The perturbative matching of four-quark operators between gradient flow and MSbar

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Edinburgh Gradient flow Workshop 2026

May 12, 2026

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2603.28516, 2603.28517

Collaborative Research Center TRR 257

Meson lifetimes motivation

- Lifetimes are fundamental parameters of Mesons
- Heavy B- and D-mesons
- Experimentally much more precise
- BSM sensitivity

Heavy Meson lifetimes

- Heavy quark expansion

$$\Gamma(D \to X) = \sum_i \Gamma_i \langle D|O_i|D\rangle$$

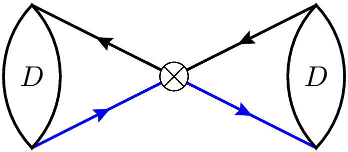
Contribution from $\Delta C = 0$ four quark operators

$$\mathcal{Q}_1 = \left(\bar{c}\gamma_\mu P_L q\right)\left(\bar{q}\gamma_\mu P_L c\right)$$

$$\mathcal{Q}_2 = \left(\bar{c}P_L q\right)\left(\bar{q}P_R c\right)$$

$$\mathcal{T}_1 = \left(\bar{c}\gamma_\mu P_L t^a q\right)\left(\bar{q}\gamma_\mu P_L t^a c\right)$$

$$\mathcal{T}_2 = \left(\bar{c}P_L t^a q\right)\left(\bar{q}P_R t^a c\right)$$



All Feynman diagrams were drawn using FeynGame

Heavy Meson lifetimes

- Heavy quark expansion

$$\Gamma(D \to X) = \sum_i \Gamma_i \langle D|O_i|D\rangle$$

Contribution from $\Delta C = 0$ four quark operators

- Wilson coefficients Γ_i perturbative

$$\Gamma_i = \Gamma_i^{(0)} + \frac{\alpha_s}{4\pi}\Gamma_i^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 \Gamma_i^{(2)} + \dots$$

- Matrix elements $\langle D|O_i|D\rangle$ non-perturbative

Difficulties on the lattice

- Operators mix under renormalization

$$O_i = \sum_j Z_{ij} O_j^B$$

- Mixing severly complicates lattice determination

$$\mathcal{O}^R = Z_1 \mathcal{O}_1^B + Z_2 \frac{1}{a} \mathcal{O}_2^B$$

- Heavy Meson lifetimes: Sum rules

Solution: gradient flow

GF+SFTX

- Short-flow-time expansion

$$\tilde{O}_i(\tau) \sim \sum_j \zeta_{ij}(\tau) O_j \quad \Rightarrow \quad O_i \sim \sum_j \zeta_{ij}^{-1}(\tau) \tilde{O}_j(\tau)$$

- Perturbative coefficients ζ_{ji}

$$\langle B|O_i|B\rangle (\mu) = \zeta_{ji}^{-1}(\mu, \tau) \langle B|\tilde{O}_i|B\rangle (\tau)$$

- Mixing shifted to the perturbative calculation

Renormalization

- Short-flow-time expansion

$$\zeta_{ij}^{R} = Z_{ik}^{-1}\zeta_{kj}^{R}$$

- Ansatz $\overline{\text{MS}}$ NNLO

$$Z_{ij} = \delta_{ij} + \alpha_s\left(\frac{1}{\epsilon}\gamma_{0,ij}\right) + \alpha_s^2\left(\frac{1}{2\epsilon^2}[\gamma_{0,ik}\gamma_{0,kj} - \beta_0\gamma_{0,ij}] + \frac{1}{2\epsilon}\gamma_{1,ij}\right)$$

- Important checks:
  - Z flowtime independent
  - γ agrees with literature

Calculating the mixing matrix

- Projectors $P_n[X]$ such that
  [Gorishny, Larin, Tkachov 1983; Gorishny, Larin 1986]

$$P_n[\mathcal{O}_m] = \delta_{nm}$$

  for all orders
- Usage on SFTX
  [Harlander, Kluth, Lange 2019]

$$\tilde{\mathcal{O}}_n(t) \approx \sum_m \zeta^B_{nm} \mathcal{O}^B_m + \dots$$

  leads to

$$\boxed{P_n[\tilde{\mathcal{O}}_m] = \xi^B_{mn}}$$

Form of the Projectors

$$P_n[\mathcal{O}_m] = \delta_{nm}$$

- General form

$$P_n[X] = \sum_k \Pi_k(\partial_p, \partial_m) \langle f_k | X | i_k \rangle \Big|_{p=m=0}$$

- Only has to hold at tree level

Operator basis

- $\Delta B = 0$ operators

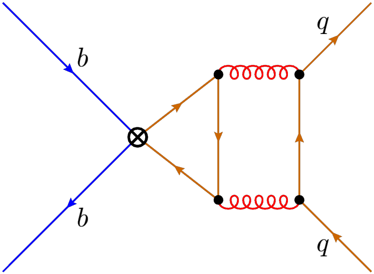
$$
\begin{aligned}
\mathcal{Q}_1 &= \left(\bar{c}\gamma_\mu P_L q\right)\,\left(\bar{q}\gamma_\mu P_L c\right)\\
\mathcal{Q}_2 &= \left(\bar{c} P_L q\right)\,\left(\bar{q} P_L c\right)\\
\mathcal{T}_1 &= \left(\bar{c}\gamma_\mu P_L t^a q\right)\,\left(\bar{q}\gamma_\mu P_L t^a c\right)\\
\mathcal{T}_2 &= \left(\bar{c} P_L t^a q\right)\,\left(\bar{q} P_L t^a c\right)
\end{aligned}
$$

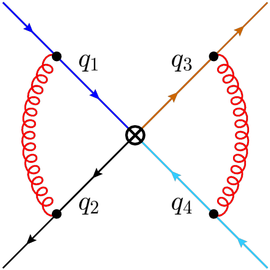
- Not a closed set of operators under renormalization

Perturbative calculation

$\left(\bar{c}\Gamma q\right)\ \left(\bar{q}\Gamma c\right)$

$\left(\bar{q_1}\Gamma q_2\right)\ \left(\bar{q_3}\Gamma q_4\right)$





Lifetime differences

- Mixing with penguin operators and lower dimensional operators drops out

$$\begin{aligned}
\mathcal{Q}_1 &= \left(\bar{q}_1 \gamma_\mu P_L q_2\right) \left(\bar{q}_3 \gamma_\mu P_L q_4\right) \\
\mathcal{Q}_2 &= \left(\bar{q}_1 P_L q_2\right) \left(\bar{q}_3 P_L q_4\right) \\
\mathcal{T}_1 &= \left(\bar{q}_1 \gamma_\mu P_L t^a q_2\right) \left(\bar{q}_3 \gamma_\mu P_L t^a q_4\right) \\
\mathcal{T}_2 &= \left(\bar{q}_1 P_L t^a q_2\right) \left(\bar{q}_3 P_L t^a q_4\right)
\end{aligned}$$

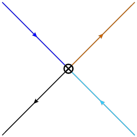
- Basis of operators smaller

- Mixing matrix of $\mathcal{Q}_1$ and $\mathcal{T}_1$ known [Harlander, Lange 2023]

Projectors four-quark operators

$$\mathcal{Q}_1 = \left(\bar{q}_1 \gamma_\mu P_L q_2\right) \left(\bar{q}_3 \gamma_\mu P_L q_4\right)$$

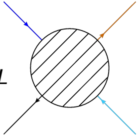
- Feynman rule



$$\sim \gamma_\mu P_L \otimes \gamma^\mu P_L$$

- Projector

$$P \sim \gamma_\mu P_L \otimes \gamma^\mu P_L$$



Flowed OPE

$$\Gamma(D \to X) = \sum_i \Gamma_i(\mu) \underbrace{\zeta_{ji}^{-1}(\mu,\tau) \langle D|\tilde{O}_i|D\rangle(\tau)}_{\langle D|O_i|D\rangle(\mu)}$$

Three ingredients:

- Mixing matrix
- Flowed matrix elements
- Wilson coefficients

Flowed OPE

$$\Gamma(D \to X) = \sum_i \Gamma_i(\mu) \underbrace{\zeta_{ji}^{-1}(\mu,\tau) \langle D|\tilde{O}_i|D\rangle(\tau)}_{\langle D|O_i|D\rangle(\mu)}$$

Three ingredients:

- Mixing matrix $\leftarrow$ this talk
- Flowed matrix elements
- Wilson coefficients

Flowed OPE

$$\Gamma(D \to X) = \sum_i \Gamma_i(\mu) \underbrace{\zeta_{ji}^{-1}(\mu,\tau) \langle D|\tilde{O}_i|D\rangle(\tau)}_{\langle D|O_i|D\rangle(\mu)}$$

Three ingredients:

- Mixing matrix $\leftarrow$ this talk
- Flowed matrix elements $\leftarrow$ Antonios' talk
- Wilson coefficients

[Black, Harlander, JK, Lange, Rago, Shindler, Witzel 2026]

Flowed OPE

$$\Gamma(D \to X) = \sum_i \Gamma_i(\mu) \underbrace{\zeta_{ji}^{-1}(\mu,\tau) \langle D|\tilde{O}_i|D\rangle (\tau)}_{\langle D|O_i|D\rangle(\mu)}$$

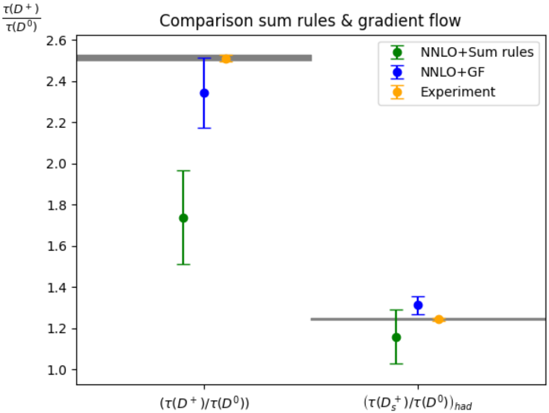
Three ingredients:

- Mixing matrix $\leftarrow$ this talk
- Flowed matrix elements $\leftarrow$ Antonios' talk
- Wilson coefficients $\leftarrow$ [Moretti, Nierste, Reeck, Steinhauser 2026]

[Black, Harlander, JK, Lange, Rago, Shindler, Witzel 2026]

Results D lifetime ratios

$$D^0 = \bar{u}c \qquad D^+ = \bar{d}c \qquad D_s^+ = \bar{s}c$$



[Moretti, Nierste, Reeck, Steinhauser 2026]

Outlook

- Absolute lifetimes
  - Eye diagrams
  - Lower dimensional operators
- B mesons

SFTX of non-singlet twist-two operators

Collaborators: Robert Harlander and Andrea Shindler

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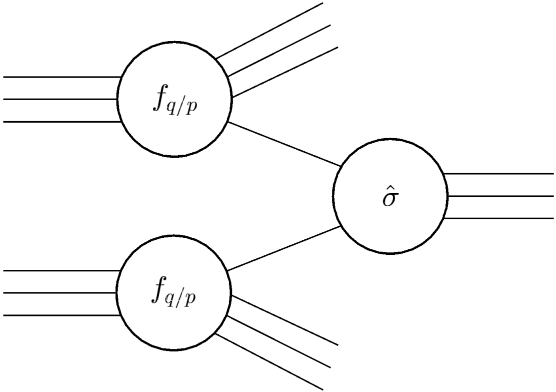
Factorization



$$\sigma = \sum_{a,b} \int_0^1 dx_a \, dx_b \, f_{a/p}(x_a, \mu) \, f_{b/p}(x_b, \mu) \, \hat{\sigma}_{ab} + \ldots$$

PDFs

- Mellin moments

$$\langle x^{n-1}\rangle(\mu) = \int dx\, x^{n-1}\, f_{q,H}(x,\mu)$$

- Non-perturbative matrix elements

$$\langle x^{n-1}\rangle \Leftrightarrow \langle H|\hat{\mathcal{O}}_{\{\mu_1\ldots\mu_n\}}|H\rangle$$

- Twist-two operators, traceless and symmetric

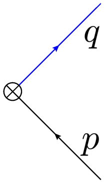
$$\hat{\mathcal{O}}_{\{\mu_1\ldots\mu_n\}} = \overline{\Psi}_a \gamma_{\{\mu_1} \overleftrightarrow{D}_{\mu_2} \ldots \overleftrightarrow{D}_{\mu_n\}} \Psi_b - \text{traces}$$

Projectors PDF moments

Twist-two non-singlet operators

$$\hat{\mathcal{O}}_{\{\mu_1\dots\mu_n\}} = \overline{\Psi}_a \gamma_{\{\mu_1} \overleftrightarrow{D}_{\mu_2} \dots \overleftrightarrow{D}_{\mu_n\}} \Psi_b - \text{traces}$$

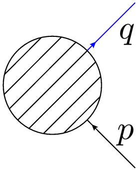
first Feynman rule



$$\sim \gamma_{\{\mu_1} (q-p)^{n-1}_{\mu_2\dots\mu_n\}}$$

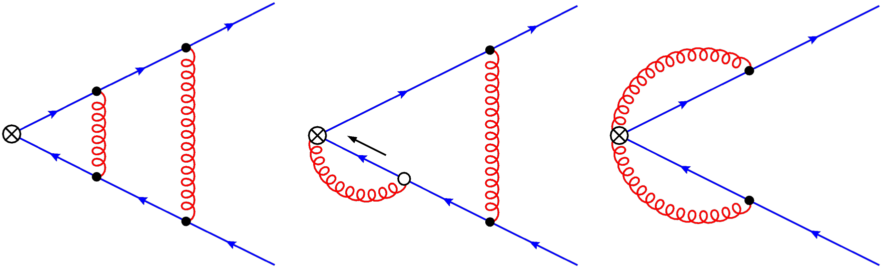
Projector

$$P \sim \gamma_{\{\mu_1} (\partial_q - \partial_p)^{n-1}_{\mu_2\dots\mu_n\}}$$



Diagrams Twist-two non-singlet

NNLO up to $n = 6$



Results for PDFs of Pions $\rightarrow$ Andreas' talk

[Francis, . . . , Harlander, . . . , JK,. . . , Shindler,. . . 2025]

Setup for the calculation

- `qgraf` [Nogueira 1991]
- `tapir` [Gerlach, Herren, Lang 2022]
- `exp` [Harlander, Seidensticker, Steinhauser 1998, Seidensticker 1999]
- `FORM` [Vermaseren 1989]
- `Kira` [Maierhöfer, Usovitsch, Uwer 2017; Klappert, Lange, Maierhöfer, Usovitsch 2020]
- `FireFly` [Klappert, Lange 2019], [Klappert, Klein, Lange 2020]
- `ftint` [Harlander, Nellopoulos, Olsson, Wesle 2024]
  - `pySecDec` [Borowka, Heinrich, Jahn, Jones, Kerner, Schlenk, Zirke 2017]

Setup for the calculation

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Automization

- Lagragian Feynman rules already implemented
- Operator Feynman rules

`frules`

[Harlander, Geuskens (unpublished)]

- Automatic implementation in the setup

`prepsetup`

Conclusion

- GF+SFTX method grants access to observables
- Need for SFTX of higher dimensional operators
- Automization of perturbative calculation

The QCD gradient flow

- QCD extended by flow time τ
  [Lüscher 2010]

$$B_\mu(\tau = 0, x) = A_\mu(x)$$

- Flow equations describe $\tau > 0$ behavior

$$\partial_\tau B_\mu = D_\nu G_{\nu\mu}$$

  flowed field strength tensor

$$G^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + f^{abc} B^b_\mu B^c_\nu$$

- Note that $[\tau] = -2$

Flowed quark fields

- The flow equations for the flowed quark field χ are

$$\partial_\tau \chi = \triangle \chi - \kappa \partial_\mu B_\mu^a T^a \chi,$$
$$\partial_\tau \overline{\chi} = \overline{\chi} \overleftarrow{\triangle} + \kappa \overline{\chi} \partial_\mu B_\mu^a T^a,$$

$$\chi^i(\tau = 0, x) = \psi^i(x)$$

Bag parameters

- Matrix elements parametrized by bag parameters

$$B \sim \frac{\langle D|\mathcal{O}_i|D\rangle}{f_H^2}$$

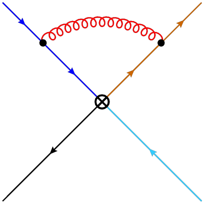
- Decay constant f_H

- Z_χ drops out

Evanescent operators

- Consider 1 loop diagrams of

$$\mathcal{Q}_1 \sim \gamma_\mu P_L \otimes \gamma^\mu P_L$$



$$\sim \gamma_{\mu\nu\rho} P_L \otimes \gamma^{\mu\nu\rho} P_L$$

- In $D = 4 - 2\varepsilon$

$$E_Q^1 = \gamma_{\mu\nu\rho} P_L \otimes \gamma^{\mu\nu\rho} P_L - 16\gamma_\mu P_L \otimes \gamma^\mu P_L = \mathcal{O}(\varepsilon)$$

- Operator mixing with E_Q^1
- Where $\gamma_{\mu\ldots\nu} = \gamma_\mu \ldots \gamma_\nu$

Evanescent operators

$$E \overset{D\to 4}{=} 0$$

- Mix into the physical operators

$$\begin{pmatrix} \mathcal{O} \\ E \end{pmatrix}_R = \begin{pmatrix} Z_{\mathcal{O}\mathcal{O}} & Z_{\mathcal{O}E} \\ Z_{E\mathcal{O}} & Z_{EE} \end{pmatrix} \begin{pmatrix} \mathcal{O} \\ E \end{pmatrix}_B$$

- Part of operator basis
- $Z_{E\mathcal{O}}$ includes finite terms chosen such that

$$E_R = \mathcal{O}(\varepsilon)$$

Ratios of Twist-two operators

- Flowed twist two operators

$$Z_\chi \mathcal{O}^n_{\{\mu_1 \ldots \mu_n\}}(\tau) = Z_\chi \bar{\Psi}_a \gamma_{\{\mu_1} \overleftrightarrow{D}_{\mu_2} \ldots \overleftrightarrow{D}_{\mu_n\}} \Psi_b$$

- Finite, but need Z_χ
- Instead work with ratios

$$R(t) = \frac{\langle x^n \rangle}{\langle x \rangle}(\tau)$$

- Completely finite for $\tau > 0$