Lattice determination of bag parameters using gradient flow

Antonio Rago
Quantum Theory Center
University of Southern Denmark

In collaboration with:
M. Black, R. Harlander, J. Kohnen, F. Lange, A. Shindler, O. Witzel

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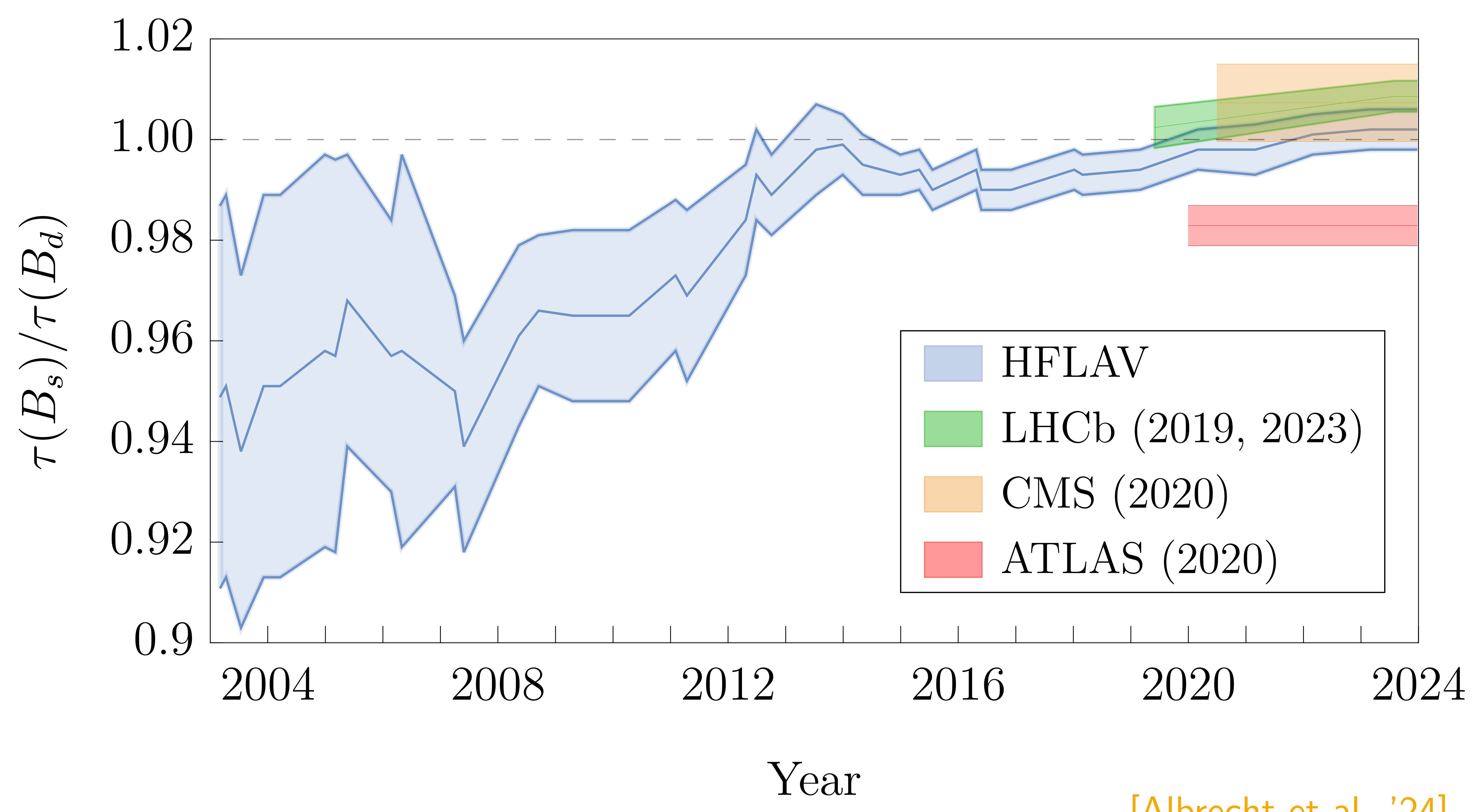
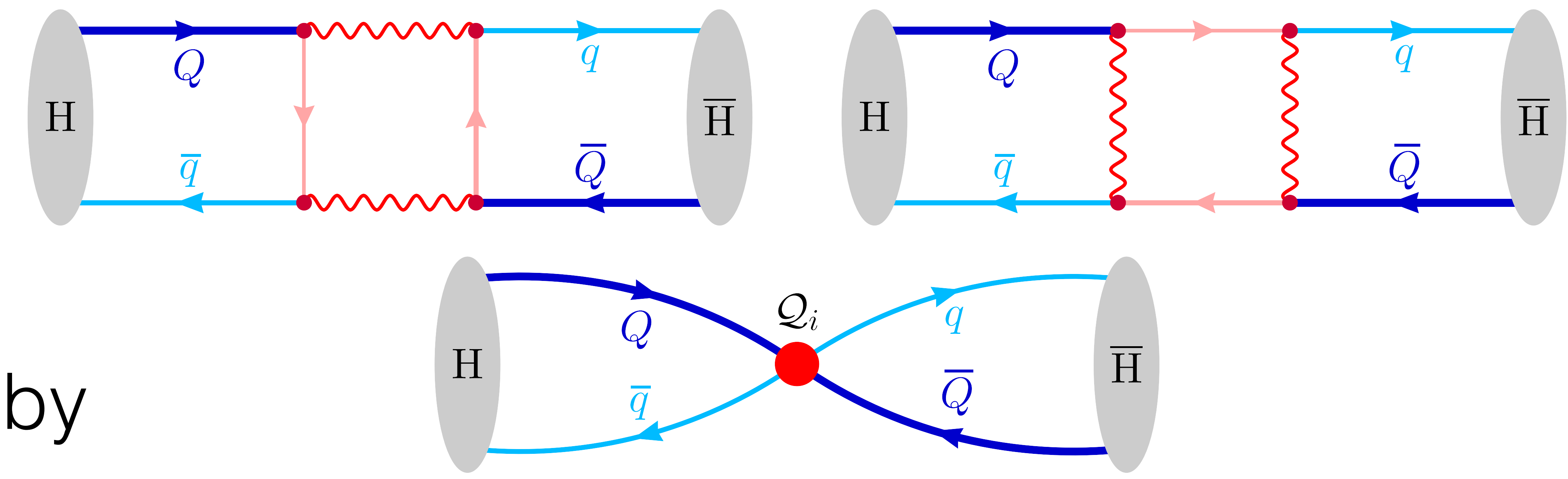
A case of interest: Heavy-Light Meson mixing and lifetimes

- Lifetimes of heavy-light mesons containing one heavy quark Q is particularly instructive as there are very precise experimental estimates.
- B mesons are mesons composed of a bottom quark and either an up (B_+), down (B_0), strange (B_{0s}) or charm quark (B_{+c})
- D mesons are mesons composed of a charm quark and either an up (D_+), down (D_0), strange (D_{0s})
- They are identified as key observables for probing New Physics
- Theory must match the experimental precision



[Albrecht et al. '24]

Meson Mixing



- The time evolution of the mixing is described by

$$i\frac{d}{dt}\begin{pmatrix}|H(t)\rangle\\|\overline{H}(t)\rangle\end{pmatrix}=\left(\hat{M}-\frac{i}{2}\hat{\Gamma}\right)\begin{pmatrix}|H(t)\rangle\\|\overline{H}(t)\rangle\end{pmatrix}.$$

- The off-diagonal elements can be expressed as

$$2M_H\left(M_{12}-\frac{i}{2}\Gamma_{12}\right)=\langle H|\mathcal{H}^{\Delta Q=2}|\overline{H}\rangle+\sum_n\frac{\langle H|\mathcal{H}^{\Delta Q=1}|n\rangle\langle n|\mathcal{H}^{\Delta Q=1}|\overline{H}\rangle}{M_H-E_n+i\epsilon},$$

In the SM, this yields a contribution ΔM from only one operator for $\Delta Q=2$

$$\mathcal{Q}_1=(\bar{Q}\gamma_\mu(1-\gamma_5)q)(\bar{Q}\gamma_\mu(1-\gamma_5)q)$$

These interactions are inherently long distance and very relevant for the D mesons they are negligible for B meson.

[Di Carlo, F. Erben, Hansen 2025]

$\Delta Q = 2$ operator basis

Mass difference of neutral mesons $\Delta M_q \; (q = d, s)$ governed by $\Delta Q = 2$ four-quark operators

$$
\begin{aligned}
\mathcal{Q}_1^q &= \bar{Q}^\alpha \gamma^\mu (1-\gamma_5) q^\alpha \, \bar{Q}^\beta \gamma_\mu (1-\gamma_5) q^\beta, & \langle \mathcal{Q}_1^q \rangle &= \langle \bar{H} | \mathcal{Q}_1^q | H \rangle = \frac{8}{3} f_H^2 M_H^2 \mathcal{B}_1^q, \\
\mathcal{Q}_2^q &= \bar{Q}^\alpha (1-\gamma_5) q^\alpha \, \bar{Q}^\beta (1-\gamma_5) q^\beta, & \langle \mathcal{Q}_2^q \rangle &= \langle \bar{H} | \mathcal{Q}_2^q | H \rangle = -\frac{5 M_H^2}{3(m_h + m_q)^2} f_H^2 M_H^2 \mathcal{B}_2^q, \\
\mathcal{Q}_3^q &= \bar{Q}^\alpha (1-\gamma_5) q^\beta \, \bar{Q}^\beta (1-\gamma_5) q^\alpha, & \langle \mathcal{Q}_3^q \rangle &= \langle \bar{H} | \mathcal{Q}_3^q | H \rangle = \frac{M_H^2}{3(m_h + m_q)^2} f_H^2 M_H^2 \mathcal{B}_3^q, \\
\mathcal{Q}_4^q &= \bar{Q}^\alpha (1-\gamma_5) q^\alpha \, \bar{Q}^\beta (1+\gamma_5) q^\beta, & \langle \mathcal{Q}_4^q \rangle &= \langle \bar{H} | \mathcal{Q}_4^q | H \rangle = \left[\frac{2 M_H^2}{(m_h + m_q)^2} + \frac{1}{3} \right] f_H^2 M_H^2 \mathcal{B}_4^q, \\
\mathcal{Q}_5^q &= \bar{Q}^\alpha (1-\gamma_5) q^\beta \, \bar{Q}^\beta (1+\gamma_5) q^\alpha, & \langle \mathcal{Q}_5^q \rangle &= \langle \bar{H} | \mathcal{Q}_5^q | H \rangle = \left[\frac{2 M_H^2}{3(m_h + m_q)^2} + 1 \right] f_H^2 M_H^2 \mathcal{B}_5^q.
\end{aligned}
$$

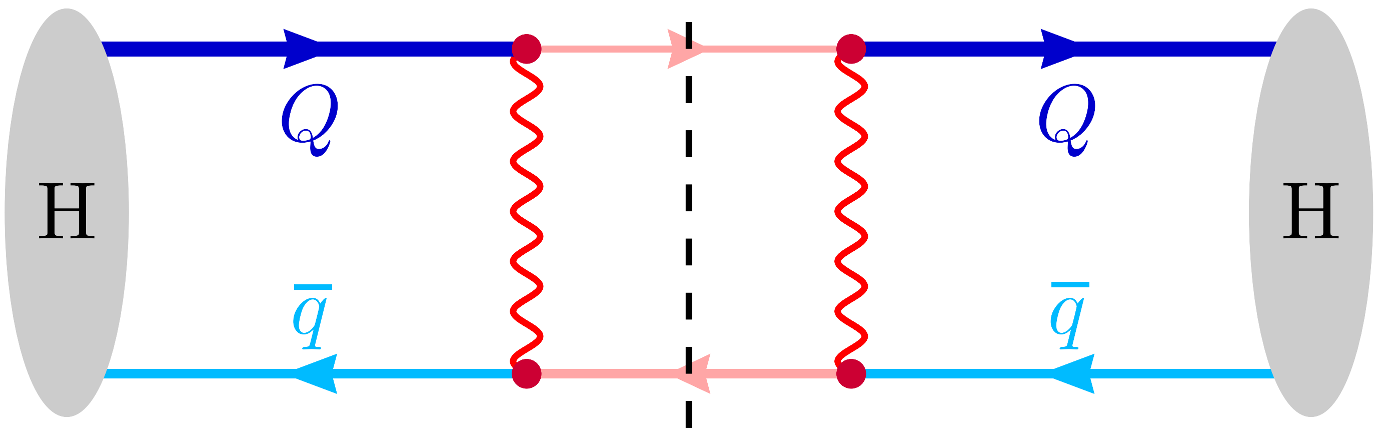
In the SM, only $\mathcal{Q}_1^q$ contributes to $\Delta M_q \; (q = d, s)$

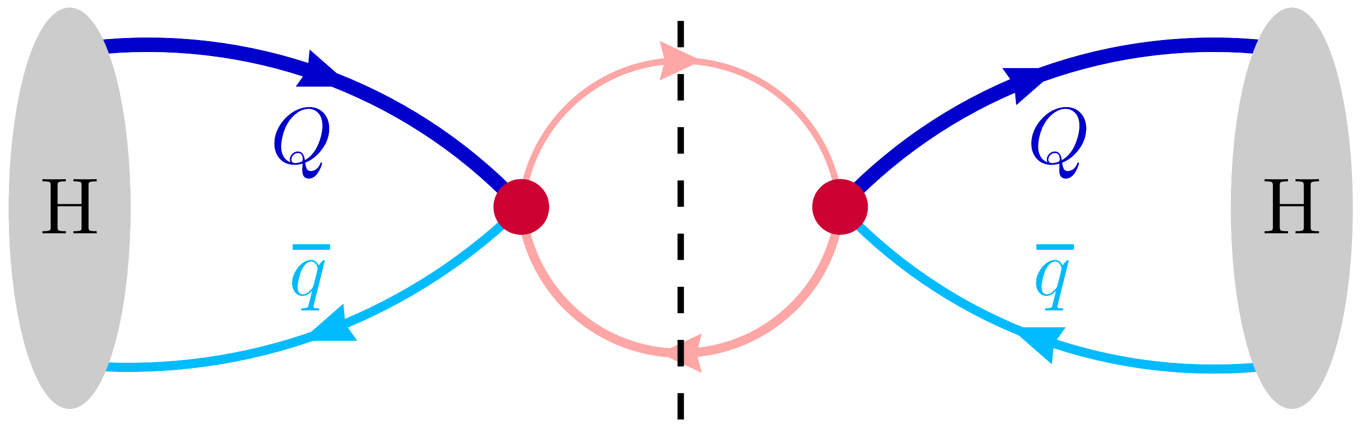
In vacuum insertion approximation (VIA) $\mathcal{B}_1^q$ is 1

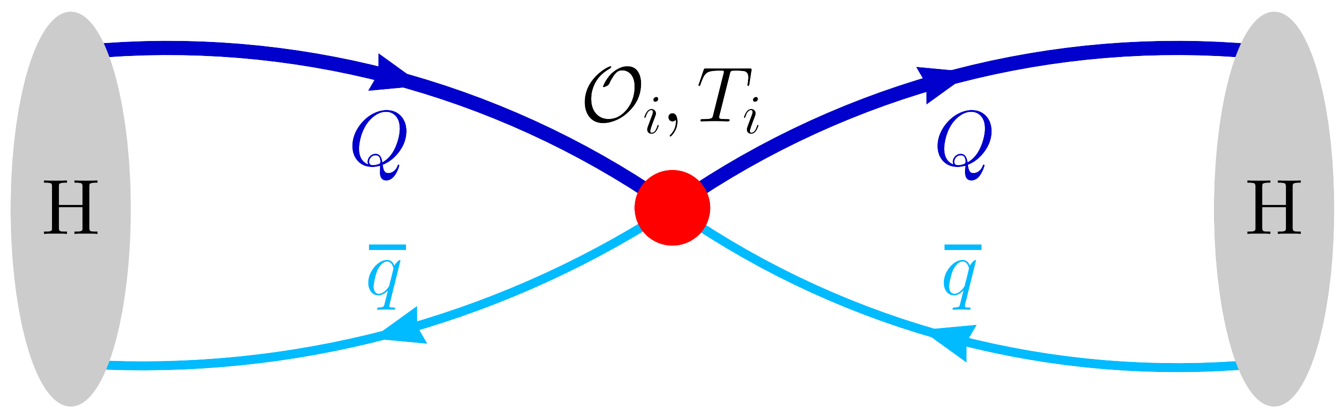
Meson lifetimes

$$\Gamma_H = \frac{1}{2M_H}\sum_n \int_{\mathrm{PS}} (2\pi)^4 \delta^{(4)}(p_H - p_n) |\langle n | \mathcal{H}_{\mathrm{eff}}^{\Delta Q=1} | H \rangle|^2,$$ which thanks to the optical theory

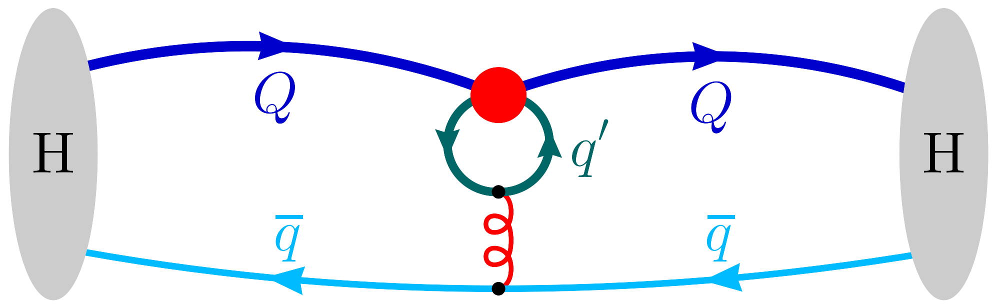
$$\Gamma_H = \frac{1}{2M_H} \mathrm{Im}\langle H | \mathcal{T}^{\Delta Q=0} | H \rangle.$$ were the forward scattering ampl. $$\mathcal{T}^{\Delta Q=0} = \mathrm{i} \int d^4x\, \mathrm{T}\left\{ \mathcal{H}_{\mathrm{eff}}^{\Delta Q=1}(x)\, \mathcal{H}_{\mathrm{eff}}^{\Delta Q=1}(0) \right\}$$

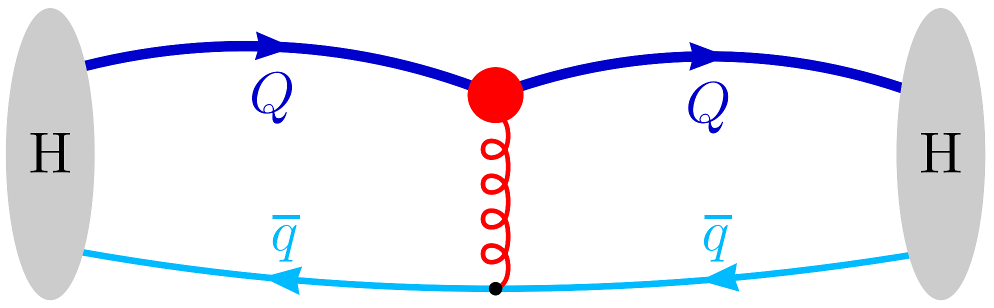






Caveat, the above is sufficient for ratio of lifetimes, for absolute measures one needs also:





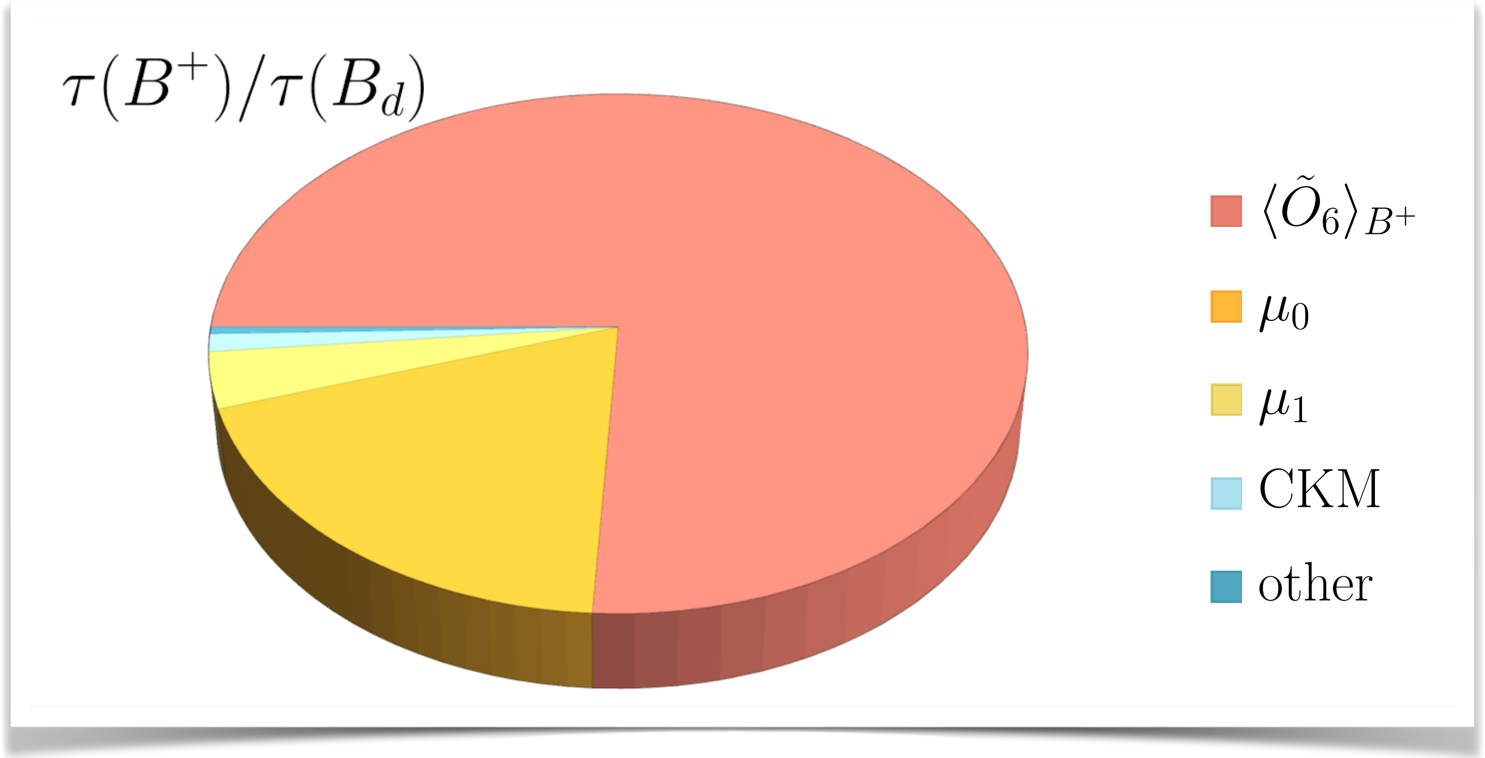
Meson lifetimes

- For the evaluation of Heavy mesons lifetimes, the most used approach is the Heavy Quark Expansion.

$$\Gamma_H = \Gamma_3 + \Gamma_5 \frac{\overline{\langle \mathcal{O}_5 \rangle}}{m_Q^2} + \Gamma_6 \frac{\overline{\langle \mathcal{O}_6 \rangle}}{m_Q^3} + \cdots + 16\pi^2 \left[\tilde{\Gamma}_6 \frac{\overline{\langle \mathcal{O}_6 \rangle}}{m_Q^3} + \tilde{\Gamma}_7 \frac{\overline{\langle \mathcal{O}_7 \rangle}}{m_Q^4} + \cdots \right],$$

Where $\overline{\langle \mathcal{O}_d \rangle}$ identify 2 quark operators and $\overline{\langle \mathcal{O}_d \rangle}$ 4 quark ones.

- $\overline{\langle \mathcal{O}_6 \rangle}$ are leading uncertainties for lifetimes of the B.



$\Delta Q = 0$ operator basis

For lifetimes, the dimension-6 $\Delta Q = 0$ operators are:

$$\mathcal{O}_1^H = \bar{b}^\alpha \gamma^\mu (1-\gamma_5) q^\alpha \, \bar{q}^\beta \gamma_\mu (1-\gamma_5) b^\beta , \qquad \langle \mathcal{O}_1^H \rangle = \langle H | \mathcal{O}_1^H | H \rangle = f_H^2 M_H^2 B_1^H ,$$

$$\mathcal{O}_2^H = \bar{b}^\alpha (1-\gamma_5) q^\alpha \, \bar{q}^\beta (1-\gamma_5) b^\beta , \qquad \langle \mathcal{O}_2^H \rangle = \langle H | \mathcal{O}_2^H | H \rangle = \frac{M_H^2}{(m_Q + m_q)^2} f_H^2 M_H^2 B_2^H ,$$

$$T_1^H = \bar{b}^\alpha \gamma^\mu (1-\gamma_5) (T^a)^{\alpha\beta} q^\beta \, \bar{q}^\gamma \gamma_\mu (1-\gamma_5) (T^a)^{\gamma\delta} b^\delta , \qquad \langle T_1^H \rangle = \langle H | T_1^H | H \rangle = f_H^2 M_H^2 \epsilon_1^H ,$$

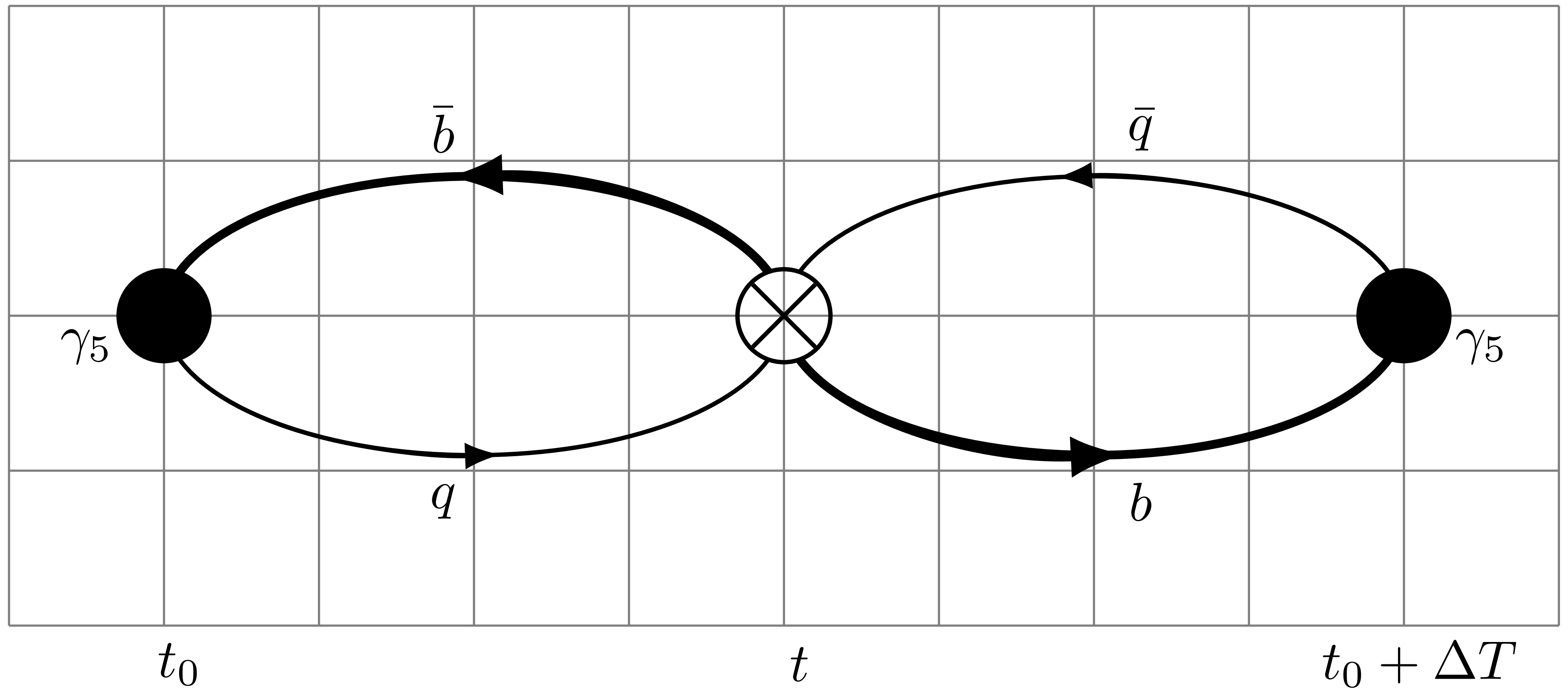
$$T_2^H = \bar{b}^\alpha (1-\gamma_5) (T^a)^{\alpha\beta} q^\beta \, \bar{q}^\gamma (1-\gamma_5) (T^a)^{\gamma\delta} b^\delta , \qquad \langle T_2^H \rangle = \langle H | T_2^H | H \rangle = \frac{M_H^2}{(m_Q + m_q)^2} f_H^2 M_H^2 \epsilon_2^H .$$

In vacuum insertion approximation (VIA) B are 1 and ϵ are 0

Four-quark matrix elements determined in lattice QCD simulations

- $\Delta B = 2$ well-studied by several groups
- Preliminary $\Delta K = 2$ for kaon mixing with gradient flow [Suzuki et al. '20][Taniguchi, Lattice '19}]
- $\Delta B = 0$ exploratory studies from 20 years ago
  - Contributions from statistically-noisy diagrams
  - Mixing with lower dimension operators in renormalisation
- [Lin, Detmold, Meinel '22] spectator effects in b hadrons
  - Focus on lifetime ratios for both B mesons and Λ_b baryon
  - Isospin breaking, $\langle B|\mathcal{O}^d - \mathcal{O}^u|B\rangle$
  - Position-space renormalisation + perturbative matching to $\overline{\mathrm{MS}}$
- Our work
  - Goal is individual $\Delta B = 0$ matrix elements for B mesons
  - Non-perturbative gradient flow renormalisation
  - Perturbative matching to $\overline{\mathrm{MS}}$ i n short-flow-time expansion

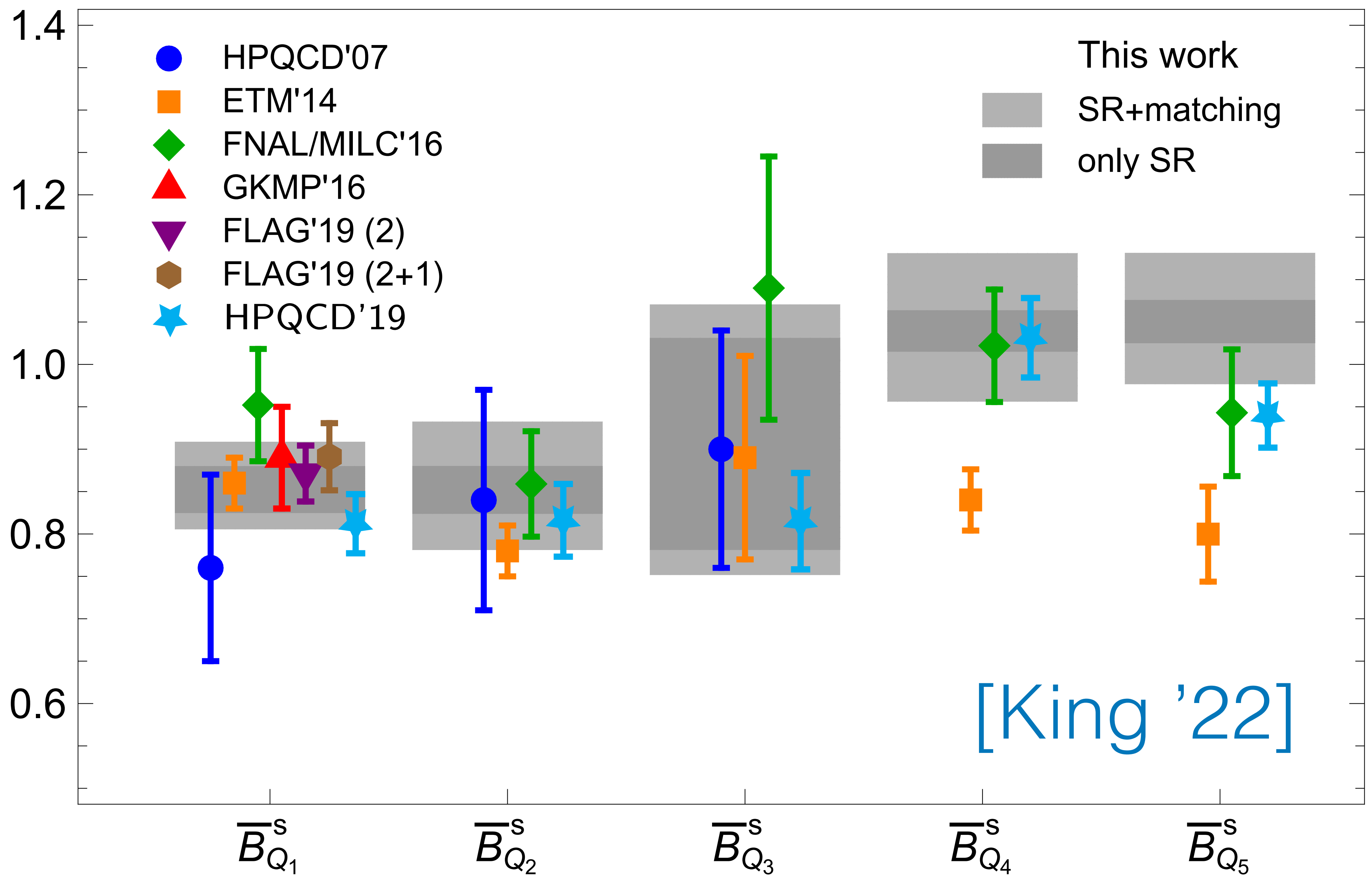
$\Delta B = 2$ operators - Lattice Sketch



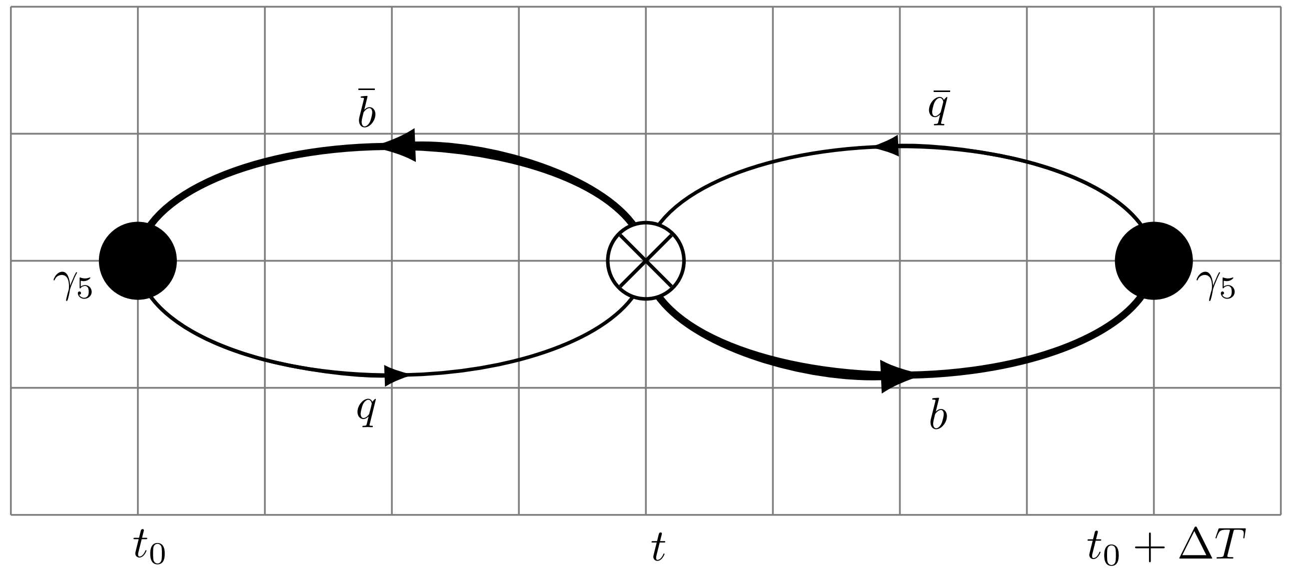
$$B_q \to \langle Q_6 \rangle \to \bar{B}_q$$

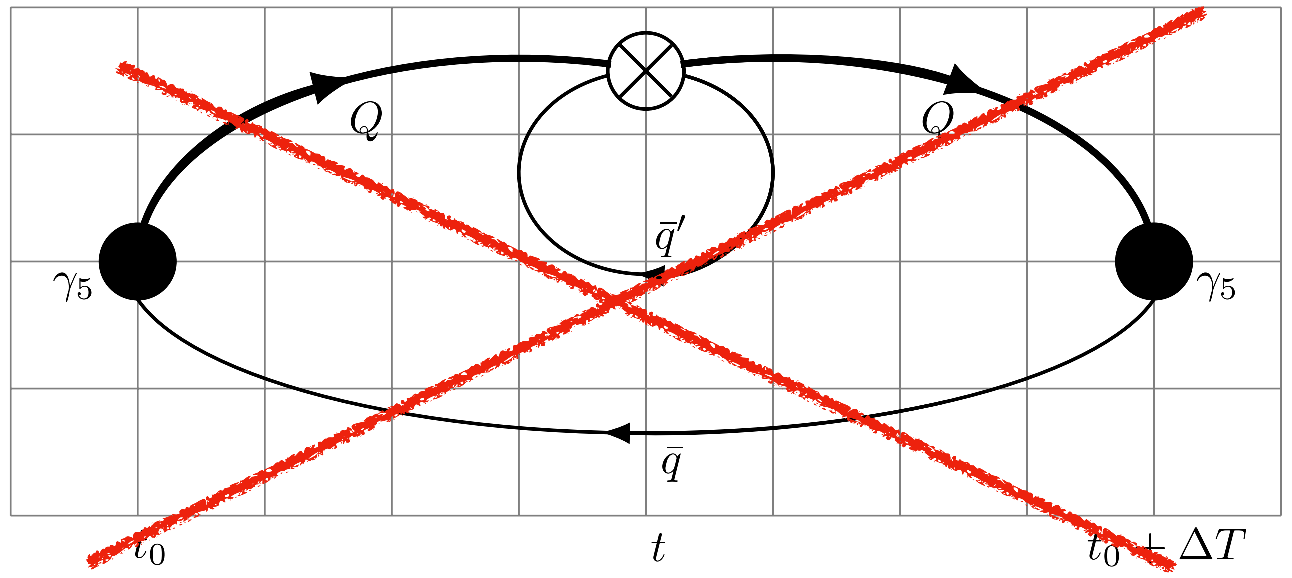
- More widely studied
- No disconnected terms
- No eye diagrams

- Bag parameters well-studied on the lattice and with QCD sum rules
- Ongoing work by RBC/UKQCD and JLQCD [Boyle et al '21][sang, Lattice '23]
- Dimension-7 matrix elements calculated for first time [HPQCD '19]



$\Delta B = 0$ operators - Lattice Sketch



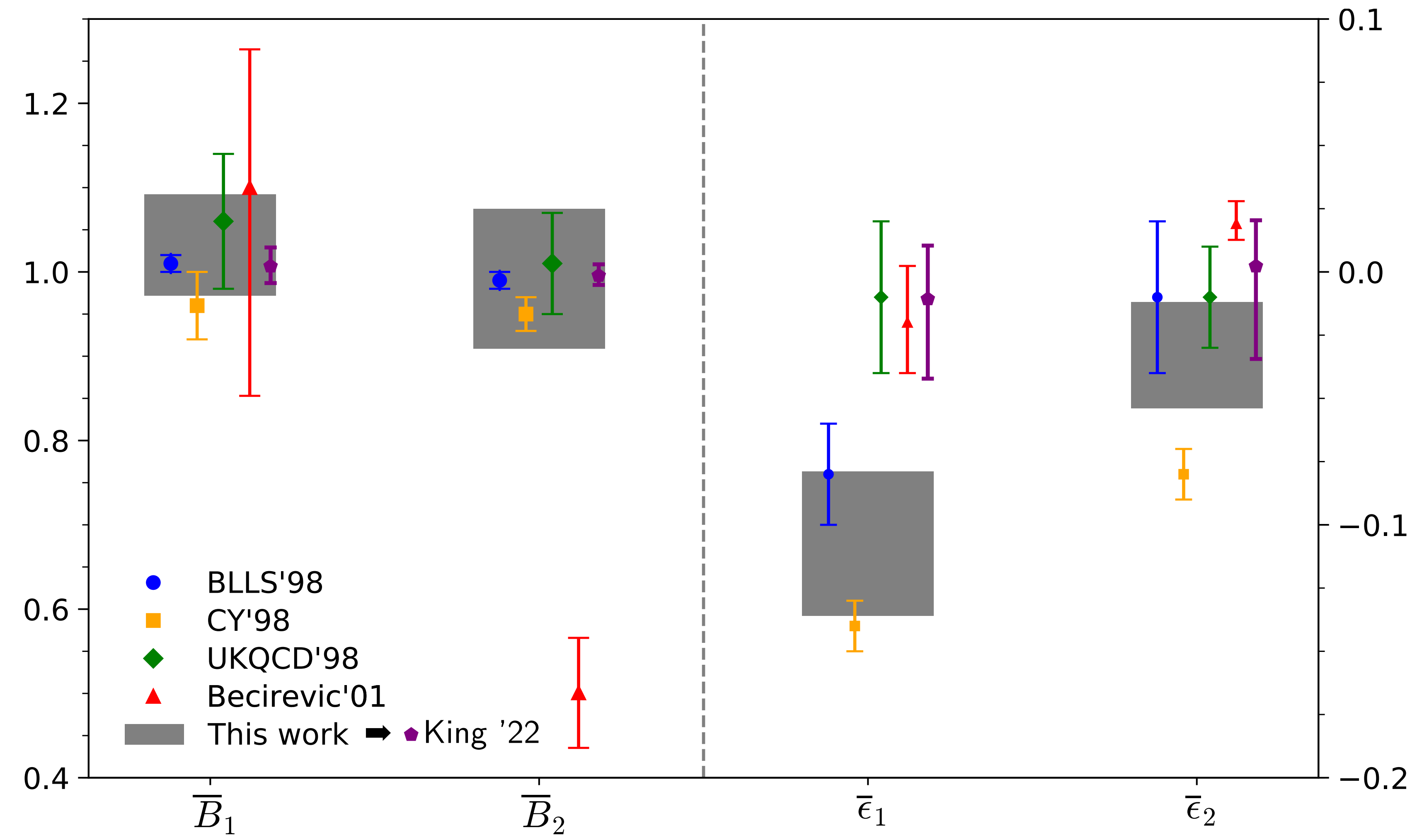


~~… + lower dimension insertions~~

$$B_q \to \langle Q_6 \rangle \to B_q$$

- First diagram is pretty similar to the previous case.
- "Eye" diagrams and gluons-disconnected are statistically very noisy and have not been included in any lattice computation so far.
- Issue of mixing with lower dimensional operators that gives rise to power divergencies.
- Only the topologies relevant for life time ratio have been studied so far.

…our proposal is to use the gradient flow

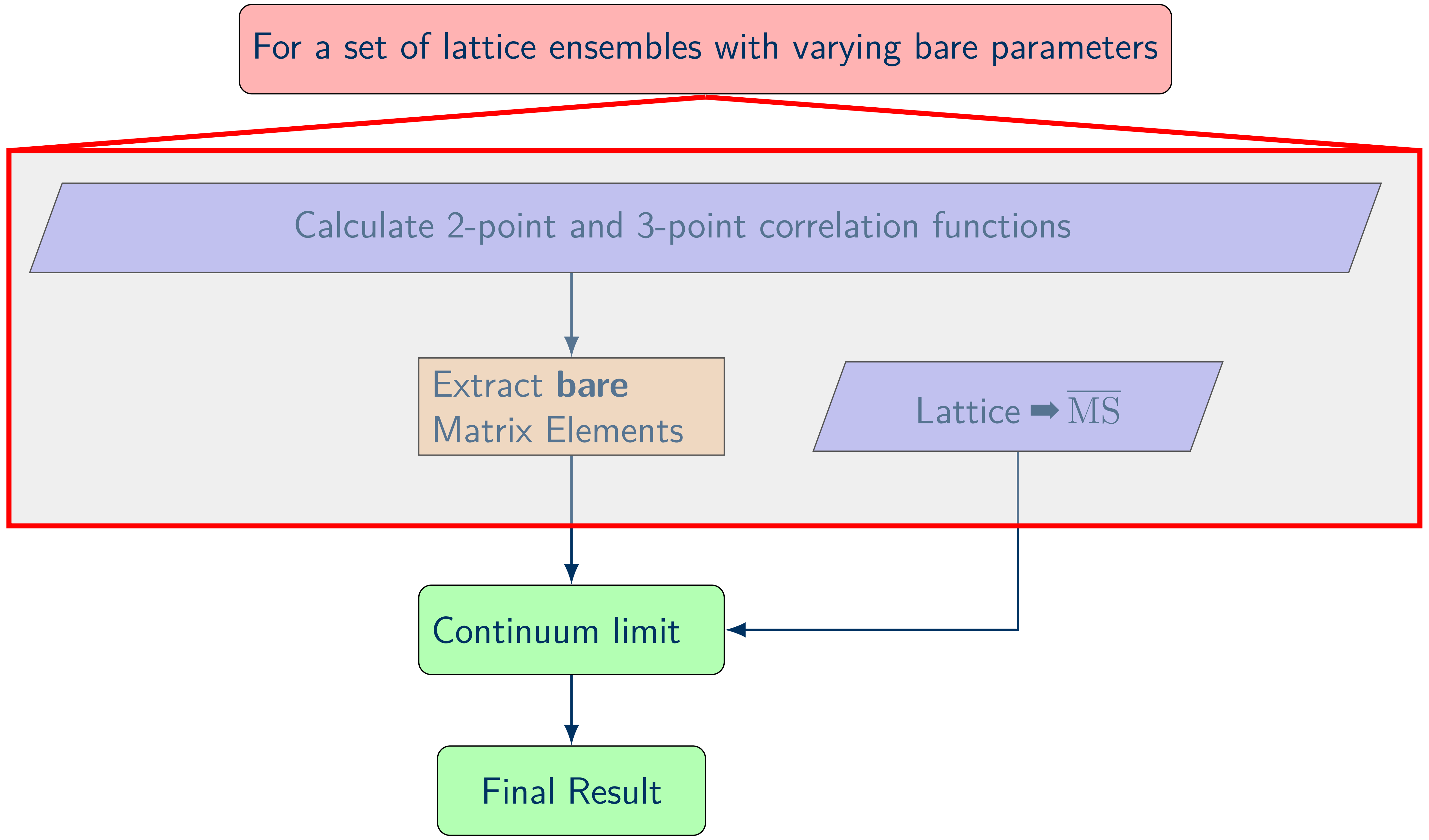


Strategy (for the foreseeable future)

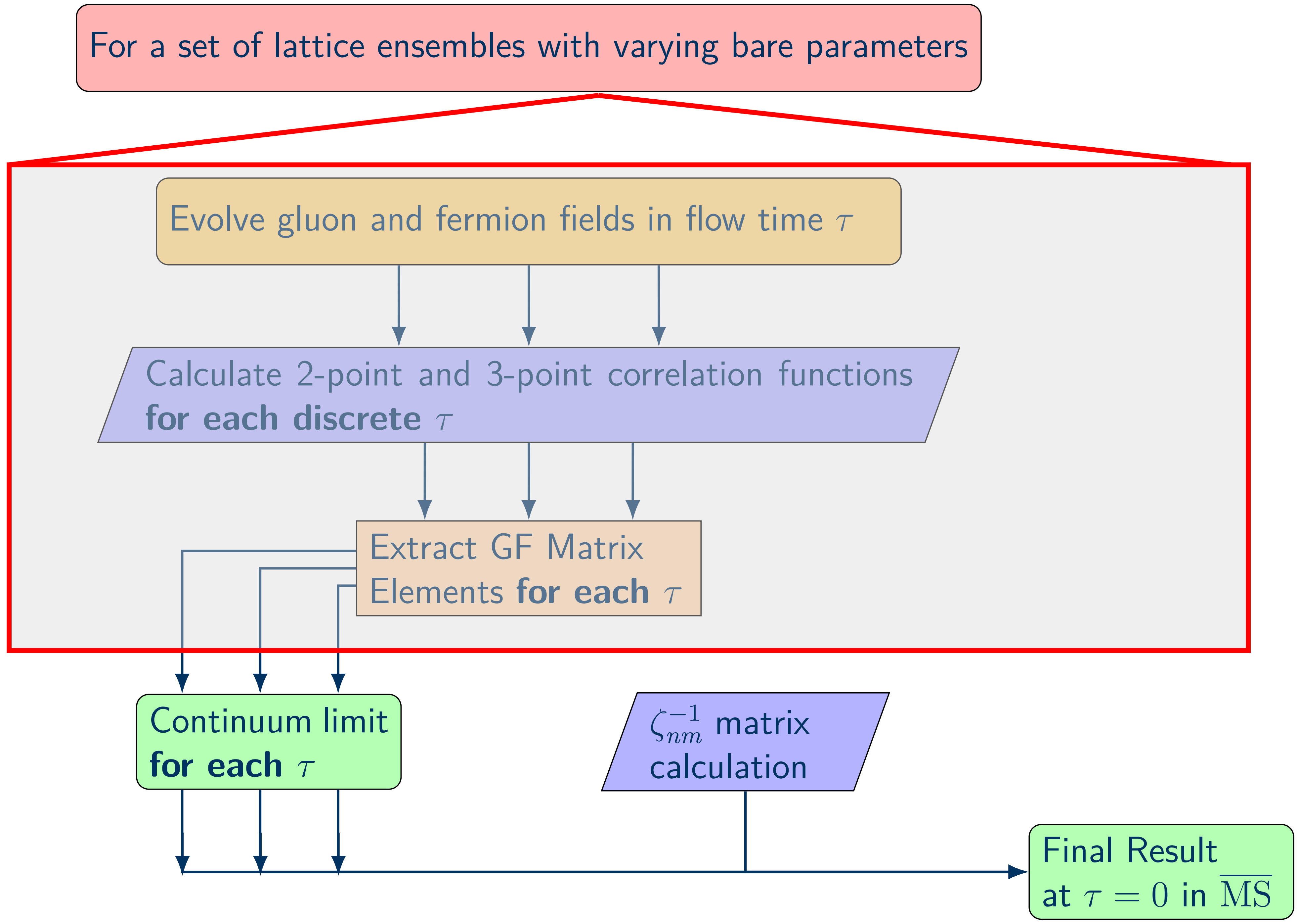
- Complete exploratory studies in simplified setup without additional extrapolations
  - simulate physical charm and strange quarks
  - test case for gradient flow renormalisation and short-flow-time expansion procedure
- Use $\Delta Q = 2$ matrix elements for further validation of method
  - neutral charm-strange meson: proxy to short-distance D^0 mixing (+ spectator effects)
- Pioneer $\Delta Q = 0$ matrix element calculation for lifetime ratio
- Run full-scale simulations for *B* meson mixing and lifetimes
  - simulate at multiple charm-like masses to extrapolate to *b*
  - consider both light and strange spectators
- Tackle additional contributions for absolute lifetimes
  - Evaluate \`eye' and lower dimensional insertions diagram

The procedure

Traditional:



Gradient flow:



Configurations

- We use RBC/UKQCD's 2+1 flavour DWF + Iwasaki gauge action ensembles
-

	L	T	a^{-1}/GeV	am_l^{sea}	am_s^{sea}	M_π/MeV	$\mathrm{srcs} \times \mathrm{N_{conf}}$
C1	24	64	1.7848	0.005	0.040	340	32×101
C2	24	64	1.7848	0.010	0.040	433	32×101
M1	32	64	2.3833	0.004	0.030	302	32×79
M2	32	64	2.3833	0.006	0.030	362	32×89
M3	32	64	2.3833	0.008	0.030	411	32×68
F1S	48	96	2.785	0.002144	0.02144	267	24×98

- For strange quarks tuned to physical value, $am_q \ll 1$
- For heavy *b* quarks, $am_q > 1$ while manageable for physical *c* quarks instead

The procedure for $\Delta Q = 2$

- Evaluate:

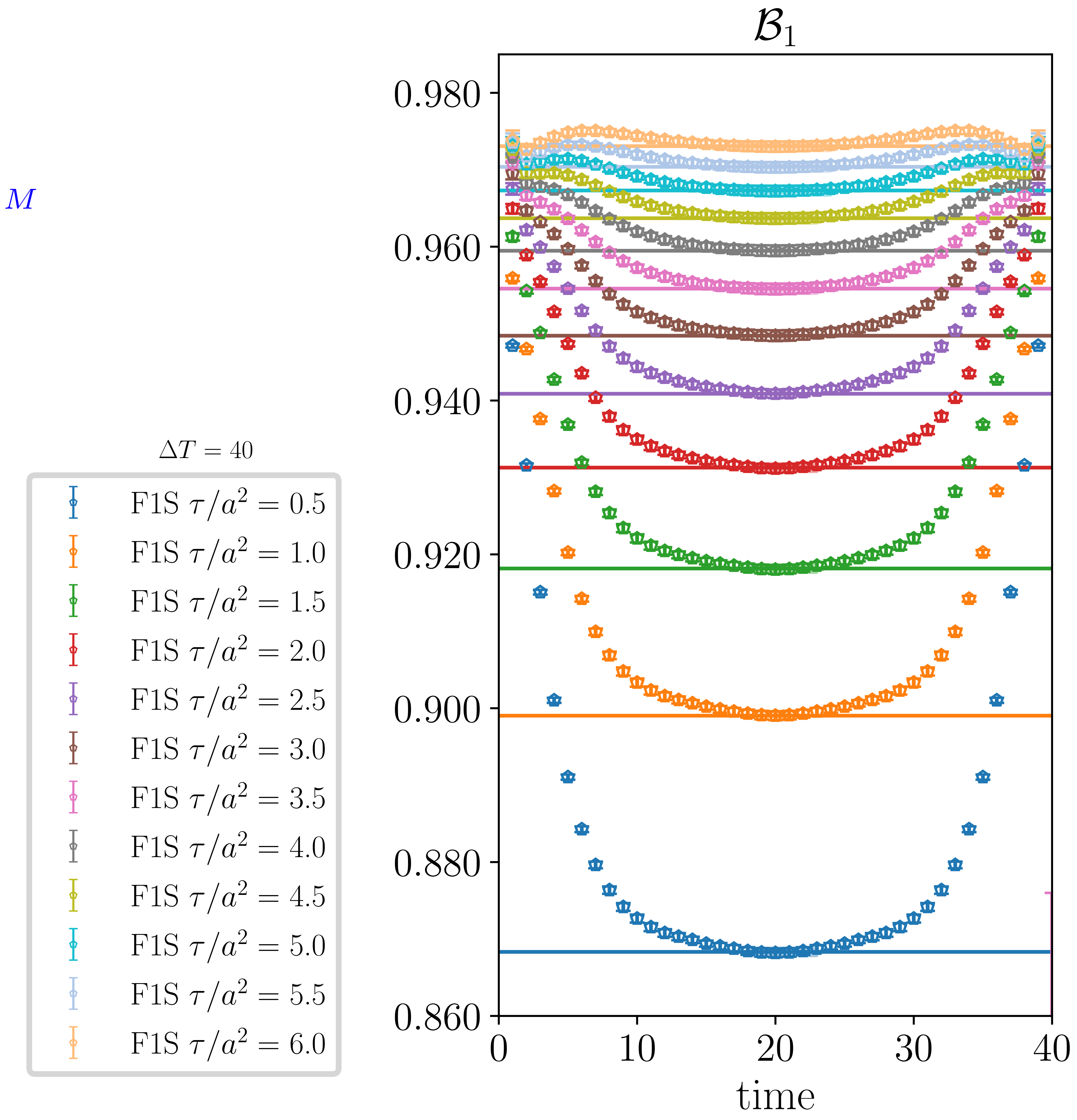
$$C^{3\text{pt}}_{Q_i}(t, \Delta T, \tau) = \sum_{n,n'} \frac{\langle P_n | Q_i | \bar{P}_{n'} \rangle(\tau)}{4 M_n M_{n'}} e^{-(\Delta T - t) M_n} e^{-t M_{n'}} \underset{t_0 \ll t \ll t_0 + \Delta T}{\Longrightarrow} \frac{\langle P \rangle^2}{4 M^2} \langle Q_i \rangle(\tau) e^{-\Delta T\, M}$$

- Rescale with two-point correlation function and obtain:

$$\mathcal{B}_i = \frac{\langle Q_i \rangle}{\eta_i m^2 f^2}$$

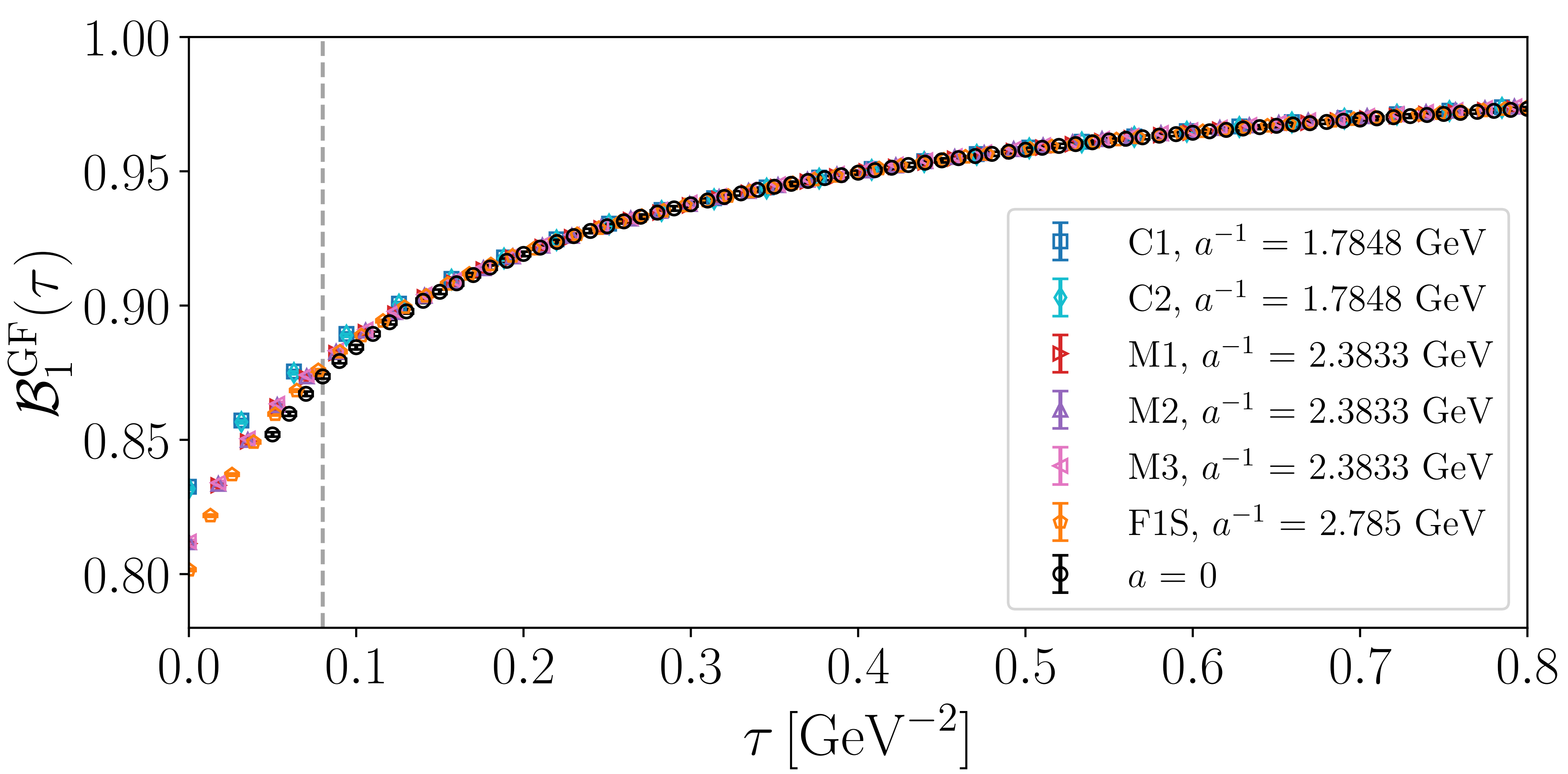
- Find a plateaux as function of t and ΔT at fixed τ in a global fit.

- Combine the lattice results with the perturbative ζ_i

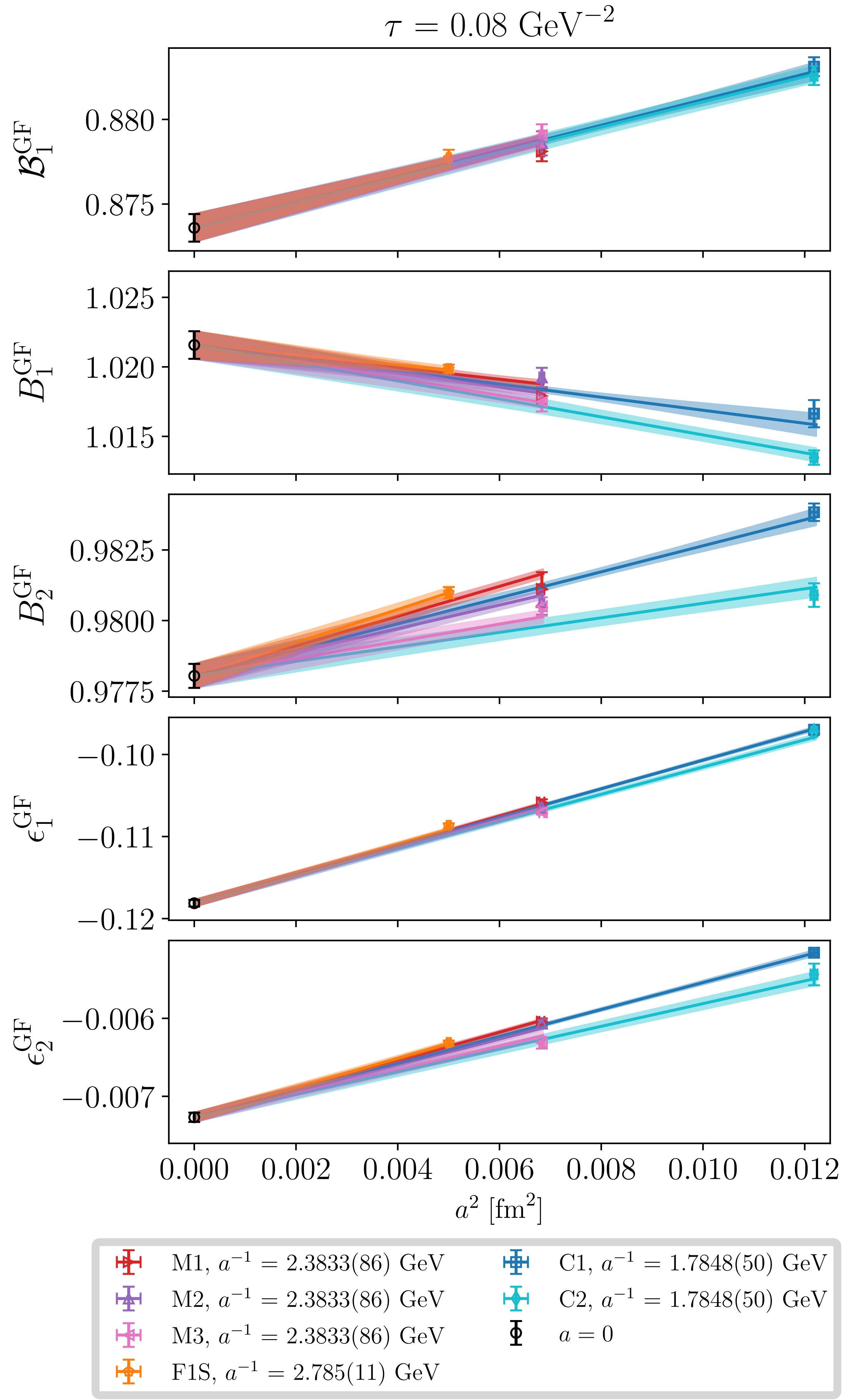


The continuum limit for $\Delta Q = 2$ an more

- Find a plateaux as function of t and ΔT at fixed τ
- Perform global fits for t and ΔT



- Perform the continuum limit at fixed τ (in physical unit)
- Very mild continuum limit, however…



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The zero flow time limit: $\tau \to 0$

Identify the intercept is a “window problem”, which affects in some form every matching scale procedure

- We take advantage of the scale dependance of the methodology:

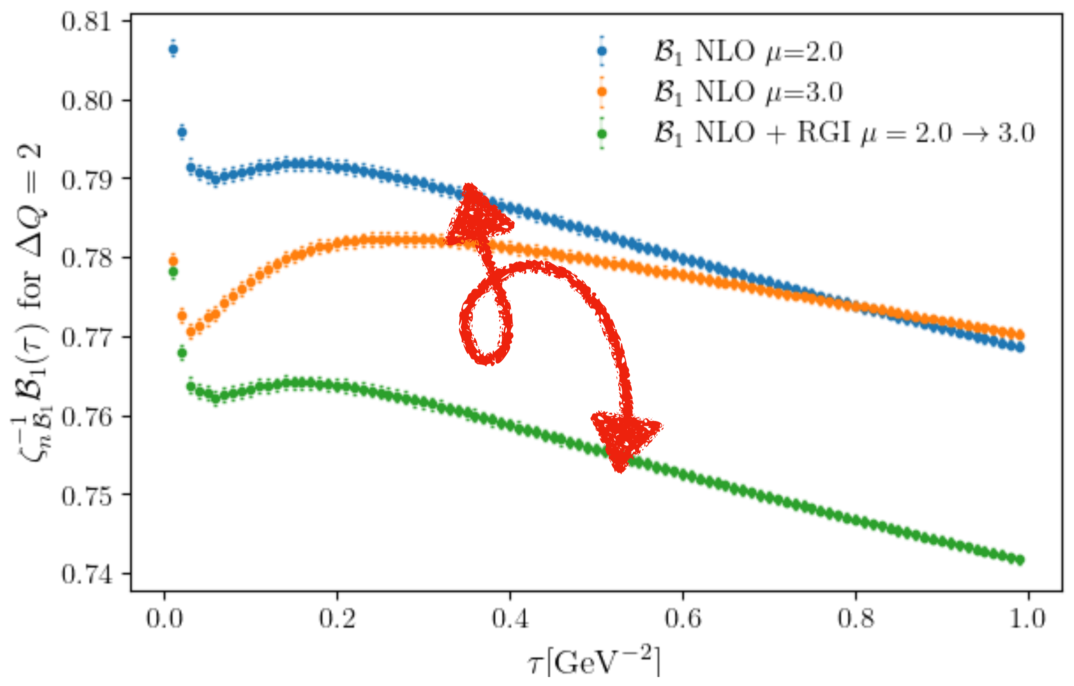
$$\mu^2 \frac{\partial}{\partial \mu^2} \zeta(\tau, \mu) = -\gamma(\alpha_s(\mu))\zeta(\tau, \mu)\,,$$

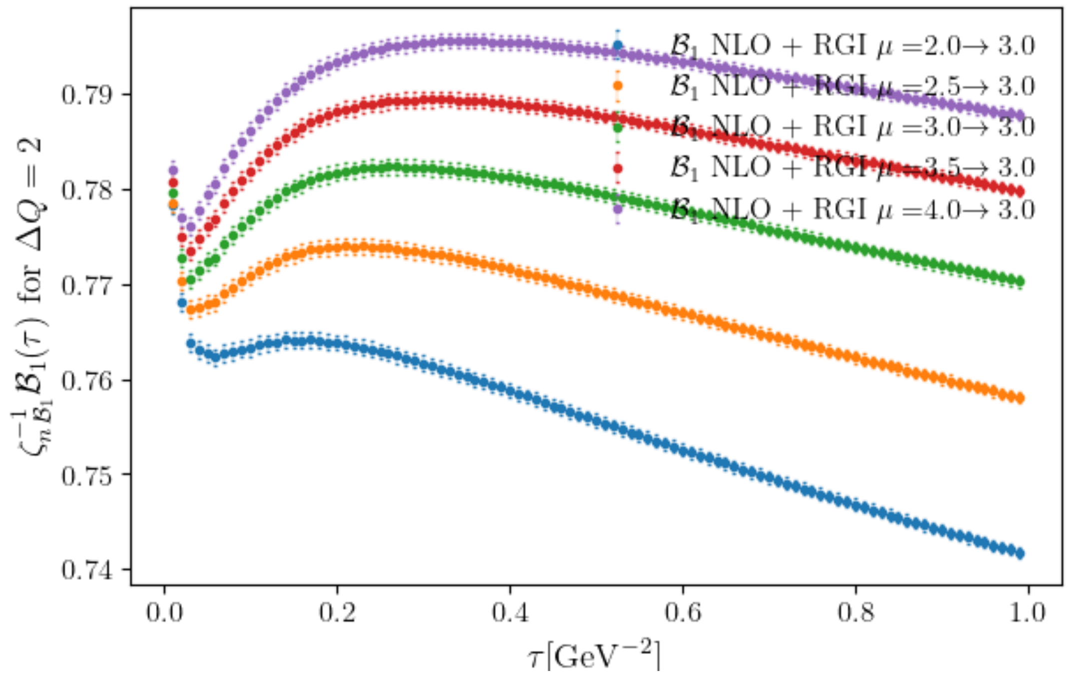
We can impose invariance of our results under scale shift e.g. from $\mu = 2.0 \to 3.0\,\mathrm{Gev}$

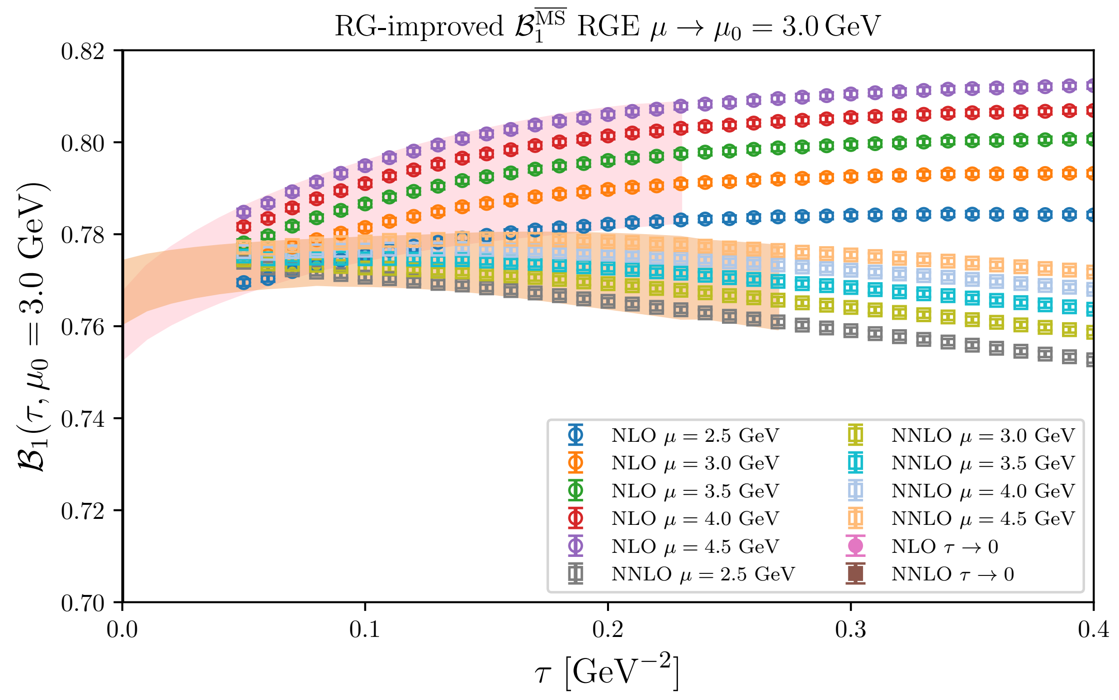
- We also use the flowed anomalous dimension to improve the perturbative behavior of our operators and limit the effects of logarithms

$$\tau \frac{d}{d\tau} \tilde{\mathcal{O}}(\tau) = \tilde{\gamma} \tilde{\mathcal{O}}(\tau) \quad \text{with} \quad \tilde{\gamma} = \left(\tau \frac{d}{d\tau} \zeta\right) \zeta^{-1}$$

so we can run any flow time at to a single reference value $\tau \to \tau_0 = e^{-\gamma_E}/2\mu^2$







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Limits and analysis

Many scales need to be considered:

- The smearing radius must be larger of the lattice spacing of each ensembles used, but small enough not to be sensitive to higher dimensional operators.

$$0.08/\mathrm{GeV}^2 \leq \tau < .035/\mathrm{GeV}^2$$

- Scales of expected convergence of the perturbative expansion

$$2\,\mathrm{GeV} \leq \mu \leq 4\,\mathrm{GeV}$$

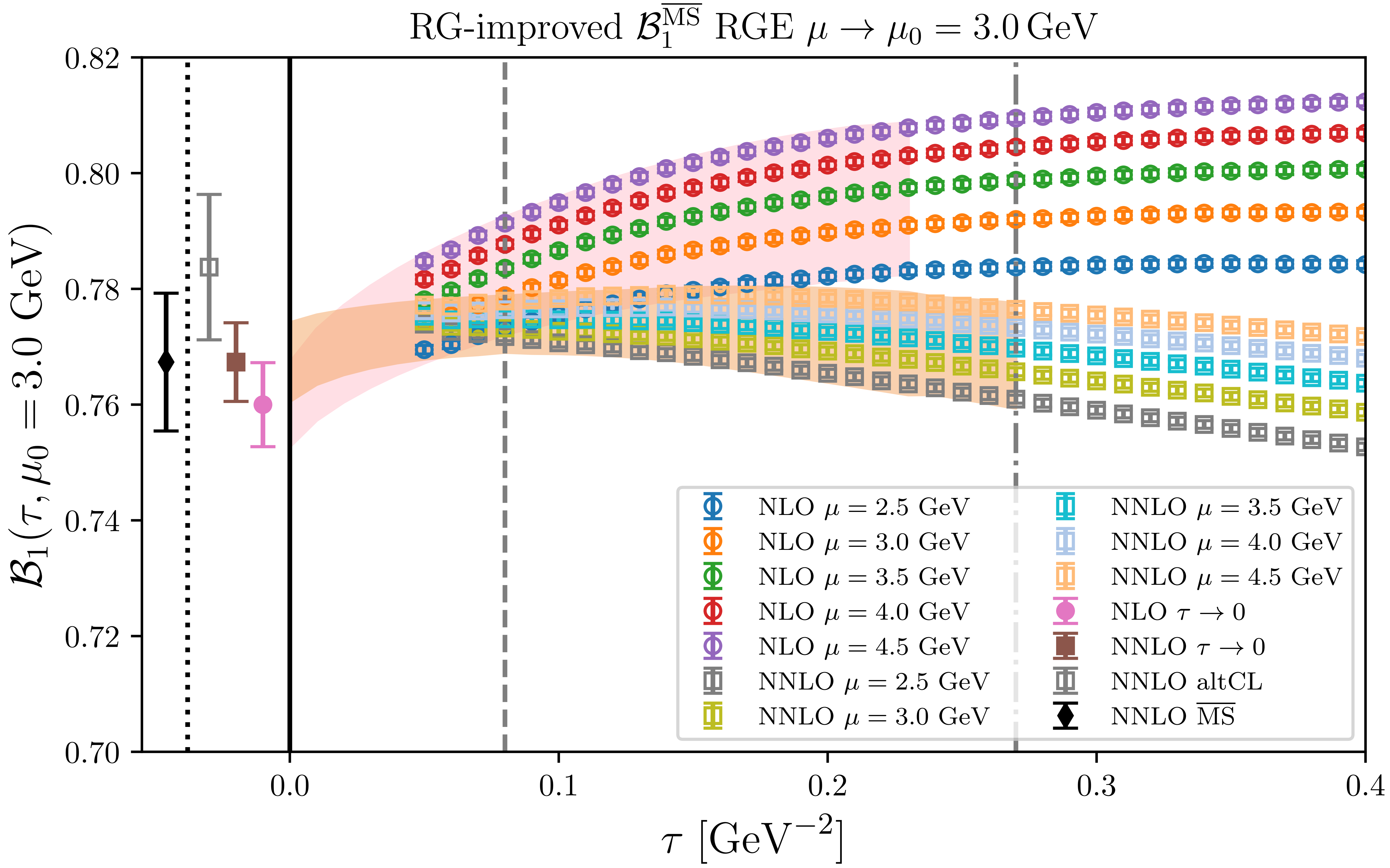
- Assumption for the validity of the SFTX

$$\tau_0 < 1/(8m_c^2)$$

- Our final results are obtained by performing a single global fit to $B(\mu \to \mu_0)$ for a range of initial values of μ using the functional form

$$f(\tau,\mu) = c_0 + \tau \left(c_1^{(\mu)} + c_2^{(\mu)} \ln(\mu^2 \tau) \right) \,,$$

All in one: Mixing



Our final value is shown in black and accounts for

- NNLO central value and statistical uncertainty (GF)
- Continuum limit systematic (CL)
- D_s mistuning + other systematics (OS)
- Perturbative order (PT)

$$\mathcal{B}_1^{\overline{\mathrm{MS}}}(3\,\mathrm{GeV}) = 0.7673(68)_{\mathrm{GF}}(82)_{\mathrm{CL}}(37)_{\mathrm{PT}}(49)_{\mathrm{OS}}$$
$$= 0.7673(123)$$

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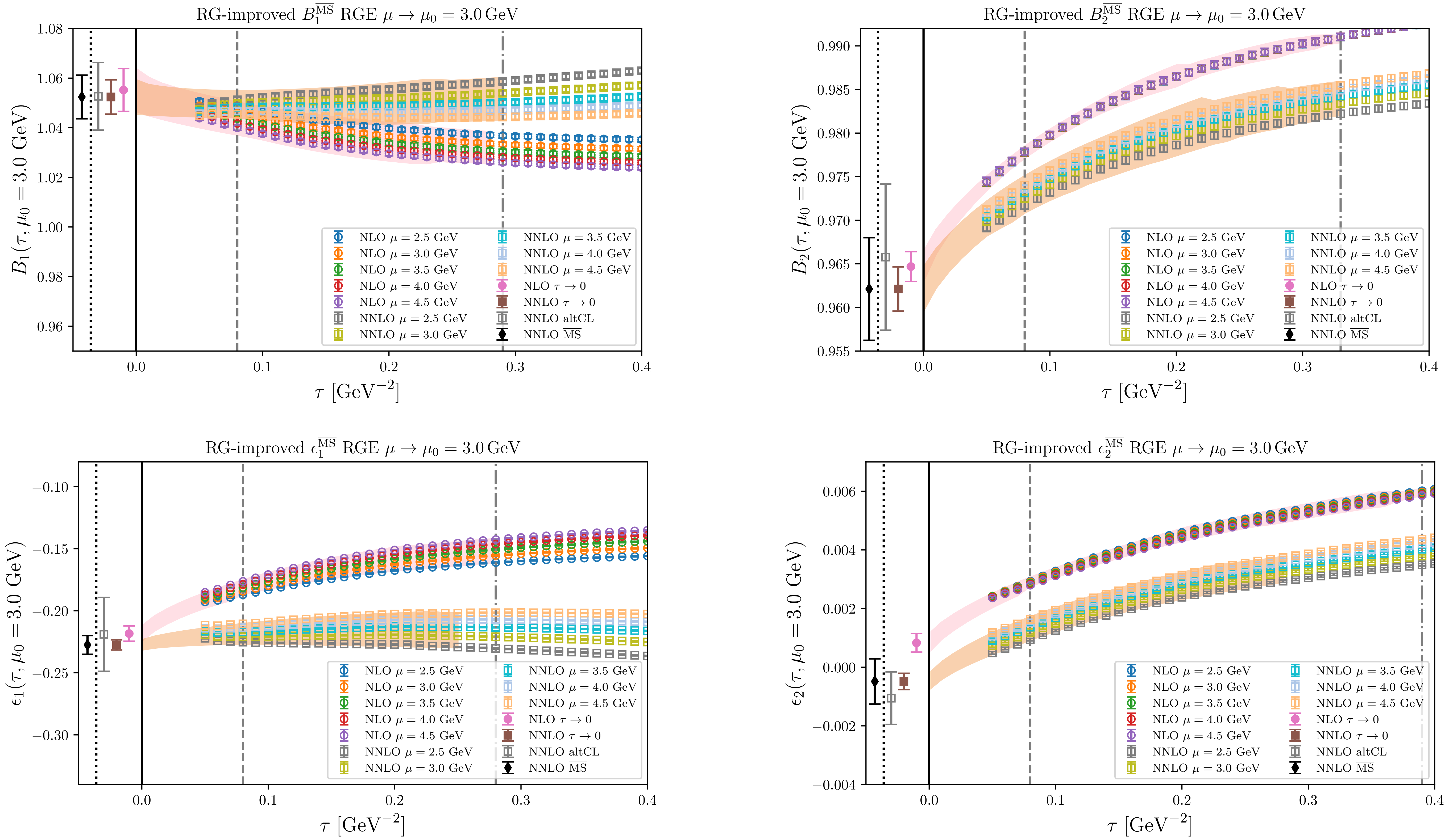
All in one: Lifetimes



$$
\begin{aligned}
B_1^{\overline{\rm MS}}(3\,{\rm GeV}) &= \quad 1.0524(69)_{\rm GF}(\ 1)_{\rm CL}(14)_{\rm PT}(67)_{\rm OS} \\
&= \quad 1.0524(97), \\
B_2^{\overline{\rm MS}}(3\,{\rm GeV}) &= \quad 0.9621(25)_{\rm GF}(18)_{\rm CL}(13)_{\rm PT}(62)_{\rm OS} \\
&= \quad 0.9621(70), \\
\epsilon_1^{\overline{\rm MS}}(3\,{\rm GeV}) &= -0.2275(41)_{\rm GF}(42)_{\rm CL}(46)_{\rm PT}(14)_{\rm OS} \\
&= -0.2275(76), \\
\epsilon_2^{\overline{\rm MS}}(3\,{\rm GeV}) &= -0.0005(\ 3)_{\rm GF}(\ 0)_{\rm CL}(\ 7)_{\rm PT}(\ 0)_{\rm OS} \\
&= -0.0005(\ 8).
\end{aligned}
$$

Note: that these results are scheme dependent

$\tau(D^+)/\tau(D^0)$: sum rules = $1.737^{+0.203}_{-0.260}$, GF+SFTX = 2.344 ± 0.170, experiment = 2.510 ± 0.015

$\tau(D_s)/\tau(D^0)$: sum rules = $1.157^{+0.124}_{-0.016}$, GF+SFTX = 1.311 ± 0.042, experiment = 1.222 ± 0.006

Francesco Moretti, Ulrich Nierste, Pascal Reeck, Matthias Steinhauser
arxiv: April 2026

Results and Summary

- Composite operators can be represented in terms of positive flowtime operators via the small flowtime expansion.
- We applied this methodology to the study of four-quark operators for heavy quarks
- Four-quark matrix elements are strongly-desired quantities (in particular $\Delta B = 0$)
- They are very hard to study: standard renormalisation introduces mixing with operators of lower mass dimension
- Problem that can be overcome by the fermionic gradient flow renormalisation procedure
- We calculate $\Delta Q = 2$ matrix elements as a test case for the short-flow-time expansion
- Preliminary studies of $\Delta Q = 0$ matrix elements for lifetime ratio.