Non-perturbative determination of the strong coupling $\alpha_s(m_Z)$ from four flavor lattice QCD using gradient flow

Fermilab Lattice & MILC Collaborations

Yash Mandlecha

MICHIGAN STATE UNIVERSITY

Workshop on
Standard model parameters and observables from gradient flow
University of Edinburgh, UK | May 14, 2026

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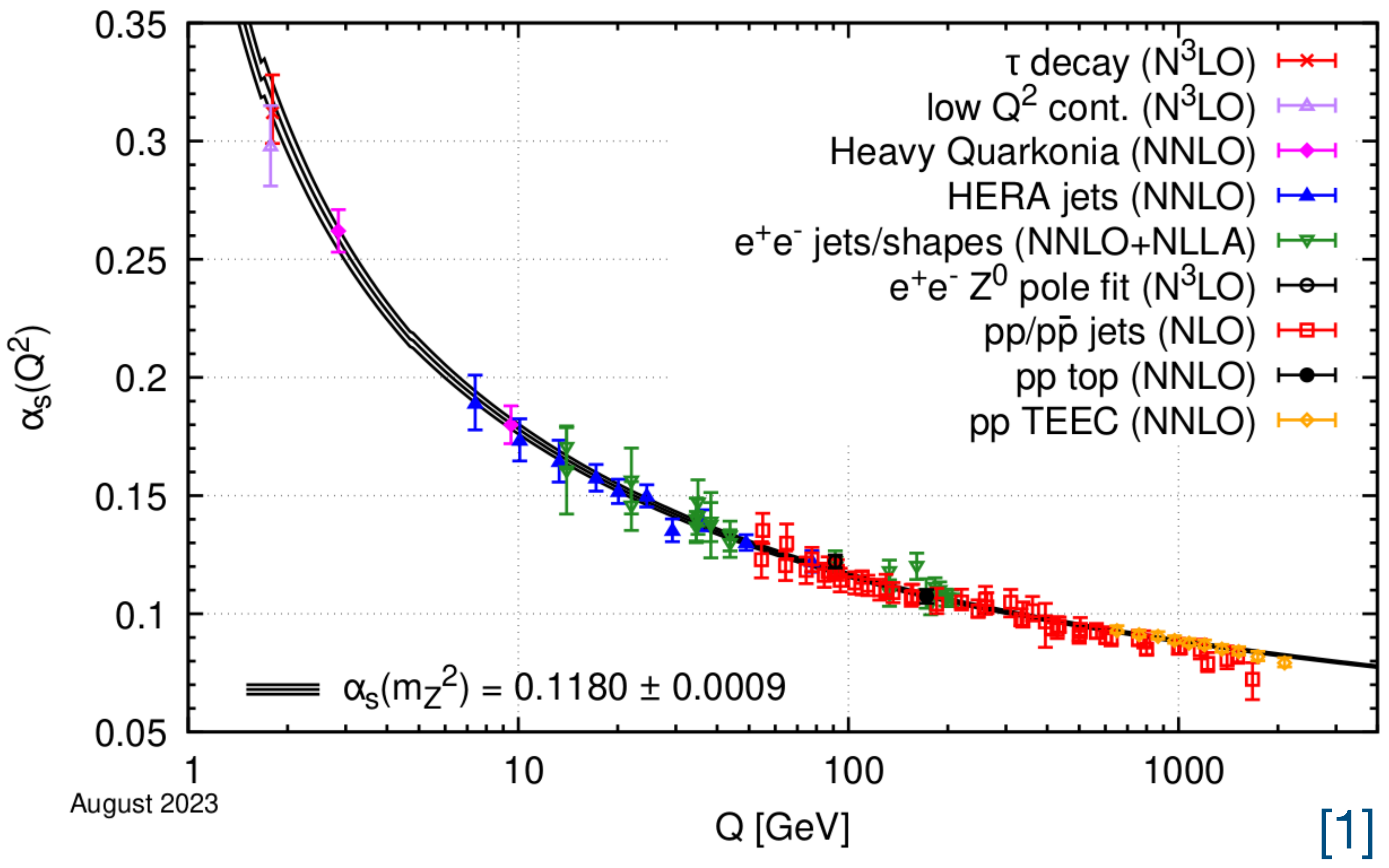
On behalf of the Fermilab Lattice & MILC Collaborations

Overview

- Motivation & Background
- Gradient flow
- Continuous β-function methodology
- Analysis pipeline
- Current status
- Future directions

Strong coupling $\alpha_s(m_Z)$

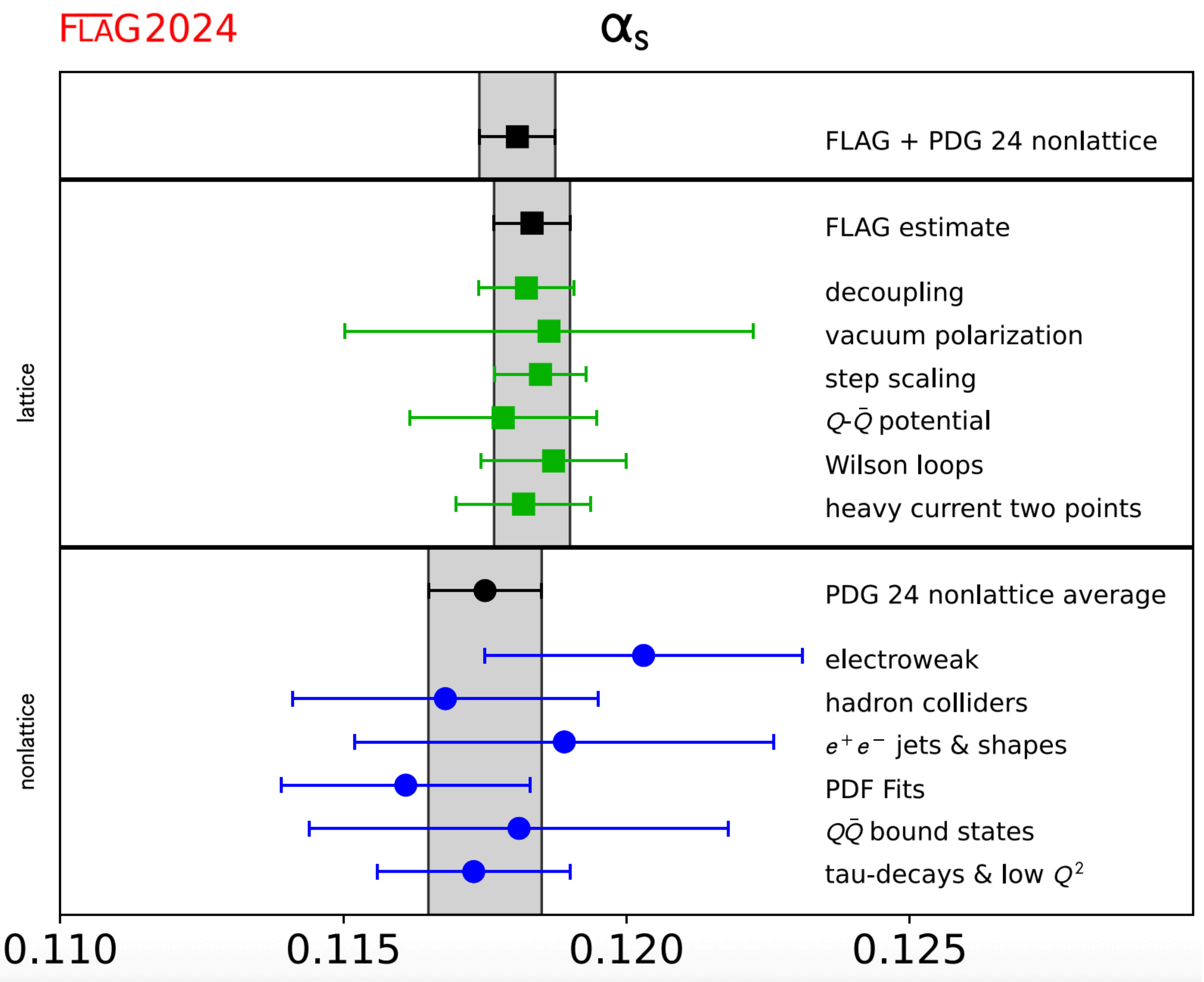
- Fundamental parameter of the Standard model
  - Affects QCD processes [1]
- Important source of uncertainty
  - top-quark mass, Yukawa coupling, key Higgs production channels [1, 2, 3]
- Future precision goal:
  - Global average: $\alpha_s(m_Z) = 0.1179(9)[0.8\%]$ [1]
  - Phenomenology target: $\lesssim 0.2\ \%$ [2, 3]
  - Lattice QCD currently achieves $\sim 0.5\ \%$ precision [4]
- Essential for LHC precision predictions, EIC QCD structure studies, and global PDF/QCD consistency. [3, 5]



[1]

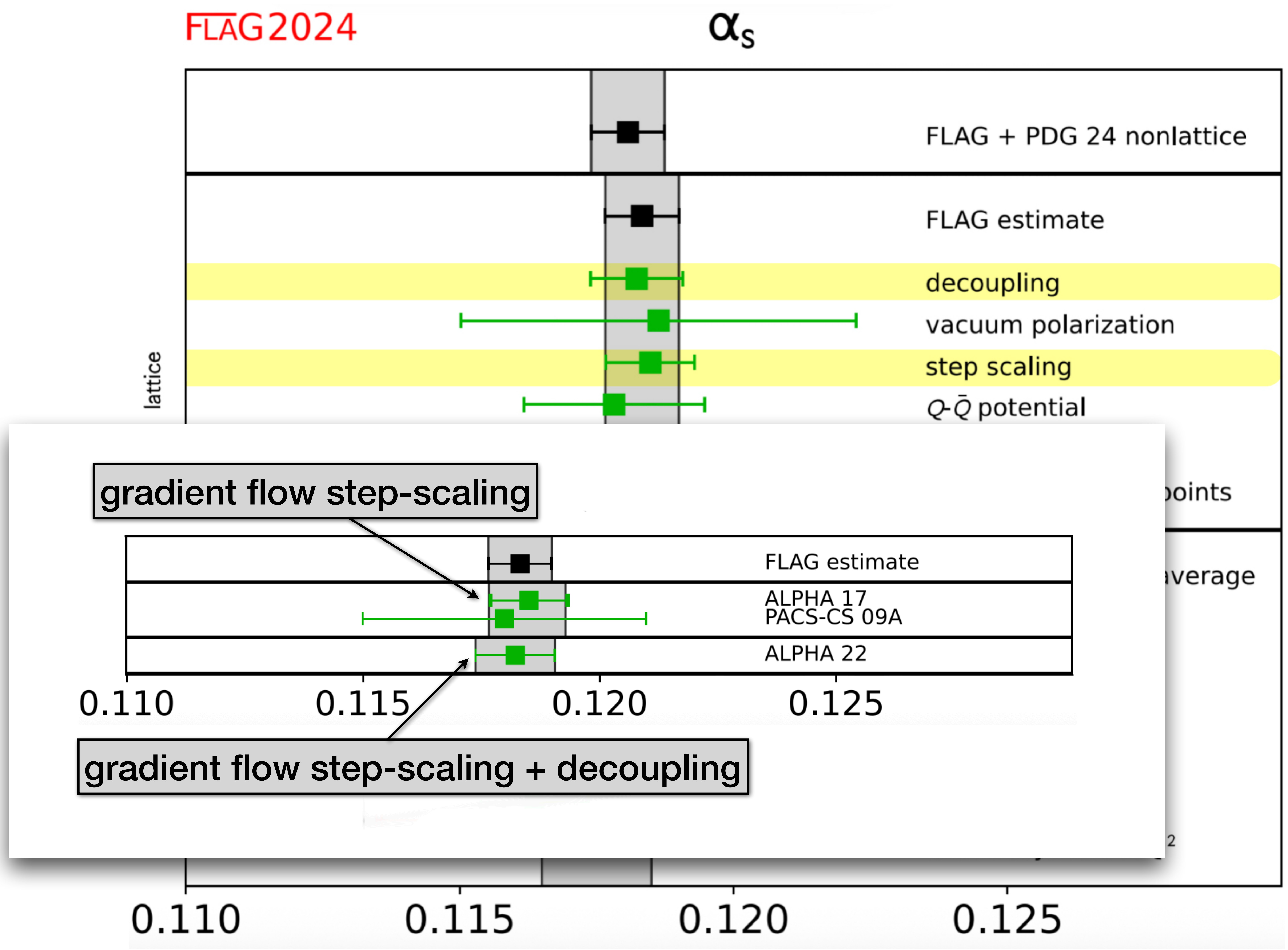
[1] Navas et al. PRD 110 (2024) 030001
[2] Snowmass 2021, 2209.10758
[3] Snowmass 2022, 2203.08271
[4] [FLAG2024, arXiv:2411.04268]
[5] Kutz et al. PRD 110 (2024) 074004

$\alpha_s(m_Z)$ from lattice gauge theory



[FLAG2024, arXiv:2411.04268]

$\alpha_s(m_Z)$ from lattice gauge theory



[FLAG2024, arXiv:2411.04268]

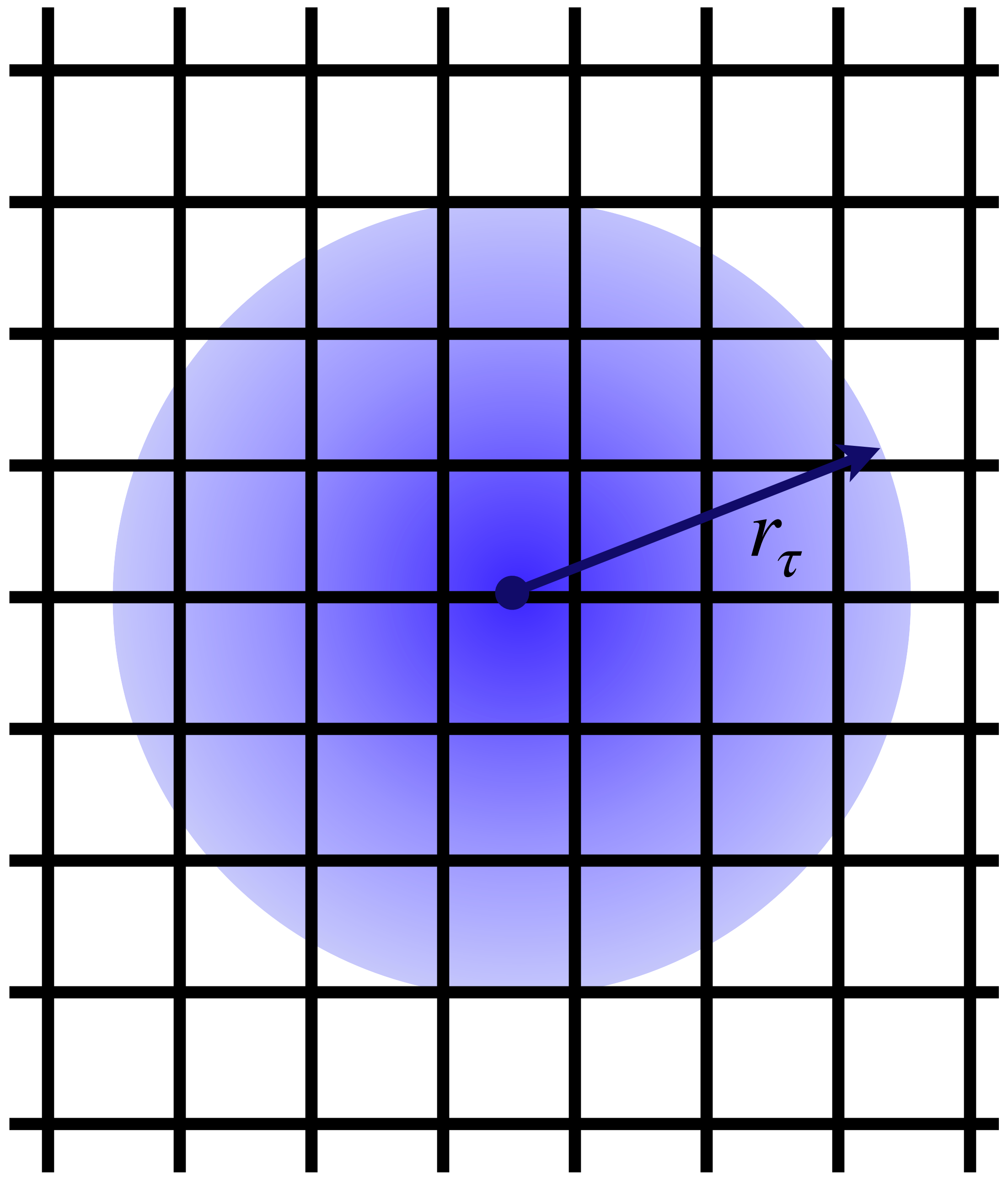
Gradient flow β-function

$r_\tau \equiv \sqrt{8\tau}$

$\mu^{-1} \sim r_\tau$

- Gradient flow: continuous smearing transformation
  - Evolves the gauge fields along a fictitious flow time τ. [1]
  - Describes a renormalization group (RG) transformation [2]
- RG β-function encodes how the $g^2(\mu)$ varies with μ.
- The renormalized gradient-flow coupling
  - Finite volume, with bare coupling $\beta_b \equiv 10/g_0^2$,
  - In terms of flowed Yang–Mills energy density $E(\tau)$

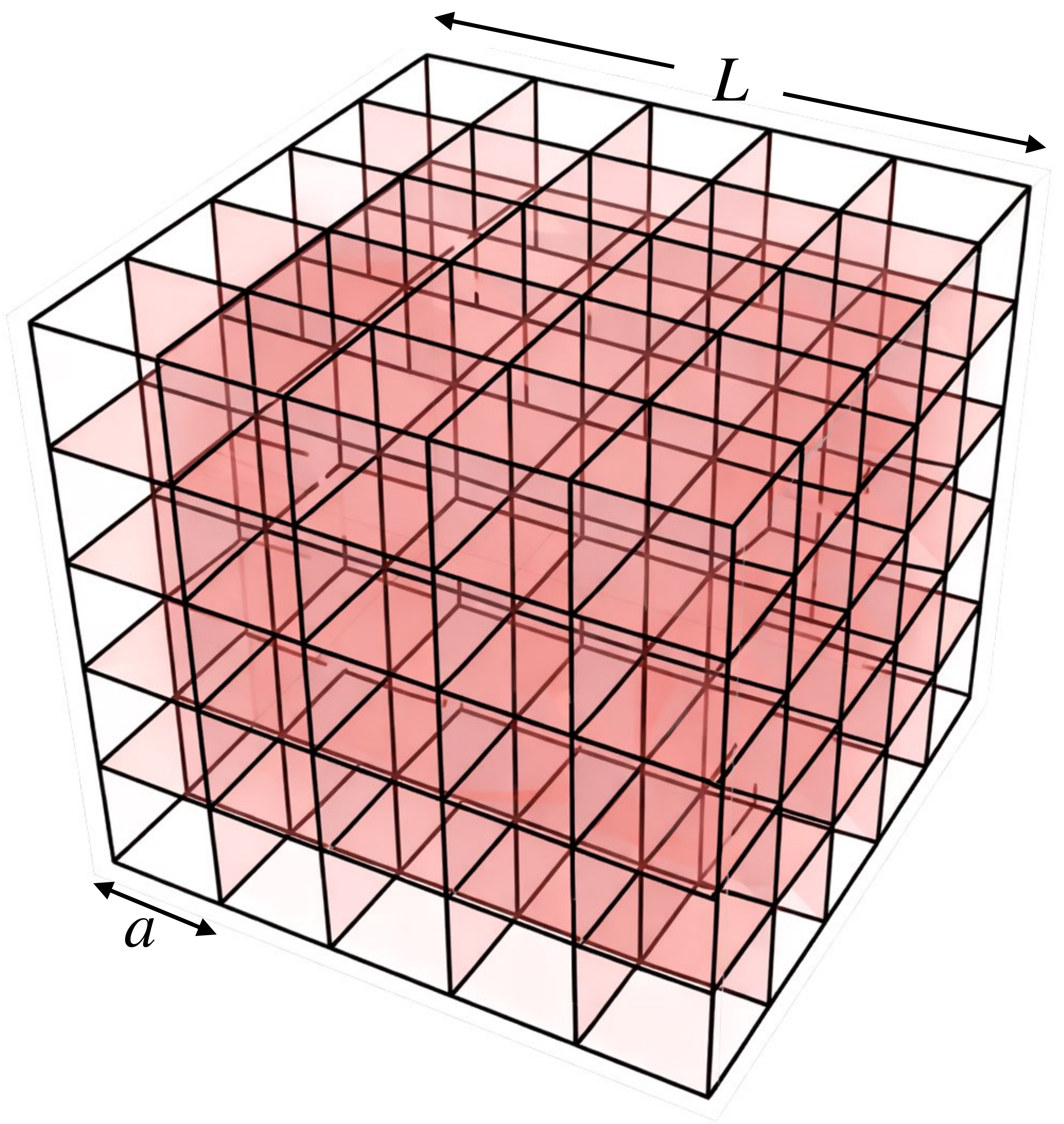
$$g^2_{\rm GF}(\tau;\beta_b) \propto \tau^2 \langle E(\tau) \rangle_{\beta_b}$$



- GF β-function: $\beta_{\rm GF}(\tau;\beta_b) \equiv -\tau \dfrac{dg^2_{\rm GF}(\tau;\beta_b)}{d\tau}$ [3]

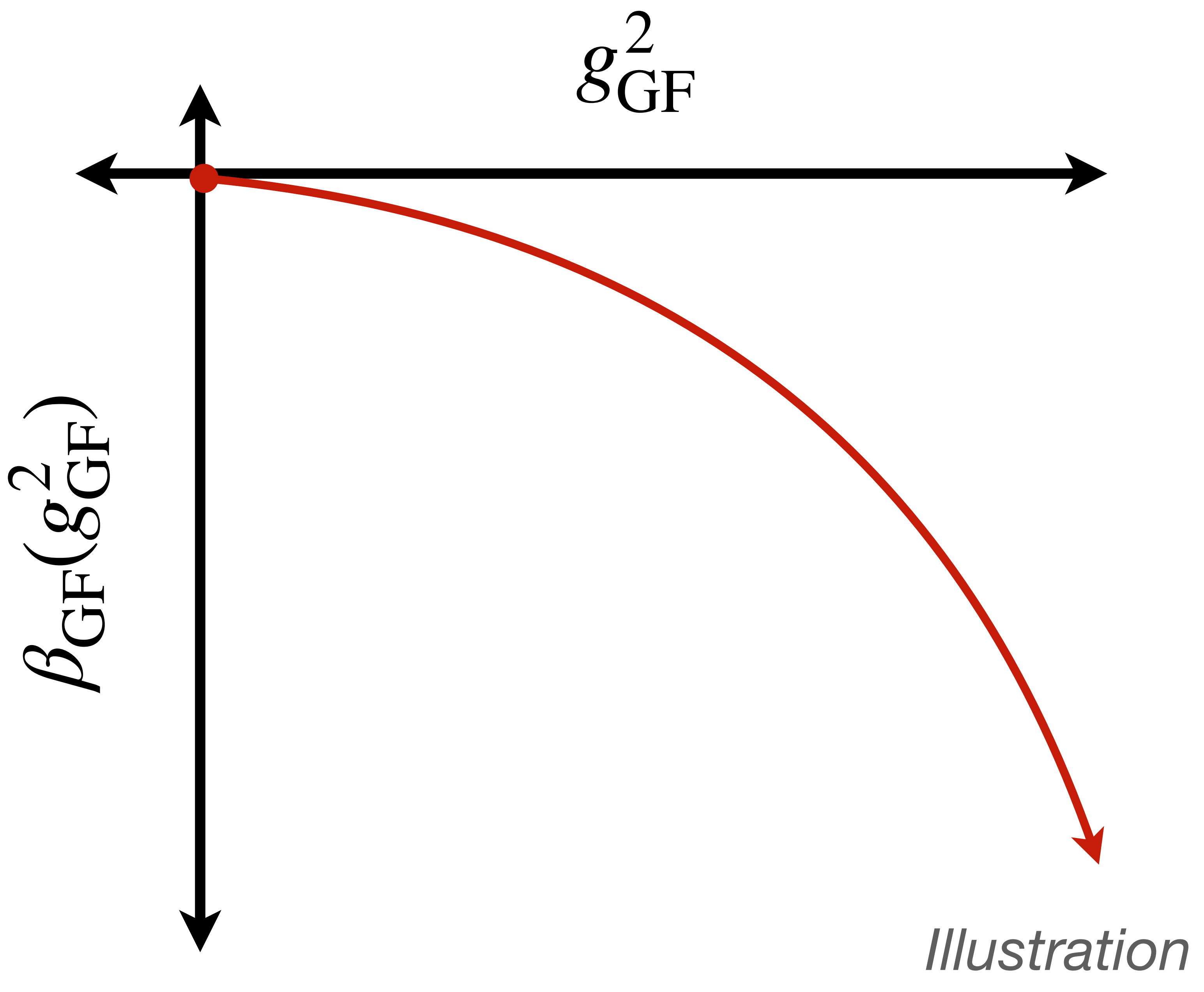
[1] [Luescher, M., JHEP 08 (2010) 71]
[2] [Carosso, A., et. al., PRL 121 (2018) 201601]
[3] [Hasenfratz A., Witzel O., Phys.Rev.D 101 (2020) 3, 034514]

Continuous β-function method



Finite volume lattice

1. Finite-volume $g_{\rm GF}^2$ & β-function
2. Chiral ($am_f \to 0$) limit*
3. Infinite volume ($a/L \to 0$) limit
4. Continuum ($a^2/\tau \to 0$) limit



Infinite volume & continuum

*In the strong coupling (confined) regime

[1] [Fodor, Z. et. al., JHEP (2014) 018]
[2] [Hasenfratz, A., Peterson, C.T., PRD 108, 014502 (2023)]
[3] [Hasenfratz A., Witzel O., Phys.Rev.D 101 (2020) 3, 034514]

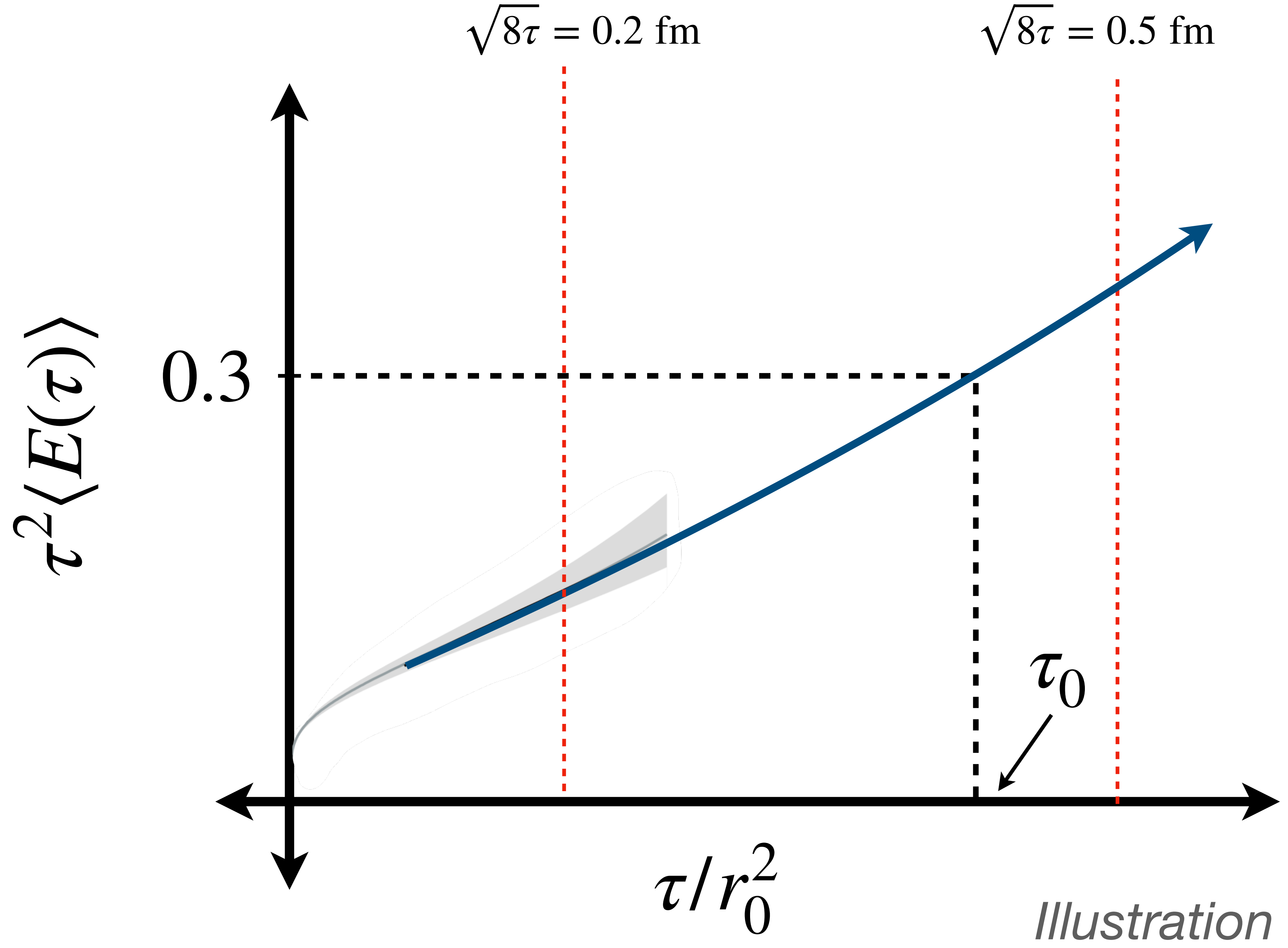
From $\beta_{\mathrm{GF}}(g^2_{\mathrm{GF}}) \to \sqrt{8\tau_0}\Lambda_{\overline{\mathrm{MS}}}$

- **Schematically:**

$$8\tau\Lambda^2_{\mathrm{GF}} \sim \exp\left[-\int^{g^2_{\mathrm{GF}}(\tau)} \frac{\mathrm{d}x}{\beta_{\mathrm{GF}}(x)}\right]$$

- **Integrate** $g^2_{\mathrm{GF}} = 0 \to g^2_{\mathrm{GF}}(\tau_0{}^*)$
- $\Lambda_{\mathrm{GF}} \to \Lambda_{\overline{\mathrm{MS}}}$: **known exactly at** 1**-loop**
  - **Requires matching with perturbation theory**
- $\alpha_s(m_Z)$ **and** $\Lambda_{\overline{\mathrm{MS}}} = \Lambda_{\mathrm{QCD}}$ **are interchangeable.**
  - **Calculating** $\Lambda_{\overline{\mathrm{MS}}}$ **is easier (in our case)**

$${}^*\langle \tau^2 E(\tau)\rangle\Big|_{\tau=\tau_0} \equiv 0.3 \quad [1]$$



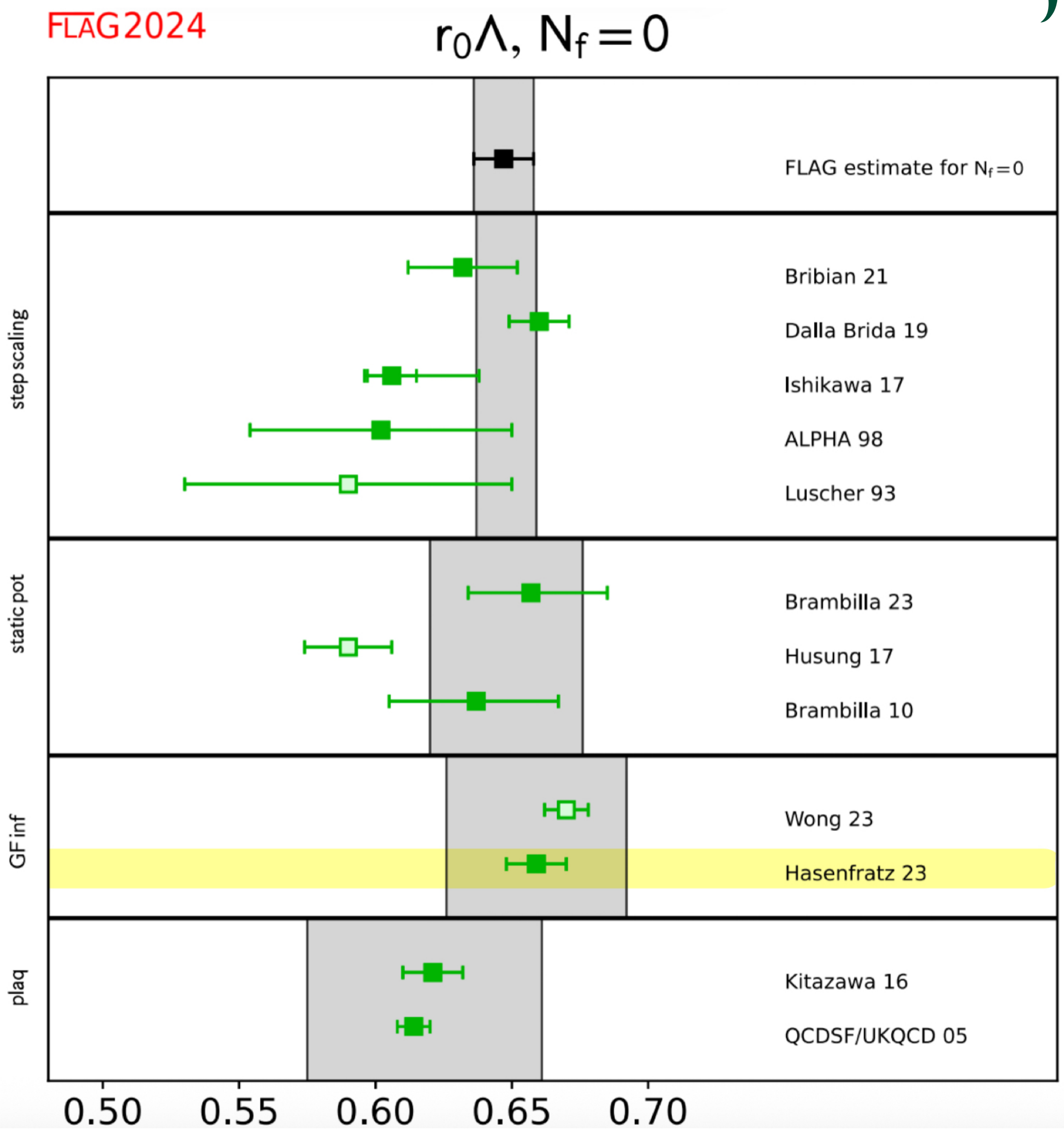
[1] [Luescher, M., JHEP 08 (2010) 71]
[2] [Hasenfratz, A., Peterson, C.T., PRD 108, 014502 (2023)]

$\sqrt{8\tau_0}\Lambda_{\overline{\rm MS}}$ at $N_f = 0$

- Using the continuous-β function method at $N_f = 0$.
- For $N_f = 0$ flavors,

  $\alpha_s(m_Z) = 0.08286(28)[0.3\%]$

  uses $\sqrt{8\tau_0}\Lambda_{\overline{\rm MS}} = 0.622(10)[1.6\%]$ [1]
- The goal is to reach comparable precision.



[2]

[1] [Hasenfratz, A. et. al., PRD 108, 014502 (2023)]
[2] [FLAG2024, arXiv:2411.04268]

$\alpha_s(m_Z)$ from HISQ at $N_f = 4$

- Targeting the $\alpha_s(m_Z)$
  - $N_f = 4$ using highly improved staggered quark (HISQ) action [1]
  - Switching to hypercubic (HYP) action at strong coupling (future).
  - Similar approach as $N_f = 0$
  - Additional chiral $am_f \to 0$ limit step (at strong coupling)
  - τ_0 in the chiral limit: an ongoing investigation
- $\sqrt{8\tau_0}\Lambda^{(4)}_{\overline{\rm MS}} \to \alpha_s^{(4)}(m_Z)$: τ_0 + run coupling to 5-loop $\alpha_s^{(4)}(\mu)$ [2]
- $\alpha_s^{(4)}(m_Z) \to \alpha_s^{(5)}(m_Z)$: perturbative decoupling
  - Alternatively, $\sqrt{8\tau_0}\Lambda^{(4)}_{\overline{\rm MS}} \to \sqrt{8\tau_0}\Lambda^{(5)}_{\overline{\rm MS}}$ for our case

[1] [Follana et al., Phys. Rev. D 75 (2007) 054502]
[2] [Herzog, F., JHEP 02 (2017) 090]

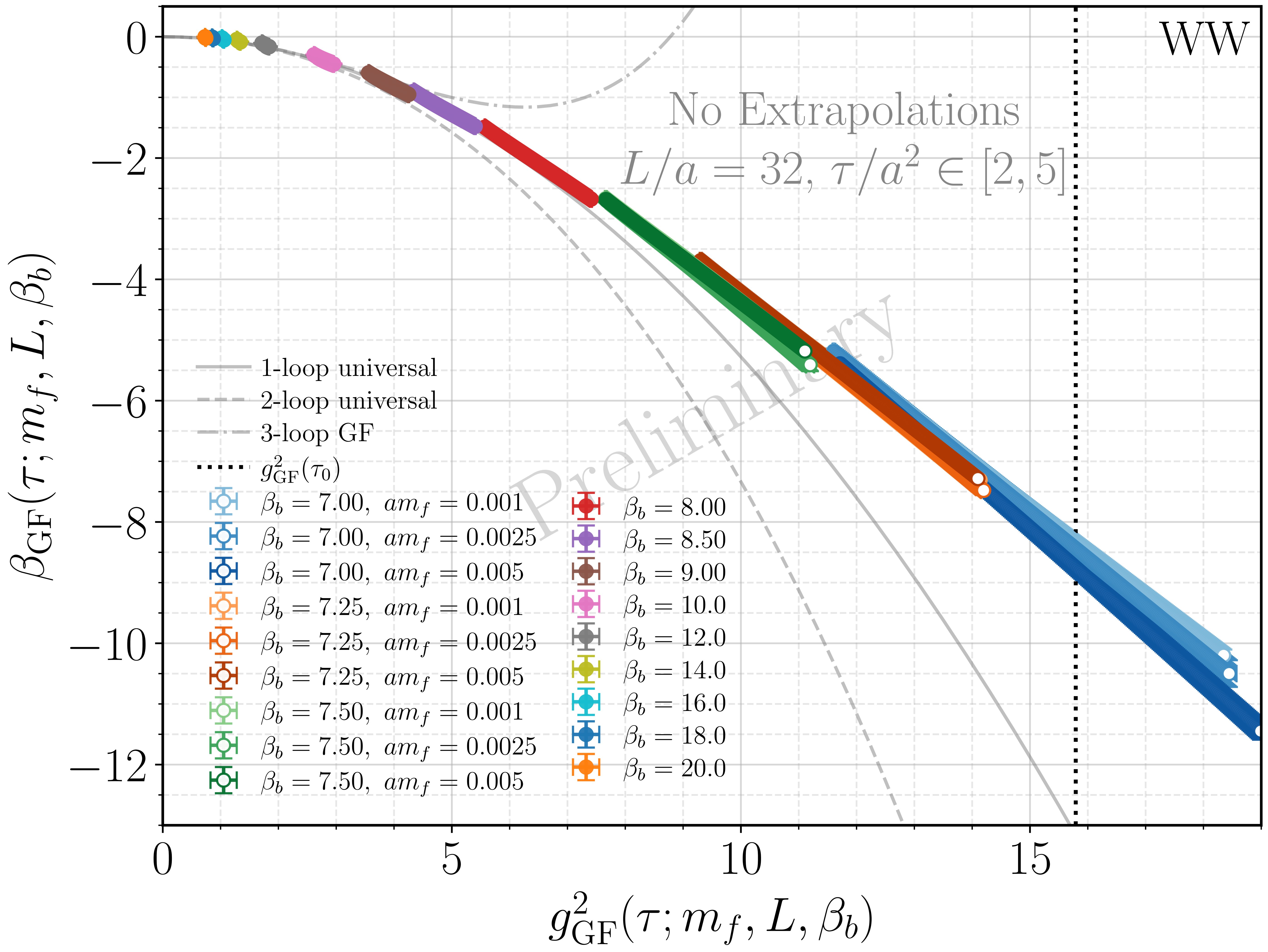
Gradient flow β-function

$L/a = 32$, Wilson flow (W), Wilson operator (W)

- Weak couplings over $8.0 \leq \beta_b \leq 20.0$ (9 total)
  - Massless $am_f = 0$
- Volumes: $20 \leq L/a \leq 48$, $(L/a)^3 \times (2L/a)$ (5 total)
- Strong couplings over $7.00 \leq \beta_b \leq 7.50$ (3 total)
  - Three bare masses

$am_f = (1.0, 2.5, 5.0) \times 10^{-3}$

[Harlander, Neumann, JHEP 06 (2016) 161]



Gradient flow β-function

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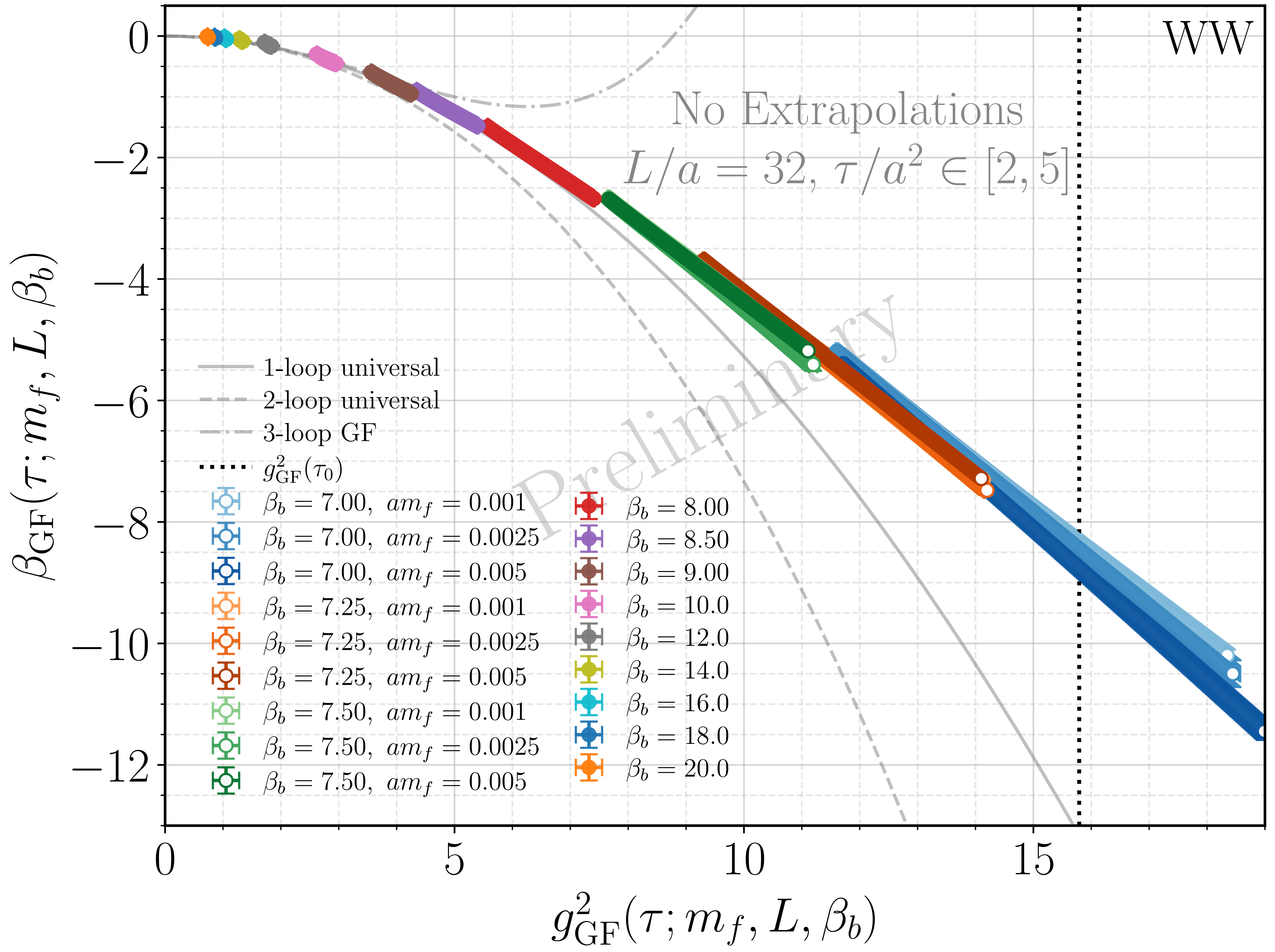
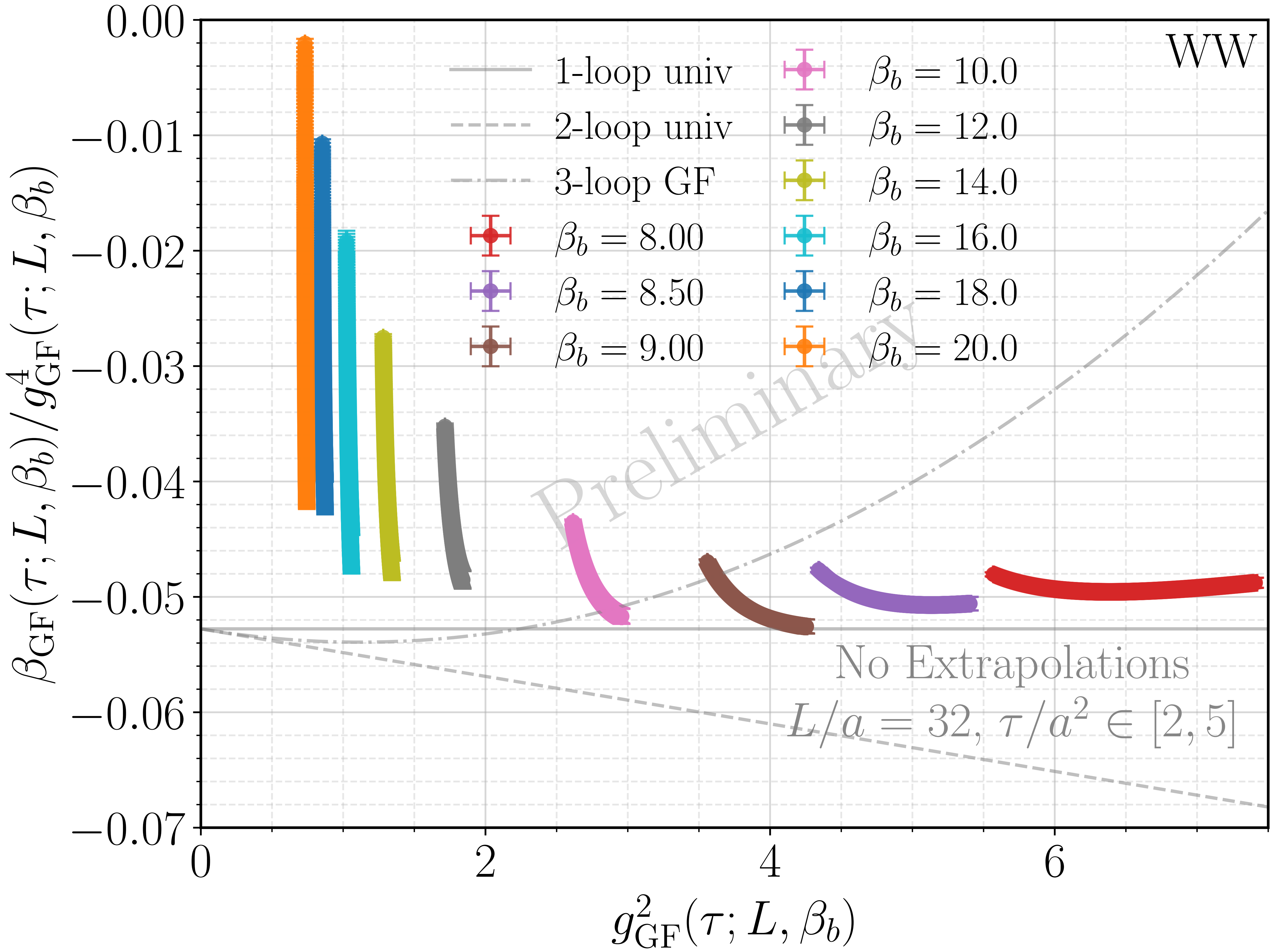




[Harlander, Neumann, JHEP 06 (2016) 161]

Chiral Extrapolation

At strong coupling for $N_f = 4$

- We simulate our strongest couplings with HISQ action currently
  - $\beta_b = 7.00,\ 7.25$, and 7.50, with $am_f = (1.0,\ 2.5,\ 5.0) \times 10^{-3}$.
- For sufficiently small fermion masses, the gradient-flow coupling is expected to depend linearly on am_f at fixed τ [1]

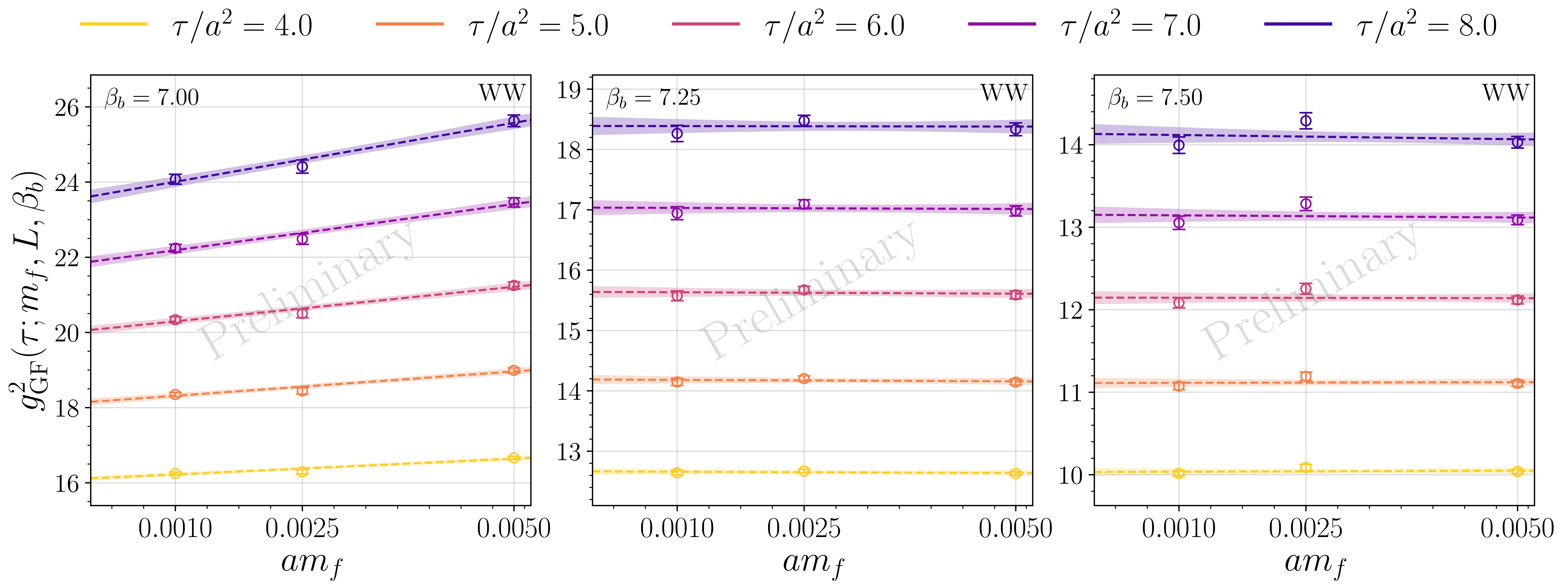
$$g^2_{\mathrm{GF}}(\tau; m_f, L, \beta_b) = g^2_{\mathrm{GF}}(\tau; 0, L, \beta_b) + A(\tau; L, \beta_b)\, am_f + \mathcal{O}(a^2 m_f^2)$$

- The renormalized coupling $g^2_{\mathrm{GF}}(\tau; L, \beta_b)$ must also be evaluated in the massless limit at strong couplings.

[1] [Bär, Golterman, Phys. Rev. D 89 (2014) 034505]

Chiral Extrapolation

From $N_f = 4$ results



Extrapolate $g_{\rm GF}(\tau; m_f, L, \beta_b)$ linearly in $am_f \to 0$ at fixed β_b and τ/a^2.

Infinite volume extrapolation

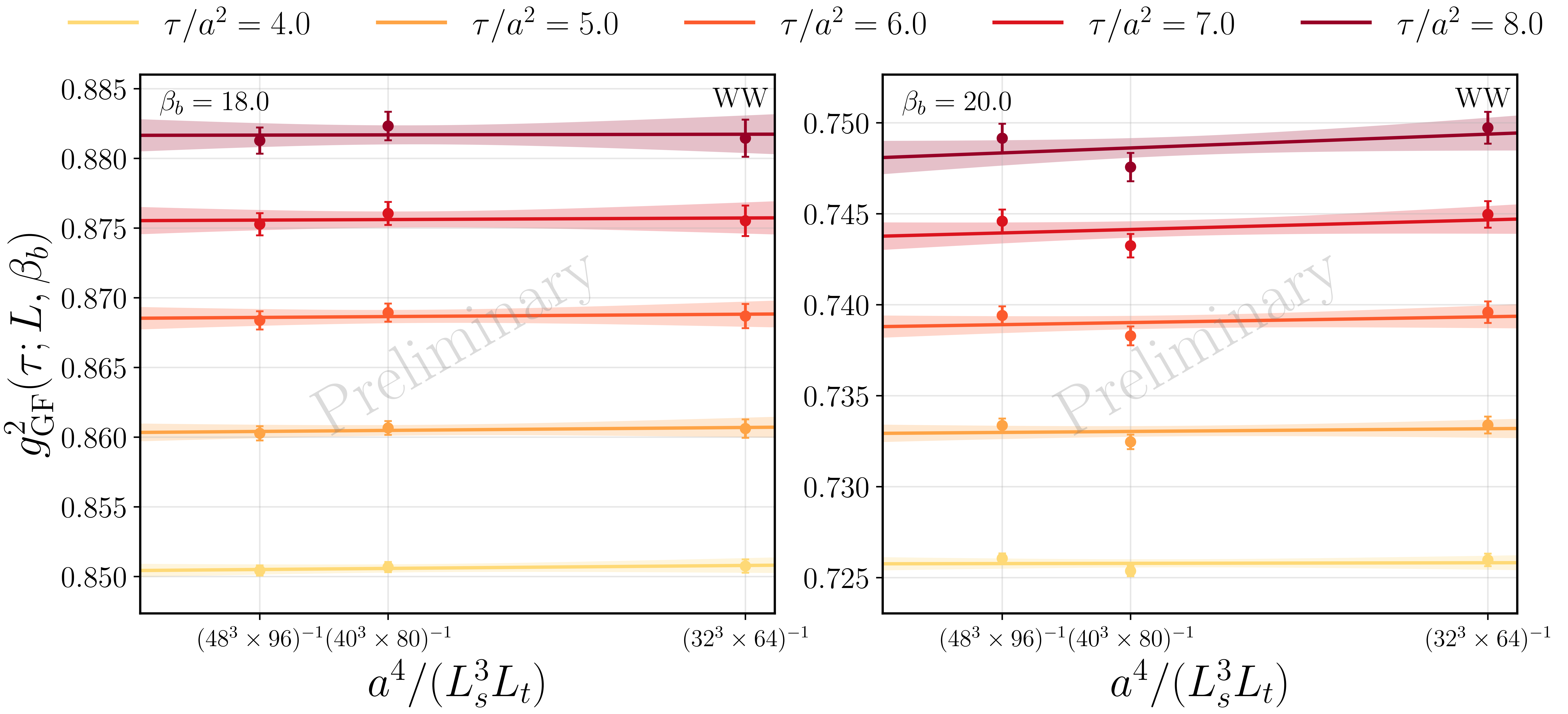
At weak coupling

- Finite-volume effects are expected to go like τ^2/L^4. [1]
  - Investigate transition to different leading behavior at stronger couplings.
- Weakly coupled ensembles probe smaller flow times
  - Decrease in flow time mitigates the effect of the smaller physical box size L.
- Additional volume might be added to the analysis to have better control over infinite volume extrapolation at the weaker couplings.

[1] [Fodor et al., JHEP 11 (2012) 007]
[2] [Fodor et al., JHEP 09 (2014) 018]

Infinite volume extrapolation

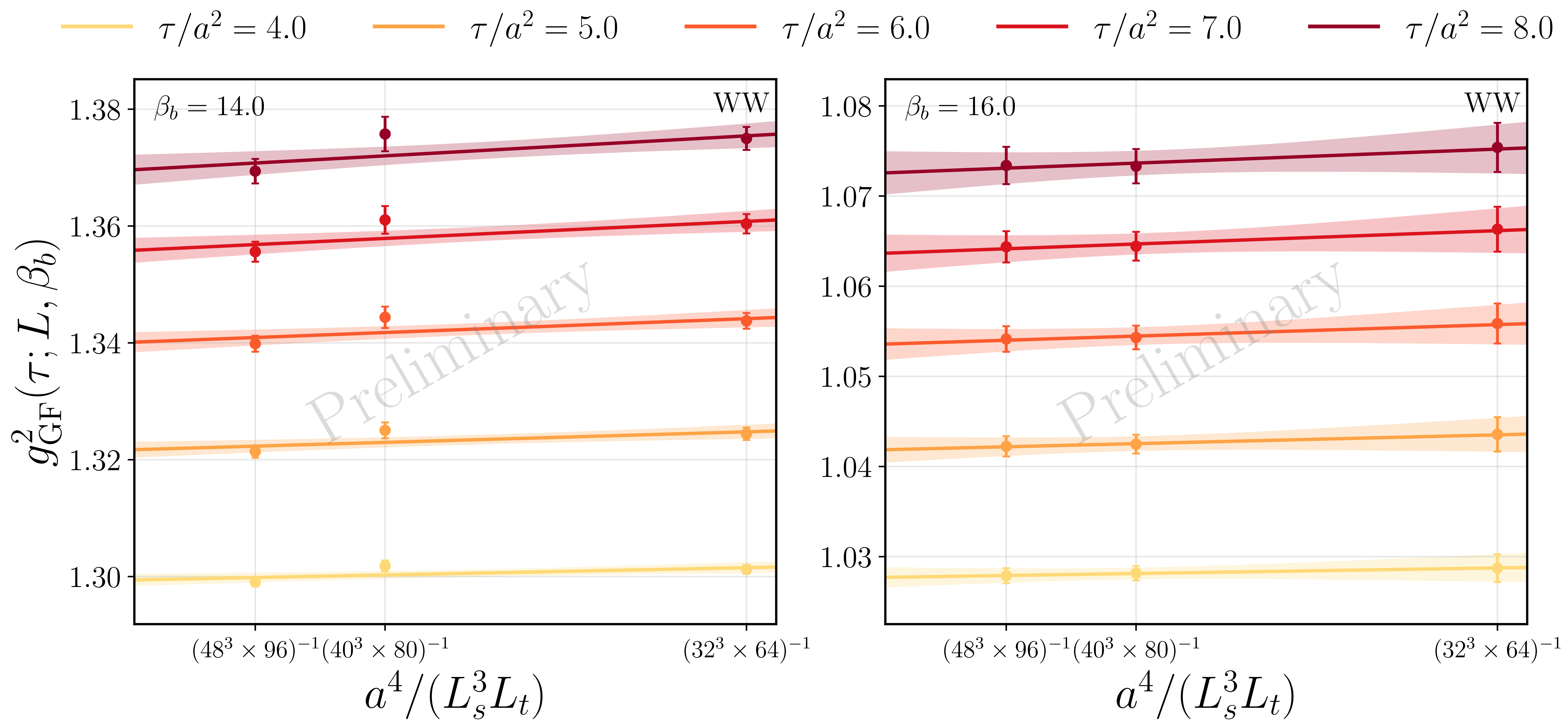
At weak coupling



Extrapolate $g^2_{\mathrm{GF}}(\tau; L, \beta_b)$ linearly in $a^4/(L_s^3 L_t) \to 0$ at fixed β_b and τ/a^2 (largest 3 volumes).

Infinite volume extrapolation

At weak coupling



Extrapolate $g^2_{\mathrm{GF}}(\tau; L, \beta_b)$ linearly in $a^4/(L_s^3 L_t) \to 0$ at fixed β_b and τ/a^2 (largest 3 volumes).

Continuum Extrapolation*

- The $L/a = 32$ data is used as a proxy for the infinite-volume data
  - Until the full infinite-volume analysis data is available.
  - The figures shown are *preliminary*.
- The continuum limit is obtained at fixed g^2_{GF} from a linear extrapolation in $a^2/\tau \to 0$, within a chosen flow-time window.
  - Wilson, clover, and Symanzik operator definitions are compared.
- This necessitates the interpolation, performed on $L/a = 32$ data at each bare coupling β_b.

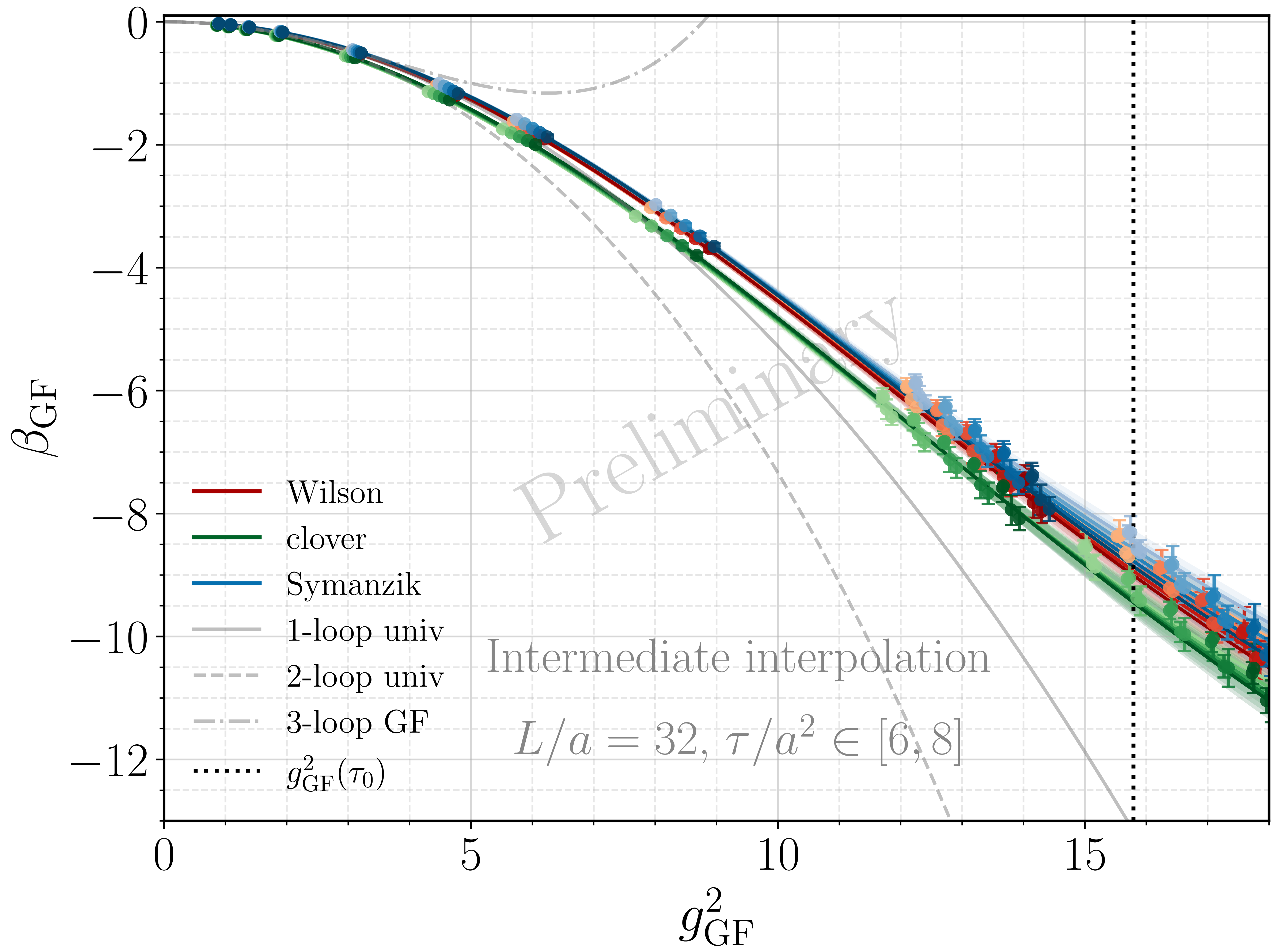
Intermediate Interpolation

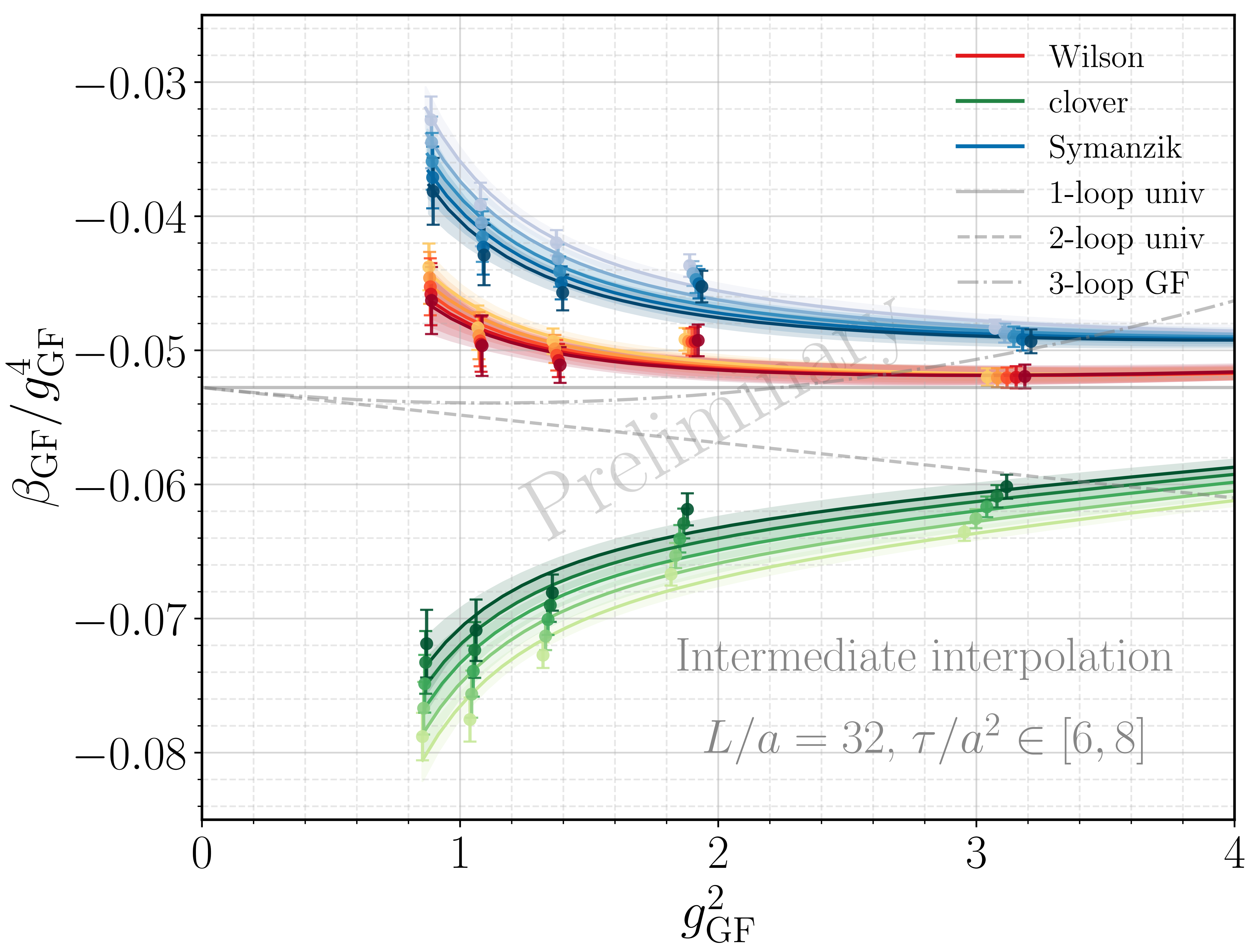
- At each fixed flow time τ/a^2, the beta function is interpolated as a function of the renormalized coupling $g^2_{\mathrm{GF}}(\tau)$.
- Adopted ansatz:

$$\beta_{\mathrm{GF}}(g^2_{\mathrm{GF}}, \beta_b) = p_0 + g^4_{\mathrm{GF}}\left(p_1 + p_2 g^2_{\mathrm{GF}} + p_3 g^4_{\mathrm{GF}} + p_4 g^6_{\mathrm{GF}}\right)$$

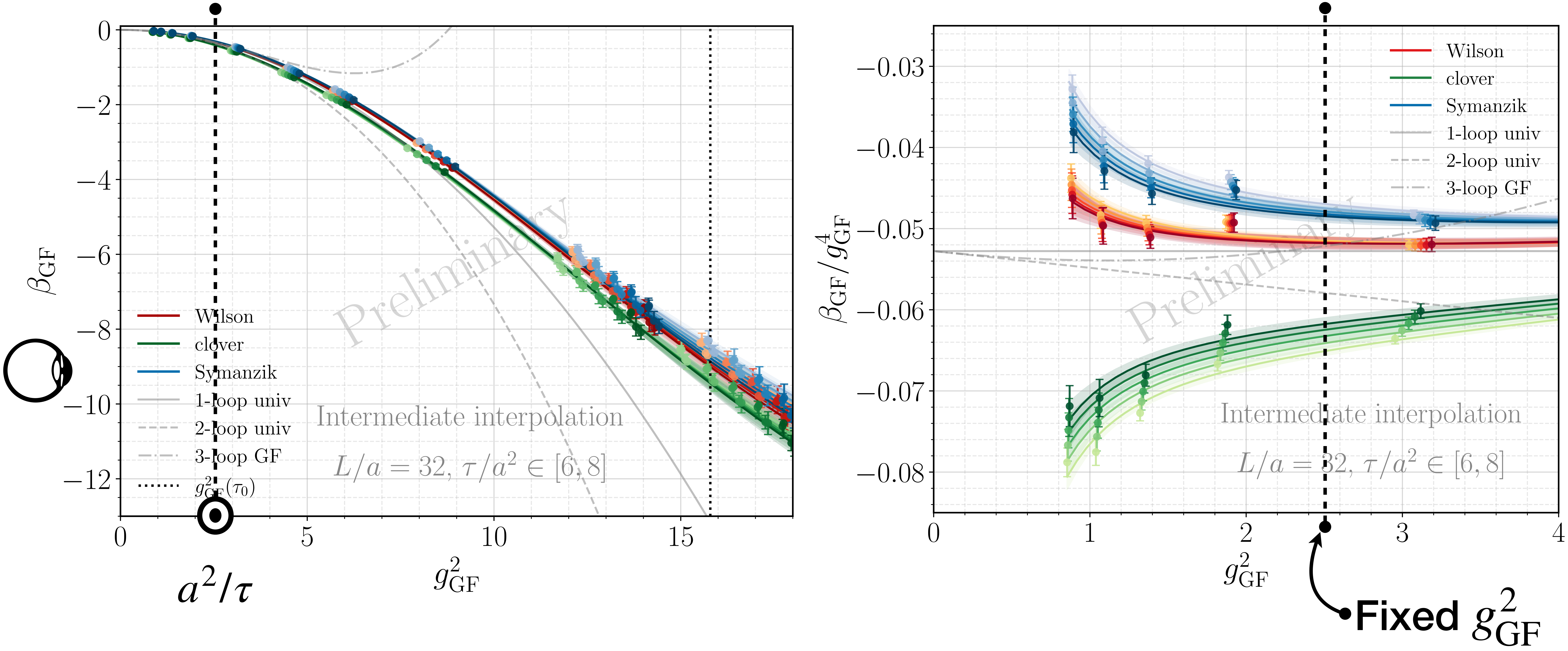
  - The fit ansatz is motivated by the perturbative behavior of the continuum β-function.
- The fitted coefficients p_i, $i \in [0,4]$ are determined independently at each fixed flow time.

Intermediate Interpolation



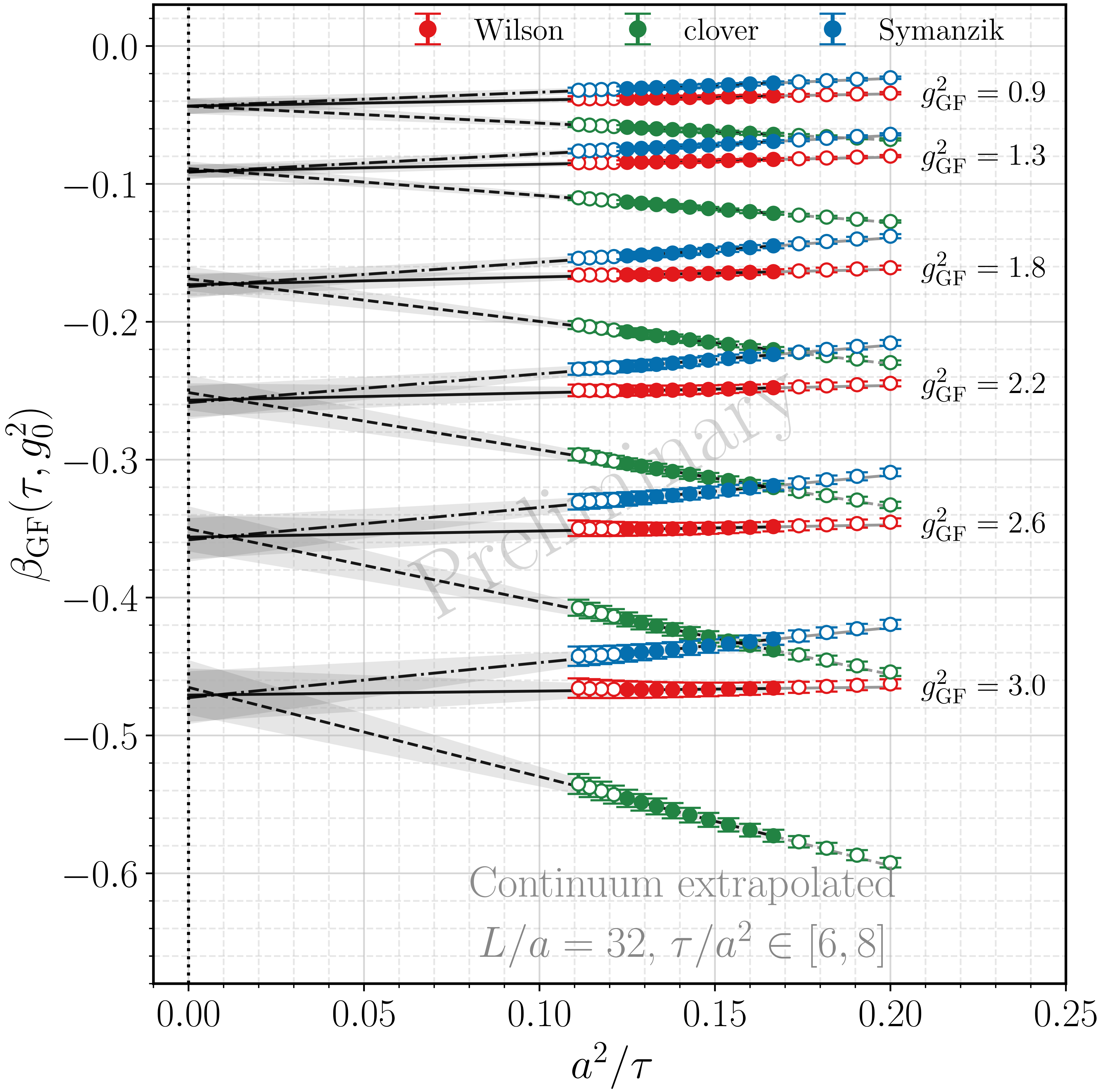


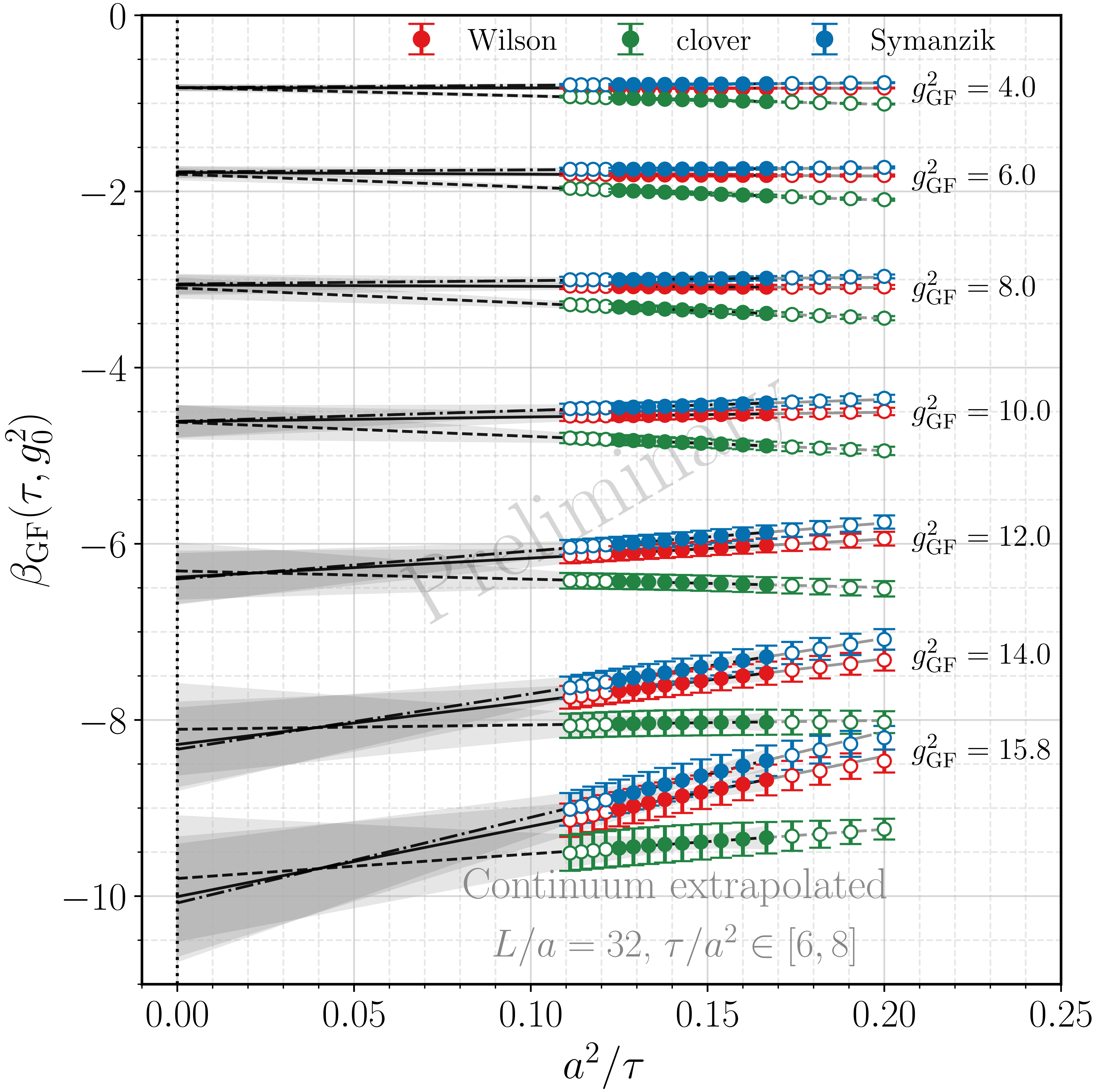
Intermediate Interpolation



Continuum Extrapolation*

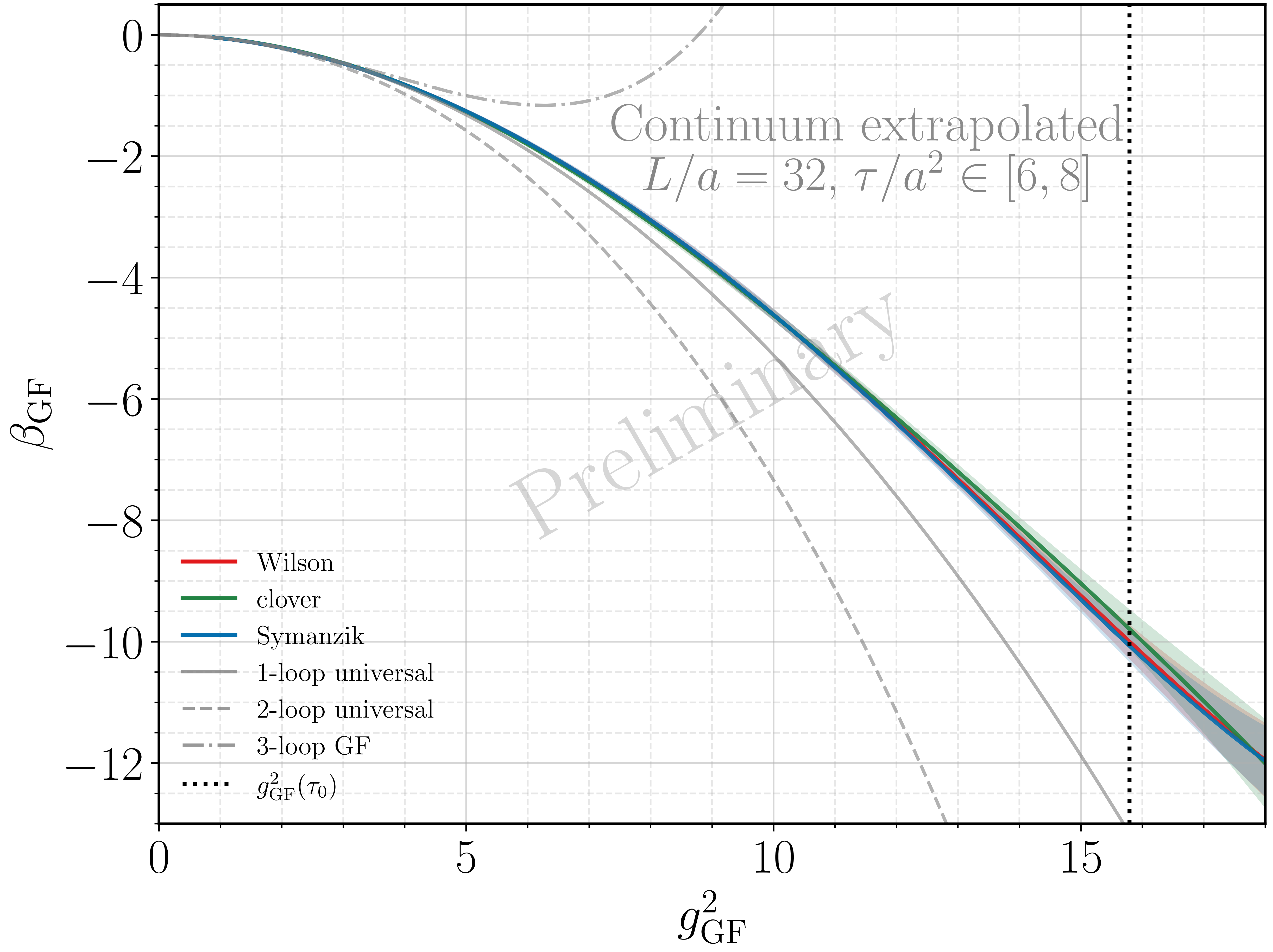
Without tree-level normalization

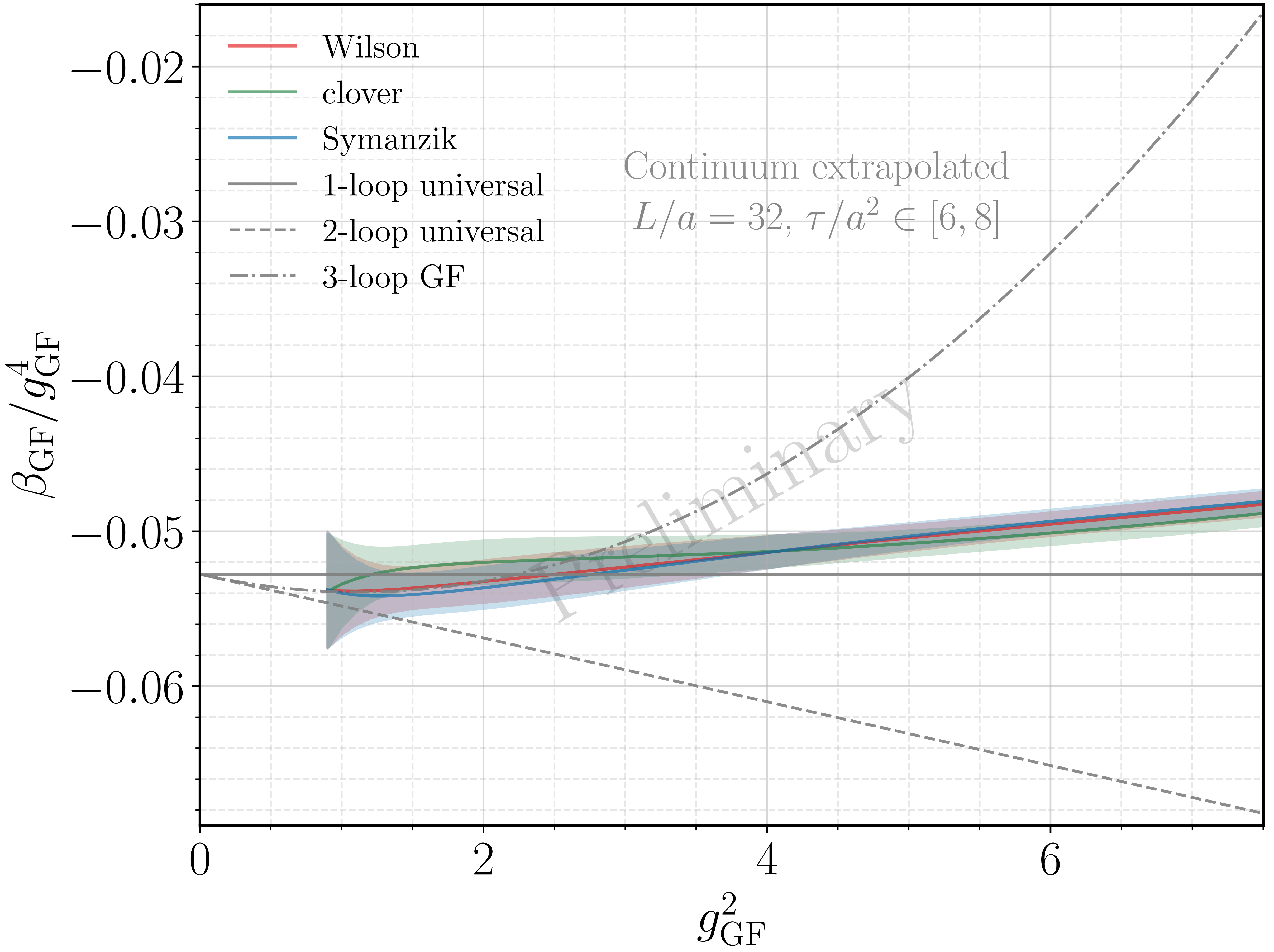




Gradient flow β-function

After continuum extrapolation*





[Harlander, Neumann, JHEP 06 (2016) 161]

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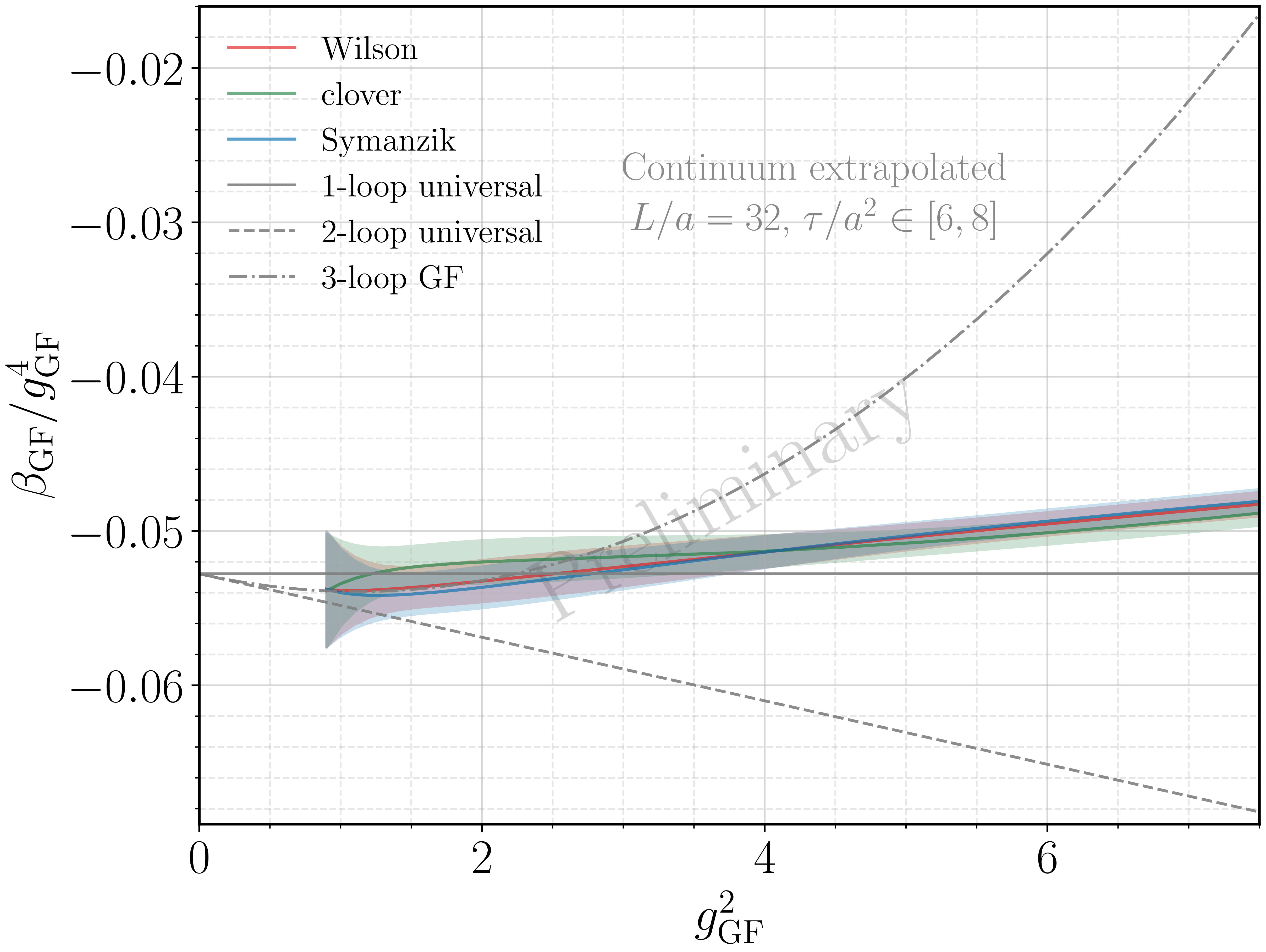

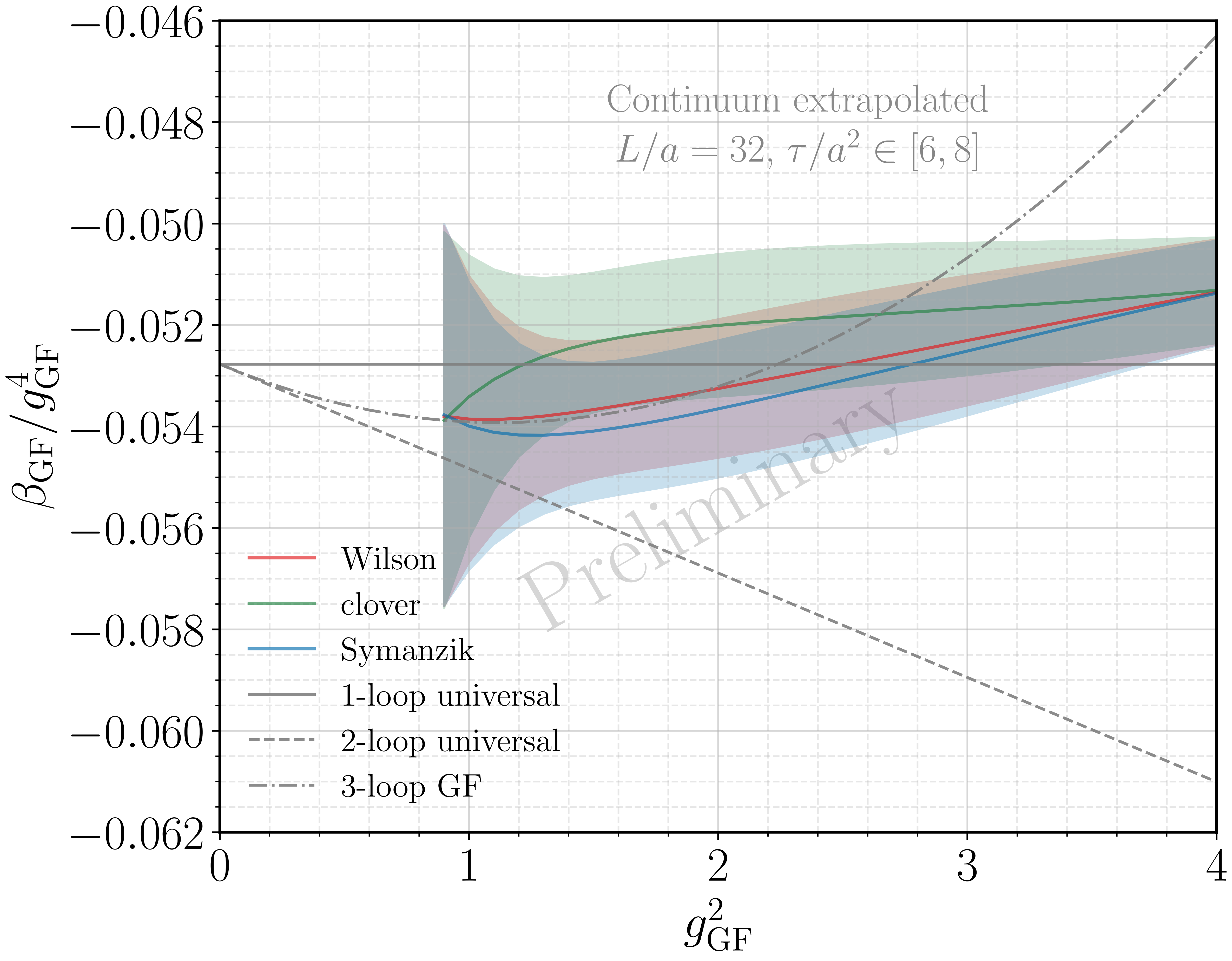

[Harlander, Neumann, JHEP 06 (2016) 161]

Future goals

- Continue the analysis steps while the data with ensembles at all volumes is completing.
- Obtain infinite-volume extrapolations over strong and weak couplings.
- Obtain the continuum extrapolations with the infinite-volume data.
- Estimate the Λ-parameter in the gradient-flow and $\overline{\mathrm{MS}}$ schemes.
- Running a new set of simulations with HYP action at strong coupling
  - Analysis on the HYP & HISQ ensembles will be done separately
  - Final result will be obtained by continuing from overlap of the continuum results.

Quark masses & fermion flow for staggered fermions

- Can also be extended to a fully gauge-invariant determination of quark masses and anomalous dimensions, opening a parallel precision program beyond the beta function.
- Significant progress made with the fermion flow implementation in Quantum EXpressions (QEX).

See talks by

Akhil Chauhan
11:30 am, May 12
Nathan Mackey
9:30 am, May 14

Acknowledgements

USQCD

US Lattice Quantum Chromodynamics

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Michigan State University
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Centre for Theoretical Physics
Higgs

The University of Edinburgh

Thank you for your time.

Questions?

Backup slides

Accomplishments so far

- Generated gauge configurations at weak couplings
  - for several volumes $(L/a)^3 \times (2L/a)$ with $N_f = 4$ massless flavors.
- Also equilibrated ensembles at strong couplings at three bare masses.
- Involved with the analysis steps of the data, generating the flowed data.
- Chiral extrapolation for the strong coupling regime.
- Examined infinite-volume extrapolations for weakest couplings $\beta_b = 14, 16, 18, 20$.

[Mandlecha et al., PoS LATTICE2025 (2025) 207]

Gradient flow

- Gradient flow: continuous smearing transformation
  - evolves the fields along a fictitious flow time τ. [1]

$$\frac{dU_\mu(x,\tau)}{d\tau} = -g_0^2 \frac{\partial S(U_\mu)}{\partial U_\mu(x,\tau)} \qquad U_\mu(x,0) = U_\mu(x)$$

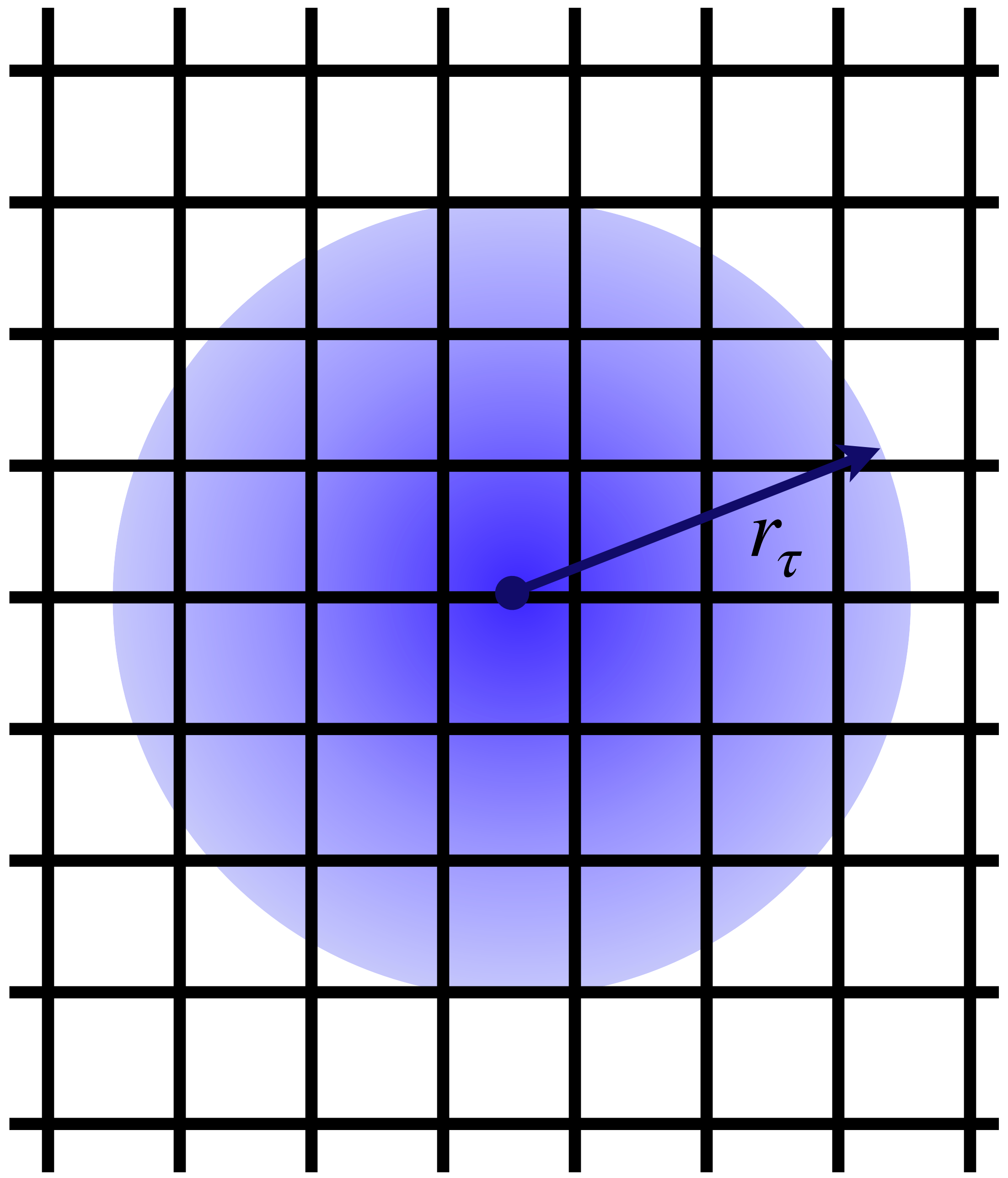
- Suppresses UV fluctuations above resolution scale

$\mu \approx 1/r_\tau$

- Describes a Renormalization Group transformation
  - Coarse-graining at level of expectation values

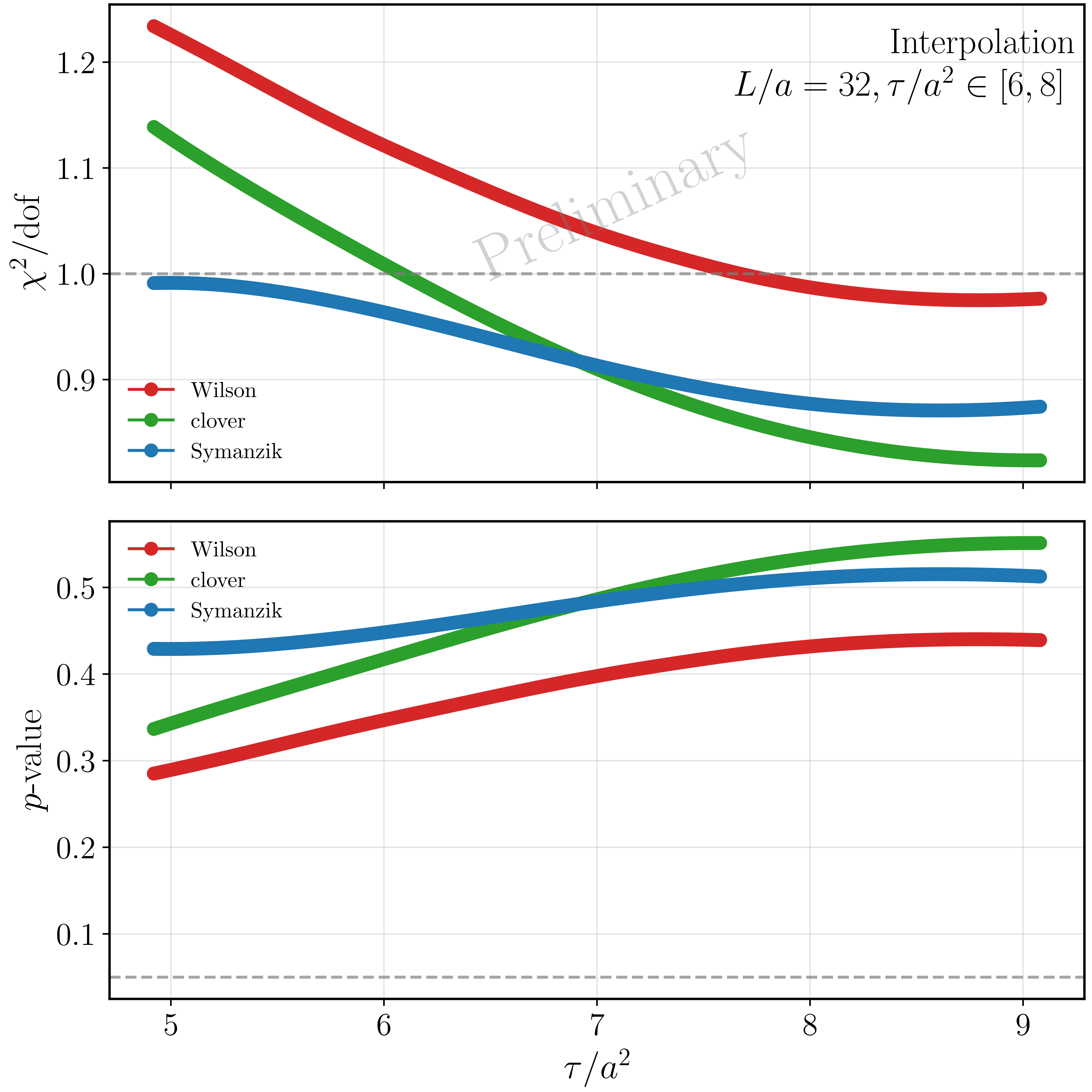
$$\mu^{-1} \sim \sqrt{8\tau}$$

$$r_\tau \approx \sqrt{8\tau}$$



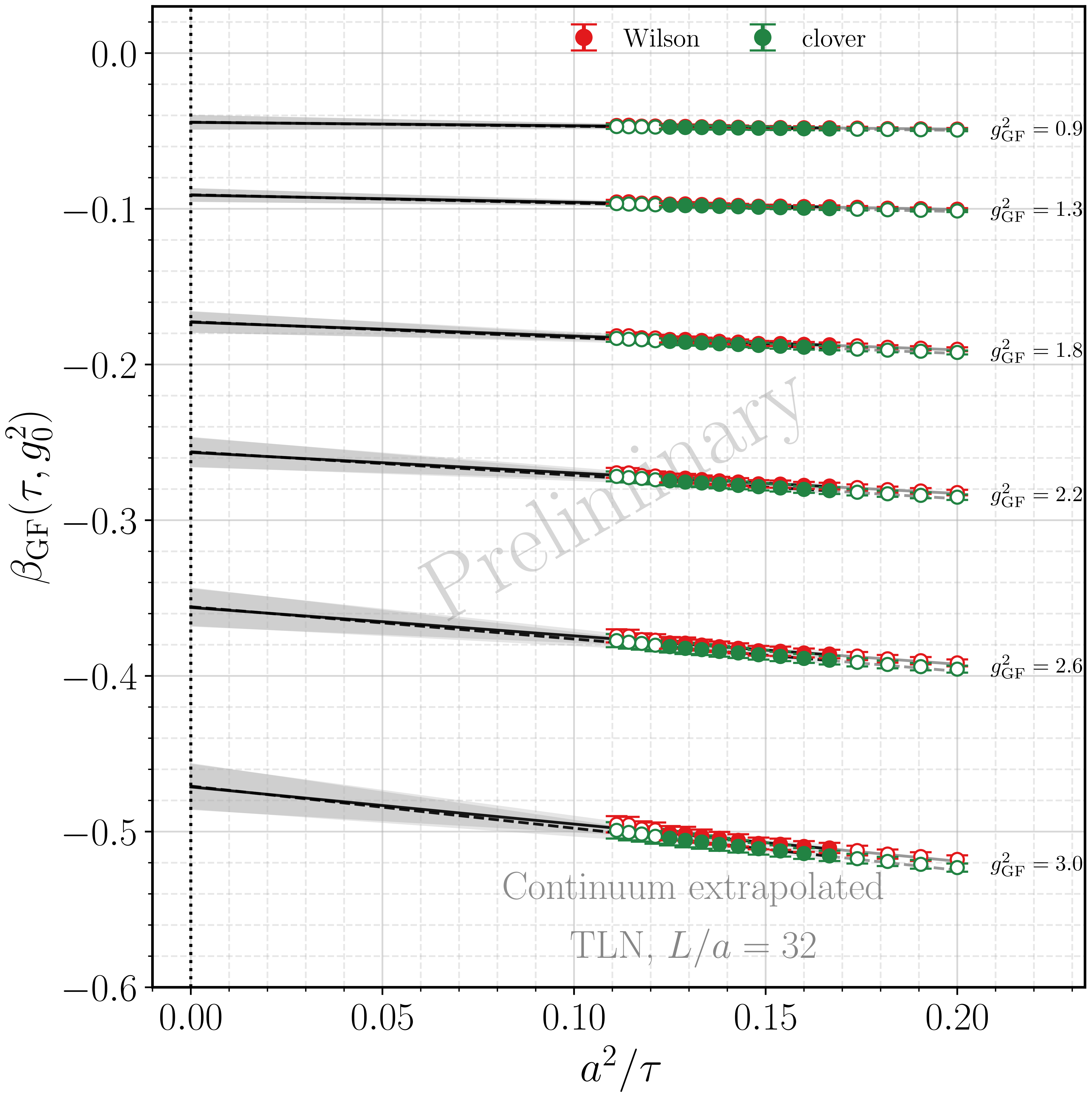
[1] [Luescher, M., JHEP 08 (2010) 71]
[2] Hasenfratz A., Witzel O., Phys.Rev.D 101 (2020) 3, 034514

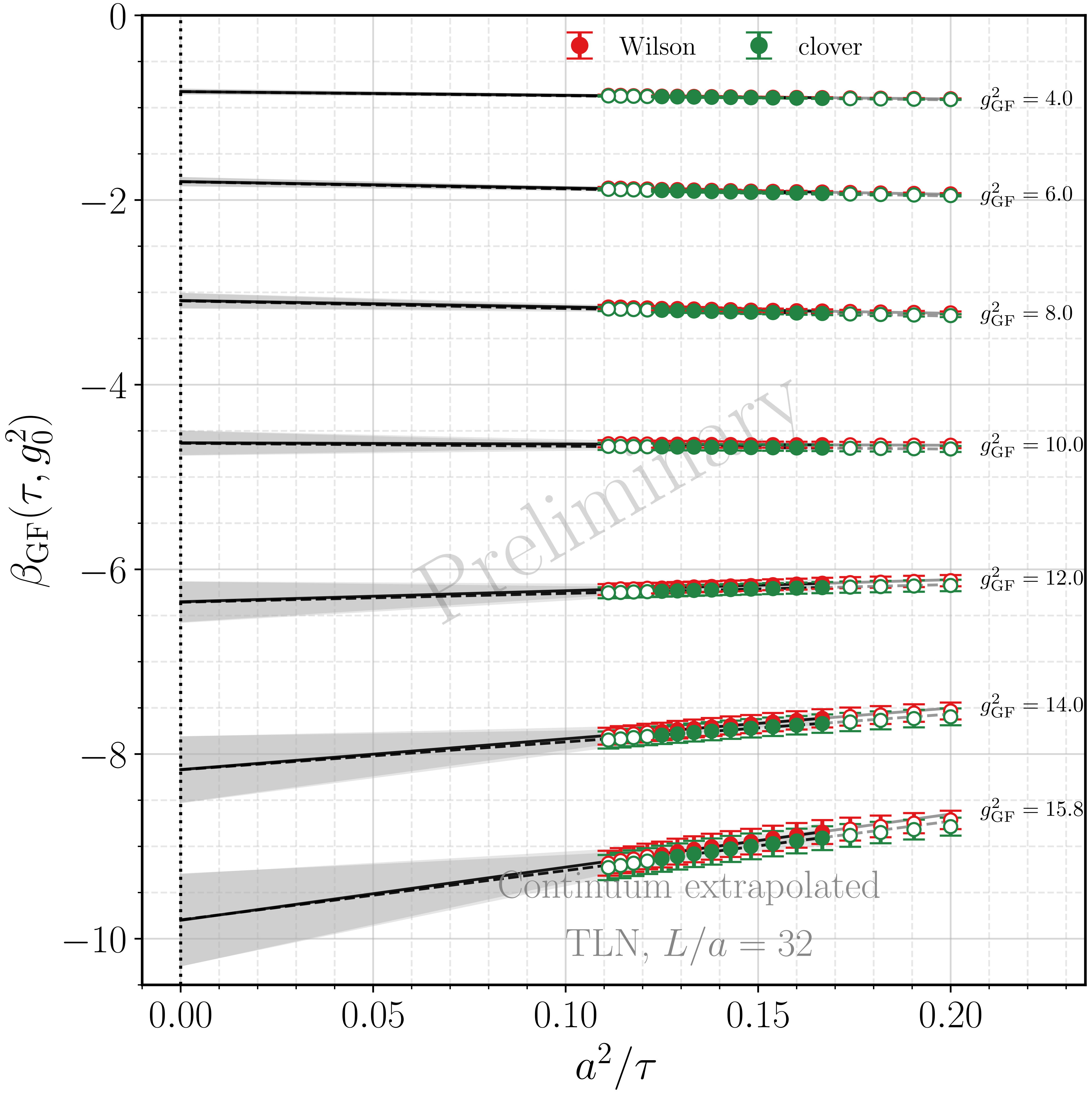
Intermediate Interpolation: Quality of fits



Continuum Extrapolation*

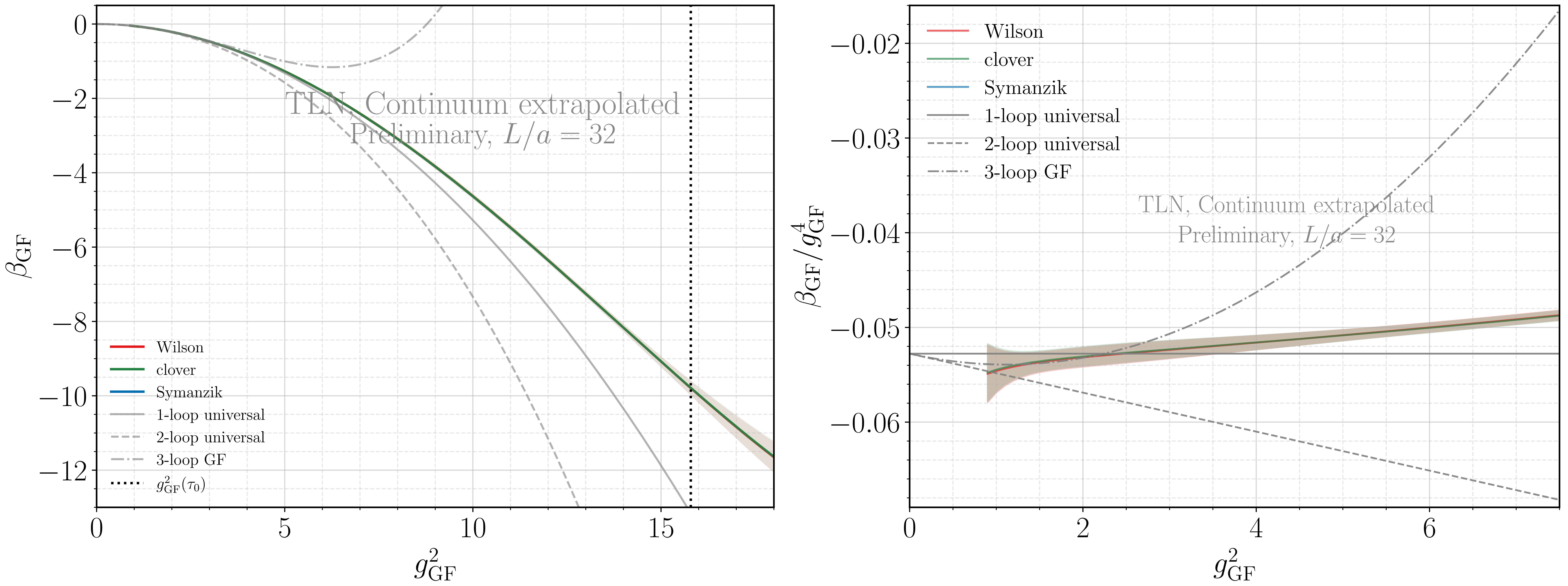
With tree-level normalization





Gradient flow β-function

After continuum extrapolation* with tree-level normalization



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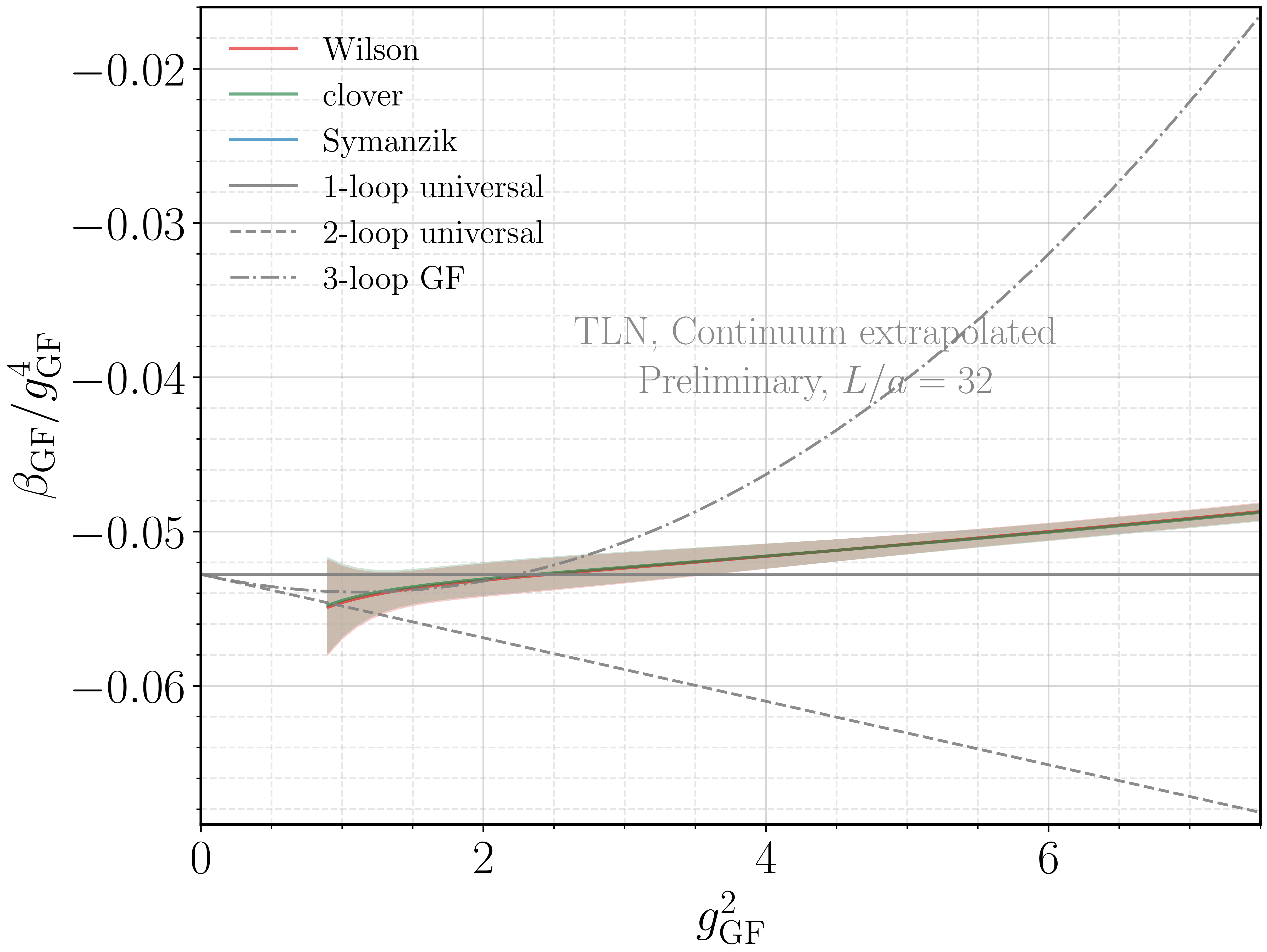

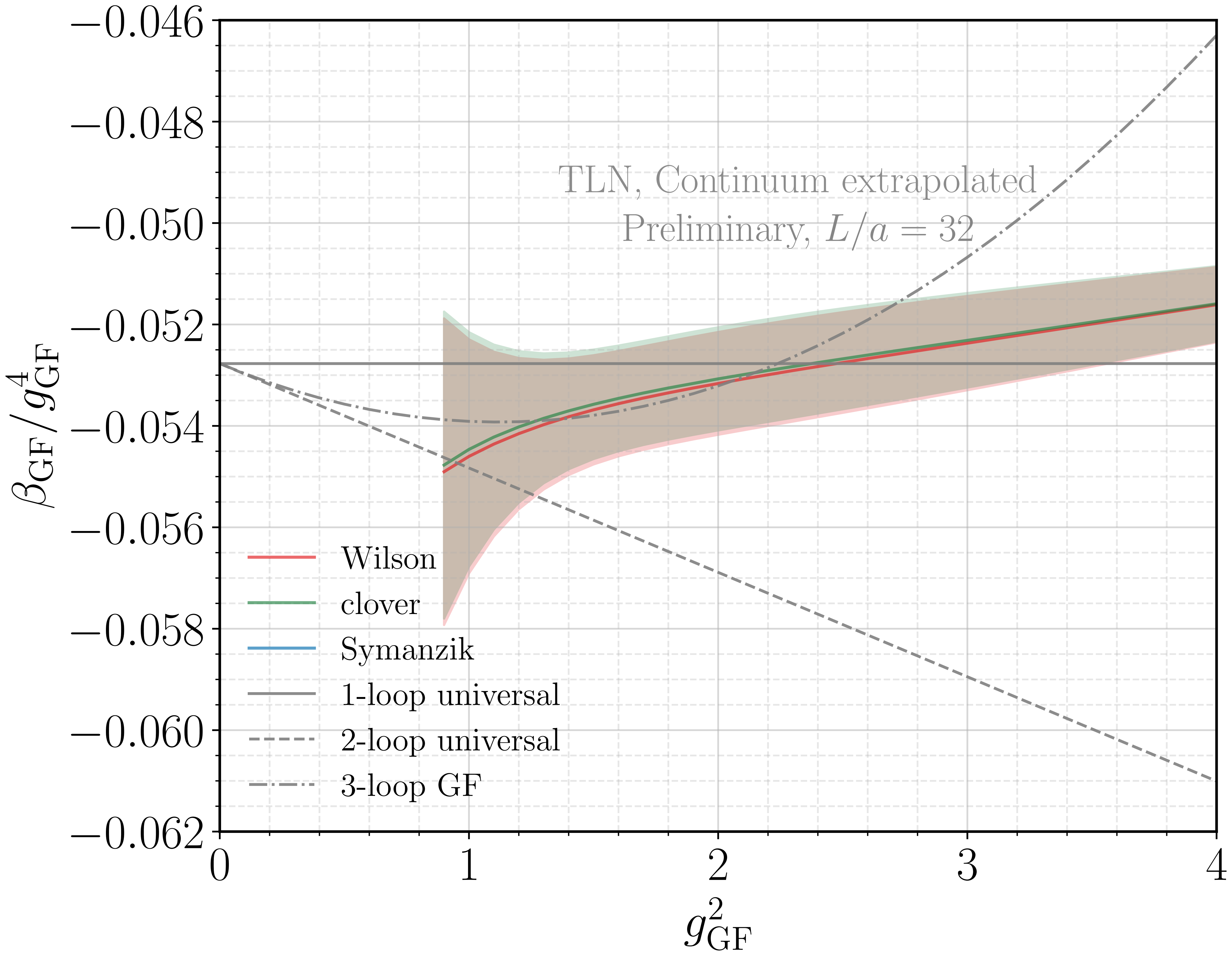

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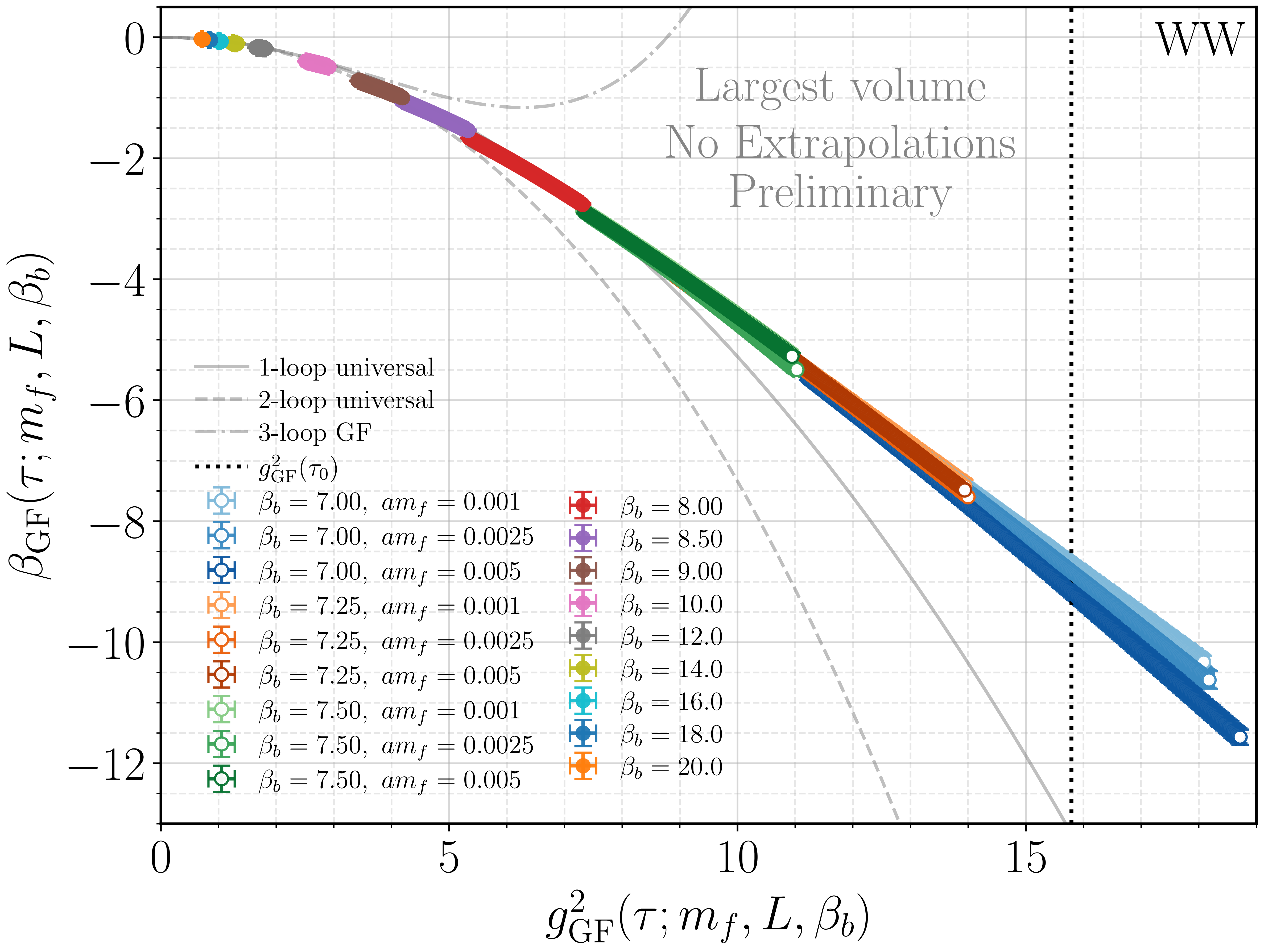
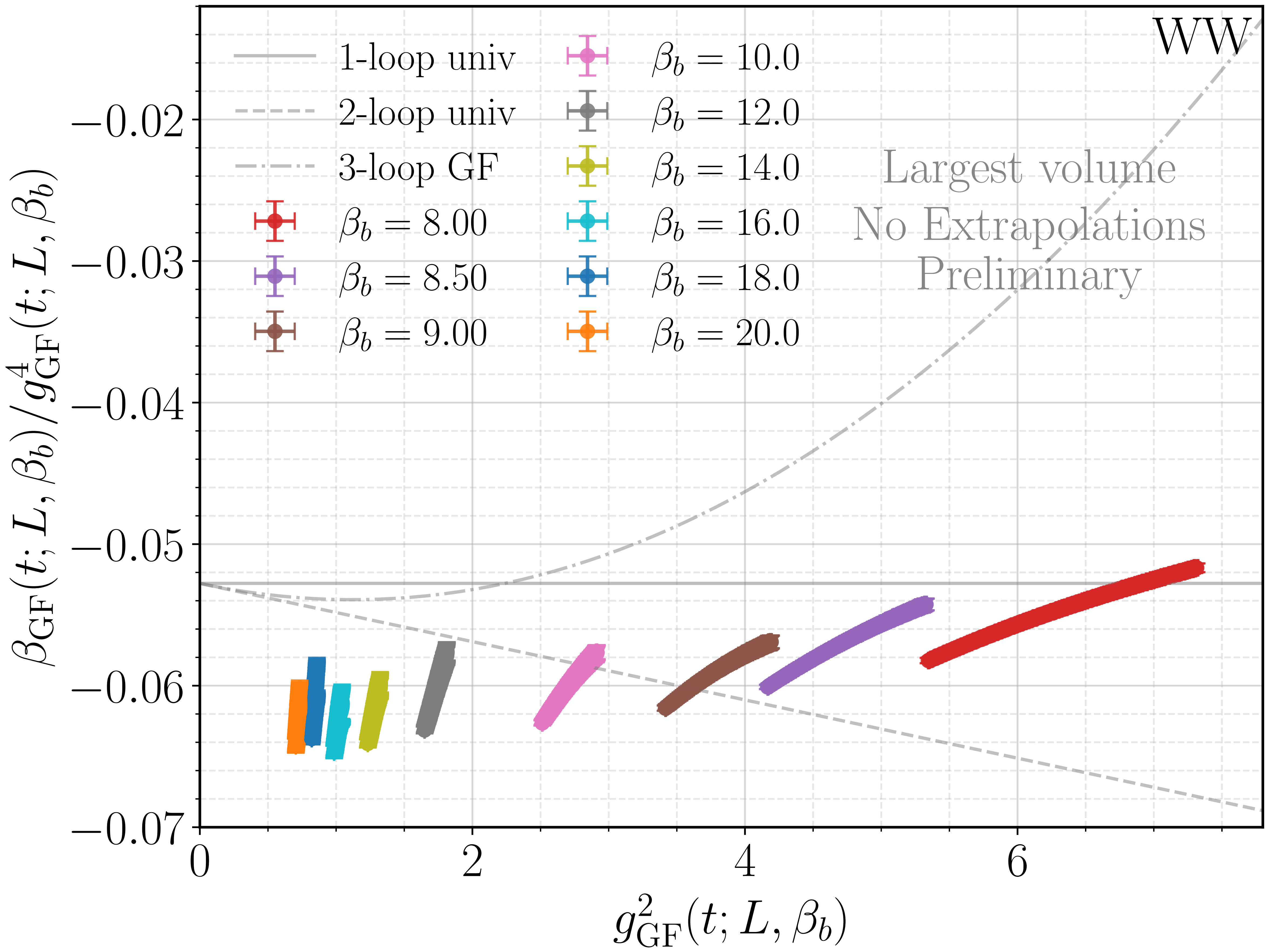
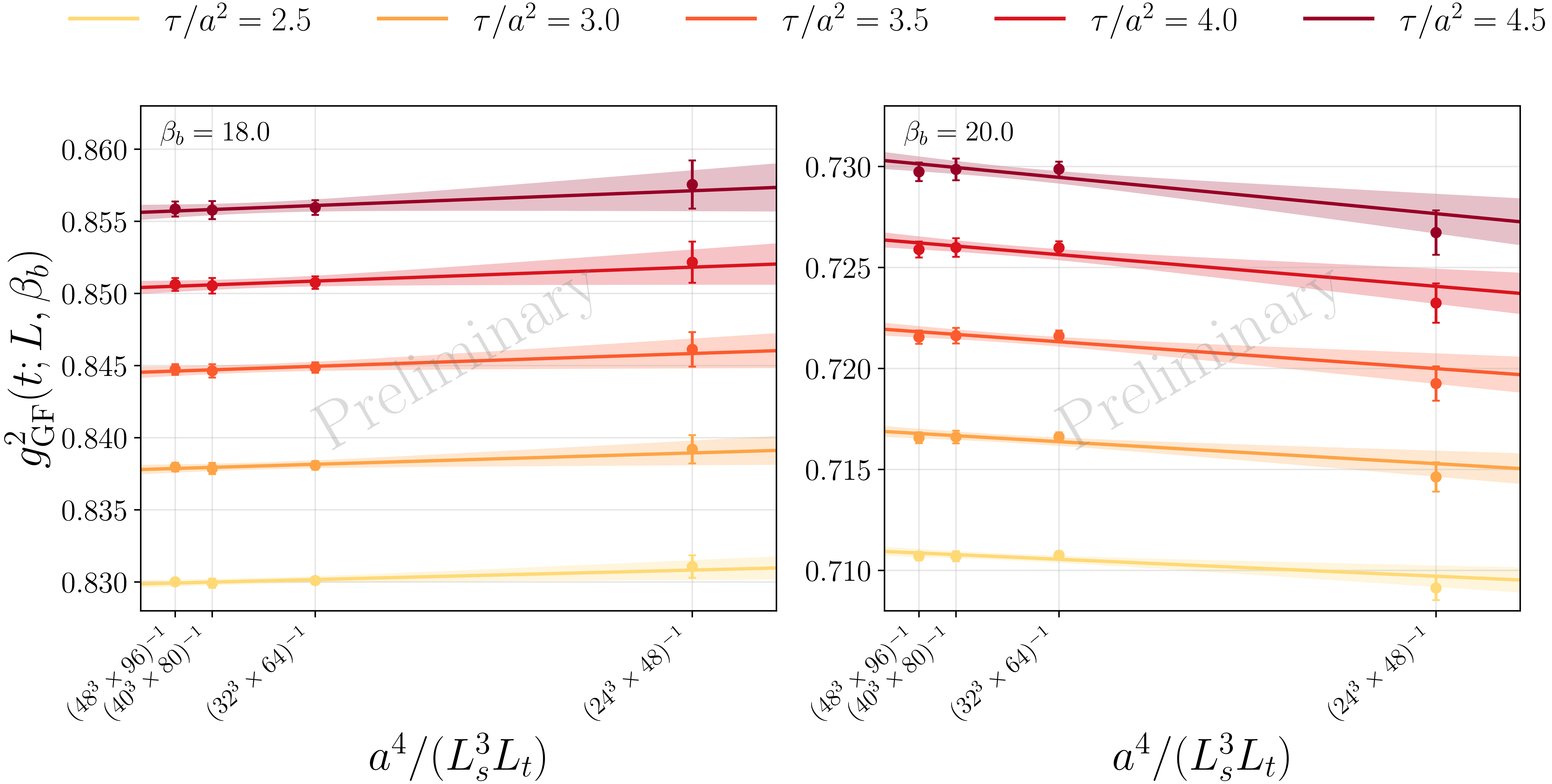




[Harlander, Neumann, JHEP 06 (2016) 161]

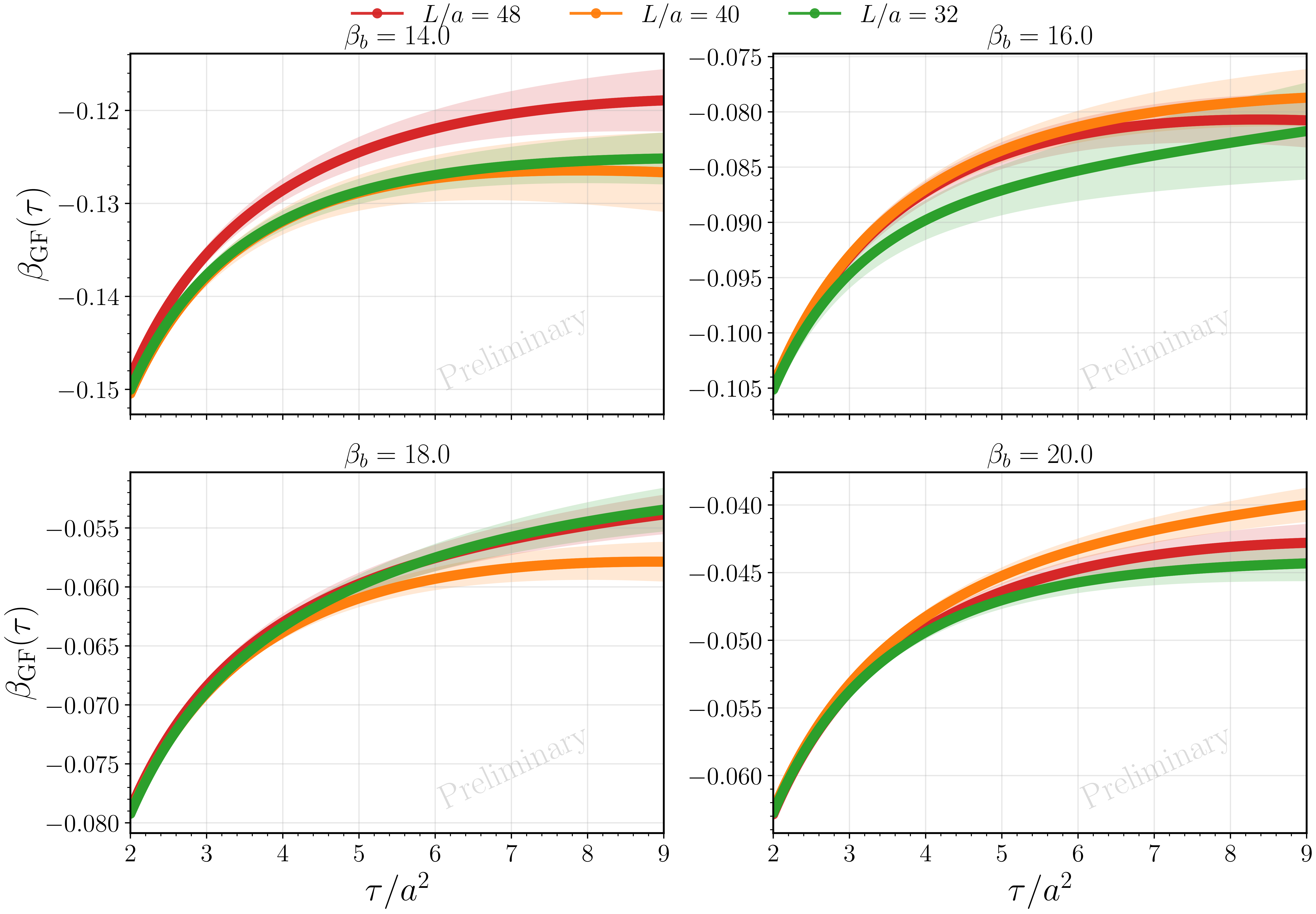
Infinite volume extrapolation

At weak coupling

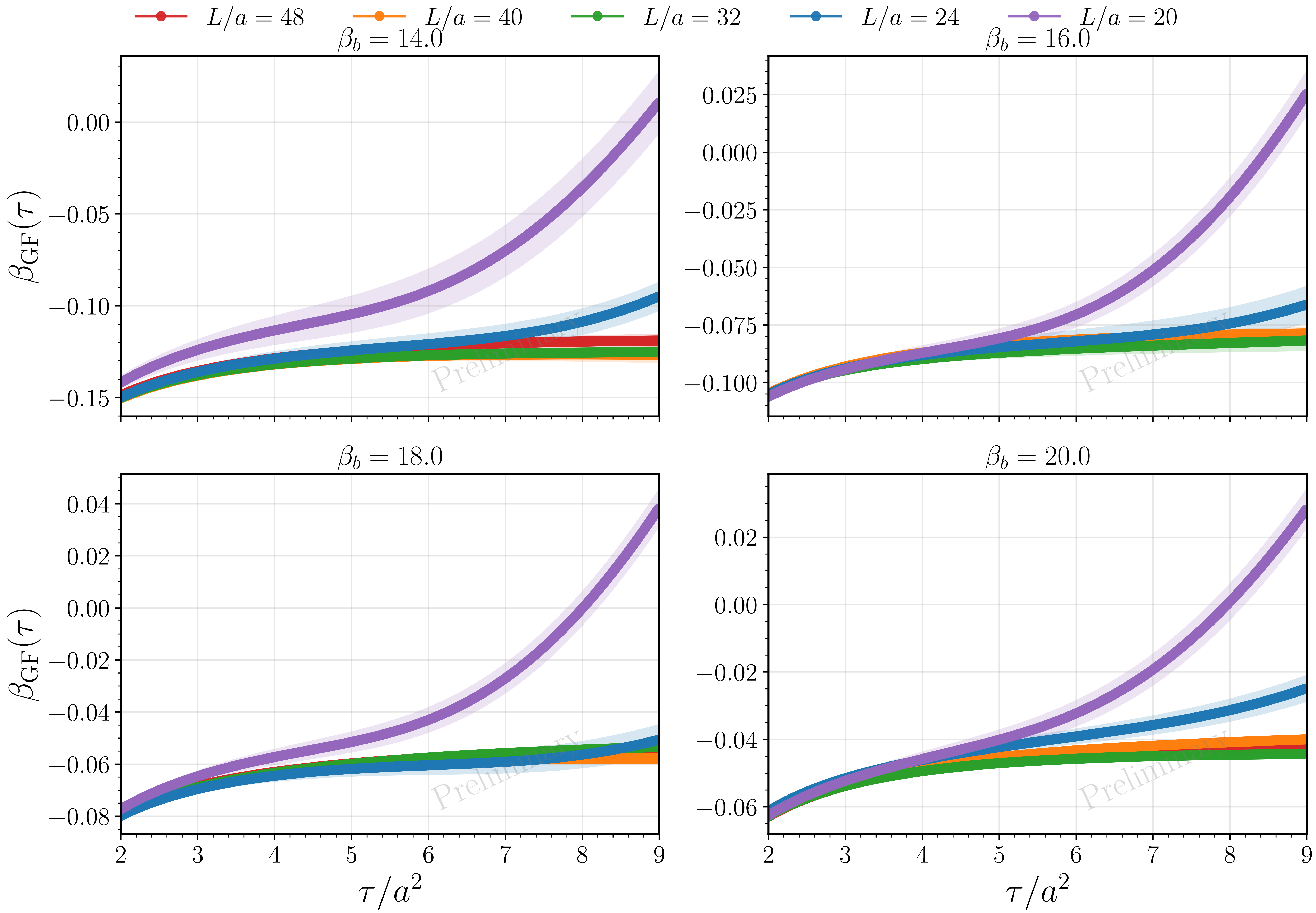


Extrapolate $g^2_{\mathrm{GF}}(\tau; L, \beta_b)$ linearly in $a^4/(L_s^3 L_t) \to 0$ at fixed β_b and τ/a^2 (largest 4 volumes).

$\beta_{\mathrm{GF}}(\tau)$ vs τ/a^2 for largest volumes



$\beta_{\mathrm{GF}}(\tau)$ vs τ/a^2 for all volumes



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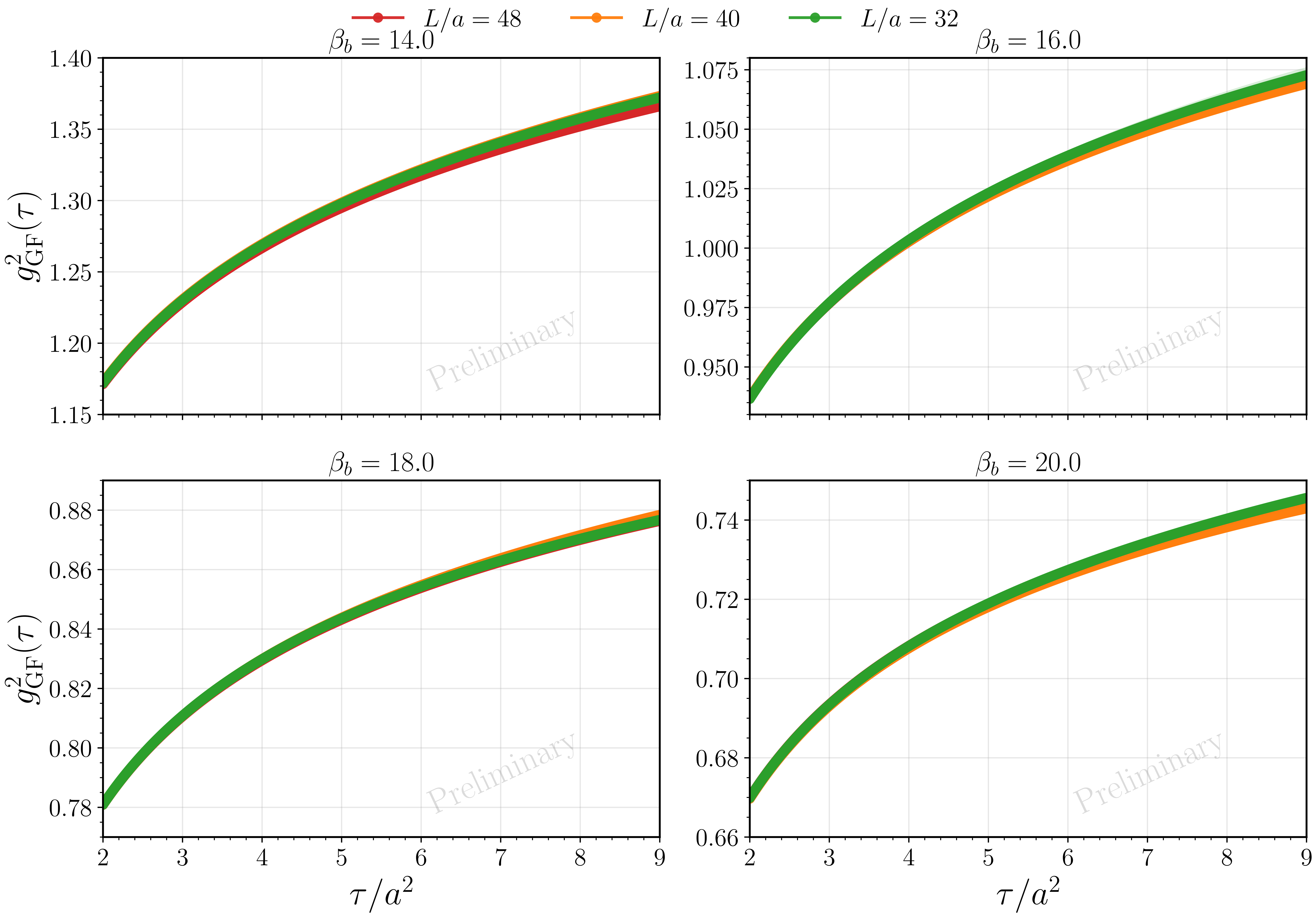

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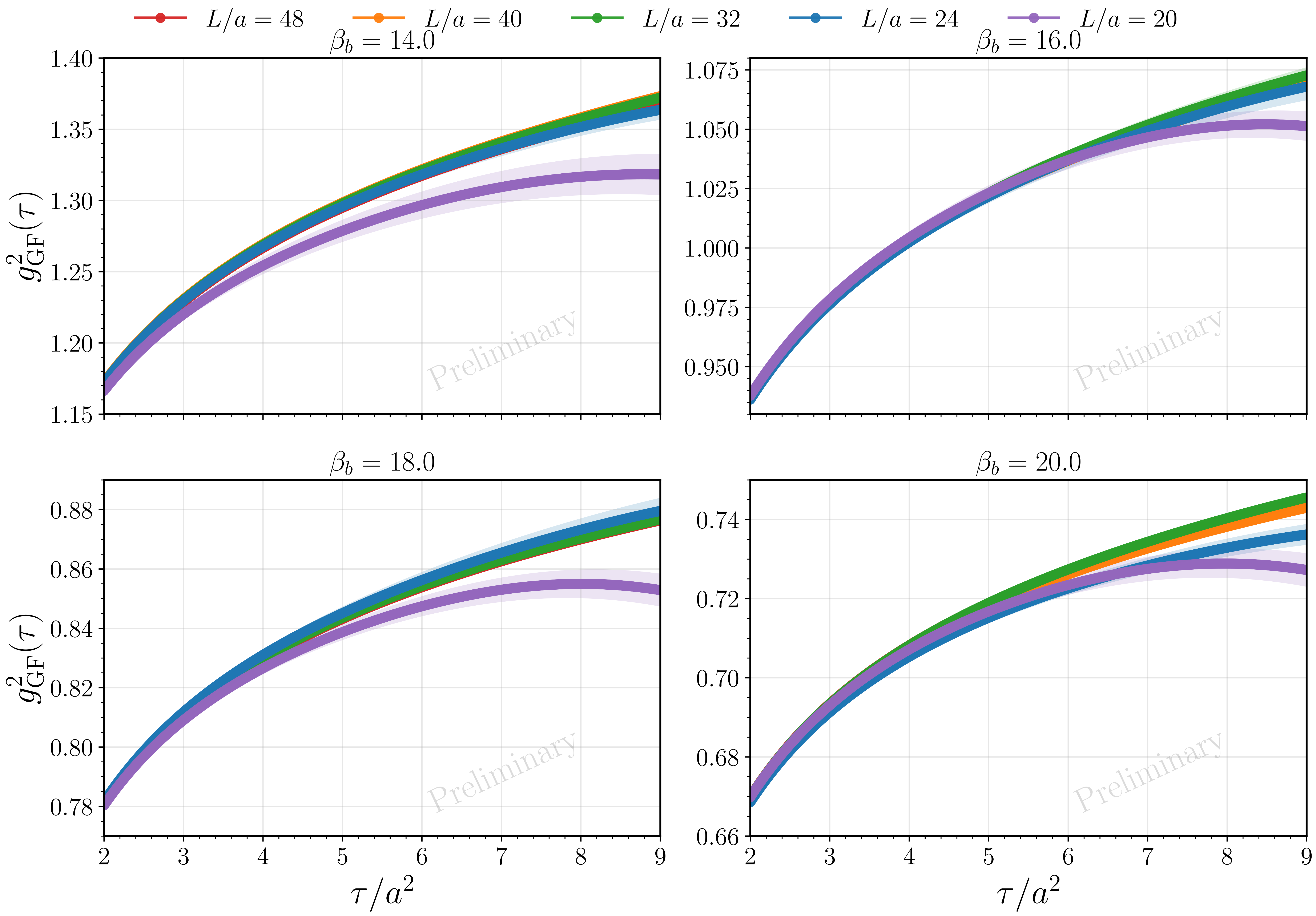

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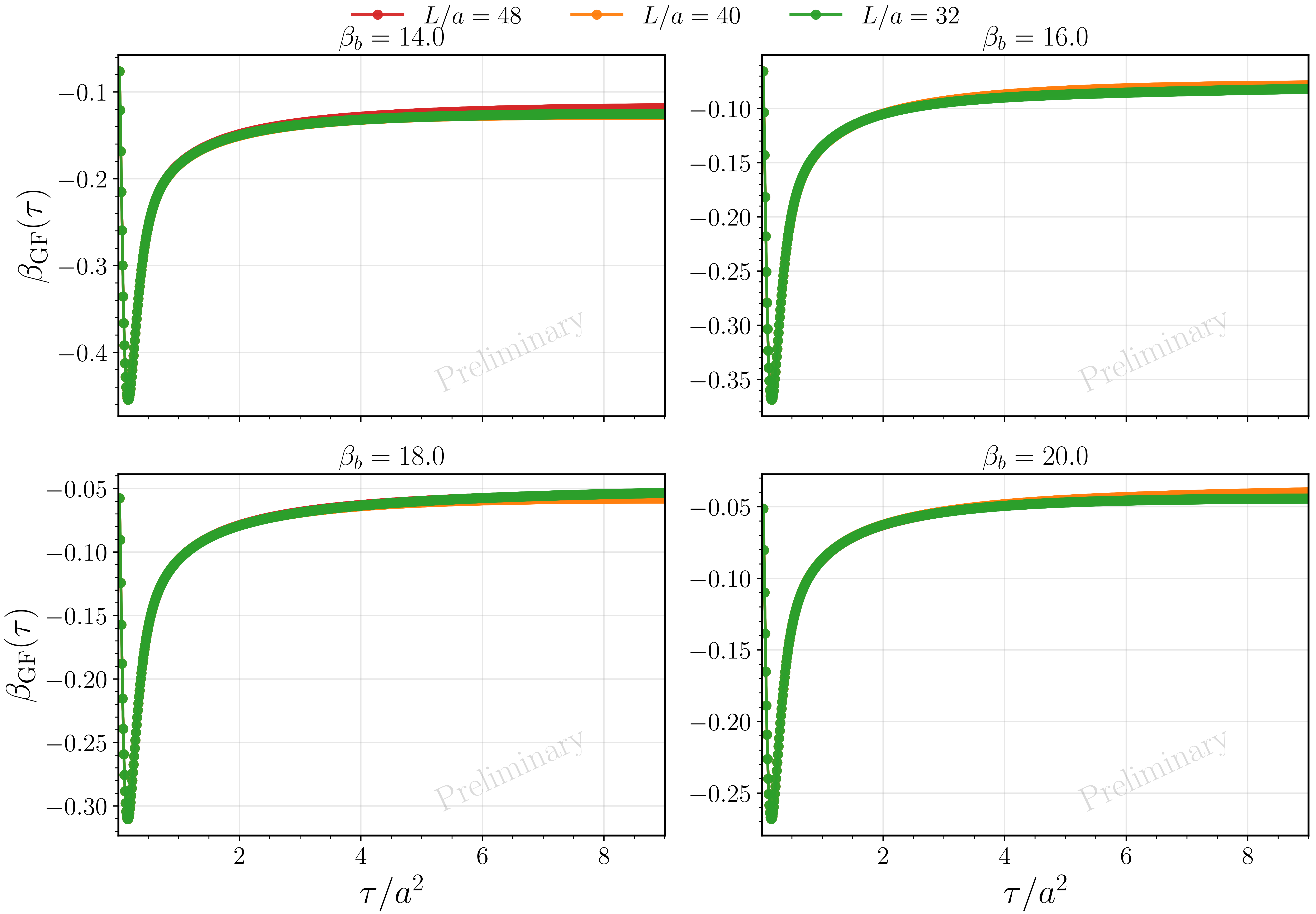
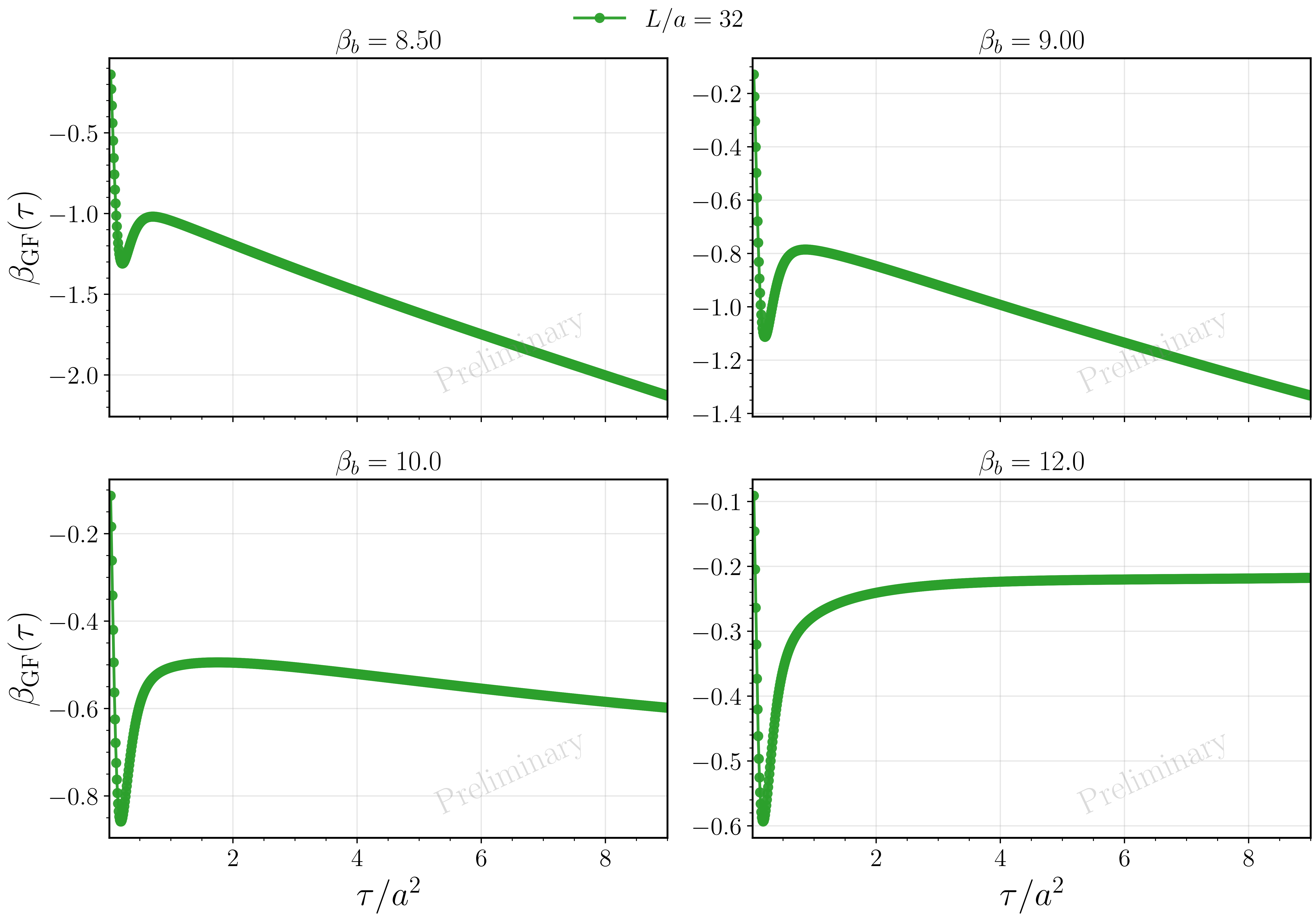
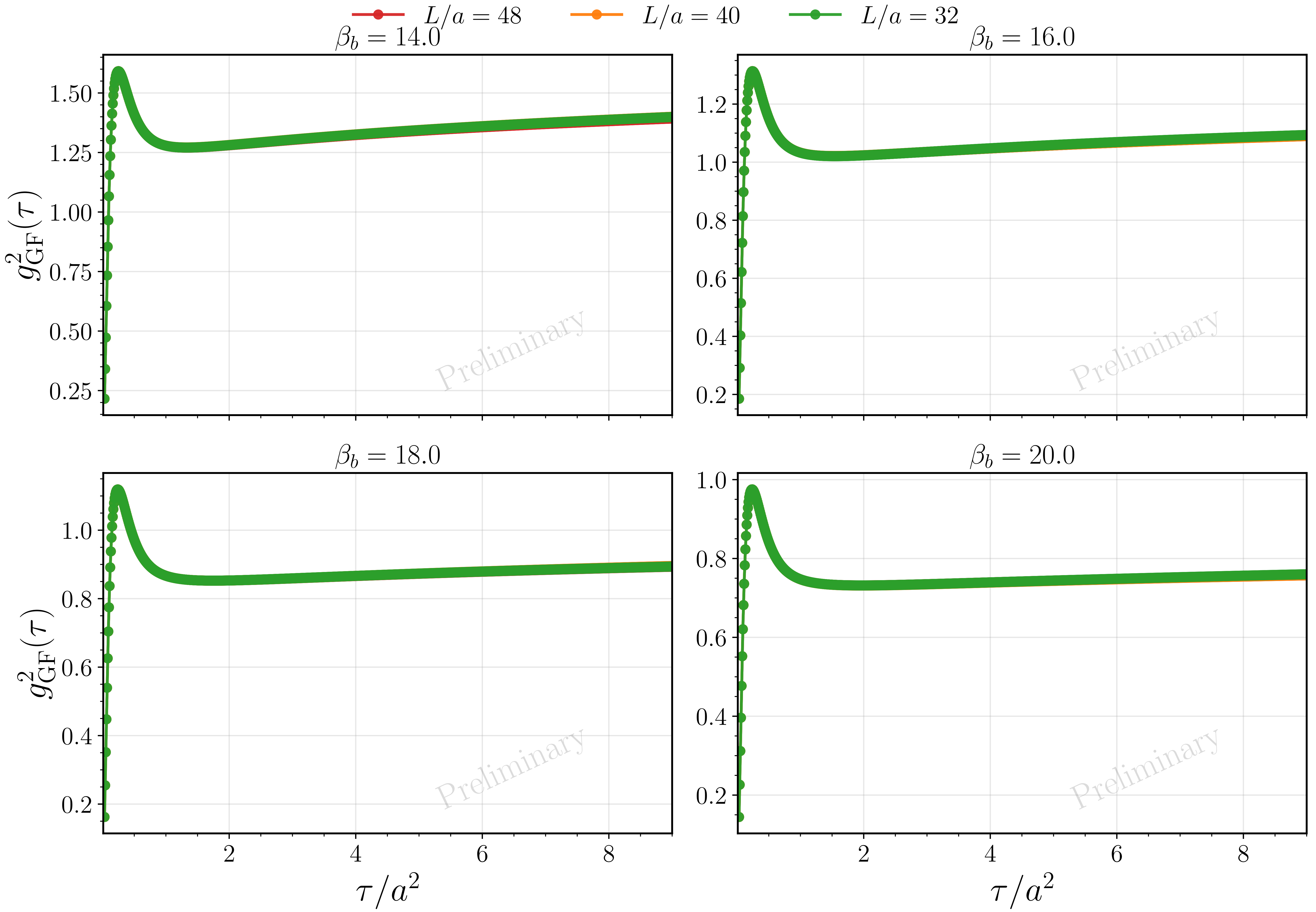


$\beta_{\mathrm{GF}}(\tau)$ vs τ/a^2 for $L/a = 32$

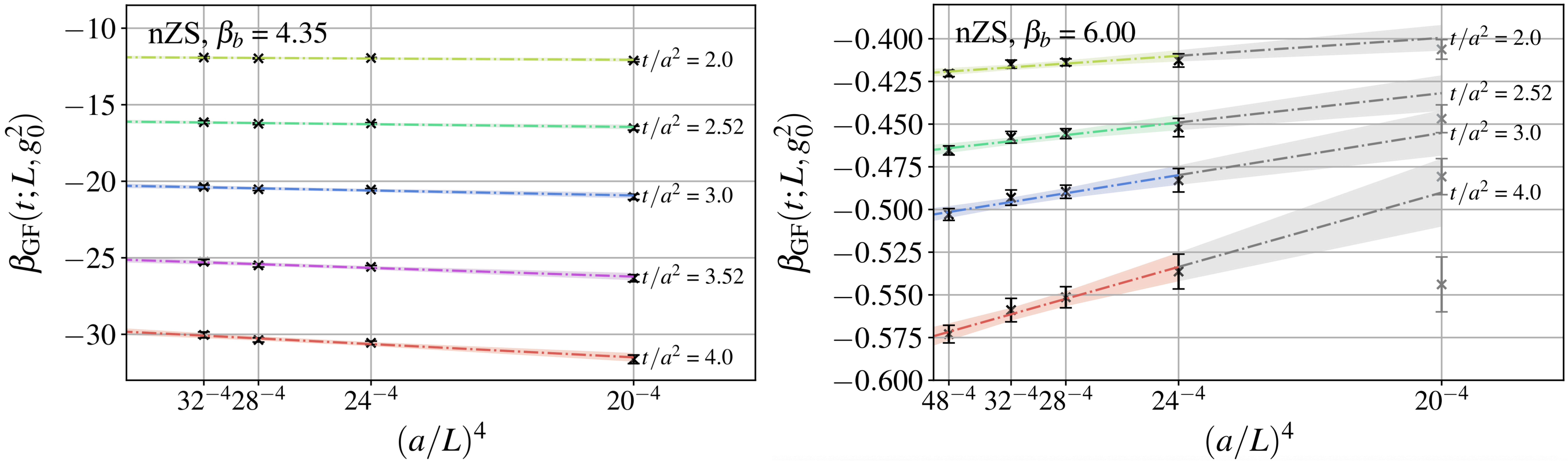


$g^2_{\mathrm{GF}}(\tau)$ vs τ/a^2 for largest volumes (full range)



Infinite volume extrapolation

From $N_f = 0$ results

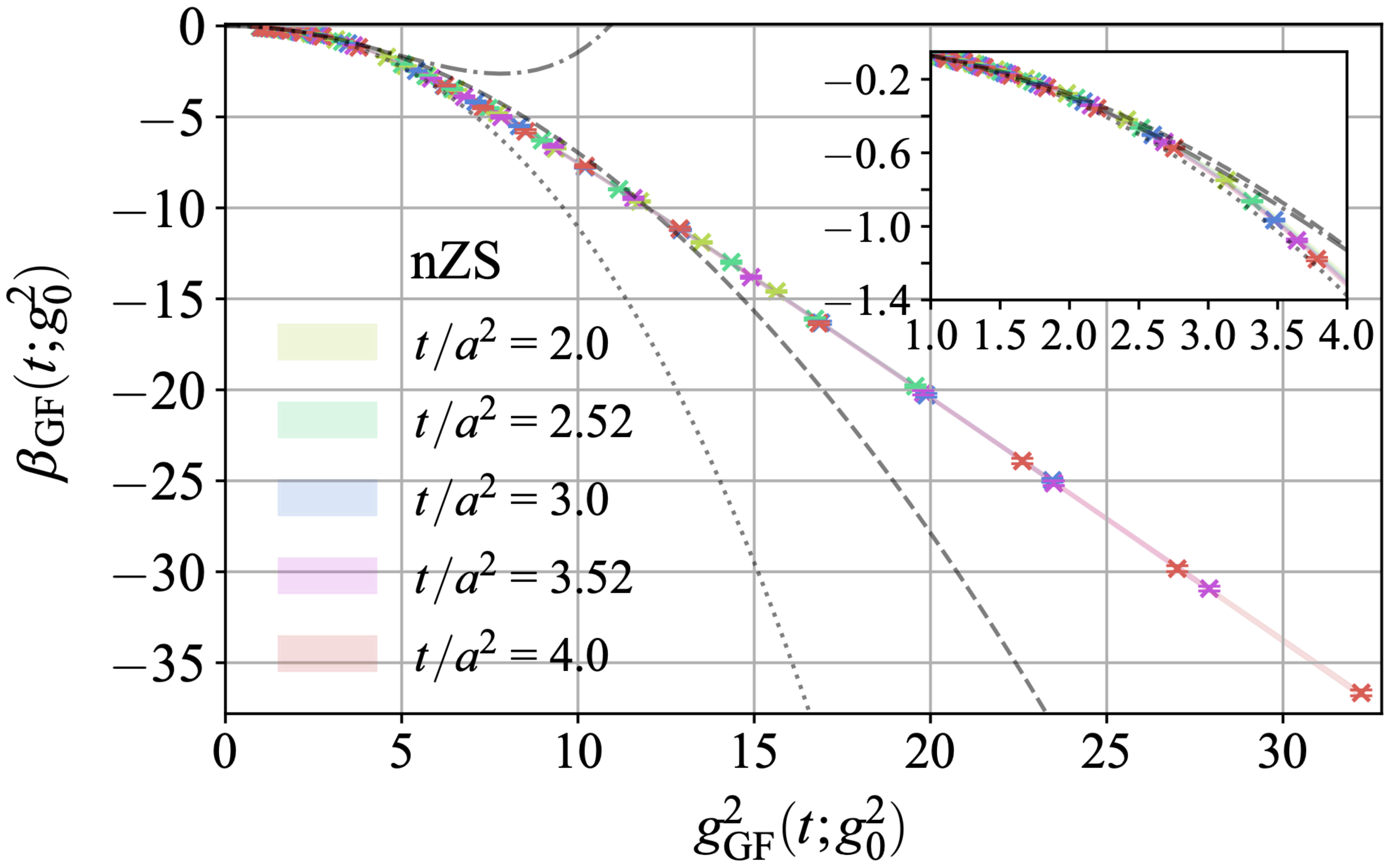


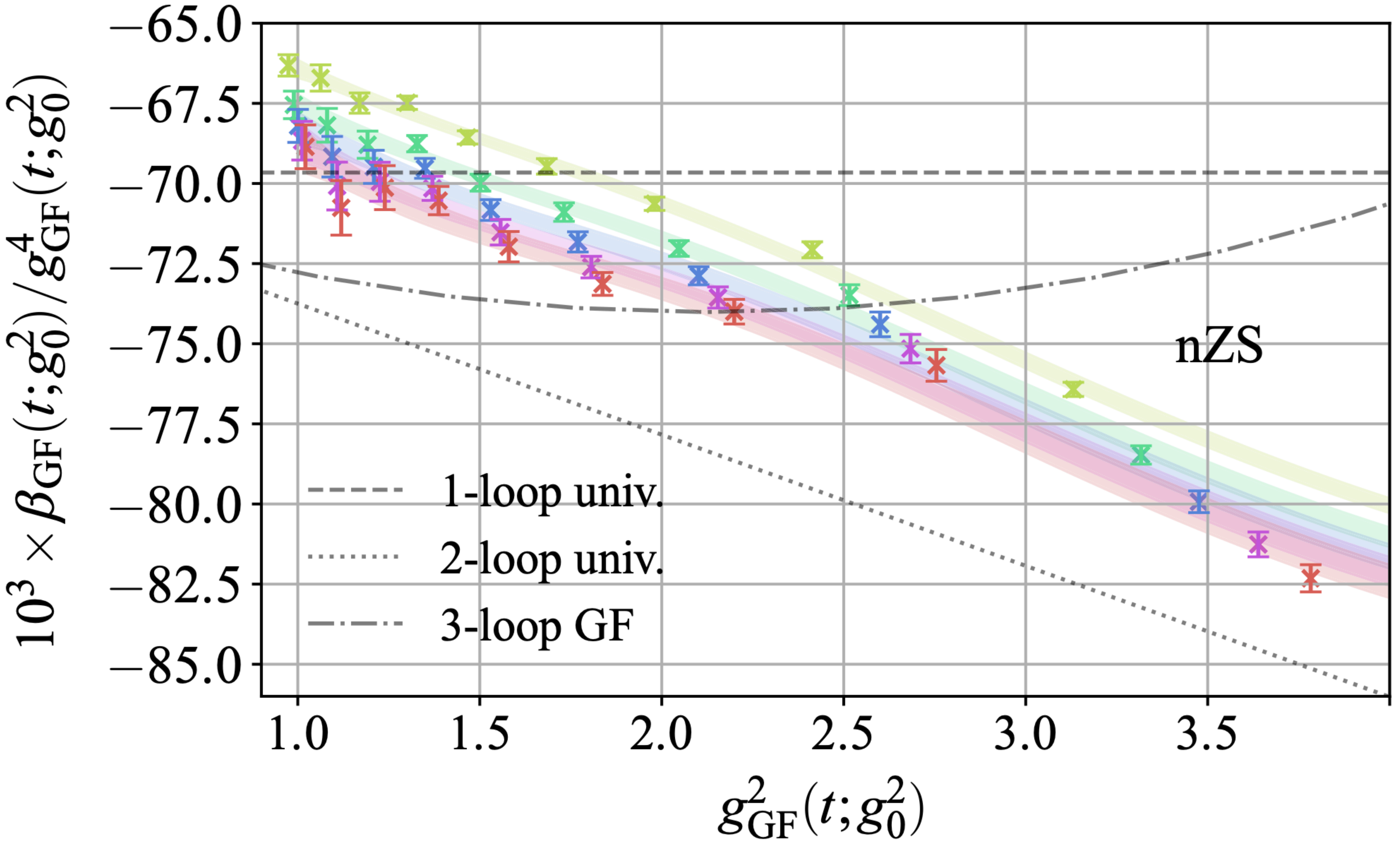
Extrapolate $\beta_{GF}(\tau; L, g_0^2)$ linearly in $(a/L)^4 \to 0$ at fixed β_b and τ/a^2.

[Hasenfratz, A., Peterson, C.T., PRD 108, 014502 (2023)]

Intermediate interpolation

From $N_f = 0$ results



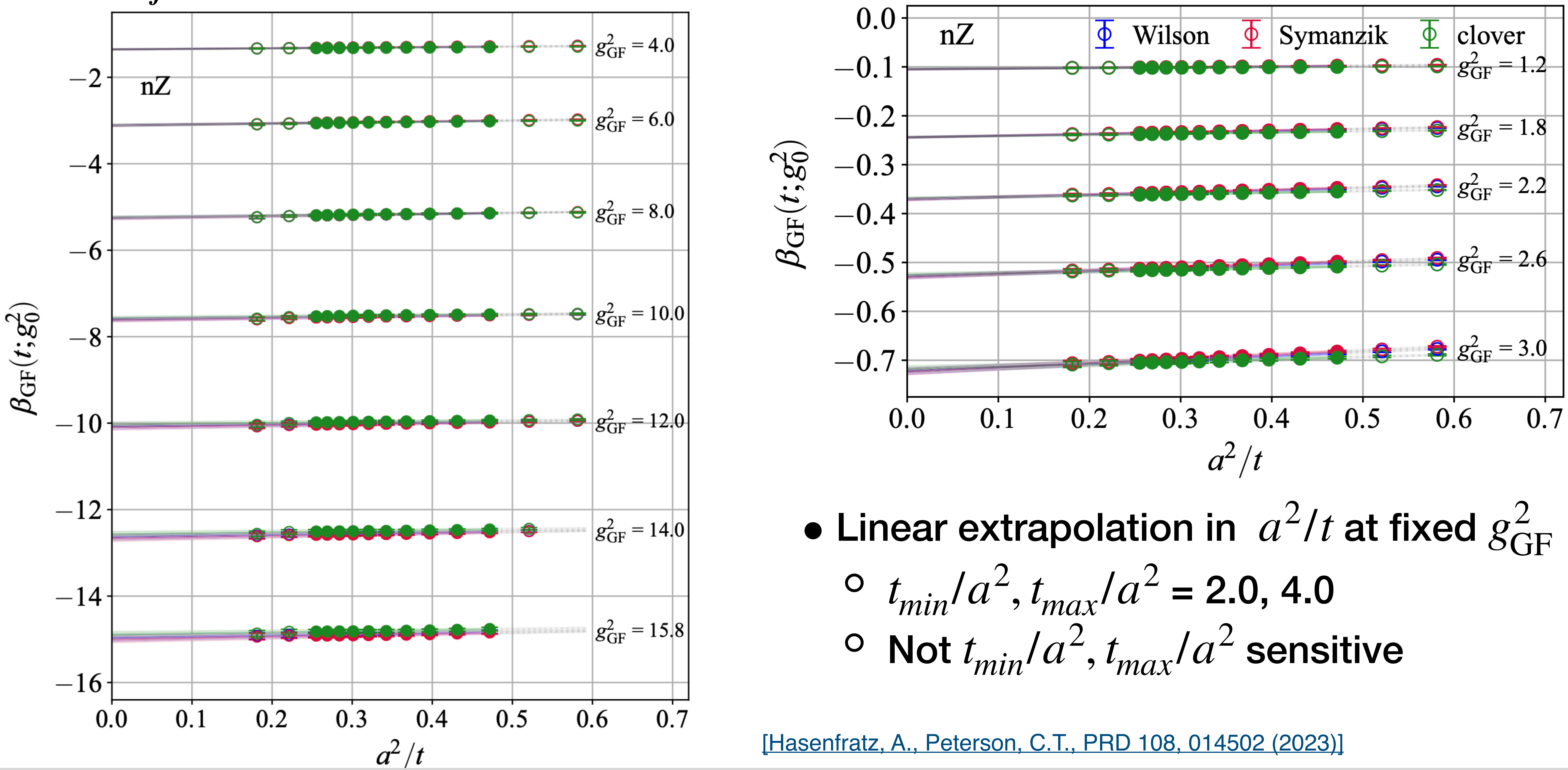


Now, continuum limit can be taken at fixed g^2_{GF}

[Hasenfratz, A., Peterson, C.T., PRD 108, 014502 (2023)]

Continuum extrapolation

From $N_f = 0$ results



- Linear extrapolation in a^2/t at fixed g^2_{GF}
  - $t_{min}/a^2, t_{max}/a^2$ = 2.0, 4.0
  - Not $t_{min}/a^2, t_{max}/a^2$ sensitive

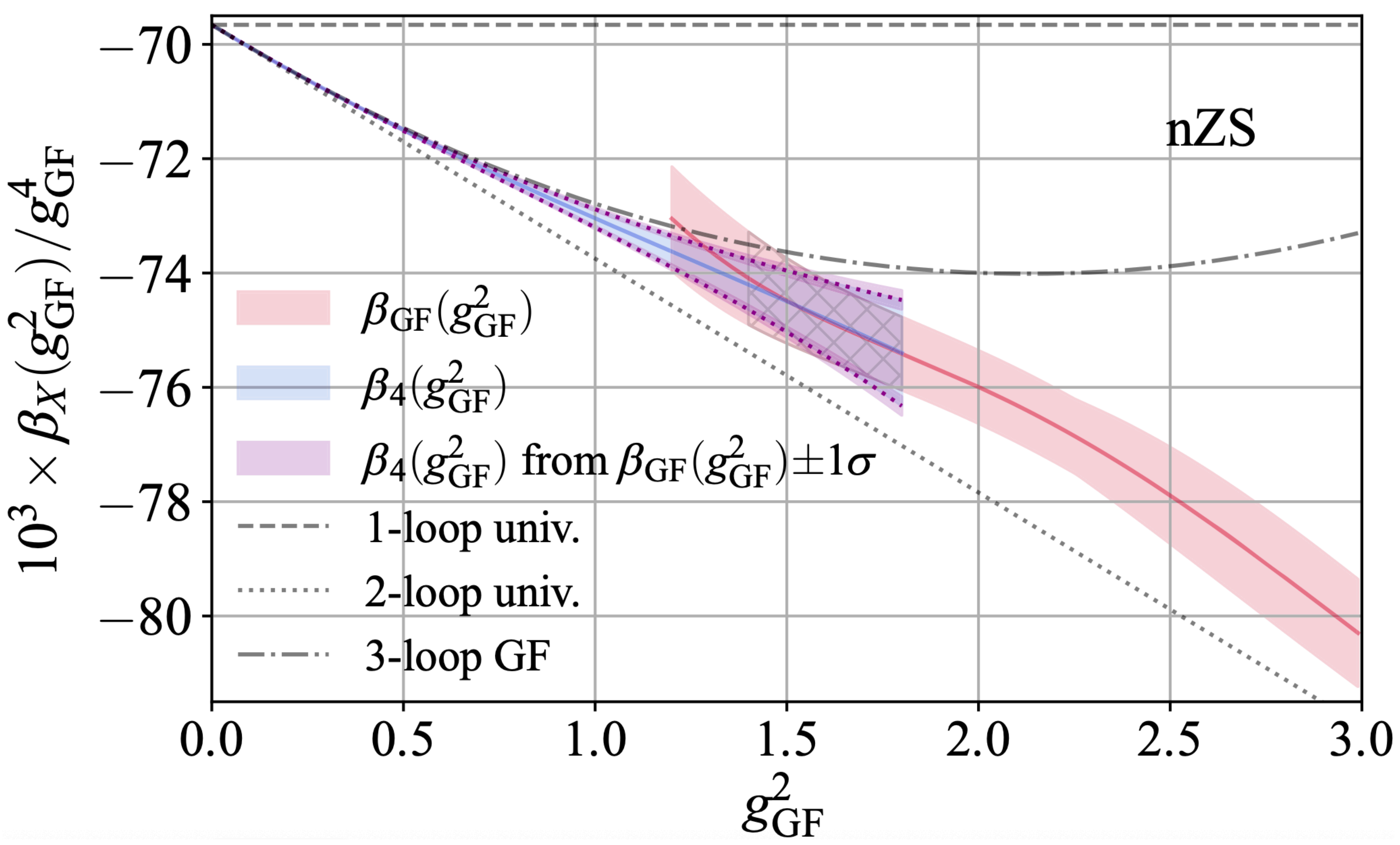
[Hasenfratz, A., Peterson, C.T., PRD 108, 014502 (2023)]

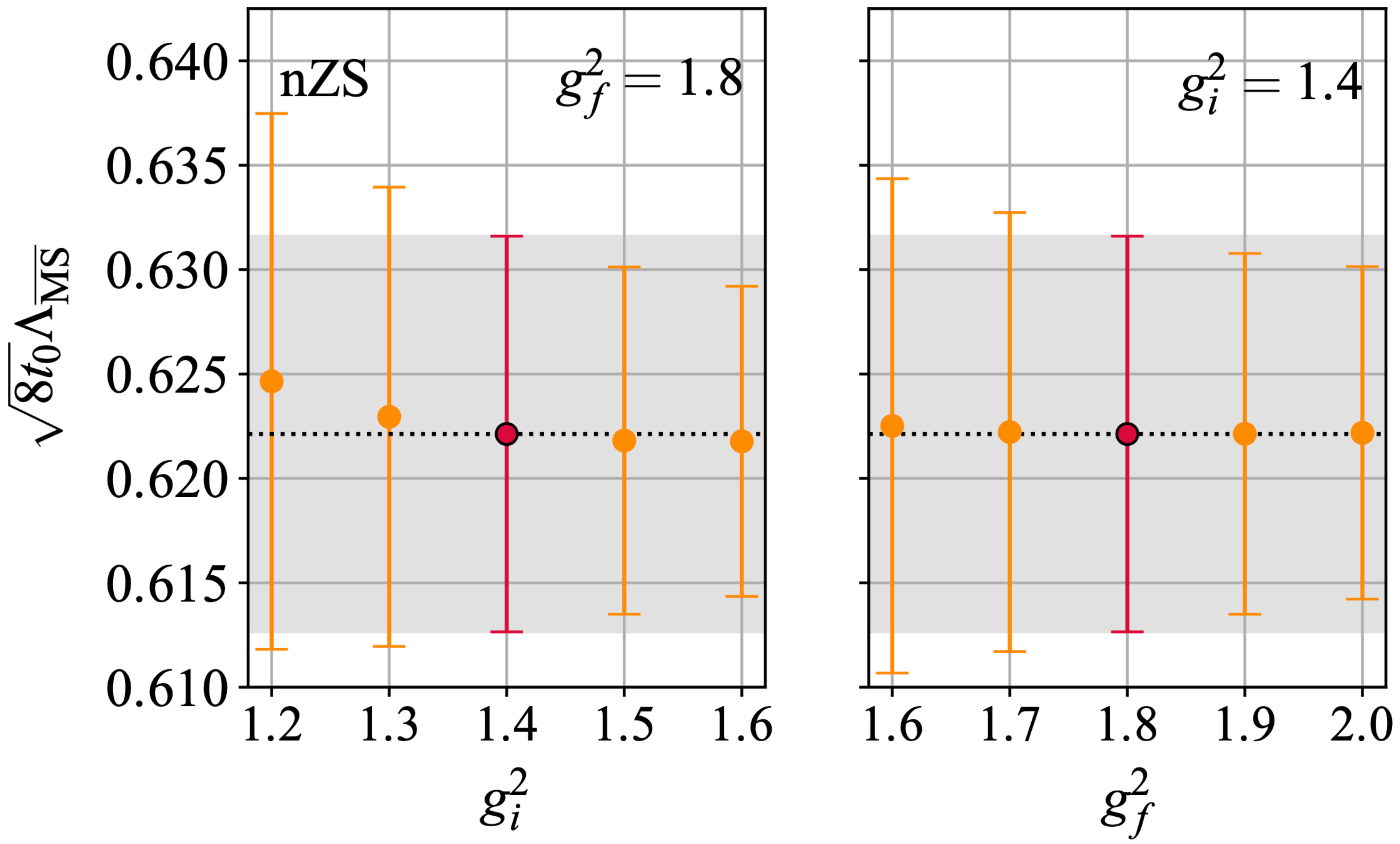
Effective β-function & Λ-parameter

- Λ-parameter requires β_{GF} down to $g^2_{\mathrm{GF}} = 0$
  - Parameterize

$$\beta_4(g^2_{\mathrm{GF}}) = -\, g^4_{\mathrm{GF}} \sum_{I=0}^{3} b_i g^{2i}_{\mathrm{GF}}$$

  - b_0, b_1, b_2 = perturbative values
  - b_3 determined by matching to β_{GF} over $1.4 \leq g^2_{\mathrm{GF}} \leq 1.8$
  - Systematic uncertainties associated with matching range under control





[Hasenfratz, A., Peterson, C.T., PRD 108, 014502 (2023)]

Quark masses & fermion flow for staggered fermions

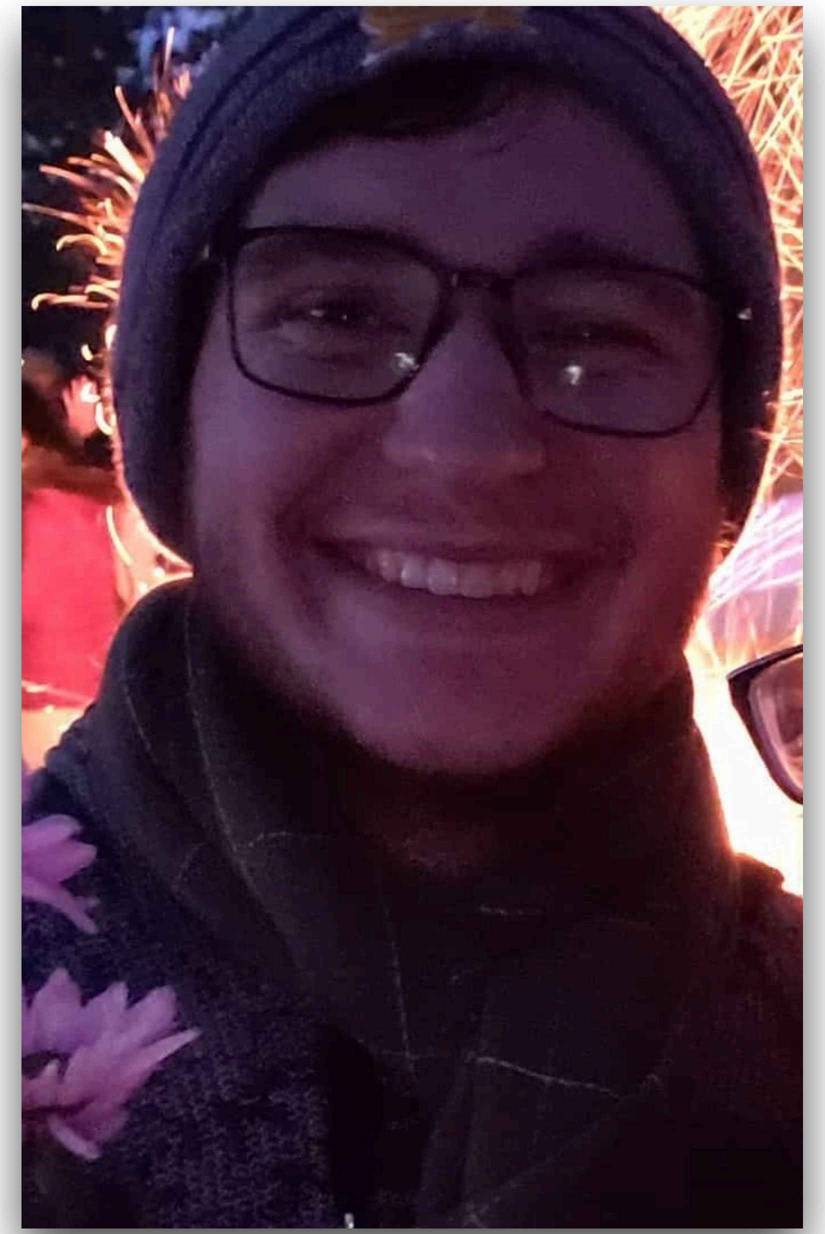
- Running quark masses from $\gamma_m(g^2)$ in the GF scheme
- γ_m from gradient-flowed pseudoscalar / vector correlators

$$\gamma_m(\tau; n_4) = -2\frac{\mathrm{d}}{\mathrm{d}\log\tau}\log\left(\frac{C_\tau^{\mathcal{S}}(n_4)}{C_\tau^{\mathcal{V}}(n_4)}\right)$$

- Renormalized masses from the flowed PCAC relation
- $\overline{\mathrm{MS}}$ conversion: short-flow-time matching + $\tau \to 0$ extrapolation
  - gauge-invariant, helps avoid the window problem.
- Significant progress made with the fermion flow implementation in Quantum EXpressions (QEX).

$\alpha_s(m_Z)$ from HISQ ($N_f = 4$)

Fermilab Lattice and MILC Collaborations



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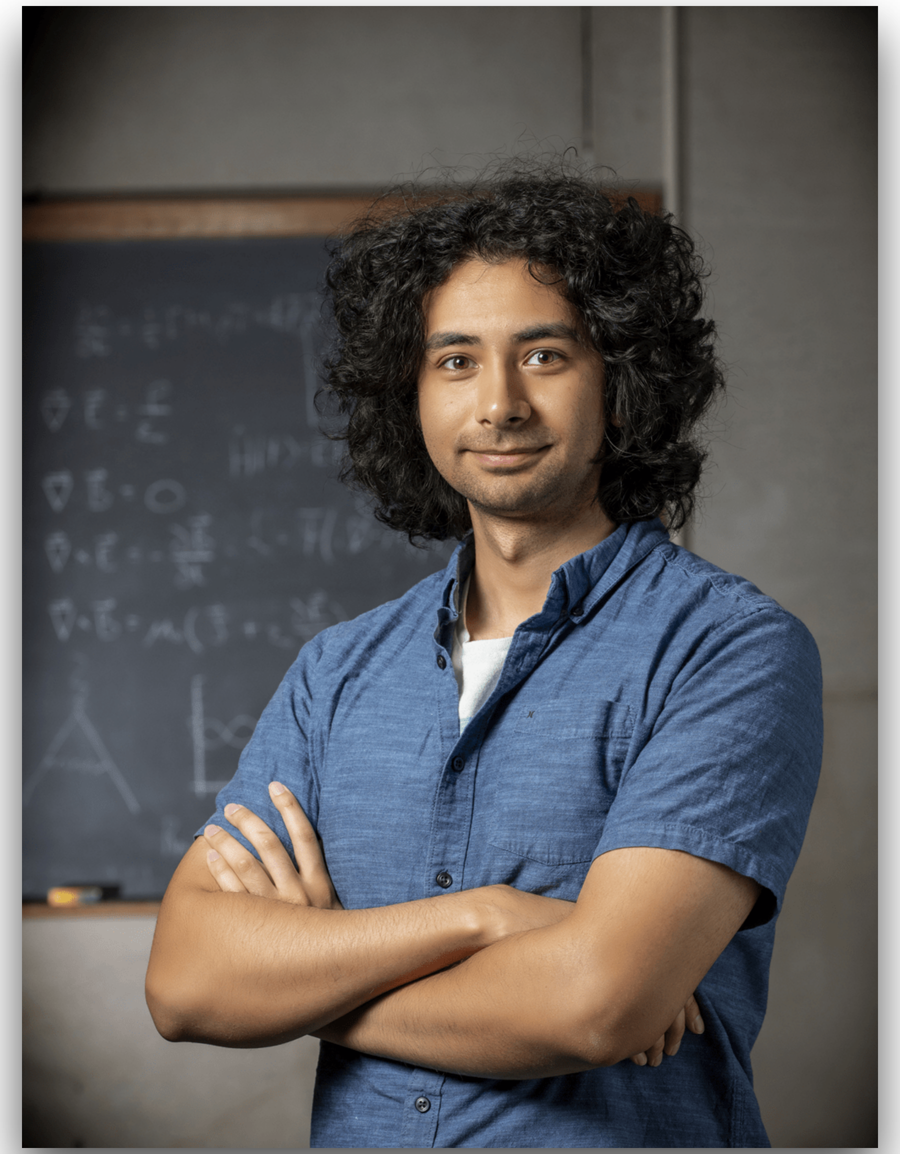

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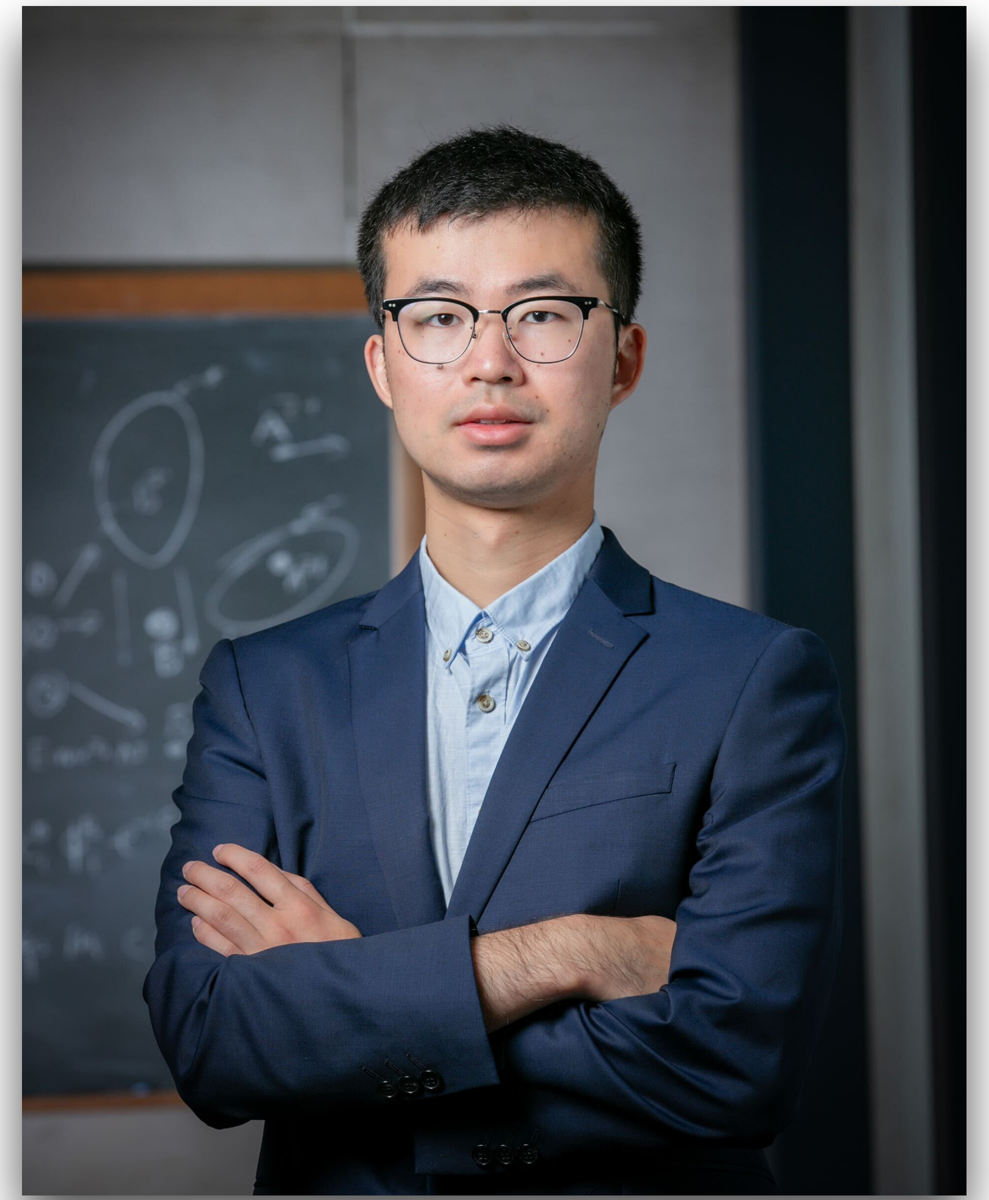

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