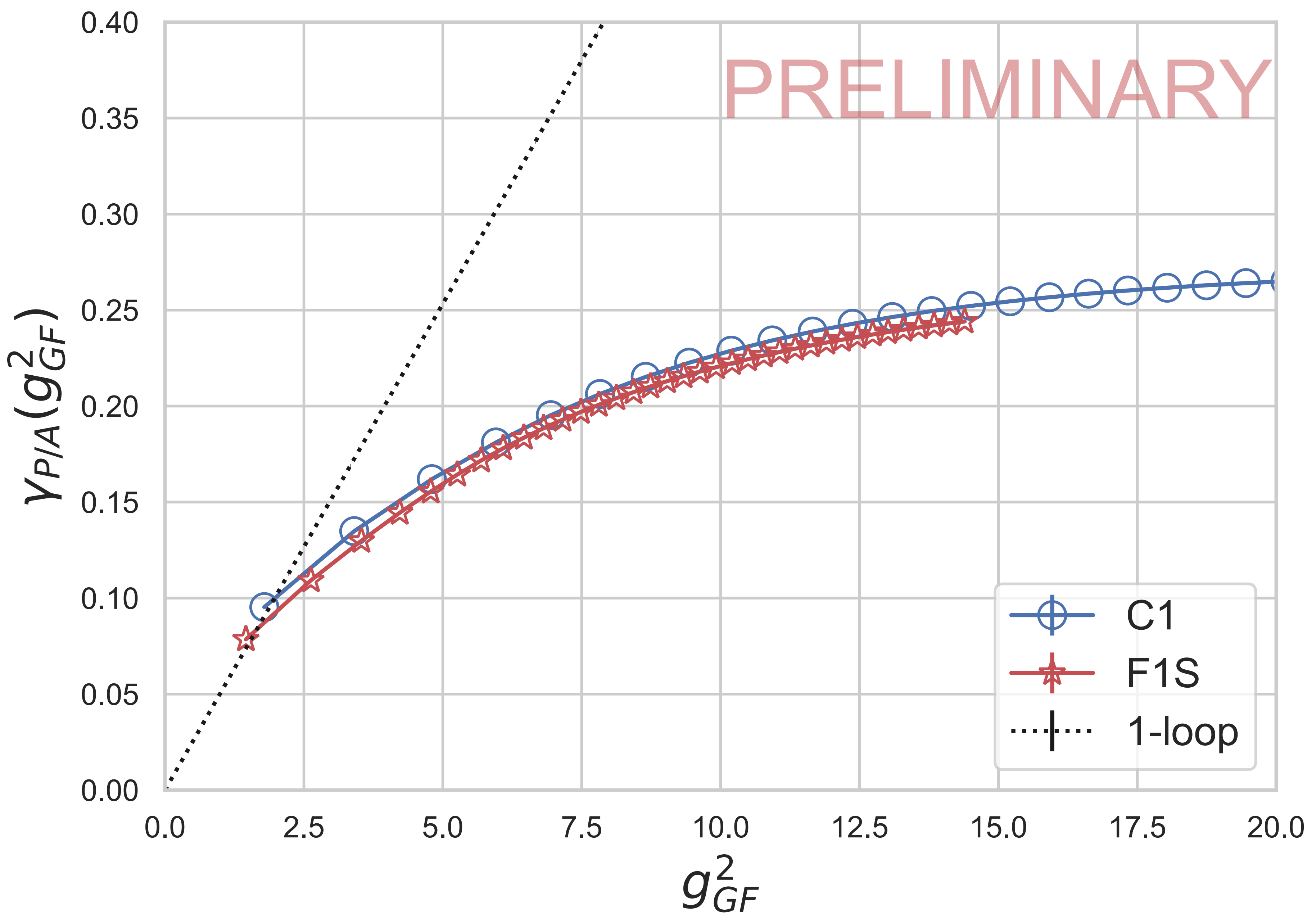
Can we determine $C(\tau^*, \tau)$?

Direct measurement breaks down around $\tau \sim 0.08\mathrm{GeV}^{-2}$ even on F1S. Needs finer lattices

Anomalous dimension - examples

2+1 flavor DWF fermions, $O \to P$ (mass anomalous dimension)
Alternative: determine $\gamma_{P/A}(g^2_{GF})$ in small volume simulations

$$C(\tau^*, \tau) = \exp\left[-\int_{g^2_{\rm GF}(\tau)}^{g^2_{\rm GF}(\tau^*)} dg^2 \, \frac{\gamma_{O/V}(g^2)}{\beta_{\rm GF}(g^2)}\right]$$



Small-volume simulations can fill in the gap between $g^2_{GF} \in (0,5)$.
Combine with perturbative γ to predict $C(\tau^* \to 0,\tau)$

In summary

- GF is a nonperturbative renormalization scheme
- Combined with SFTX it can predict matrix elements of $\overline{MS}$ renormalized operators
- using simple flowed lattice correlators.

There are many interesting talks to come in the next four days

Stay tuned !

Extra slides

Renormalization factors

GF renormalized operator : $\widetilde{O}^{(A)}_{GF}(\tau) = \widetilde{Z}^{(A)}_{\chi}(\tau) O(\tau)$, $\langle \widetilde{O}^{(A)}_{GF}(\tau) O(0) \rangle = \widetilde{Z}^{(A)}_{\chi}(\tau) \langle O(\tau) O(0) \rangle$

- Define renormalization factor $\widetilde{Z}^{(A)}_{O}(\tau) :.\ \langle \widetilde{O}^{(A)}_{GF}(\tau) O(0) \rangle = \widetilde{Z}^{(A)}_{O}(\tau) \langle O(0) O(0) \rangle$
- at $t \to \infty$ $\widetilde{Z}^{(A)}_{O}(\tau) = \widetilde{Z}^{(A)}_{\chi}(\tau) \dfrac{\langle O(\tau) O(0) \rangle}{\langle O(0) O(0) \rangle}$ is independent of the probe,
- $\widetilde{Z}^{(A)}_{O}(\tau)$ connects bare correlators to renormalized ones; it is fully determined by flowed-unflowed correlators
- $\overline{MS}$ scheme is similar: $\langle \widetilde{O}_{\overline{MS}}(\mu) O(0) \rangle = \lim_{\tau \to 0} (\mathring{\zeta}_O^{-1} \mathring{\zeta}_V)(\mu;\tau)\, \widetilde{Z}^{A)}_{\chi}(\tau) \langle O(\tau) O(0) \rangle$

$= \lim_{\tau \to 0} (\mathring{\zeta}_O^{-1} \mathring{\zeta}_V)(\mu;\tau)\, \widetilde{Z}^{A)}_{O}(\tau) \langle O(0) O(0) \rangle$ for the ground state

Renormalization factors - example

The renormalized quark mass is

$$\widetilde{m}^{(A)}_{GF}(\tau) = \frac{1}{\widetilde{Z}^{(A)}_P(\tau)} m_{\text{bare}} = \frac{1}{Z^{(A)}_\chi} \frac{\langle P(0)A_0(0)\rangle}{\langle P(\tau)A_0(0)\rangle} m_{\text{bare}}$$

probe: A_0 couples to P

$$= \frac{1}{Z_A} \frac{\langle A_0(\tau)A_0(0)\rangle}{\langle A_0(0)A_0(0)\rangle} \frac{\langle P(0)A_0(0)\rangle}{\langle P(\tau)A_0(0)\rangle} m_{\text{bare}}$$

$$= \frac{1}{Z_A} \frac{\langle A_0(\tau)A_0(0)\rangle}{\langle P_0(\tau)A_0(0)\rangle} \frac{\langle P(0)A_0(0)\rangle}{\langle A_0(0)A_0(0)\rangle} m_{\text{bare}}$$

notice bare and renormalized PCAC forms

$$= \frac{2\widetilde{m}^{(A)}_{GF}(\tau)}{2M_{PS}} \frac{2M_{PS}}{2m_{\text{bare}}} m_{\text{bare}} = \widetilde{m}^{(A)}_{GF}(\tau)$$

In the V-scheme we get different lattice form but identical result in the $\tau \to 0$ limit