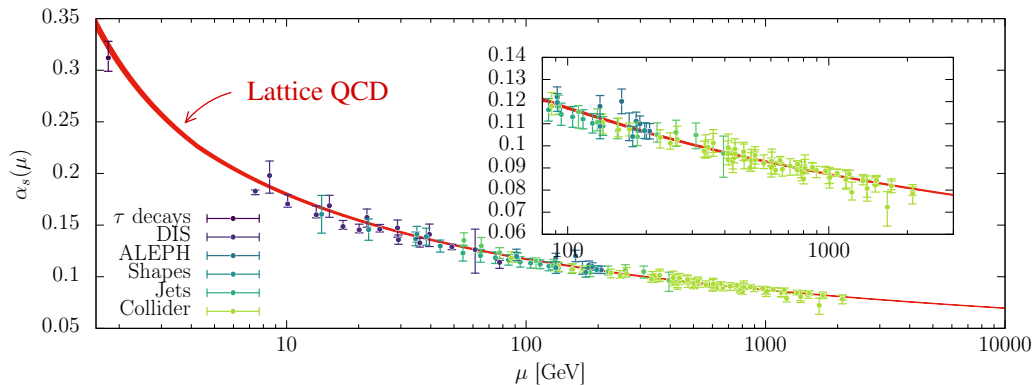


# Demystifying lattice QCD computations of $\alpha_s$

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$$\underbrace{\frac{1}{3}(2f_K + f_\pi), M_\pi, M_K, M_D, M_B}_{\text{physics input}} \longrightarrow \alpha_s(M_Z) = 0.11876(58)$$

## Outline

- \* Perturbative QCD  $\subset$  Lattice QCD
- \* Asymptotic freedom & the continuum limit
- \* If we had infinite computing power . . .
- \* A strange world: the femto-universe
- \* Stepping up the energy scale
- \* Heavy quark decoupling revisited

## Lattice QCD

= QCD at imaginary time with a UV regularisation

Lattice action = a discretised version of

$$S = \int d^4x \left\{ \frac{1}{4g_0^2} F_{\mu\nu}^a F_{\mu\nu}^a + \sum_{q=u,d,s,\dots} \bar{q} (\not{D} + m_{q,0}) q \right\}$$

### Parameters

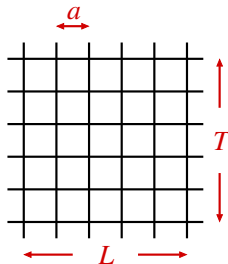
$g_0$ : bare (i.e. unrenormalised) coupling

$m_{q,0}$ : bare quark masses

$a$ : lattice spacing

$L$ : spatial size of the lattice

$T$ : time extent of the lattice



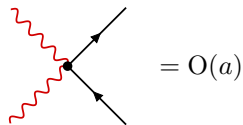
## Perturbative QCD $\subset$ Lattice QCD

### Lattice perturbation theory

- Momenta are restricted to the Brillouin zone

$$|p_\mu| \leq \pi/a$$

- Propagators & vertices are not rational functions
- Vertices with any number of gluon legs
- Gauge fixing as usual, exact BRS symmetry



PhD of Thomas Reisz (1987/88):

*Power-counting theorem for lattice Feynman diagrams*

*Proof of the renormalisability of lattice QCD*

Renormalisation proceeds as usual

$$g^2 = g_0^2 Z, \quad Z = 1 + \{z_{1,0} + z_{1,1} \ln(a\mu)\} g_0^2 + \dots$$

Minimal subtraction:  $z_{1,0} = z_{2,0} = \dots = 0$

$\Rightarrow$  *Continuum limit exists and coincides with standard perturbative QCD up to a finite renormalisation of the parameters and fields*

$$\text{In particular: } \alpha \equiv \frac{g^2}{4\pi} = \alpha_s + c_1 \alpha_s^2 + \dots$$

↑  
computable

## Asymptotic freedom & the continuum limit

At fixed  $g$  and  $\mu$ ,

$$g_0^2 = g^2 - \{z_{1,0} + z_{1,1} \ln(a\mu)\}g^4 + \dots$$

depends on the lattice spacing

The RGE then leads to the exact asymptotic expression

$$g_0^2 \underset{a\mu \rightarrow 0}{=} -\frac{1}{2b_0 \ln(a\Lambda)} + \dots$$

$$\Lambda = \mu\phi(g) = k\Lambda_{\overline{\text{MS}}}$$

↑  
exactly calculable

⇒ *Provides a link between the lattice theory and collider physics*

## If we had infinite computing power ...

### How meson masses are computed in LQCD

- Choose  $g_0$ , the quark masses  $am_{q,0}$  and a large lattice
- Introduce interpolating fields for the mesons, e.g.

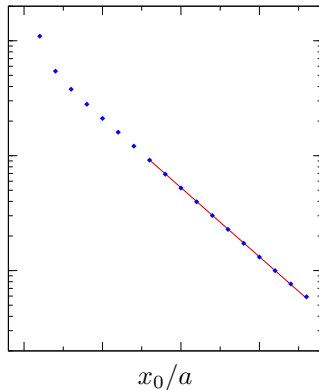
$$\pi^+(x) = \bar{d}(x)\gamma_5 u(x), \quad \pi^-(x) = \bar{u}(x)\gamma_5 d(x)$$

- Compute

$$a^3 \sum_{\mathbf{x}} \langle (\pi^+(x)\pi^-(0)) \rangle_{x_0 \rightarrow \infty} \propto e^{-M_\pi x_0}$$

using numerical simulations

$\Rightarrow aM_\pi, aM_K, \dots$  at the chosen values of  $g_0$  and  $am_{q,0}$   
( $af_\pi$  and  $af_K$  are obtained similarly)



## What is the lattice spacing in fm?

Fix  $g_0$  and tune the quark masses  $am_{0,u}, am_{0,d}, \dots$  such that

$$M_\pi/f_\pi, M_K/f_\pi, \dots$$

assume their physical values. Then

$$a = \frac{af_\pi}{f_\pi} = af_\pi \times 1.513(2) \text{ fm}$$

## Computation of $\alpha_s$

Recall

$$g_0^2 \underset{a\mu \rightarrow 0}{=} -\frac{1}{2b_0 \ln(a\Lambda)} + \dots \quad \text{and thus } a \ll 1 \text{ fm if } g_0 \ll 1$$

$\Rightarrow$  May extract  $\Lambda$  (in MeV) at these couplings

$$\Rightarrow \Lambda_{\overline{\text{MS}}} \Rightarrow \alpha_s(M_Z) = \frac{1}{8\pi b_0 \ln(M_Z/\Lambda_{\overline{\text{MS}}})} + \dots$$

## The “window problem”

Should have

- ▶  $L \geq 6 \text{ fm}$  to suppress finite-volume effects
  - ▶  $1/a = O(100) \text{ GeV}$  for  $g_0$  to be sufficiently small
- }  $L/a = O(3000)$

In practise

$$L/a \leq 128 \dots 256 \Rightarrow 1/a \leq 4.2 \dots 8.4 \text{ GeV} \text{ @ } L = 6 \text{ fm}$$

$$\text{Computational effort} \propto 1/a^{6\dots 7}$$

## A strange world: the femto-universe

Consider QCD in an  $L^4$  box with  $L \ll 1$  fm

- \* Perturbative regime at scale  $\mu = 1/L$
- \* No hadrons — a world of quarks and gluons



James D. Bjorken

*QCD in the box renormalises in the same way as in infinite volume with  $L$ -independent renormalisation constants*

Holds for various boundary conditions

- Periodic in space & time
- Periodic in space, Dirichlet in time

## QCD on femto-lattices

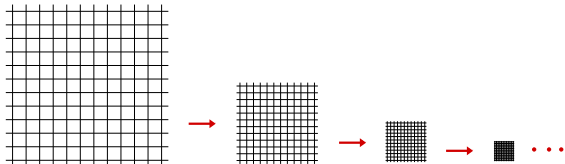
Can be simulated straightforwardly

⇒ At sufficiently small  $L$  and  $a \ll L$  given in fm

$$g_0^2 = -\frac{1}{2b_0 \ln(a\Lambda)} + \dots \rightarrow \Lambda_{\overline{\text{MS}}} [\text{MeV}] \rightarrow \alpha_s(M_Z)$$

How to connect QCD at large  $L$  to the femto-universe

- Start from a lattice with large  $L$
- Set  $m_q = 0$  at fixed  $g_0$  (i.e. fixed  $a$ )
- Use “step scaling” to go from  $L, a \rightarrow L/2, a/2 \rightarrow \dots$



## Step scaling

Lüscher, Weisz & Wolff, Nucl. Phys. B359 (1991) 221

Each cycle proceeds in two steps

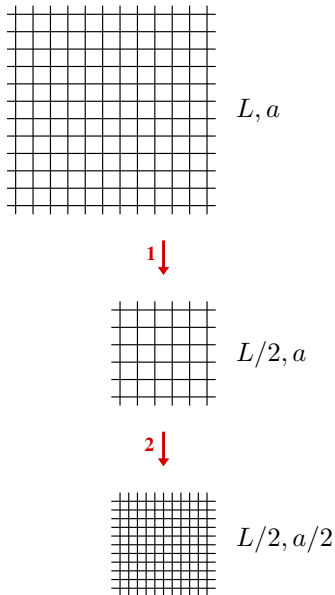
### 1. Size reduction

Reduce  $L \rightarrow L/2$  at fixed  $g_0$  and thus fixed  $a$  [fm]

### 2. Renormalisation step

Reduce  $a \rightarrow a/2$  and adjust  $g_0$  such that  
“the physics is unchanged”

$\Rightarrow$  box size  $L/2$  [fm] remains the same



## Couplings used in the renormalisation step

$\alpha_{\text{SF}}$ : Schrödinger-functional coupling

Lüscher, Narayanan, Weisz & Wolff (1992)

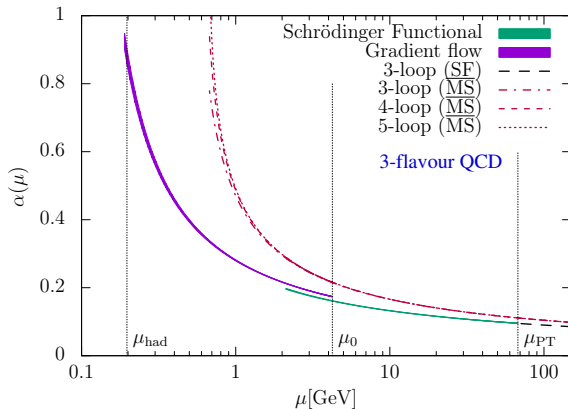
$\alpha_{\text{GF}}$ : based on the YM gradient flow

Fritzsch & Ramos (2013)

## Properties

- \* Renormalisation scale  $\mu = 1/L$
- \* Non-perturbatively defined
- \*  $\alpha_{\overline{\text{MS}}} = \alpha + k_1 \alpha^2 + \dots$

Bruno et al. (ALPHA Collab.), PRL 119 (2017) 102001



$$\Rightarrow \Lambda_{\overline{\text{MS}}}^{(3)} = 341(12) \text{ MeV}$$

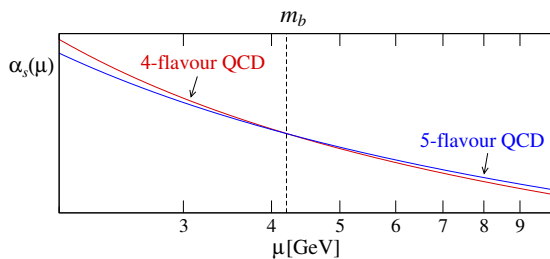
## How can the heavy quarks be included?

Another window problem . . .

$$m_c = 1.27 \text{ GeV}, \quad M_D = 1.87 \text{ GeV}, \quad c\bar{c} \text{ threshold} \sim 3 \text{ GeV}$$

$$m_b = 4.18 \text{ GeV}, \quad M_B = 5.28 \text{ GeV}, \quad b\bar{b} \text{ threshold} \sim 9 \text{ GeV}$$

$\alpha_s$  depends on the number of quark flavours



$$\rightarrow \alpha_s^{(4)}(M_Z) < \alpha_s^{(5)}(M_Z)$$

## Heavy quark decoupling

Appelquist & Carazzone (1975)

At energies  $E \ll m_b$

$$\text{QCD}^{(5)} = \text{QCD}^{(4)} + \mathcal{O}(E^2/m_b^2)$$

↑

with adjusted parameters

In particular,

$$g^2|_{4\text{-flavour QCD}} = g^2 + \{a_1 + b_1 \ln(m_b/\mu)\}g^4 + \dots$$

is known up to 4 loops

Bernreuther & Wetzel (1982)

⋮

Chetyrkin, Kühn & Sturm (2005)

Schröder & Steinhauser (2005)



$$\frac{\Lambda_{\overline{\text{MS}}}^{(4)}}{\Lambda_{\overline{\text{MS}}}^{(5)}} = \begin{array}{cccc} & \text{2-loop} & \text{3-loop} & \text{4-loop} \\ = & 1.4255 & -0.0308 & -0.0029 & -0.0005 \end{array}$$

$$\frac{\Lambda_{\overline{\text{MS}}}^{(3)}}{\Lambda_{\overline{\text{MS}}}^{(4)}} = 1.1875 - 0.0323 - 0.0060 - 0.0024$$

## Non-perturbative and $E^2/m_c^2$ effects on $\Lambda_{\overline{\text{MS}}}^{(4)}$

... were shown to be on the level of a fraction of the error of  $\Lambda_{\overline{\text{MS}}}^{(3)}$

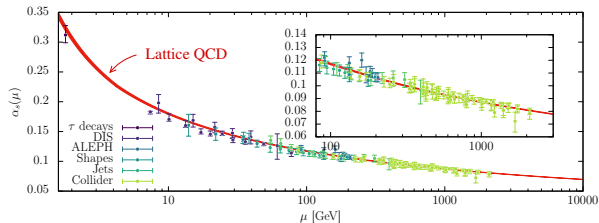
Bruno et al. (ALPHA Collab.), Phys. Rev. Lett. 114 (2015) 102001

Athenodorou et al. (ALPHA Collab.), Nucl. Phys. B943 (2019) 114612

→ talk by Alberto Ramos later today

## Concluding remarks

$$\underbrace{\frac{1}{3}(2f_K + f_\pi), M_\pi, M_K, M_D, M_B}_{\downarrow}$$
$$\alpha_s^{(5)}(M_Z) = 0.11876(58)$$



- ★ *Computation links low-energy hadron to collider physics*
- ★ *A beautiful piece of evidence that QCD is the theory of the strong interactions at all energies*

*Backup slides*

## Renormalisation in finite volume

*The singularities of propagators on an  $L^4$  box with periodic boundary conditions do not depend on  $L$*

In infinite volume

$$G(x, y; \infty) \underset{x \rightarrow y}{\propto} \frac{1}{(x - y)^2} + \dots$$

but

$$(-\square + m^2) \{G(x, y; L) - G(x, y; \infty)\} = 0 \Rightarrow \text{difference is regular at } x = y$$

Example: massive  $\phi^4$  theory

$$\text{Diagram} = G(x, y; L)^2 = G(x, y; \infty)^2 + \text{integrable}$$

Kopper, Müller & Reisz, *Annals Henri Poincaré* 2 (2001) 387 [ $\phi^4$  to all orders]

Bode, Weisz & Wolff, *Nucl. Phys. B* 576 (2000) 517 [QCD with SF bc up to 2 loops]

## Definition of $\alpha_{\text{SF}}$

Lüscher, Narayanan, Weisz & Wolff, Nucl. Phys. B384 (1992) 168

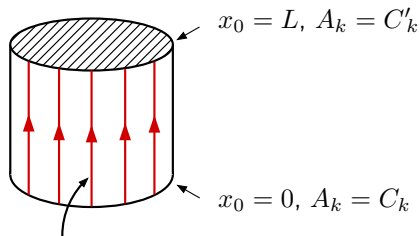
Consider an  $L^4$  cylindrical space-time

Free energy of the induced field

$$-\ln \mathcal{Z} = \underbrace{\frac{1}{g_0^2} \Gamma_0[E]}_{\text{classical action}} + \Gamma_1[E] + g_0^2 \Gamma_2[E] + \dots$$

SF coupling at renormalisation scale  $\mu = 1/L$

$$g_{\text{SF}}^2 = -\frac{\partial \Gamma_0}{\partial E} \left\{ \frac{\partial \ln \mathcal{Z}}{\partial E} \right\}^{-1}$$
$$= g_{\overline{\text{MS}}}^2 + \dots$$



induced chromo-electric  
field  $E = (C' - C)/L$

## Definition of $\alpha_{\text{GF}}$

Fritzsche & Ramos, JHEP 10 (2013) 008

Consider an  $L^4$  cylindrical space-time

Yang–Mills gradient flow

$$\partial_t B_\mu = D_\nu G_{\nu\mu}, \quad B_\mu|_{t=0} = A_\mu$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$$

GF coupling at renormalisation scale  $\mu = 1/L$

$$\alpha_{\text{GF}} = \text{constant} \times \left\{ t^2 \langle G_{kl}^a G_{kl}^a \rangle \right\}_{x_0=L/2, \sqrt{8t}=0.3 \times L}$$

$$= \alpha_{\overline{\text{MS}}} + \dots$$

