Lattice Determination of Λ_{QCD}

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Standard Model parameters and observables from gradient flow
2026

Introduction

- Goal: High precision determination of the strong coupling $\alpha_S(m_Z)$ at the Z-pole.
- The determination of $\alpha_S(m_Z)$, equivalently $\mu^{-1}\Lambda_{\overline{MS}}$, requires integration of the inverse β-function from perturbative regime to the scale of hadron physics at strong coupling.

$$\mu^{-1}\cdot\Lambda_{\overline{MS}} = (b_0\bar{g}^2)^{-b_1/2b_0^2} \cdot \exp(-1/2b_0\bar{g}^2)\cdot\exp\left(-\int_0^{\bar{g}} dx[1/\beta(x)+1/(b_0x^3)-b_1/(b_0^2x)]\right)$$

- The integration beyond the perturbative regime requires non-perturbative calculation of the β-function, which can be done on lattice
- On the lattice:
  - g^2 and β-function are not defined in $\overline{MS}$ scheme, but the conversion to $\Lambda_{\overline{MS}}$ is known
  - μ^{-1} is $\sqrt{8t_0}$ where $\bar{g}^2 = g^2(\mu)$ is known

Introduction

Alpha collaboration

RG in finite physical volume, running scale set with linear size of the physical volume

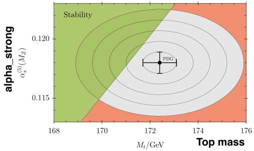
- requirement to run at exactly zero fermion mass is approximated with tuning process (wilson fermions)
- need to switch for scale matching between finite volume scale and infinite volume physical scale
- sampling of the β-function to connect $\Lambda_{\overline{MS}}$ with a physical scale is broken up into three ranges including intermediate SF range without gradient flow renormalization scheme
- Symanzik effective action is invoked investigating the a/L ->0 and a->0 limits at fixed c and at fixed physical volume
- challenge: tuning explicit chiral symmetry breaking of wilson fermions

LatHC

RG directly applied in infinite lattice volume
RG scale set with gradient flow time

- requirement to run at exactly zero fermion mass is set without tuning (staggered fermions)
- no need to switch for scale matching between finite volume SF scale and infinite volume gradient flow scale t
- sampling of $\beta(g^2(t)) = tdg^2/dt$ to connect $\Lambda_{\overline{MS}}$ with a physical scale is broken up into a week-to-intermediate coupling range and the strong coupling range to reach the t_0 scale without change in scheme
- control on infinite lattice volume limit at fixed lattice spacing
- infrared Dirac spectrum and split goldstone spectrum with taste breaking

SM vacuum stability



Introduction

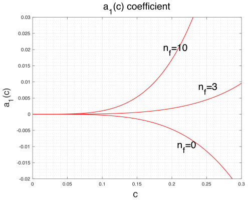
at weak coupling in chiral limit
with zero mode dynamics

$$c = \sqrt{8t}/L \qquad g_c^2 = \frac{16\pi^2\langle t^2 E(t)\rangle}{3(1+\delta(c))}$$

$$\delta(c) = \vartheta^4(e^{-1/c^2}) - 1 - \frac{\pi^2 c^4}{3}$$

$$g_c^2 = g_{\overline{MS}}^2\left(1 - a_1(c) g_{\overline{MS}} + O(g_{\overline{MS}}^2)\right)$$

- tree-level finite volume correction
- next order initiated by Dani Nogradi
  JK checks and extensions



- $a_1 \approx c^4$ for small c
- zero mode becomes irrelevant in $c \to 0$ limit
- small perturbative volume is embedded in chirally broken infinite volume phase (like bag model)
- this is always how we do perturbation theory in infinite volume
- $a_1(c, a^4/L^4)$ is incorporated in weak coupling analysis when $a^4/L^4 \to 0$

at strong coupling in chiral limit

- control of infinite volume limit at strong coupling comes from the massless pion spectrum locked up in finite volume.
- chiral perturbation theory with massless pions in finite volume is the right expansion
  (think about Leutwyler's rotator model)
- chiPT on the gradient flow in finite volume m=0 limit
  (Baer and Golterman p-regime infinite volume)
- scale invariance and its breaking are controlled by F:
  $< E(t) >_L = < E(t) >_\infty \cdot (1 + c_4/L^4 + c_6/F^2L^6 + \ldots)$
- theoretical analysis of the large-L limit in continuum space-time with dim-reg and with lattice regulator at fixed lattice spacing

- infinite volume limit at fixed lattice spacing a is controlled at weak coupling →
- intermediate coupling consistent extension from weak or strong coupling?
- ← infinite volume limit at fixed lattice spacing is controlled at strong coupling

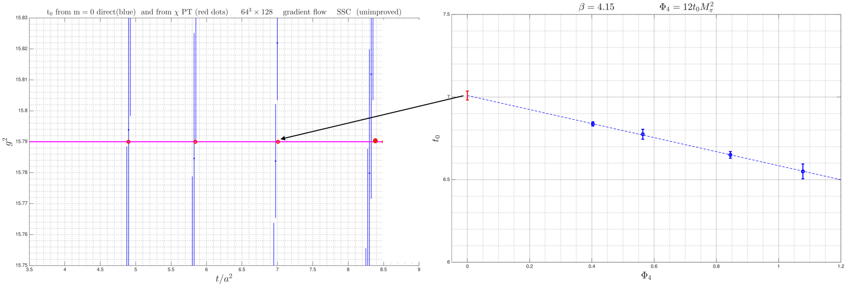
Introduction

We introduced the method in 2017 and continued developing it since

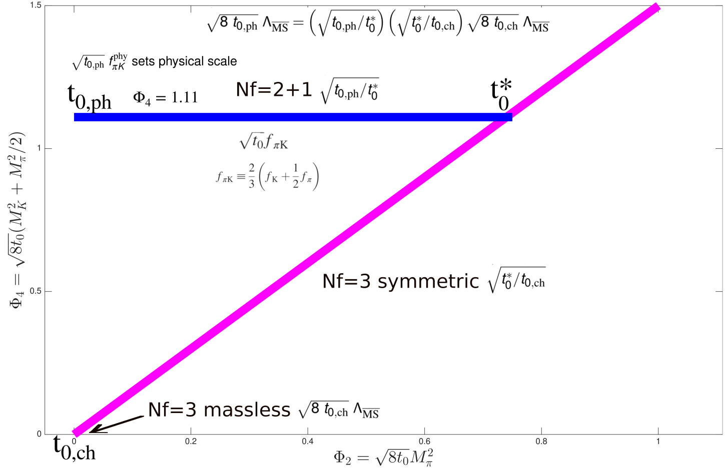
first applications in BSM theories

e-print: 1711.04833

the foundation of the method: to match the chiral limit from the p-regime to direct calculation in the chiral limit on infinite lattices at fixed lattice spacings before the a—> 0 limit is taken



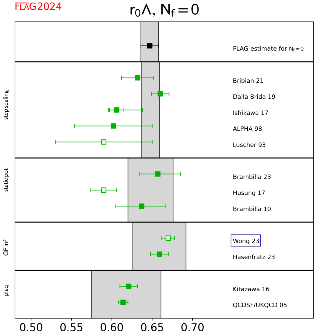
Introduction



Physical scale is set for $\sqrt{8\, t_{0,\rm ph}} \Rightarrow \Lambda_{\overline{\rm MS}}$ can be obtained in MeV

Introduction

- In past years, we tested this strategy on BSM models such as models with $N_f = 2$ in sextet representation, $N_f = 10, 12$ in fundamental representation.
- In Gradient Flow 2025, we discussed in details about how we computed in the case $N_f = 0$ as pilot study for application on the massless $N_f = 3$ model



Methodology

- Simulation details :
  - Symanzik improved $SU(3)$ gauge action with 4-stout ($\rho = 0.12$) smeared staggered fermions
  - Boundary Conditions:
    - Gauge: periodic
    - Fermion:
      massless $N_f = 3$: anti-periodic in spatial, anti-periodic in temporal
      $N_f = 3$ and $N_f = 2+1$: periodic in spatial, anti-periodic in temporal
- In this talk, we focus on:
  - obtaining $\sqrt{8\ t_{0,\mathrm{ch}}}\ \Lambda_{\overline{\mathrm{MS}}}$ in massless $N_f = 3$
  - g^2 scheme: Tree-level improved SSC
    (Flow: Symanzik, Action: Symanzik, Observable: Clover)[2203.15847]

$$g_c^2 = \frac{16\pi^2 \langle t^2 E(t) \rangle}{3(1+\delta(c))},\ c \equiv \sqrt{8t}/L$$

$$\delta(c) = \vartheta^4(e^{-1/c^2}) - 1 - \frac{\pi^2 c^4}{3}$$

Methodology

- In our previous pilot studies, we measured $\beta(g^2) \equiv t\, dg^2/dt$ with local numerical derivatives in each ensemble

- In addition, we measure :

  $$s^2 \equiv t_2/t_1 \equiv \exp\left(\int_{g^2(t_1)}^{g^2(t_2)} dg^2/\beta(g^2)\right)$$

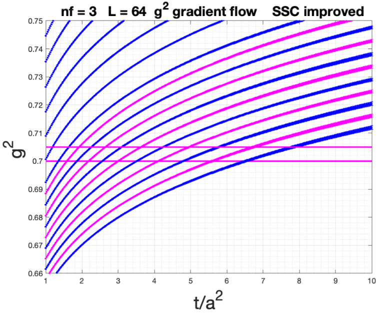
  for a chosen set of $\{g^2(t_1), g^2(t_2)\}$ pairs

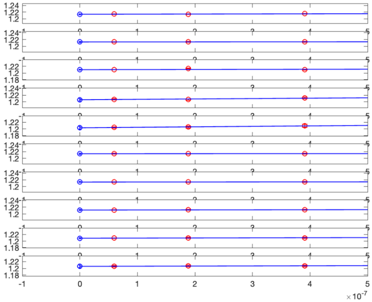
- It is expected that s^2 is more precise than the local β, thus we focus on using s^2 in this analysis

Methodology for massless $N_f = 3$

For each pair of g^2:

- a^2/t_1 and $s^2 = t_2/t_1$ are measured at each L/a and $6/g_0^2$
- interpolation among $6/g_0^2$ values are obtained as additional data points
- At each $6/g_0^2$, infinite volume limit is taken: $s^2(L/a) = s^2(\infty) + c_1(L/a)^{-4}$

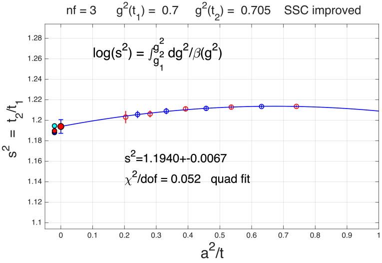


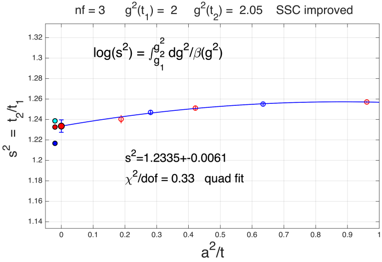


Methodology for massless $N_f = 3$

For each pair of g^2:

- a^2/t_1 and $s^2 = t_2/t_1$ are measured at each L/a and $6/g_0^2$
- interpolation among $6/g_0^2$ values are obtained as additional data points
- infinite volume limit at each $6/g_0^2 \Rightarrow$ continuum limit

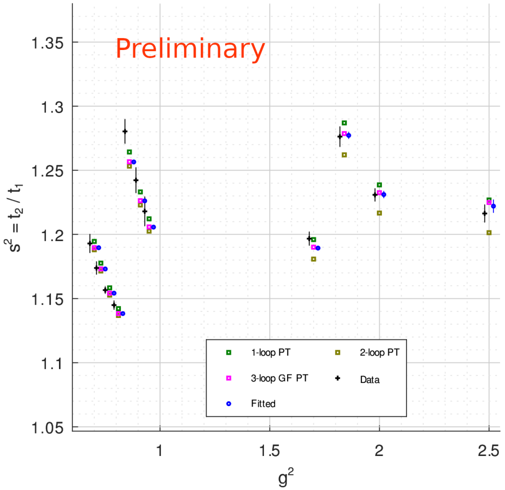




Then β-function is parameterized and fitted with data

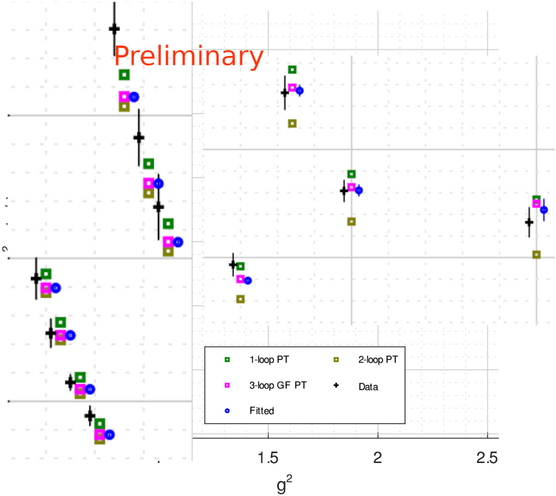
Preliminary Results

In the perturbative region, $\beta(g^2) \equiv \beta_{\text{3-loop,GF}}(g^2) + b_{\text{4-loop}}\ g^{10}$



Preliminary Results

In the perturbative region ($g^2 < g^2_{\rm ref}$), $\beta(g^2) \equiv \beta_{\text{3-loop,GF}}(g^2) + b_{\text{4-loop}}\ g^{10}$



Preliminary Results

- For any t,

  $\sqrt{8\ t}\ \Lambda_{\overline{\mathrm{MS}}} \propto \exp(-\int_0^{g(t)}[-g\ \beta^{-1}(g^2)+1/(b_0\ g^3)-b_1/(b_0^2\ g)]dg)$

- Simulation cannot reach to arbitrarily small g^2

  $\Rightarrow$ There is a minimal reachable g^2_{match}

- $\Rightarrow$ In the region $g^2 < g^2_{\mathrm{match}}$,

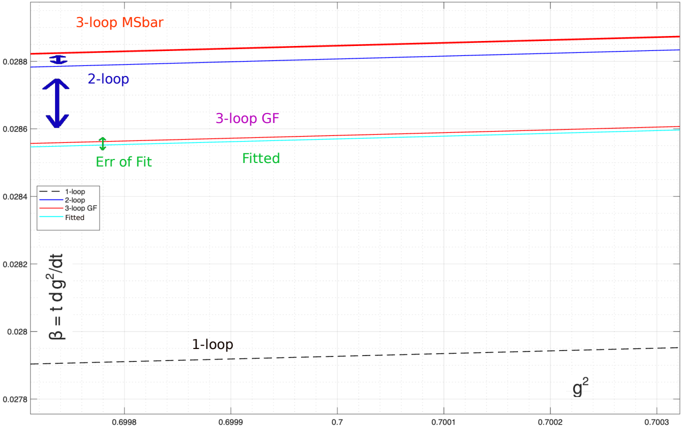
  β is taken to be the highest known n-loop β of PT. ($n = 3$ for GF)

- $\Rightarrow$

$$\sqrt{8\ t}\ \Lambda_{\overline{\mathrm{MS}}} \propto \exp(-\int_0^{g_{\mathrm{match}}} [-g\ \beta^{-1}_{3-\mathrm{loop,GF}}(g^2)+1/(b_0\ g^3)-b_1/(b_0^2\ g)]\ dg$$
$$-\int_{g_{\mathrm{match}}}^{g^2(t)} [-g\ \beta^{-1}_{\mathrm{fitted}}(g^2)+1/(b_0\ g^3)-b_1/(b_0^2\ g)]\ dg)$$

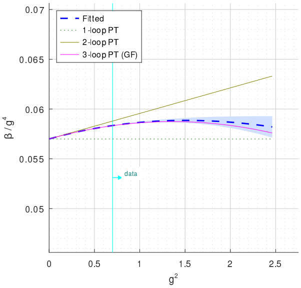
Preliminary Results

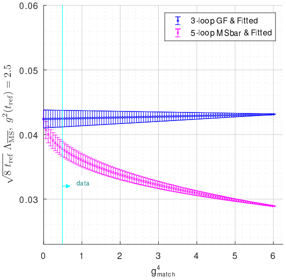
Slow convergence of β: Around our lowest possible $g^2_{\text{match}} \sim 0.7$, 2-loop and 3-loop GF are far apart compared with 3-loop $\overline{\text{MS}}$.



How reliable is the matching?

Preliminary Results





- The fitted 4-loop term is consistent with zero

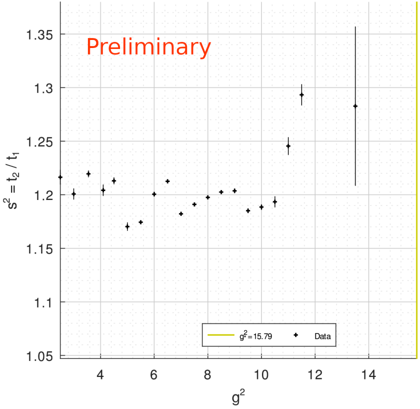
  $\Rightarrow$ β_{fitted} is indistinguishable from $\beta_{3-\mathrm{loop,GF}}$

- Result is stable against broad range of g^2_{match} from 0.7 to 2.5

  $\Rightarrow$ Matching with PT is valid and reliable

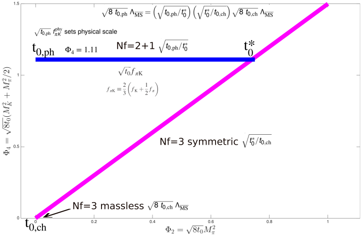
Preliminary Results

We also have data in higher g^2 regime (to be analyzed)



Fit Ansatz: $\beta(g^2) = g^4/(2\ \sum_{j=0}^{N}\ p_j\ g^{2\ j})$ for some order N with parameters p_j

Connection to physical point

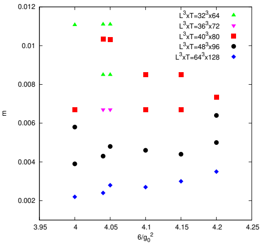


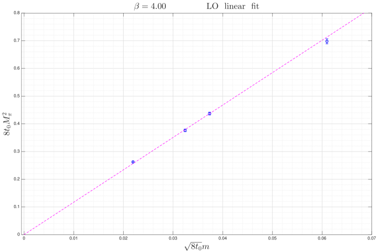
Our strategy is as follows: [M. Bruno et al, Phys. Rev. D 95, 074504 (2017)]

- At different spacings:
  - t_0 in infinite volume limit is obtained at multiple masses in symmetric massive $N_f = 3$ simulations
  - By fitting dependence of t_0 on masses with χPT, we obtain $t_0^*/t_{0,\mathrm{ch}}$
- Continuum limit of $t_0^*/t_{0,\mathrm{ch}}$ is taken
- $t_{0,\mathrm{ph}}/t_0^*$ is obtained by fitting along constant $\Phi_4 = 1.11$ computed with $N_f = 2+1$ simulations

Connection to physical point

- Ongoing analysis for symmetric $N_f = 3$

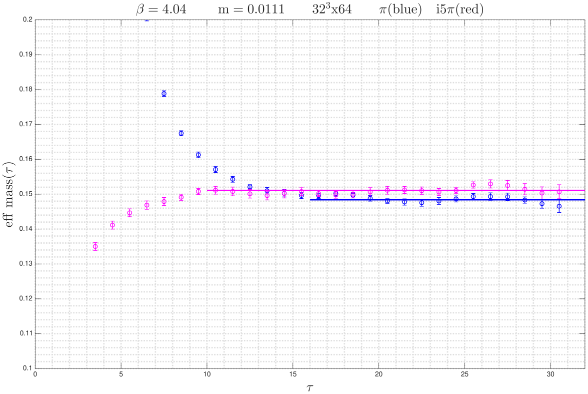




- $N_f = 2 + 1$ simulations are ongoing

Connection to physical point

Taste-breaking effect is small but needs to be taken under control



Conclusion

- We have fully developed a gradient-flow-based strategy to compute $\Lambda_{\overline{\mathrm{MS}}}$ at high precision
- We have accurate data both in weak- and strong-coupling regime in massless $N_f = 3$ model
- In perturbative regime, our data is in good contact with 3-loop PT $\Rightarrow$ the effect of the unknown higher-loop order terms is minimal
- Ongoing analysis on the weak and strong regime
- Analysis and simulations for massive $N_f = 3$ and $N_f = 2+1$ are ongoing