High-precision calculation of the quark–gluon coupling from lattice QCD

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Based on:

- (ALPHA) M. Dalla Brida R. Höllwieser, F. Knechtli, T. Korzec A. Ramos, S. Sint, R. Sommer. *High-precision calculation of the quark–gluon coupling from lattice QCD*. Nature 652 (2026) 8109, 328-334. https://www.nature.com/articles/s41586-026-10339-4

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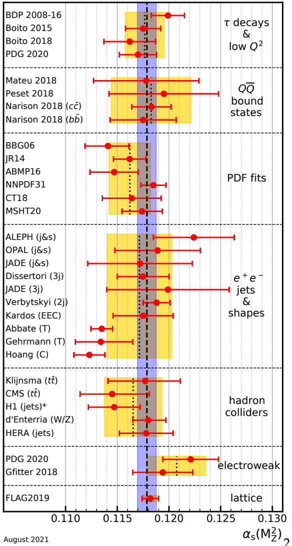
EXCELENCIA SEVERO OCHOA

Computations of α_s

$$O(Q) \overset{Q\to\infty}{\sim} \alpha_{\overline{\mathrm{MS}}}(Q) + \sum_{n=2}^{N} c_n \alpha_{\overline{\mathrm{MS}}}^{n}(Q) + \mathcal{O}(\alpha_{\overline{\mathrm{MS}}}^{N+1}(Q)) + \mathcal{O}\left(\frac{\Lambda^p}{Q^p}\right) + \dots$$

Difficulties in extracting $\alpha_{\overline{\mathrm{MS}}}$

- Difficult to compute (NP physics is difficult!)
- Difficult to estimate ($\alpha(Q)$ runs logarithmically)
- Asymptotic freedom: use large Q
  - **Pheno**: Large statistical uncertainties
  - **Lattice**: Window problem
- Solution: Finite size scaling **[M. Lüscher]**/extrapolate $L \to \infty$ **[A. Hasefratz]**
  - Systematic low $Q \iff$ statistical



Computations of α_s

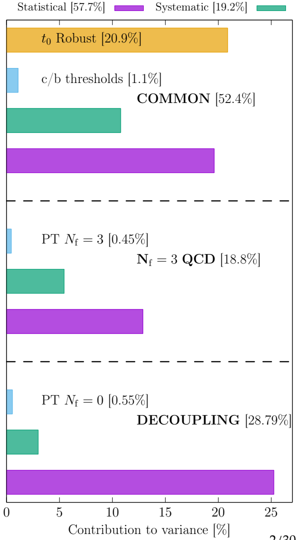
$$O(Q) \overset{Q\to\infty}{\sim} \alpha_{\overline{\mathrm{MS}}}(Q) + \sum_{n=2}^{N} c_n \alpha^n_{\overline{\mathrm{MS}}}(Q) + \mathcal{O}(\alpha^{N+1}_{\overline{\mathrm{MS}}}(Q)) + \mathcal{O}\left(\frac{\Lambda^p}{Q^p}\right) + \ldots$$

Key takes of ALPHA result [ALPHA Nature 652]

$$\alpha_{\overline{\mathrm{MS}}}(M_Z) = 0.11876(45)_{\mathrm{stat}}(25)_{\mathrm{sys}}(27)_{\mathrm{robust}}(58)_{\mathrm{tot}} \qquad [0.47\%]\,.$$

- Really: determination of $\Lambda_{\overline{\mathrm{MS}}}$ as extrapolation $g \to 0$
- Some details on the Gradient Flow
- Decoupling idea and $N_f = 0$ result
- Some details on the matching with PT

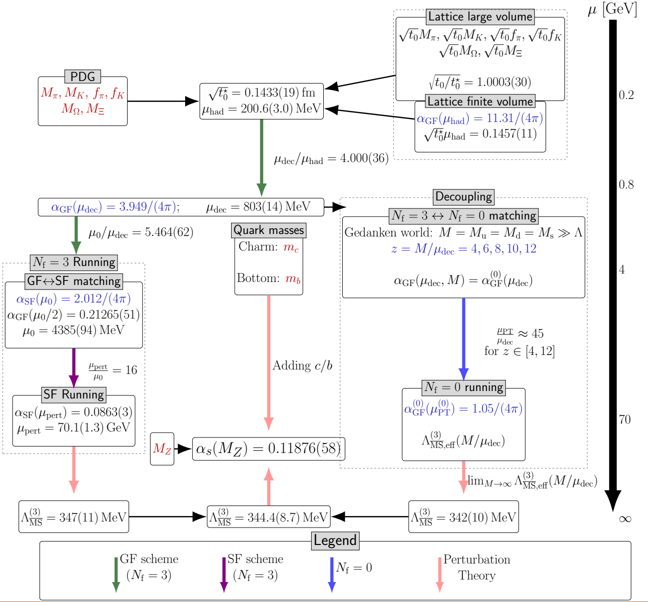
$$\Lambda_s = \mu \left[b_0 \bar{g}_s^2(\mu)\right]^{\frac{-b_1}{2b_0^2}} e^{-\frac{1}{2b_0\bar{g}_s^2(\mu)}} \exp\left\{-\int_0^{\bar{g}_s(\mu)} dx \left[\frac{1}{\beta_s(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0 x}\right]\right\}$$



The strategy

- Input: Hadron spectrum: $M_\pi, M_K, M_\Xi, f_k, f_\pi, M_\Omega, \ldots$
- Technical intermediate scale $\sqrt{t_0^\star}$
- Two FV approaches:
  - From 200 MeV to EW scale ($N_f = 3$)
  - Match QCD with YM (decoupling)
- Use PT from EW scale on: $\Lambda^{(3)} = 344.4(8.7)$ MeV
- Use PT to cross c/b thresholds: $\alpha_s(M_Z) = 0.11876(58)$. [0.47%] precision

- Conservative error
- Statistical errors dominate



Massless renormalization schemes

Computation of observables

$$O(Q,m) \overset{\alpha\to 0}{\sim} \sum_n c_n(Q/\mu, m)\alpha^n_{\overline{\rm MS}}(\mu)$$

- Coupling $\alpha_{\overline{\rm MS}}(\mu)$ defined by: "counterterms only include the divergences"
- Divergences do not depend on values of quark masses
- $\overline{\rm MS}$ is an example of a massless scheme.
- In LQCD we also prefer massless schemes: Conditions imposed at zero mass
  - Schrödinger Functional (SF)
  - (S)RI/MOM
  - Easier to define the chiral point ($m_q = 0$) than the physical point ($m_q = ??$)
  - …

Massless renormalization schemes: Tremendous advantages

- Renormalization group functions are mass independent

$$\mu \frac{d\bar{g}^2(\mu)}{d\mu} = \beta(\bar{g}, \cancel{m}) .$$

- RGI invariants that characterize the running (i.e $\Lambda, M, B_K, \ldots$) **only** exists in massless schemes

$$\Lambda_s = \mu \left[b_0 \bar{g}_s^2(\mu) \right]^{\frac{-b_1}{2b_0^2}} e^{-\frac{1}{2b_0 \bar{g}_s^2(\mu)}} \exp\left\{ -\int_0^{\bar{g}_s(\mu)} dx \left[\frac{1}{\beta_s(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0 x} \right] \right\}$$

- Precision: high loop computations available in perturbation theory

$$\beta_{\overline{\rm MS}}(\bar{g}) \overset{\bar{g} \to 0}{\sim} -\bar{g}^3 (b_0 + b_1 \bar{g}^2 + b_2^{\overline{\rm MS}} \bar{g}^4 + b_3^{\overline{\rm MS}} \bar{g}^6 + b_4^{\overline{\rm MS}} \bar{g}^8 + \text{unknown})$$

  Universal only in massless schemes

- But all quarks contribute the same!! (i.e. “top” contribution to running as large as “up”)

Decoupling of heavy quarks in massless schemes

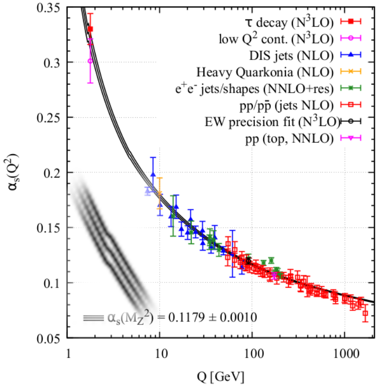
Matching between theories

- At energy scales Q just forget about all quarks with $m > Q$
- "Nice" perturtbative expressions if you only use **active** quarks
- Matching between effective theory (with **active quarks**) and fundamental theory (with **active** and heavy quarks)

$$\alpha_{\overline{\rm MS}}^{(N_f-1)}(\mu) = \alpha_{\overline{\rm MS}}^{(N_f)}(\mu) \times \left\{1 + a_1(m_h/\mu)\alpha_{\overline{\rm MS}}^{(N_f)}(\mu) + \dots\right\}$$

Abuse of language: A single $\alpha_{\overline{\rm MS}}(\mu)$ that "jumps" at quark thresholds

- $\alpha_{\overline{\rm MS}}(4\,\mathrm{GeV})$: This is the four flavor coupling
- $\alpha_{\overline{\rm MS}}(10\,\mathrm{GeV})$: This is the five flavor coupling
- $\alpha_{\overline{\rm MS}}(M_Z)$: This is the five flavor coupling



Decoupling of heavy quarks in massless schemes

$$\Lambda^{(N_f)}_{\overline{MS}} \xrightarrow{P(M/\Lambda)} \Lambda^{(N_f')}_{\overline{MS}}$$

Relation between Λ parameters

1. From $\Lambda^{(6)}_{\overline{MS}}$ determine $\alpha^{(6)}_{\overline{MS}}(\mu) = \bar{g}^2_{\overline{MS}}(\mu)/(4\pi)$ at some scale $\mu \approx m_t$

$$\frac{\Lambda^{(6)}_{\overline{MS}}}{\mu} = \left[b_0 \bar{g}^2_{\overline{MS}}(\mu)\right]^{\frac{-b_1}{2b_0^2}} e^{-\frac{1}{2b_0\bar{g}^2_{\overline{MS}}(\mu)}} \exp\left\{-\int_0^{\bar{g}_{\overline{MS}}(\mu)} dx \left[\frac{1}{\beta^{(6)}_{\overline{MS}}(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0 x}\right]\right\}$$

2. Match across the top threshold (4 loops known!)

$$\frac{\bar{g}'^2(\mu)}{4\pi} = \alpha^{(5)}_{\overline{MS}}(\mu) = \alpha^{(6)}_{\overline{MS}}(\mu) \times \left\{1 + a_1(m_t/\mu)\alpha^{(6)}_{\overline{MS}}(\mu) + \dots\right\}$$

3. Determine the Λ parameter of the 5 flavor theory

$$\frac{\Lambda^{(5)}_{\overline{MS}}}{\mu} = \left[b_0 \bar{g}'^2_{\overline{MS}}(\mu)\right]^{\frac{-b_1}{2b_0^2}} e^{-\frac{1}{2b_0\bar{g}'^2_{\overline{MS}}(\mu)}} \exp\left\{-\int_0^{\bar{g}'_{\overline{MS}}(\mu)} dx \left[\frac{1}{\beta^{(5)}_{\overline{MS}}(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0 x}\right]\right\}$$

Decoupling of heavy quarks in massless schemes

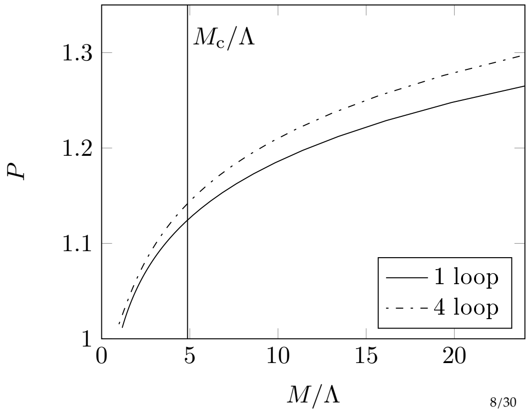
$$\frac{\Lambda_{\overline{\rm MS}}^{(N_f)}}{\Lambda_{\overline{\rm MS}}^{(N_f-1)}} = P_{N_f,N_f-1}(\Lambda/M)$$

Some numerical examples

- Start with $\Lambda_{\overline{\rm MS}}^{(6)} \approx 91.1$ MeV
- Determine $\alpha_{\overline{\rm MS}}^{(6)}(m_t) \Longrightarrow \alpha_{\overline{\rm MS}}^{(5)}(m_t)$
- Get $\Lambda_{\overline{\rm MS}}^{(5)} \approx 215$ MeV
- Determine $\alpha_{\overline{\rm MS}}^{(5)}(m_b) \Longrightarrow \alpha_{\overline{\rm MS}}^{(4)}(m_b)$
- Get $\Lambda_{\overline{\rm MS}}^{(4)} \approx 298$ MeV
- Determine $\alpha_{\overline{\rm MS}}^{(4)}(m_c) \Longrightarrow \alpha_{\overline{\rm MS}}^{(3)}(m_c)$
- Get $\Lambda_{\overline{\rm MS}}^{(3)} \approx 312$ MeV
- NO $\Lambda_{\overline{\rm MS}}^{(2)}$: PT not valid at $\mu \approx m_s < \Lambda$

Perturbative uncertainties **small**! [ALPHA '2019]

$N_q = 4,\ N_l = 3$



Decoupling of heavy quarks in massless schemes

$$\frac{\Lambda_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})}}{\Lambda_{\overline{\mathrm{MS}}}^{(N_\mathrm{f}-1)}} = P_{N_\mathrm{f},N_\mathrm{f}-1}(\Lambda/M)$$

Some numerical examples

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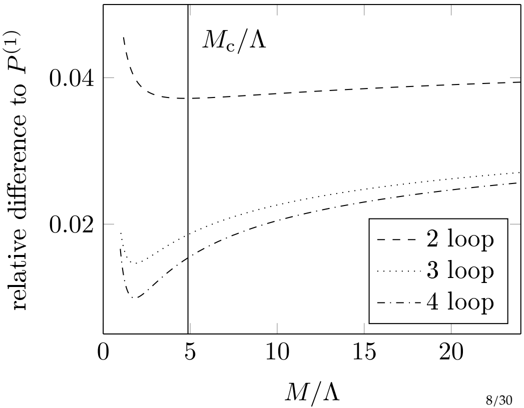


3M: A universe with three heavy degenerate quarks $(M \gg \Lambda)$

Alice uses fundamental theory

$$S_{\text{fund}}[A_\mu, \psi, \bar{\psi}] = \int \mathrm{d}^4x \left\{ -\frac{1}{2g^2} \mathrm{Tr}\left(F_{\mu\nu}F_{\mu\nu}\right) + \sum_{i=1}^{3} \bar{\psi}_i(\gamma_\mu D_\mu + M)\psi_i \right\}$$

Bob uses effective theory

$$S_{\text{eff}}[A_\mu] = -\frac{1}{2g_{\text{eff}}^2} \int \mathrm{d}^4x \, \{\mathrm{Tr}\left(F_{\mu\nu}F_{\mu\nu}\right)\} + \frac{1}{M^2} \sum_k \omega_k \int \mathrm{d}^4x \, \mathcal{L}_k^{(6)} + \ldots$$

3M: A universe with three heavy degenerate quarks $(M \gg \Lambda)$

Alice uses fundamental theory

$$S_{\text{fund}}[A_\mu, \psi, \bar{\psi}] = \int \mathrm{d}^4x \left\{ -\frac{1}{2g^2} \mathrm{Tr}\left(F_{\mu\nu}F_{\mu\nu}\right) + \sum_{i=1}^{3} \bar{\psi}_i(\gamma_\mu D_\mu + M)\psi_i \right\}$$

Bob uses effective theory

$$S_{\text{eff}}[A_\mu] = -\frac{1}{2g^2_{\text{eff}}} \int \mathrm{d}^4x \, \{\mathrm{Tr}\left(F_{\mu\nu}F_{\mu\nu}\right)\} + \cancel{\frac{1}{M^2}\sum_k \omega_k \int \mathrm{d}^4x \, \mathcal{L}_k^{(6)} + \ldots}$$

Decoupling

- Dimensionless "low energy quantities" $\sqrt{t_0}/r_0, w_0/\sqrt{8t_0}, r_0/w_0, \ldots$ from effective theory

$$\frac{\mu_1^{\text{fund}}(M)}{\mu_2^{\text{fund}}(M)} = \frac{\mu_1^{\text{eff}}}{\mu_2^{\text{eff}}} + \mathcal{O}\left(\frac{\mu^2}{M^2}\right)$$

Simplest application of decoupling

Bob can compute Λ

- Determine (pure gauge)

$$\sqrt{8t_0} \times \Lambda_{\overline{\mathrm{MS}}}^{(0)}$$

- Match QCD (3M) and Yang-Mills with $P_{3,0}(M/\Lambda)$. Use scale determined by Alice $\sqrt{t_0(M)}$ in MeV.

$$\Lambda_{\overline{\mathrm{MS}}}^{(3)} = \frac{1}{\sqrt{8t_0(M)}} \times \sqrt{8t_0} \times \Lambda^{(0)} \times \frac{1}{P_{3,0}(M/\Lambda^{(3)})} + \mathcal{O}(\alpha^4(m^\star)) + \mathcal{O}(M^{-2})\,.$$

- Main point: Exact relation in $M \to \infty$
  - Implicit equation for $\Lambda^{(3)}$.
  - Running in pure gauge $\sqrt{8t_0} \times \Lambda^{(0)}$
  - Perturbative matching factor $P_{3,0}(M/\Lambda^{(3)})$
  - A scale in physical units $\sqrt{8t_0(M)}$
- We do **NOT** live in 3M ... But we can simulate it!

$$\sqrt{8t_0(M)} = \sqrt{8t_0^{\mathrm{phys}}} \times \lim_{a\to 0} \frac{\sqrt{8t_0(M)}/a}{\sqrt{8t_0^{\mathrm{phys}}}/a}$$

- Hard multiscale problem M/Λ wants to be large, aM wants to be small: **"Solved" in finite volume!**

Our setup: Choices optimized to be able to simulate heavy quarks

$$\Lambda^{(3)} = \mu_{\rm dec}(M) \times \frac{\Lambda^{(0)}}{\mu_{\rm dec}} \times \frac{1}{P(\Lambda/M)} + \mathcal{O}(\alpha^4(m^\star)) + \mathcal{O}\left(\frac{\mu_{\rm dec}}{M}\right) + \mathcal{O}\left(\frac{\mu_{\rm dec}^2}{M^2}\right)$$

- Work in finite volume schemes with Schrödinger Functional boundary conditions: $T \times L^3$ with Dirichlet bcs. in time. ($\mu \sim 1/L$): “Only” two scales.
- Use Gradient Flow couplings

$$\bar{g}^2(\mu) = \mathcal{N}^{-1}(c, a/L)\, t^2 \langle E(t) \rangle \Big|_{\mu^{-1} = \sqrt{8t} = cL}.$$

- Fix $\bar{g}^2(\mu_{\rm dec})|\Big|_{N_f=3, M=0, T=L} = 3.95$. This defines $\mu_{\rm dec} \sim 800$ MeV
- Small volume $\Longrightarrow$ We can simulate heavy quarks $M \gtrsim 10$ GeV (i.e. $a \sim 30\ \text{GeV}^{-1}$)
- Matching condition ($\{N_f = 3, M\} \leftrightarrow \{N_f = 0\}$) between massive scheme and effective theory

$$\bar{g}^2(\mu_{\rm dec}(M))\Big|_{N_f=3, M, T=2L} = \bar{g}^2(\mu_{\rm dec})\Big|_{N_f=0, T=2L}.$$

  Matching: QCD in a finite volume!
- Convenient variable: $z = M/\mu_{\rm dec}$

Our setup: Choices optimized to be able to simulate heavy quarks

$$\Lambda^{(3)} = \mu_{\rm dec}(M) \times \frac{\Lambda^{(0)}}{\mu_{\rm dec}} \times \frac{1}{P(\Lambda/M)} + \mathcal{O}(\alpha^4(m^\star)) + \mathcal{O}\left(\frac{\mu_{\rm dec}}{M}\right) + \mathcal{O}\left(\frac{\mu_{\rm dec}^2}{M^2}\right)$$

Exact relation in the limit $M \to \infty$

- Define $\rho^{\rm eff} = \Lambda^{(3)}_{\rm eff}/\mu_{\rm dec}(M)$. Implicit equation and matching condition

$$\rho^{\rm eff} = \frac{\Lambda^{(0)}}{\mu_{\rm dec}} \times \frac{1}{P(\rho^{\rm eff}/z)}\,; \qquad \bar{g}^2(\mu_{\rm dec}(M))\Big|_{N_{\rm f}=3,M,T=2L} = \bar{g}^2(\mu_{\rm dec})\Big|_{N_{\rm f}=0,T=2L}\,.$$

- We have $\lim_{M\to\infty} \rho^{\rm eff} = \Lambda^{(3)}/\mu_{\rm dec}(M)$ with $\mu_{\rm dec}(M) = 803(14)$ MeV .

We only need to fill in a table!

$M/\mu_{\rm dec}$	$\bar{g}^2_{\rm GFT}(\mu_{\rm dec}, M)$	$[\bar{g}^{(0)}_{\rm GF,T=L}(\mu_{\rm dec})]^2$	$\rho^{\rm eff}$	$\Lambda^{\rm eff}_{\overline{\rm MS}}$ [MeV]
…	…	…	…	…
…	…	…	…	…
…	…	…	…	…

Computing the Λ-parameter in units of $\mu_{\rm ref}$

$$\frac{\Lambda_s}{\mu_{\rm ref}} = \left[b_0 \bar{g}_s^2(\mu_{\rm ref})\right]^{\frac{-b_1}{2b_0^2}} e^{-\frac{1}{2b_0\bar{g}_s^2(\mu_{\rm ref})}} \exp\left\{-\int_0^{\bar{g}_s(\mu_{\rm ref})} dx \left[\frac{1}{\beta_s(x)} + \frac{1}{b_0x^3} - \frac{b_1}{b_0x}\right]\right\}$$

The non-perturbative extrapolation

- We need $\beta_s(x)$ in the range $x \in [0, \bar{g}_s(\mu_{\rm ref})]$ (NOT possible!)
- Instead, $\beta_s(x)$ known[a] for $x \in [\bar{g}_s(\mu_{\rm PT}), \bar{g}_s(\mu_{\rm ref})]$ and perfom an extrapolation

$$\int_0^{\bar{g}_s(\mu_{\rm ref})} dx \left[\frac{1}{\beta_s(x)} + \frac{1}{b_0x^3} - \frac{b_1}{b_0x}\right] \overset{g(\mu_{\rm PT})\to 0}{\sim} \int_{\bar{g}(\mu_{\rm PT})}^{\bar{g}_s(\mu_{\rm ref})} dx \left[\frac{1}{\beta_s(x)} + \frac{1}{b_0x^3} - \frac{b_1}{b_0x}\right] + \mathcal{O}(\bar{g}_s{}^2(\mu_{\rm PT}))$$

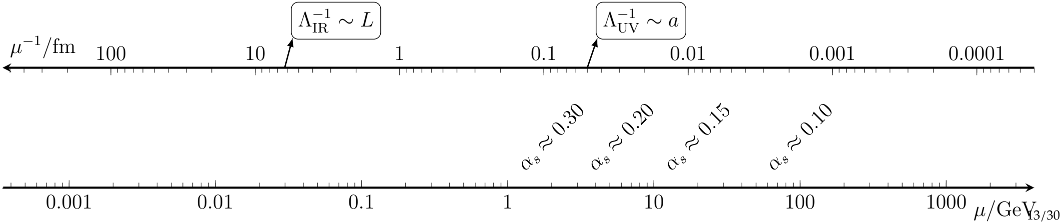
- Perturbation theory helps improving convergence

$$\begin{aligned}\int_0^{\bar{g}_s(\mu_{\rm ref})} dx \left[\frac{1}{\beta_s(x)} + \frac{1}{b_0x^3} - \frac{b_1}{b_0x}\right] \quad \overset{g(\mu_{\rm PT})\to 0}{\sim} \quad & \int_{\bar{g}(\mu_{\rm PT})}^{\bar{g}_s(\mu_{\rm ref})} dx \left[\frac{1}{\beta_s(x)} + \frac{1}{b_0x^3} - \frac{b_1}{b_0x}\right] \\ + \quad & \int_0^{\bar{g}(\mu_{\rm PT})} dx \left[\frac{1}{\beta_s{}^{(3l)}(x)} + \frac{1}{b_0x^3} - \frac{b_1}{b_0x}\right] \\ + \quad & \mathcal{O}(\bar{g}_s{}^4(\mu_{\rm PT}))\end{aligned}$$

[a] Sometimes β_s is not known but $\mu_{\rm PT}/\mu_{\rm ref} = \exp\left\{\int_{\bar{g}(\mu_{\rm PT})}^{\bar{g}_s(\mu_{\rm ref})} dx/\beta_{s(x)}\right\}$. This is equivalent.

Continuum vs. non-perturbative extrapolation

	Continuum	Non-perturbative
Corrections:	Depend on action and observable $a, a\log a, a^2, a^2\log^k a, a^4\log^k a, \ldots$	Depend **only** on scheme $\bar{g}_s{}^2(\mu_{\mathrm{PT}}), \bar{g}_s{}^4(\mu_{\mathrm{PT}}), \bar{g}_s{}^6(\mu_{\mathrm{PT}}), \ldots$
Improvement:	Symanzik EFT NP $\mathcal{O}(a)$-improvement –	Perturbation Theory $\beta_s{}^{(3\text{-loops})}(x)$ $\beta_s{}^{(4\text{-loops})}(x)$
Quality:	$a_{\max} < 0.1\,\mathrm{fm}$ $a^2_{\max}/a^2_{\min} = 4$ $[Q(a_{\min}) - Q(0)]/\sigma_Q < 3$	$\alpha_s{}^2(\mu_{\mathrm{PT}}) = \bar{g}_s{}^2(\mu_{\mathrm{PT}})/4\pi < 0.25$ Change $\bar{g}_s{}^{2k}(\mu_{\mathrm{PT}})$ by a factor 4 $[Q(a_{\min}) - Q(0)]/\sigma_Q < 3$
Cost:	$a \to a/2 \Longrightarrow \times 128$	$\alpha_s(\mu_{\mathrm{PT}}) = 0.25 \to 0.12 \Longrightarrow \times 10^6$

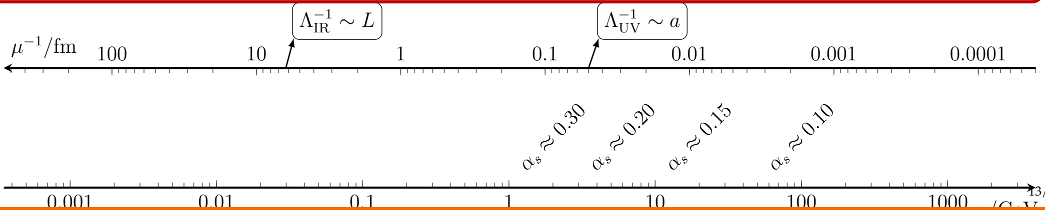


CONTINUUM VS. NON-PERTURBATIVE EXTRAPOLATION

	Continuum	Non-perturbative
	Depend on action and observable	Depend **only** on scheme

Exploring the non-perturbative extrapolation is challenging

- Very challenging to decrease $\alpha_s(\mu_{\rm PT})$ substantially
- Instead explore "truncation errors"
  - Change $\mu_{\rm PT} \to 2\mu_{\rm PT}$
  - Change order in PT
  - ...
- Every lattice (and non-lattice) determination of $\Lambda, \alpha_s, m_q, B_K, \ldots$ is quoting the result of a non-perturbative extrapolation
- Strong opinion here: *solved* using finite size scaling **[Lüscher, Weisz, Wolff '91]**.



SIMULATION DETAILS [M. DALLA BRIDA, A. RAMOS 1905.05147]

- Wilson Plaquette gauge action.
- Schrödinger Functional boundary conditions (w 2-loop c_t)
- Algorithm: Hybrid OR: $L/a \times (1\text{HB} + L/a \times \text{OR})$. Almost independent measurements of $\bar{g}^2_{\text{GF}}$.
- We always work in the scheme with $c = 0.3$. Two discretizations of $\bar{g}^2_{\text{GF}}$
  - Wilson Flow + Clover observable
  - Zeuthen Flow + Improved observable
- Measurements for $\Sigma_2(u, a/L)$ at $L/a = 8, 10, 12, 16, 24$. Factor 3 in a.
- Measurements for $\Sigma_{3/2}(u, a/L)$ at $L/a = 8, 16, 32$. Factor 4 in a.
- Topology freezing overcome by using measurements at fixed topology

$$\bar{g}^2_{\text{m,e}}(\mu) = t^2 \hat{\mathcal{N}}^{-1}_{\text{e,m}}(c, a/L) \frac{\langle E_{\text{m,e}}(t,x)\hat{\delta}_Q\rangle}{\langle \hat{\delta}_Q\rangle}\Big|_{\mu=1/\sqrt{8t},\ \sqrt{8t}=cL,\ x_0=T/2} \qquad (c = 0.3)\,.$$

with topology mesured on the flowed configuration

$$Q = -\frac{1}{16\pi^2}\sum_x \epsilon_{\mu\nu\rho\sigma}\,\text{tr}\{G^{\text{cl}}_{\mu\nu}(t,x)G^{\text{cl}}_{\rho\sigma}(t,x)\}\,, \qquad \hat{\delta}_Q = \begin{cases} 1\,, & \text{if } |Q| < 0.5, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Determination of the β-function at high energies

- Basic relation: Raw measurements for $\Sigma_s(u, a/L)$, $(s = 3/2, 2)$ obey

$$\log s = \int_{\sqrt{u}}^{\sqrt{\Sigma_s(u,a/L)}} \frac{\mathrm{d}x}{\beta_{\mathrm{GF}}(x)} + \text{cutoff effects}(s)$$

- Parametrize β-function (b_0, b_1, b_2 known from PT)

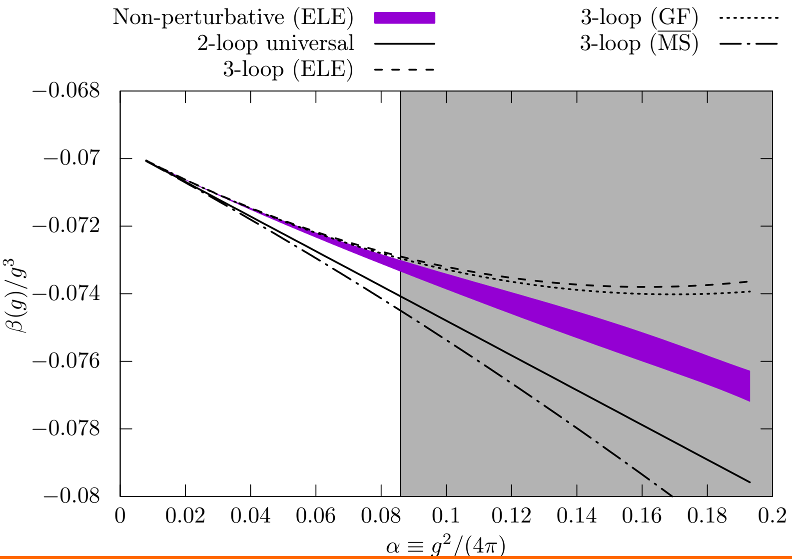
$$\beta_{\mathrm{GF}}(x) = -x^3 \left(b_0 + b_1 x^2 + b_2 x^4 + \sum_{k=3}^{n_b} p_k x^{2k} \right)$$

- Fit data to model (uncorrelated fit, but errors include correlations)

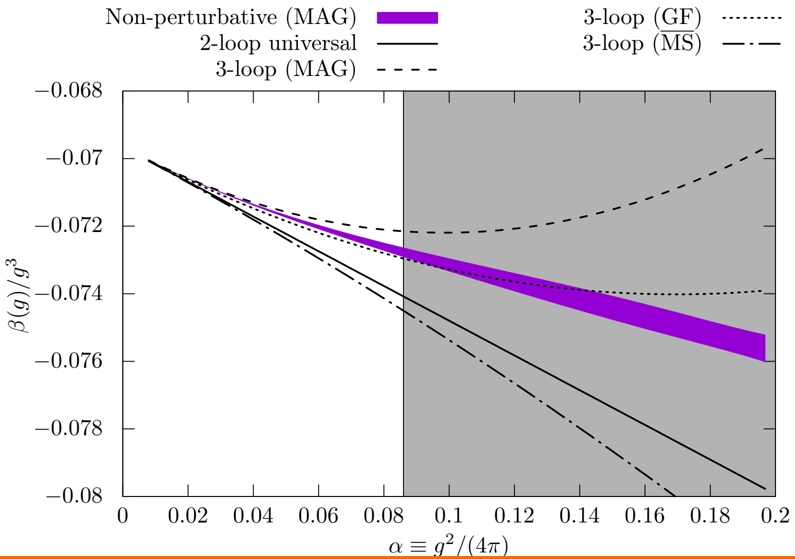
$$\chi^2 = \sum_{i \in \text{data}} \left[\frac{\log(s) + \rho^{(s)}(u_i)(a/L)^2 - F_i^{(s)}}{\delta F_i^{(s)}} \right]^2, \quad \rho^{(s)}(u) = \sum_{k=0}^{n_c} \rho_k^{(s)} u^k,$$

- Use $n_b = 4, 5$. Results insensitive to $n_c \geq 2$ (up to $n_c = 10$).
- Good fits ($\chi^2/\text{dof} \sim 0.5 - 0.9$). Compared with many other analysis.

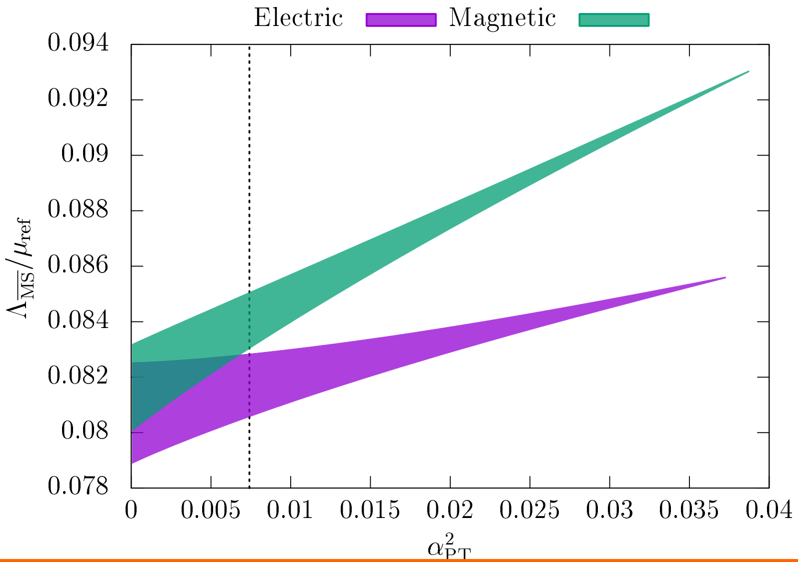
Determination of the β-function at high energies: Electric scheme



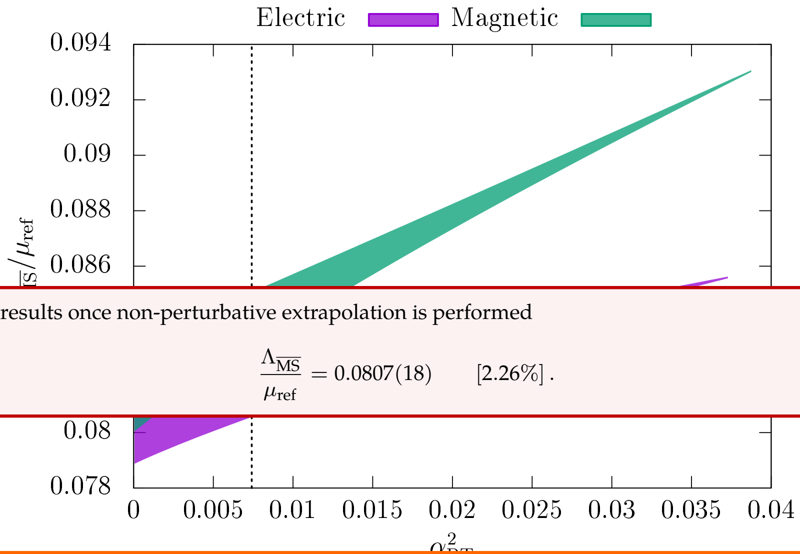
Determination of the β-function at high energies: Magnetic scheme



Approach $\alpha_{PT} \to 0$: 6% (13%) deviation at $\alpha_{PT} = 0.2$



Approach $\alpha_{PT} \to 0$: 6% (13%) deviation at $\alpha_{PT} = 0.2$



Consistent results once non-perturbative extrapolation is performed

$$\frac{\Lambda_{\overline{MS}}}{\mu_{ref}} = 0.0807(18) \qquad [2.26\%] \, .$$

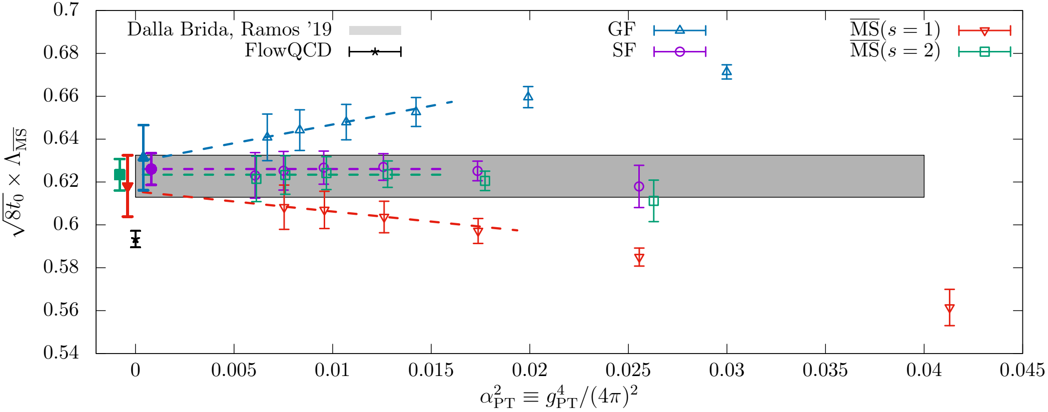
OBSERVATIONS

- 3-loop coefficients [M. Dalla Brida, M. Lüscher '17]:

$$(4\pi)^3 b_2^{\overline{\rm MS}} \approx 0.72\,, \quad (4\pi)^3 b_2^{\rm GF,e} \approx -2.0\,, \quad (4\pi)^3 b_2^{\rm GF,m} \approx -3.2\,.$$

- Electric: Contact with PT only at very high energies
- Magnetic: Significant deviation from PT even at $\bar{g}^2_{\rm GF} = 1$
- 4-loop coefficient probably large, and positive (alternating series?).
- Without finite size scaling, these schemes seem useless to extract Λ with high precision.
- Results are consistent **ONLY if NP extrapolation is performed**: Even at $\alpha_{\rm PT} = 0.09$ significant deviation.
- Extrapolation has a cost in precision: 2.6% is **not** great.
- Solved matching non-perturbatively with SF scheme

Observations



An alternative approach to the determination of $\sigma_{GF}(u)$ [A. Nada, A. Ramos. arXiv:2007.12862]

- Remember $\mu = 1/\sqrt{8t}$:

$$\sigma(u) = \bar{g}_c^2(\mu/2)\Big|_{\bar{g}_c^2(\mu)=u}$$

Fixed bare parameters $(\beta, (am_0))$ and $L/a \to 2L/a$

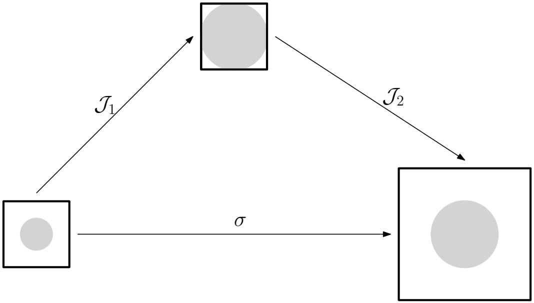
- Define functions

$$\mathcal{J}_1(u) = \bar{g}_{2c}^2(\mu/2)\Big|_{\bar{g}_c^2(\mu)=u},$$
$$\mathcal{J}_2(u) = \bar{g}_c^2(\mu/2)\Big|_{\bar{g}_{2c}^2(\mu/2)=u}.$$

- Mathematical identity

$$\sigma = \mathcal{J}_2 \circ \mathcal{J}_1 .$$

- Typical dataset
  $L/a = 8, 10, 12, 16, 20, 24, 32, 40, 48$
  - σ: require $L/a \to 2L/a$; $\sqrt{8t} \to 2\sqrt{8t}$
  - $\mathcal{J}_2$: require $L/a \to 2L/a$ fixed $\sqrt{8t}$
  - $\mathcal{J}_1$: requires $\sqrt{8t} \to 2\sqrt{8t}$ fixed L/a



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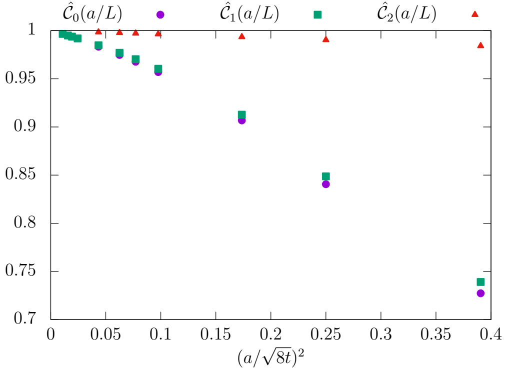


Comparison in PT:

$$\sigma \to \hat{C}_0(a/L); \mathcal{J}_1 \to \hat{C}_1(a/L); \mathcal{J}_2 \to \hat{C}_2(a/L);$$

New computation [Isabella Lione, AR; arXiv:2603.29841]

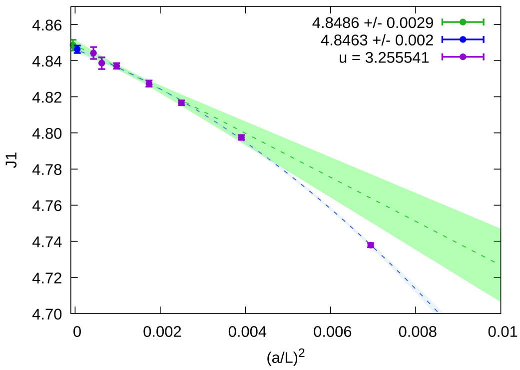


Figure: Continuum extrapolation $\mathcal{J}_1$

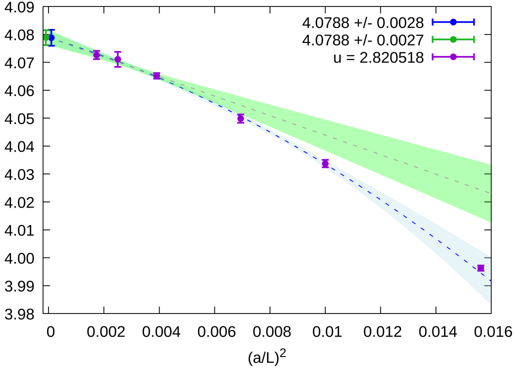


Figure: Continuum extrapolation σ

- Using Twisted boundary conditions [AR.; arXiv:1409.1445; E. Bribian et. al; arXiv:107.03747]
- $\delta = \frac{Q(a_{\min}) - Q(0)}{\sigma_Q}$; $\eta = [a_{\max}/a_{\min}]^2$

New computation [Isabella Lione, AR; arXiv:2603.29841]

Better control of systematic

Q	δ	η	N_{points}	$\Delta Q/Q$	
Σ	2.27	2.25	3	6.6e-04	u_t =2.8205
J_1	1.80	4.00	4	6.1e-04	u_t =3.2555
	1.33	5.76	5	8.6e-04	u_t =5.8401
J_2	0.35	4.00	3	4.8e-04	u_t =3.2555
	1.33	4.00	4	6.6e-04	u_t =5.8401

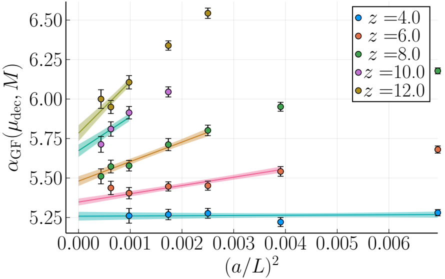
Table: The parameters $\eta = [a^2_{\text{max}}/a^2_{\text{min}}]$ (level arm) and δ (size) are used to quantify the systematics in the continuum extrapolation. In the FLAG review an extrapolation is considered good if $\eta > 2$ and $\delta < 3$. The data shows that the split combination shows smaller systematics. The fifth column displays the relative error of the extrapolated value.

- η, δ good to know for extrapolations.
- Can be used in continuous β function aproach ($a \to 0; L \to \infty$).

Filling the table

$M/\mu_{\rm dec}$	$\bar{g}^2_{\rm GFT}(\mu_{\rm dec}, M)$	$[\bar{g}^{(0)}_{\rm GF}(\mu_{\rm dec})]^2$	$\rho^{\rm eff}$	$\Lambda^{\rm eff}_{\overline{\rm MS}}$ [MeV]
4	5.258(28)	4.048(18)	0.4843(98)	388.4(10.0)
6	5.347(22)	4.103(15)	0.4523(88)	362.7(9.2)
8	5.479(31)	4.184(19)	0.4390(89)	352.1(9.1)
10	5.669(40)	4.299(24)	0.4382(92)	351.4(9.3)
12	5.780(51)	4.364(31)	0.4319(96)	346.4(9.5)
$M/\mu_{\rm dec}$	$\hat{\Gamma}_{\rm M}$	cut in z	ρ	$\Lambda_{\overline{\rm MS}}$ [MeV]
∞	0	$z \geq 4$	0.4264(91)	341.7(9.2)
∞	1	$z \geq 4$	0.4323(89)	346.3(9.1)
∞	0	$z \geq 6$	0.4264(97)	341.7(9.6)
∞	1	$z \geq 6$	0.4286(95)	343.5(9.5)

The continuum extrapolation of massive couplings



Coupling changes with M [ALPHA '23]

- Crucial determination NP b_g [ALPHA '24]
- Still: Error in $\bar{g}^2(\mu, M)$ **subdominant**

Effective field theory description of cutoff effects+mass dependence

$$\bar{g}^2(z_i, a) = C_i + p_1[\alpha_(a^{-1})]^{\hat{\Gamma}}(a\mu_{\mathrm{dec}})^2 + p_2[\alpha_(a^{-1})]^{\hat{\Gamma}'}(aM_i)^2\,.$$

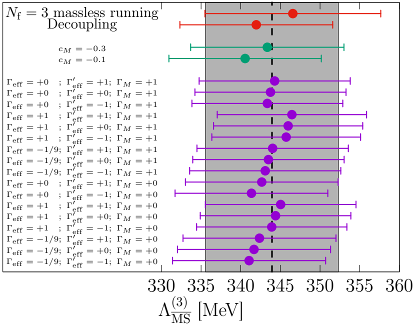
- Continuum values (our target quantity)
- Mass independent cutoff effects
- Mass dependent cutoff effects
- Loop corrections in effective theory: $-1 \le \hat{\Gamma} \le 1$ and $-1/9 \le \Gamma' \le 1$

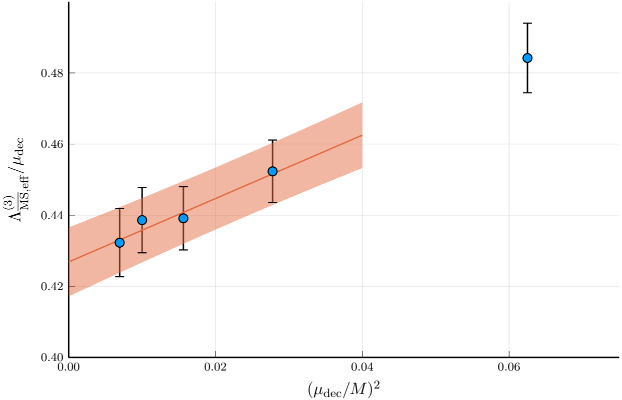
Λ coupling from decoupling

- Precise result for α_s

$$\Lambda^{(3)}_{\overline{\mathrm{MS}}} = 342(10)\ \mathrm{MeV}\,.$$

- Different models to extrapolate to continuum
- Different modesl to extrapolate $M \to \infty$
- Use of **already existing** pure gauge results





Final results

Final result for α_s

- Two strategies, consistent results

$$\text{Massless running:} \quad \Lambda^{(3)}_{\overline{\text{MS}}} = 347(11)\ \text{MeV}\,,$$
$$\text{Decoupling:} \quad \Lambda^{(3)}_{\overline{\text{MS}}} = 342(10)\ \text{MeV}\,.$$

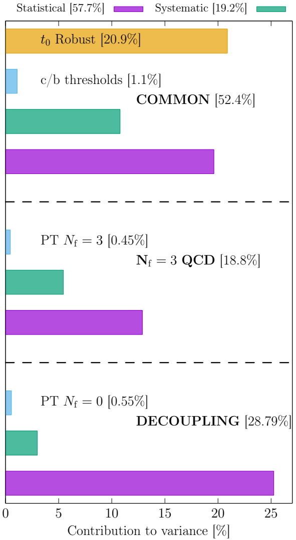
- Average

$$\Lambda^{(3)}_{\overline{\text{MS}}} = 344.4(8.7)\ \text{MeV}\,.$$

- "Crossing" c/b thresholds

$$\alpha_s(m_Z) = 0.11876(58) \quad [0.46\%]\,.$$

- Two times more precise than all pheno determinations combined!
- PT errors negligible
- Error dominated by statistics



From $\Lambda^{(3)}_{\overline{\mathrm{MS}}}$ to $\Lambda^{(5)}_{\overline{\mathrm{MS}}}$

- Exloration in PT
- Non-perturbative study comparing $N_f = 0$ and $N_f = 2$

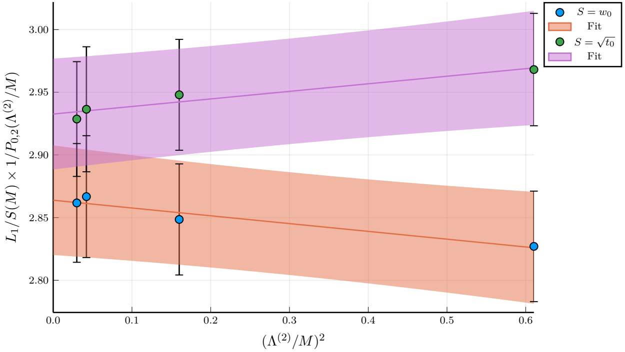
loop-orders	$P^{\mathrm{ref}}_{3,4}$	$P^{\mathrm{ref}}_{4,5}$	$P^{\mathrm{ref}}_{3,5}$	$\alpha_s^{\mathrm{ref}}(m_Z)$
5/4	0.87548	0.72143	0.63160	0.11872
loop-orders	$100 \times \delta P_{3,4}$	$100 \times \delta P_{4,5}$	$100 \times \delta P_{3,5}$	$10^5 \times \Delta\alpha_s(m_Z)$
4/3	−0.2536	0.2056	−0.3313	−5.992
3/2	−0.8503	0.5758	−1.2237	−22.20
2/1	−3.8555	1.2228	−6.8235	−126.3
SI,$m^\star$	0.0	0.0		
SI,$2m^\star$	−0.4364	−0.0702		
SI,$m^\star/2$		−0.0117		
$\overline{\mathrm{MS}}$,$\mu = \mu_h$	−0.0299	−0.0014		
$\overline{\mathrm{MS}}$,$\mu = 2\mu_h$	−0.1036	−0.0105		
$\overline{\mathrm{MS}}$,$\mu = \mu_h/2$	0.0016	0.0119		

Small corrections

- PT corrections 0.3%
- NP corrections 0.1%

From $\Lambda_{\overline{MS}}^{(3)}$ to $\Lambda_{\overline{MS}}^{(5)}$

- Exloration in PT
- Non-perturbative study comparing $N_f = 0$ and $N_f = 2$



Small corrections

- PT corrections 0.3%
- NP corrections 0.1%

CONCLUSIONS

- Determination of $\alpha_s, m_q, B_K, etc \ldots$ better understood as
  - Determination of an RGI
  - "Non-perturbative" extrapolation gives concise meaning to the target computation
- Poor convergence of GF scheme for Λ determination
  - Pure gauge: even at $\alpha \approx 0.1$, effect in Λ of 1%
  - Extrapolating $g \to 0$ has a price (larger uncertainty)
- Decoupling strategy
  - Exact relation

$$\Lambda^{(3)} = \lim_{M\to\infty} \mu(M) \times \frac{\Lambda^{(0)}}{\mu} \times \frac{1}{P(\Lambda/M)}$$

  - Running (expensive) can be done in pure gauge
- Final result, with combined strategies, statistical errors

$$\alpha_{\overline{\mathrm{MS}}}(M_Z) = 0.11876(45)_{\mathrm{stat}}(25)_{\mathrm{sys}}(27)_{\mathrm{robust}}(58)_{\mathrm{tot}} \qquad [0.47\%]\,.$$

Scale setting: The case with significant systematics

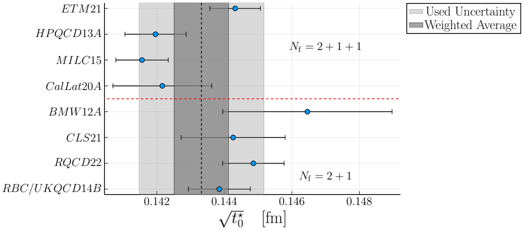
Technical scale $t_0^\star$

- $t_0^{\star 2}\langle E(t_0^\star)\rangle = 0.3$ at $m_\pi = m_K = 420\,\mathrm{MeV}$
- Input from different collaborations:

$$\sqrt{t_0^\star} \times (M_\pi, M_K, M_\Xi, M_\Omega, f_\pi, f_K, \dots)$$

- Use all results entering FLAG average
- $\chi^2/\mathrm{dof} = 2.8$
- **Robust** error: all "precise" results covered

$$\sqrt{t_0^\star} = 0.1433(7)_{\mathrm{stat}}(4)_{\mathrm{sys}}(17)_{\mathbf{robust}}(19)_{\mathrm{tot}}\ \mathrm{fm}\,.$$



- Only case of significant systematic in our work
- **Robust** error band covers all (precise) central values
- Effect propagated in all quantities.
- Our error 2.5× larger than "standard" (i.e. FLAG/PDG) error inflation
- Small effect in final error of α_s: $58 \times 10^{-4} \to 51 \times 10^{-4}$