ERIK BÄSKE · LATTICE FIELD THEORY GROUP · HUMBOLDT-UNIVERSITÄT ZU BERLIN

Isospin-breaking corrections to the Tau decay-rate

Including QED effects and quark mass splittings into the calculation of the inclusive decay-rate of the τ into hadrons

Standard model parameters and observables from gradient flow
James Clerk Maxwell Building, 12-15th May 2026

§1 · SECTION ONE

Motivation.

Why care about $\Gamma(\tau \to X\nu_\tau)$?

High precision tests of the Standard Model

THE CKM-MATRIX

The Cabibbo-Kobayashi-Maskawa matrix elements are free parameters of the SM. To compare with experimental values we need high precision lattice calculations.

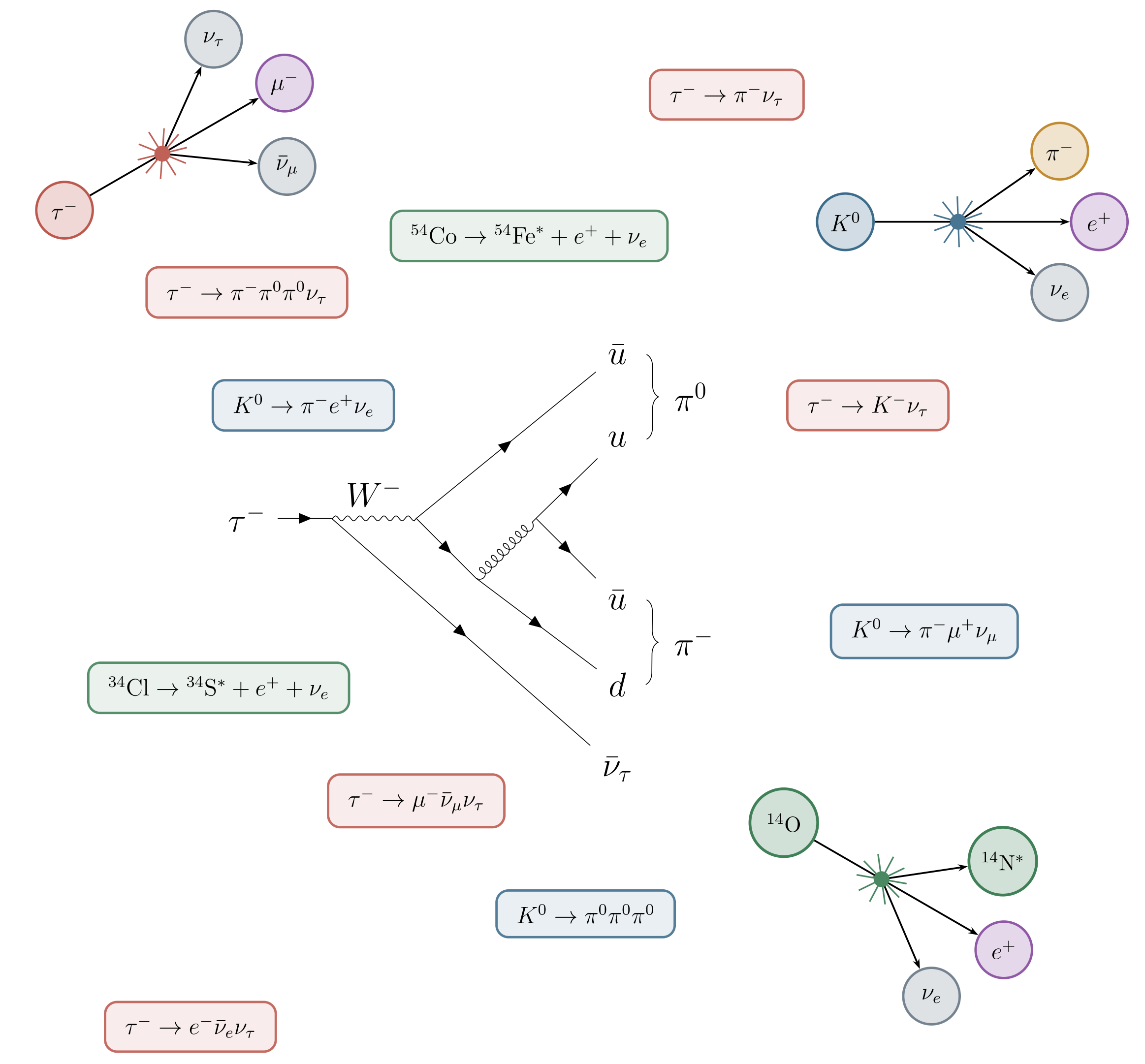
LAGRANGIAN

$$\mathcal{L}_{\text{SM}} \subset -\frac{g}{2}\left(\bar{u}_L\ \bar{c}_L\ \bar{t}_L\right)\gamma_\mu W_\mu^+ V_{\text{CKM}}\begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} \qquad V_{\text{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

Processes that depend on the CKM-matrix

- Tau decay $\tau \to X\nu$
- Semi-leptonic decays $K_L^0 \to \pi e \nu$ or $D \to \pi \ell \nu$ etc.
- β-decay
- ...

CKM-ZOO



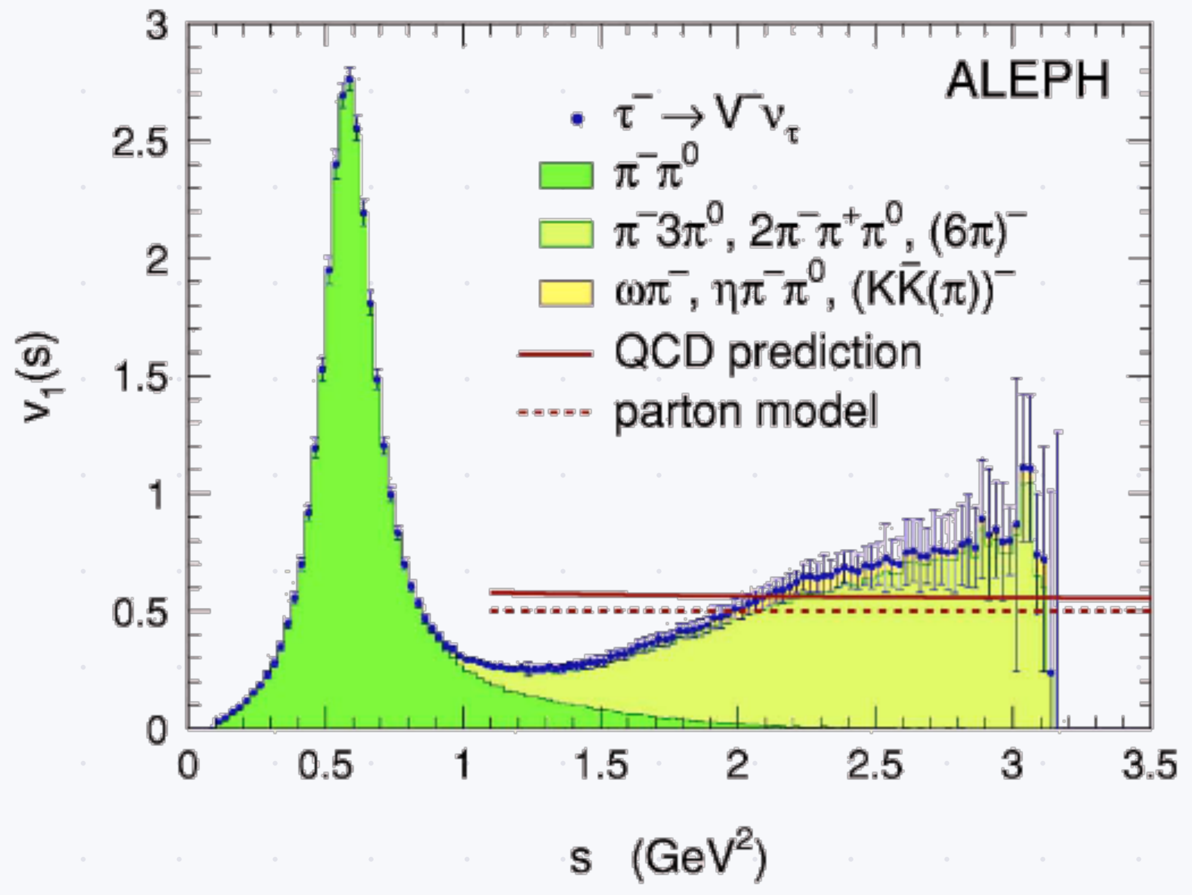
LOTS OF PROCESSES...

Which one do we choose to investigate?

[M. Davier et al., Rev. Mod. Phys. 78, 1043], [FLAG, Phys.Rev.D 113 (2026)]

Why the τ decay-rate is a prime candidate

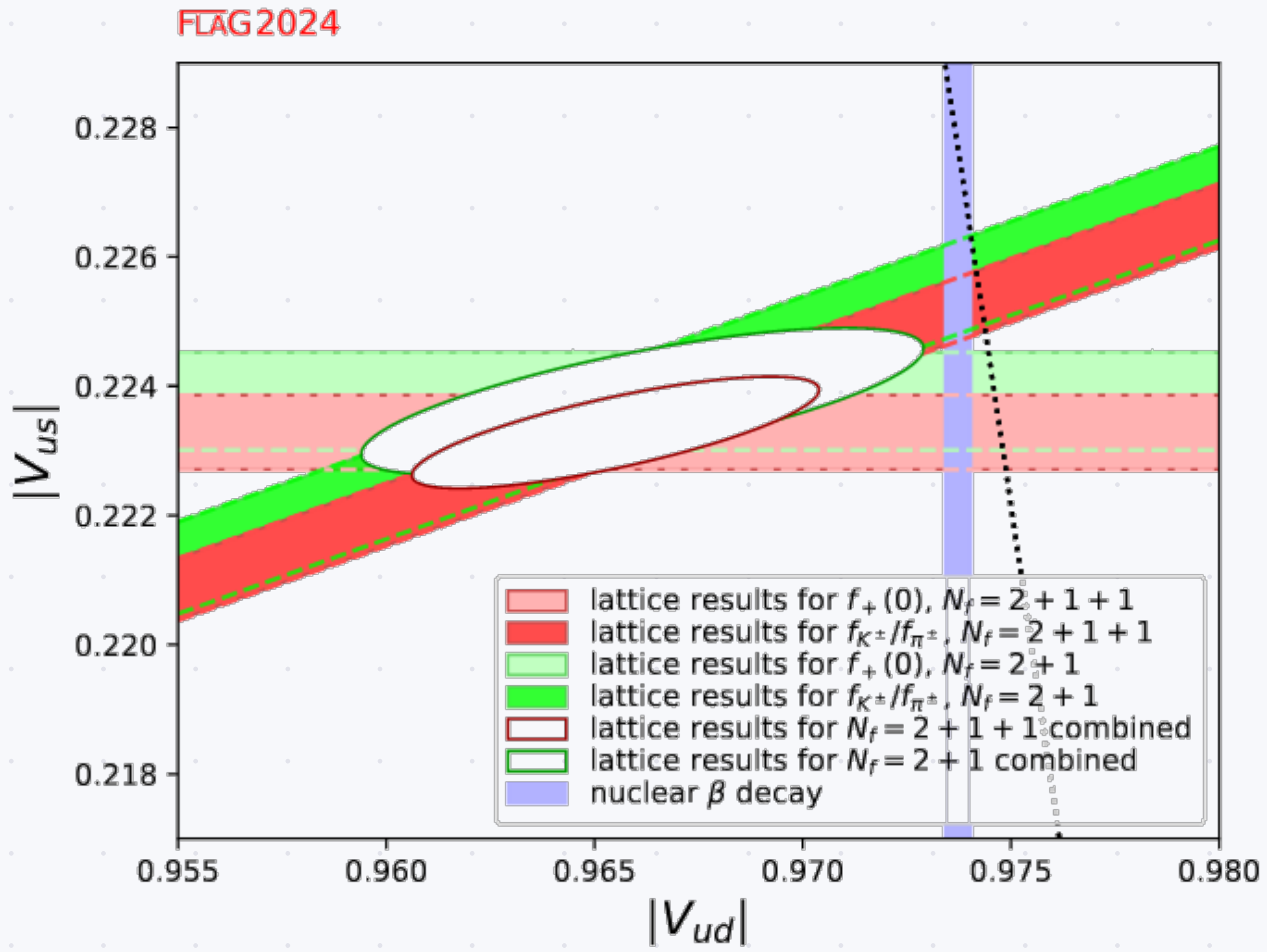
- only (known) lepton that can decay into hadrons
  $m_\tau \approx 1.7\,\mathrm{GeV}$
- intermediate scale $\Lambda_{\mathrm{QCD}} \ll m_\tau \ll m_W$ to test QCD running
- enhanced BSM physics sensitivity due to $\mathcal{O}(m_\tau^2/m_{\mathrm{NP}}^2)$
- tests lepton universality
- closely related to $(g-2)_\mu$ through the HVP
- clean initial state
- probes low energy QCD spectrum

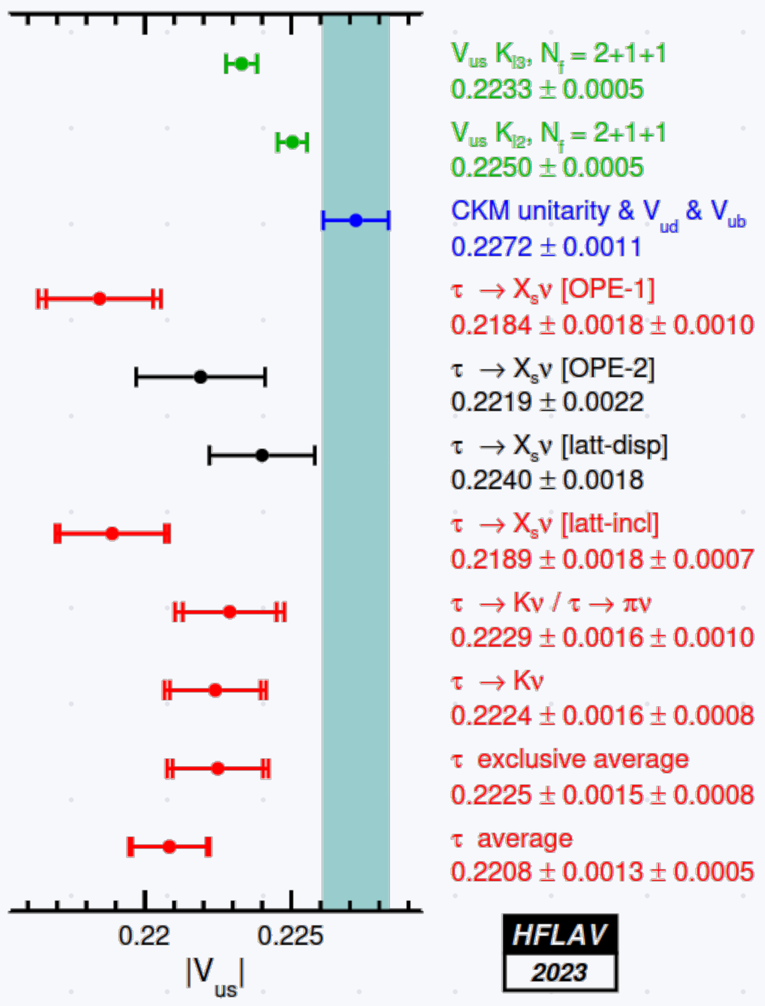


QCD RESONANCES

in the spin $J = 1$ vector channel

Current status of lattice determinations





DISCREPANCIES

Lattice results disagree ($2.8\,\sigma$) with experimentally measured β-decays and CKM-unitarity.

DISCREPANCIES

Lattice results for semi-leptonic Kaon decays and inclusive τ decays disagree ($2-3\sigma$)

Outline

§2 · SECTION TWO

The decay-rate

How to calculate a real time quantity like the decay rate from lattice data obtained in the Euclidean signature?

Old fashioned perturbation theory

TEXTBOOK DERIVATION IN THE MINKOWSKIAN

We substitute the Fermi-theory for the weak-interaction

$$H_{\mathrm{eff}} \propto \int \mathrm{d}^3x\, \mathcal{H}(\underline{x}) = \int \mathrm{d}^3x\, \bar{u}_L \gamma_\mu d_L\, \bar{\tau}_L \gamma_\mu \nu_L$$

and proceed as usual

DECAY-RATE

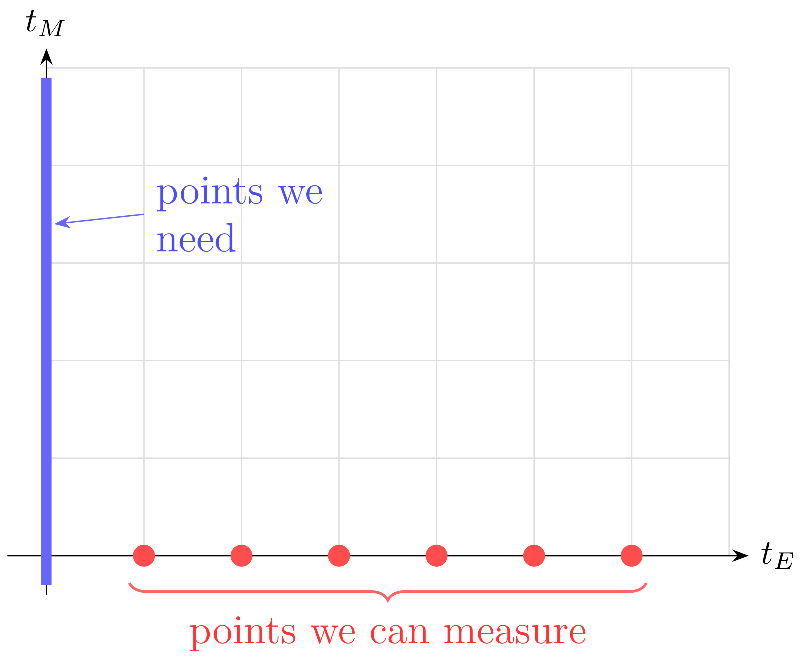
$$\Gamma(\tau \to X\nu) = \lim_{T\to\infty} \frac{1}{T} \sum_X \frac{|\langle X\nu|\tau\rangle|^2}{\langle X\nu|X\nu\rangle\langle\tau|\tau\rangle} = \frac{1}{2m_\tau}\langle\tau|\mathcal{H}(0)\,(2\pi)^4\delta^4(\hat{P}-p_i)\,\mathcal{H}(0)|\tau\rangle\left(1+\mathcal{O}(G_F^2)\right)$$

Transition to the Euclidean

Working backwards in Euclidean signature gives

$$\Gamma = \frac{1}{2m_\tau} \int \mathrm{d}^4x \, \langle \tau | \mathcal{H}(0) e^{-i(\hat{H}-m_\tau)x_0 + i\underline{\hat{P}x}} \, \mathcal{H}(0) | \tau \rangle$$
$$= \frac{1}{2m_\tau} \int \mathrm{d}^4x \, \langle \tau | \mathcal{H}(ix_0, \underline{x}) \mathcal{H}(0) | \tau \rangle$$

ANALYTIC CONTINUATION



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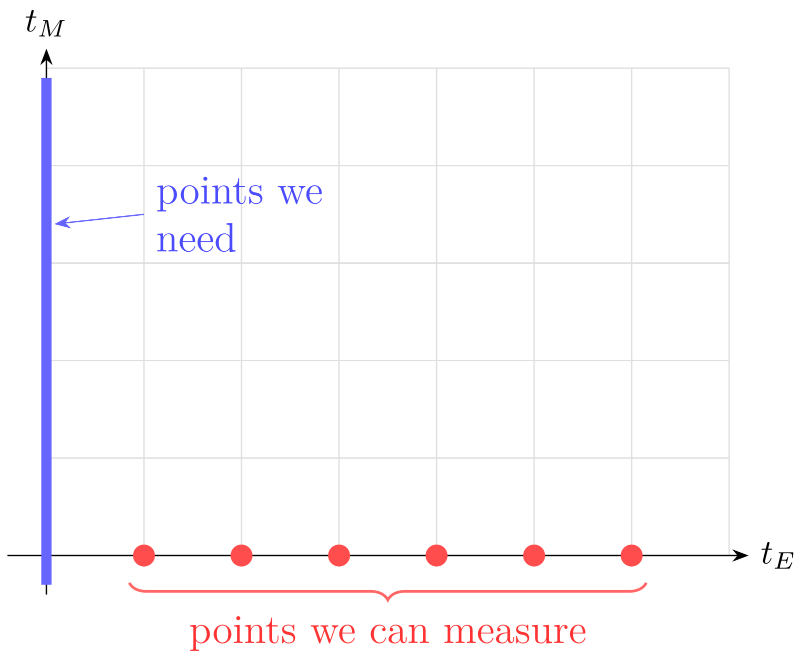


INVERSE LAPLACE TRANSFORM

What we can measure is

$$C(t) = \langle\tau|\mathcal{H}(t,\underline{0})\mathcal{H}(0)|\tau\rangle$$
$$= \int \frac{\mathrm{d}E}{2\pi} e^{-t(E-m_\tau)} \langle\tau|\mathcal{H}(0)\ (2\pi)\delta(\hat{H}-E)\ \mathcal{H}(0)|\tau\rangle$$
$$\propto \mathcal{L}[\Gamma](t)$$

but the inverse Laplace transformation $\Gamma \sim \mathcal{L}^{-1}[C]$ is *numerically ill-posed* ($\mathcal{L}^{-1}$ is not continuous)

Spectral reconstruction

We can turn the *ill-posed* problem into a merely *ill-conditioned* one by first smearing the δ-function

SMEARING

$$\delta(x) = \lim_{\sigma\to 0} \delta_\sigma(x) = \lim_{\sigma\to 0} \lim_{\varepsilon\to 0} \sum_{n=0}^{N(\varepsilon)} c_n(\sigma) e^{-nx}$$

Expanding $\delta(\hat{H} - m_\tau)$ in a basis of $e^{-an(\hat{H}-m_\tau)}$ gives the decay-rate as

$$\Gamma(\tau) = \lim_{\sigma\to 0} \lim_{\varepsilon\to 0} \sum_{n=0}^{N(\varepsilon)} c_n(\sigma)\, \frac{1}{2} \sum_r \int \mathrm{d}^3x\, \langle \tau^r(\underline{0}) | \mathcal{H}(an, \underline{x}) \mathcal{H}(0) | \tau^r(\underline{0}) \rangle$$

In some cases the smearing radius is physical, e.g., some finite detector resolution.

Computing the matrix element from lattice correlators

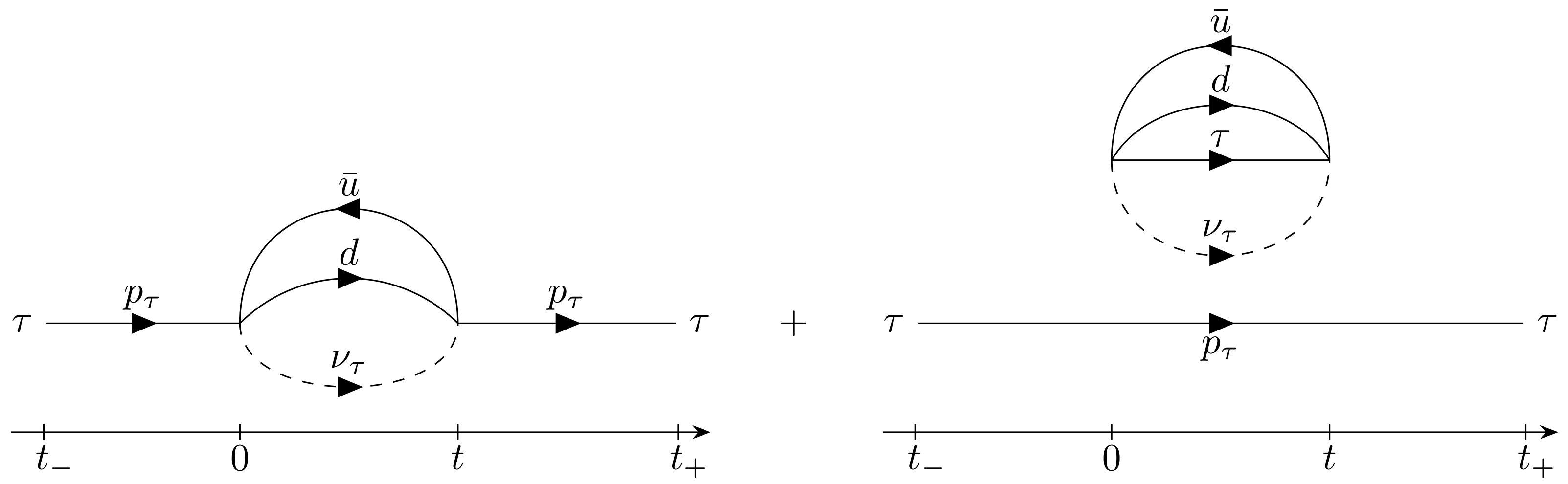
Our task is now to calculate the matrix element

$$\frac{1}{2}\sum_{r}\langle\tau^{r}(\underline{0})|\mathcal{H}(t,\underline{x})\mathcal{H}(0)|\tau^{r}(\underline{0})\rangle$$

Using the interpolating field of the τ, this becomes a 4-point function we can measure on the lattice

4-POINT FUNCTION

$$C_4(t_+,t,0,t_-) = \langle 0|\tau(t_+,\underline{p}=\underline{0})\ \mathcal{H}(t,\underline{x})\ \mathcal{H}(0)\ \bar{\tau}(t_-,\underline{p}=\underline{0})|0\rangle$$



§3 · SECTION THREE

Isospin-breaking

Non-degenerate quarks and QED

Strong and weak isospin-breaking

STRONG ISOSPIN-BREAKING

The mass differences $m_u \neq m_d$ of the quarks explicitly break the $\mathrm{SU}(2)$ flavour symmetry.

$$M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} = \frac{m_u + m_d}{2} 1 + \frac{m_u - m_d}{2} \sigma^3$$

Rough estimate: $\frac{\delta m}{\Lambda_{\mathrm{QCD}}} \approx \frac{4.7-2.2}{340} = \mathcal{O}(1\%)$

WEAK ISOSPIN-BREAKING

The different charges of the quarks $q_u \neq q_d$ imply that the $\mathrm{SU}(2)$ flavour symmetry is broken by QED-effects.

$$\bar{\psi} \left(\not{\partial} + e \begin{pmatrix} 2/3 & 0 \\ 0 & -1/3 \end{pmatrix} \not{A} \right) \psi$$

Rough estimate: $\alpha_{\mathrm{EM}} = \frac{1}{137} = \mathcal{O}(1\%)$

FLAVOUR SYMMETRIES

massless, 2-flavour QCD has an exact $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ symmetry

$$\bar{\psi} \not{D} \psi = \bar{\psi}_L \not{D} \psi_L + \bar{\psi}_R \not{D} \psi_R$$

isospin is then the diagonal subgroup $\mathrm{SU}(2)_V$

[RM123 Collaboration, Phys.Rev.D 87 (2013) 11, 114505], [M. Di Carlo et al., arXiv:2603.29016], [B. Lucini et al., JHEP 02 (2016) 076], [RC* Collaboration, Eur.Phys.J.C 80 (2020)]

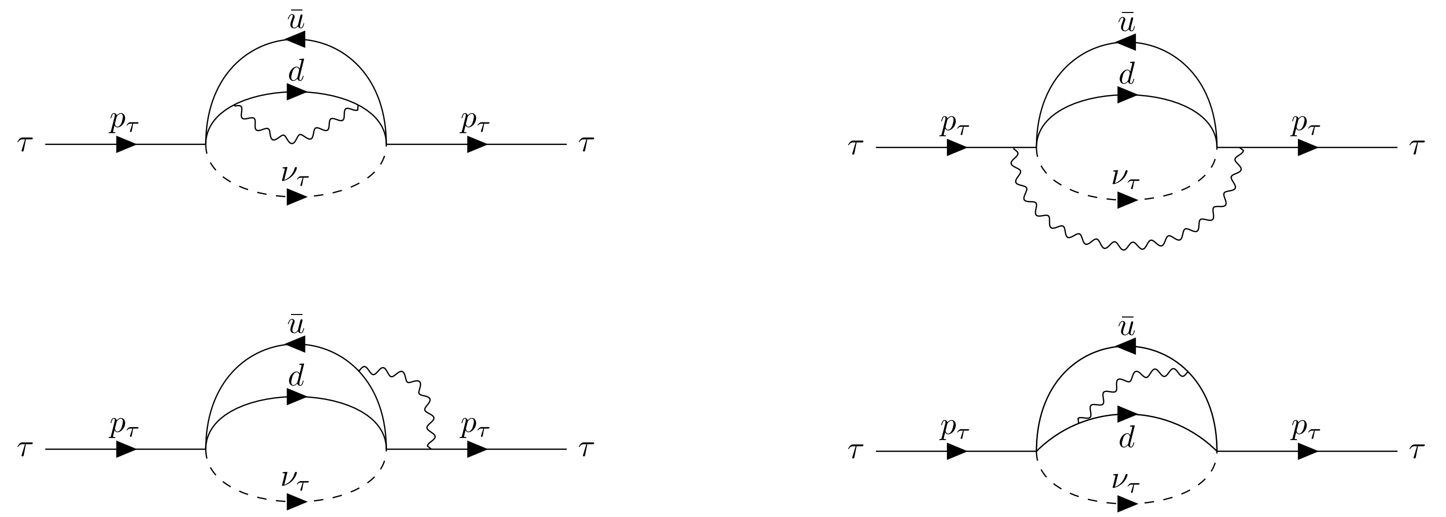
QED in a finite volume

PERTURBATIVE APPROACH

The RM123-method:

ISO-QCD EXPANSION

$$\langle O \rangle = \langle O \rangle_{\text{iso}} - \langle \Delta S \, O \rangle_{\text{iso}} + \frac{1}{2} \langle (\Delta S)^2 O \rangle_{\text{iso}} + \mathcal{O}(\alpha^2, \delta m^3)$$



pursued by our collaborators at the ETMC

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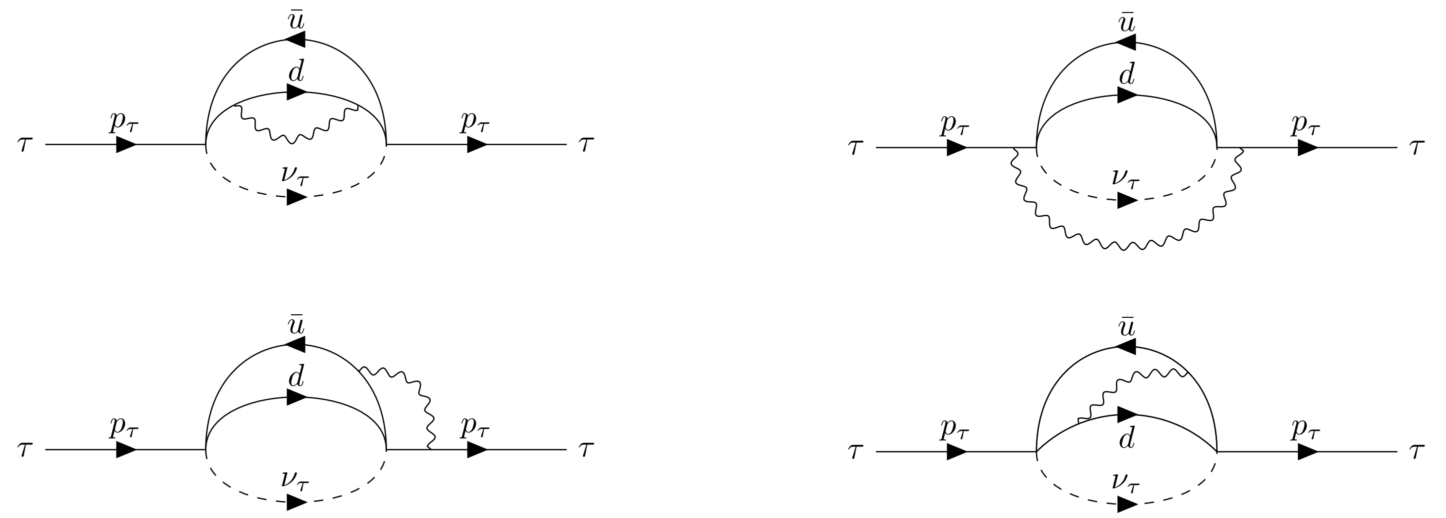


pursued by our collaborators at the ETMC

NON-PERTURBATIVE APPROACH

To allow charged states to propagate in finite volume, we use

C^* BOUNDARY CONDITIONS

$$\psi(x+\hat{L}_1/2) = \psi(x)^{\mathcal{C}} = C^{-1}\bar{\psi}(x)^T$$
$$\bar{\psi}(x+\hat{L}_1/2) = \bar{\psi}(x)^{\mathcal{C}} = -\psi^T(x)C$$
$$U_\mu(x+\hat{L}_1/2) = U_\mu(x)^{\mathcal{C}} = U_\mu(x)^*$$

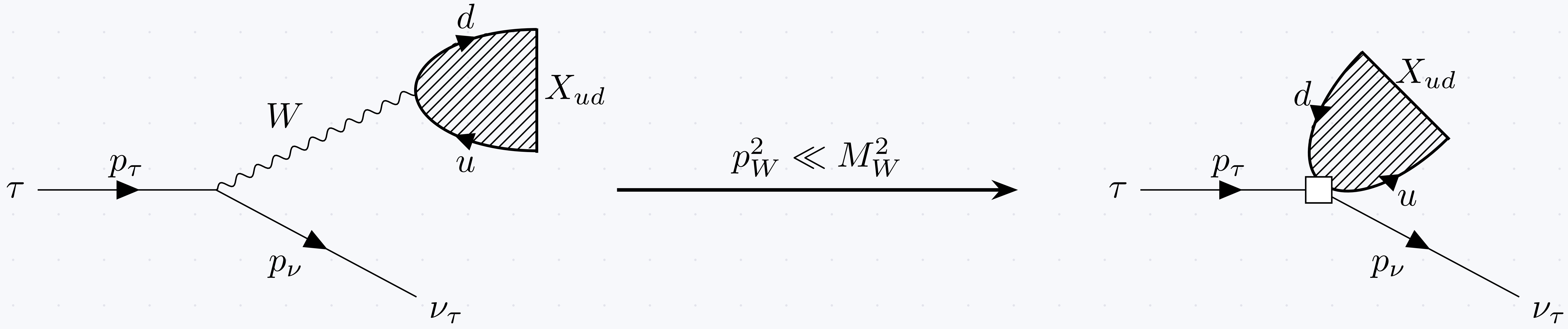
as implemented in *openQxD* by the RC* collaboration for fully dynamical QED+QCD simulations

§4 · SECTION FOUR

The effective electroweak Hamiltonian

Mixing pattern with broken symmetries

Effective weak theory



EFFECTIVE ELECTROWEAK HAMILTONIAN

$$\mathcal{H} \propto \underbrace{\bar{u}_L \gamma_\mu d_L}_{v_\mu^L} \underbrace{\bar{\tau}_L \gamma_\mu \nu_L}_{V_\mu^L} + \text{h.c.}$$

Mixing pattern

The effective Hamiltonian $\mathcal{H} \propto v^L V^L$ mixes with...

Operator \ Action	Wilson u, d, τ	Wilson u, d Overlap τ	Overlap u, d, τ
$s^L S^L = \bar{u}_R d_L \; \bar{\tau}_R \nu_L$	◆		
$s^R S^L = \bar{u}_L d_R \; \bar{\tau}_R \nu_L$	◆		
$v^L V^L = \bar{u}_L \gamma_\mu d_L \; \bar{\tau}_L \gamma_\mu \nu_L$	◆	◆	◆
$v^R V^L = \bar{u}_R \gamma_\mu d_R \; \bar{\tau}_L \gamma_\mu \nu_L$	◆	◆	
$t^L T^L = \bar{u}_R \sigma_{\mu\nu} d_L \; \bar{\tau}_R \sigma_{\mu\nu} \nu_L$	◆		

Wilson averages

New goal: renormalize only *certain* correlators instead of the full operator

Notice that the $\mathbb{Z}_2$ subgroup

$$\psi \to \gamma_5 \psi \qquad \bar{\psi} \to -\bar{\psi}\gamma_5$$

of $\mathrm{U}(1)_A$ is a spurionic symmetry

$$\bar{\psi} D(r, m, \mu) \psi = \bar{\psi}' D(-r, -m, \mu) \psi' \qquad D = \gamma \cdot \tilde{\nabla} - \frac{ar}{2} \nabla^F_\mu \nabla^B_\mu + m + i\mu\gamma_5$$

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SYMMETRY	$s^L S^L$	$s^R S^L$	$v^L V^L$	$v^R V^L$	$t^L T^L$
$\mathbb{Z}_2^A(\tau)$	$+$	$+$	$-$	$-$	$+$
$\mathbb{Z}_2^A(u)$	$+$	$-$	$-$	$+$	$+$
$\mathbb{Z}_2^A(d)$	$-$	$+$	$-$	$+$	$-$

Wilson averages

New goal: renormalize only *certain* correlators instead of the full operator

WILSON AVERAGE

$$\langle [v^L V^L]_R \rangle^{\mathrm{WA}} = \frac{1}{2}\Big(\langle [v^L V^L]_R \rangle_{r_\tau} + \langle [v^L V^L]_R \rangle_{-r_\tau} \Big)$$

$$= \left\langle \frac{Z_1(r_\tau)+Z_1(-r_\tau)}{2} s^L S^L + \frac{Z_2(r_\tau)+Z_2(-r_\tau)}{2} s^R S^L + \frac{Z_3(r_\tau)+Z_3(-r_\tau)}{2} v^L V^L + \frac{Z_4(r_\tau)+Z_4(-r_\tau)}{2} v^R V^L + \frac{Z_5(r_\tau)+Z_5(-r_\tau)}{2} t^L T^L \right\rangle_{\mathrm{cont}} + \mathcal{O}(a)$$

$$= \left\langle Z_3(r_\tau) v^L V^L + Z_4(r_\tau) v^R V^L \right\rangle_{\mathrm{cont}} + \mathcal{O}(a)$$

SYMMETRY	$s^L S^L$	$s^R S^L$	$v^L V^L$	$v^R V^L$	$t^L T^L$
$\mathbb{Z}_2^A(\tau)$	$+$	$+$	$-$	$-$	$+$
$\mathbb{Z}_2^A(u)$	$+$	$-$	$-$	$+$	$+$
$\mathbb{Z}_2^A(d)$	$-$	$+$	$-$	$+$	$-$

§5 · SECTION FIVE

Numerical Results

Some preliminary numerical results

Tuning of the τ

TUNING TO PHYSICAL TAU MASS

Using the 2-point function we extract the mass m_τ from the effective mass plateau on the $N_f = 1+2+1$ ensemble $\mathrm{A380a07b324}$ produced by the RC^* collaboration

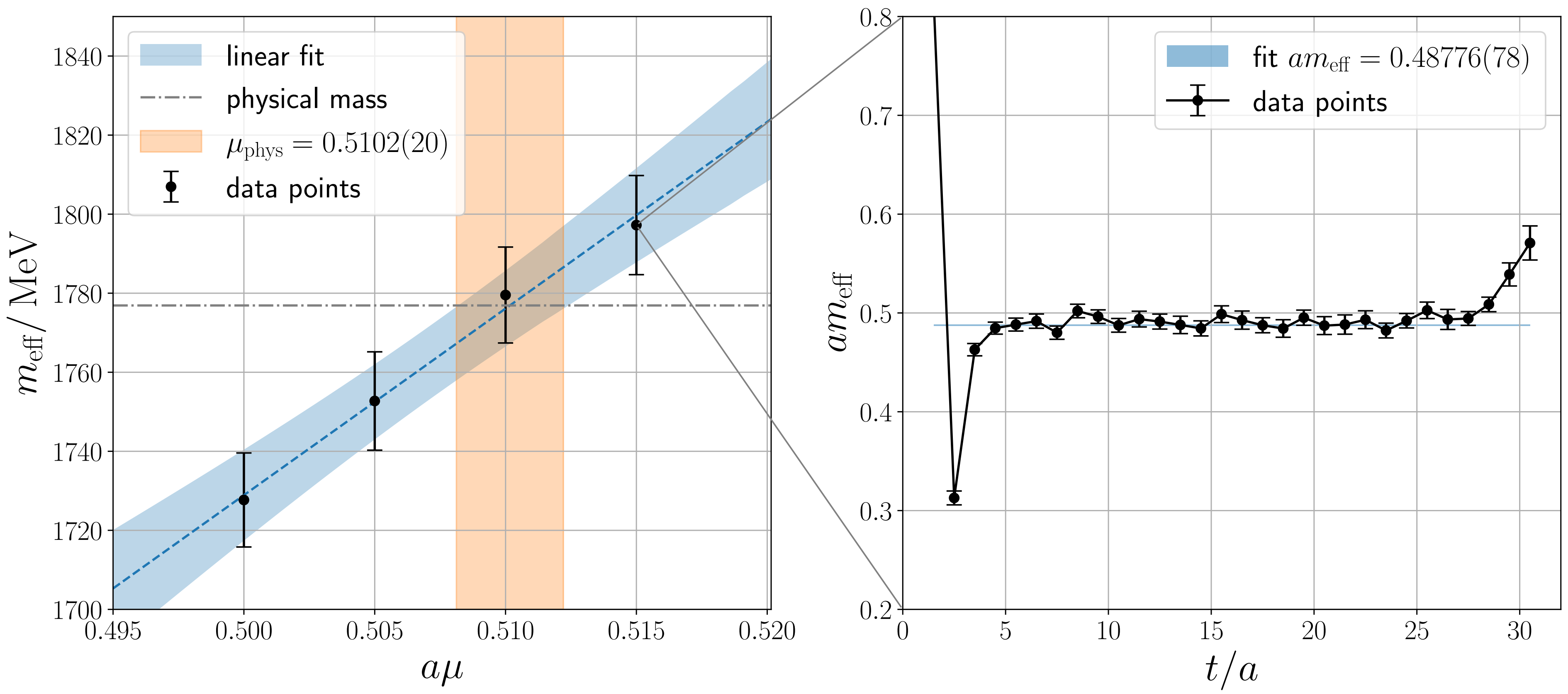
- 64×32^3
- C^* boundary conditions in all spatial directions
- $\beta = 3.24$ corresponding to $a = 0.0539(2)\,\mathrm{fm}$
- $m_\pi \approx 380\,\mathrm{MeV}$ and $m_\pi L \approx 3$
- $\alpha_{\mathrm{EM}} = 1/137$

In a partially quenched set-up with valence fermion action

- Osterwalder-Seiler (twisted mass)
- at maximal twist $m_0^\tau = m_{\mathrm{cr}}^\tau$

with statistics $N_{\mathrm{cnfg}} = 200$

PHYSICAL MASS INTERPOLATION



$$C_2(t) = \langle \tau(t)\bar{\tau}(0)\rangle = \frac{Z_\tau}{2m_\tau}\sum_r \left(e^{-tm_\tau}u^r(\underline{0})\bar{u}^r(\underline{0}) + e^{-(T-t)m_\tau}v^r(\underline{0})\bar{v}^r(\underline{0})\right) + \dots$$

Tuning of the τ

TUNING TO MAXIMAL TWIST

In a theory with mass term

$$m_0 + i\mu\gamma_5\sigma^3$$

we can use the

PCAC MASS

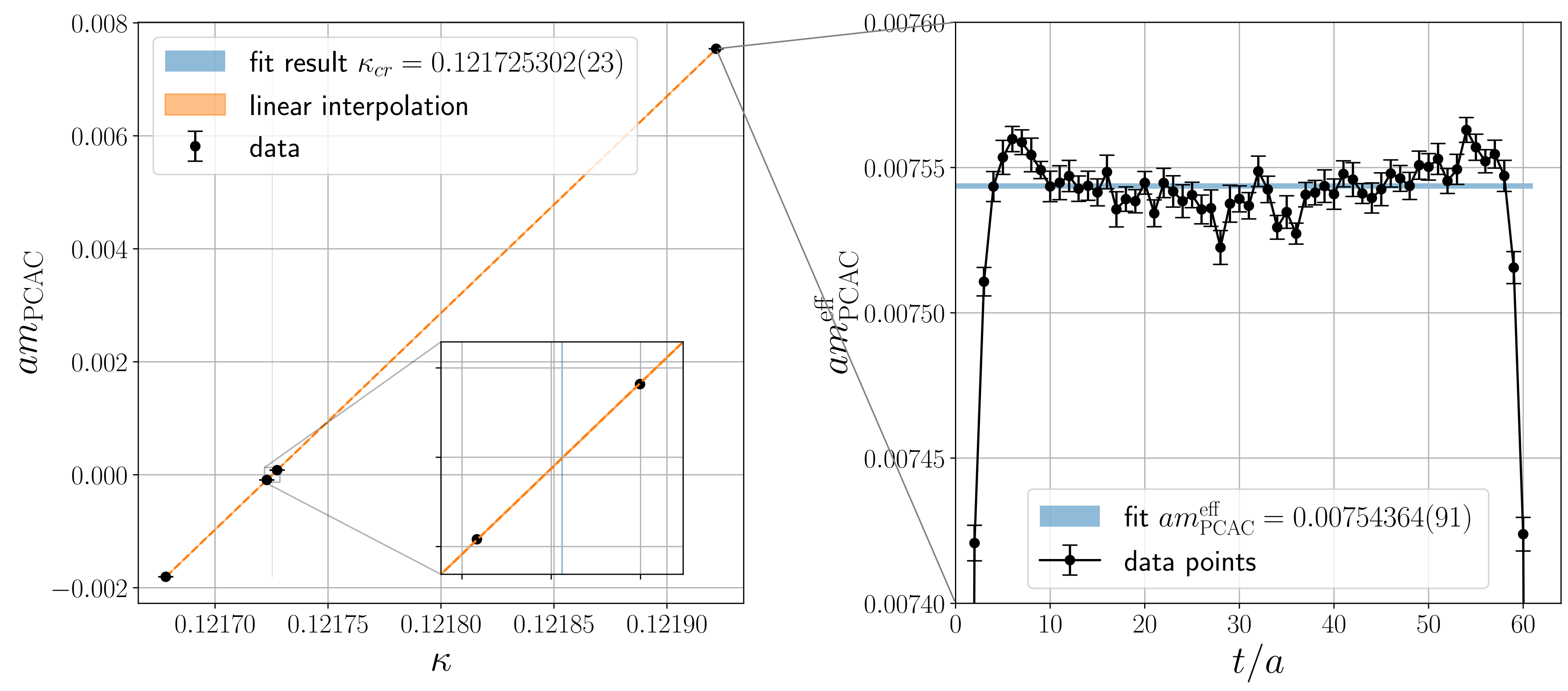
$$m_{\text{PCAC}}(t - t_0) = \frac{\langle \sum_{x,y} \partial_0^S A_0^1(t,\underline{x}) P^1(t_0,\underline{y})\rangle}{2\langle \sum_{x,y} P^1(t,\underline{x}) P^1(t_0,\underline{y})\rangle}$$

to tune to maximal twist $m_R \sim m_{\text{PCAC}} = 0$

With the CGNE solver we get the $-\mu$ solution almost for free, no extra inversions are needed

$$(D^\dagger D + \mu^2)\psi = \eta \qquad \Rightarrow \qquad \psi_{\pm\mu} \sim (D \mp i\mu\gamma_5)\psi$$

CRITICAL MASS INTERPOLATION



The 4-point function

EXCITED STATE CONTAMINATION

GOAL

Isolate external τ states

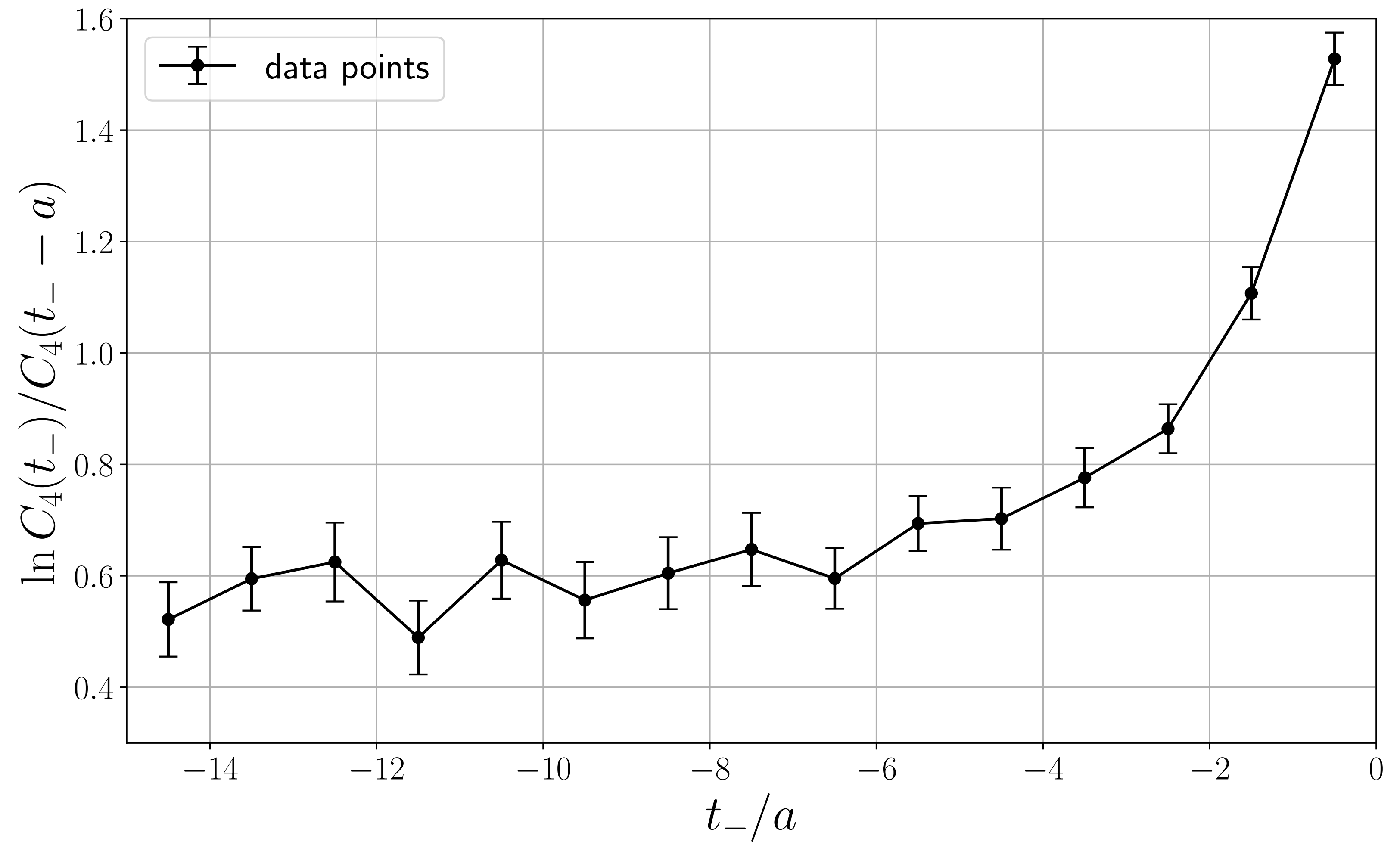
To leading order

$$C_4(t_-) \sim \langle 0|\tau(t_+)\mathcal{H}(t)\mathcal{H}(0)\bar{\tau}(t_-)|0\rangle$$
$$= \sum_n \langle 0|\tau(t_+)\mathcal{H}(t)\mathcal{H}(0)|n\rangle\langle n|\bar{\tau}(t_-)|0\rangle$$
$$\sim e^{m_\tau t_-} + \dots$$

with the remaining terms coming from excited states.
Similarly, for the $\tau(t_+)$ interpolator we have

$$C_4(t_+) \sim e^{-m_\tau t_+} + \dots$$

PLATEAU IN t_-



The 4-point function

THE MATRIX ELEMENT

To isolate the matrix element

$$\sum_r \langle \tau^r(\underline{0})|\mathcal{H}(t)\mathcal{H}(0)|\tau^r(\underline{0})\rangle$$

We first note that

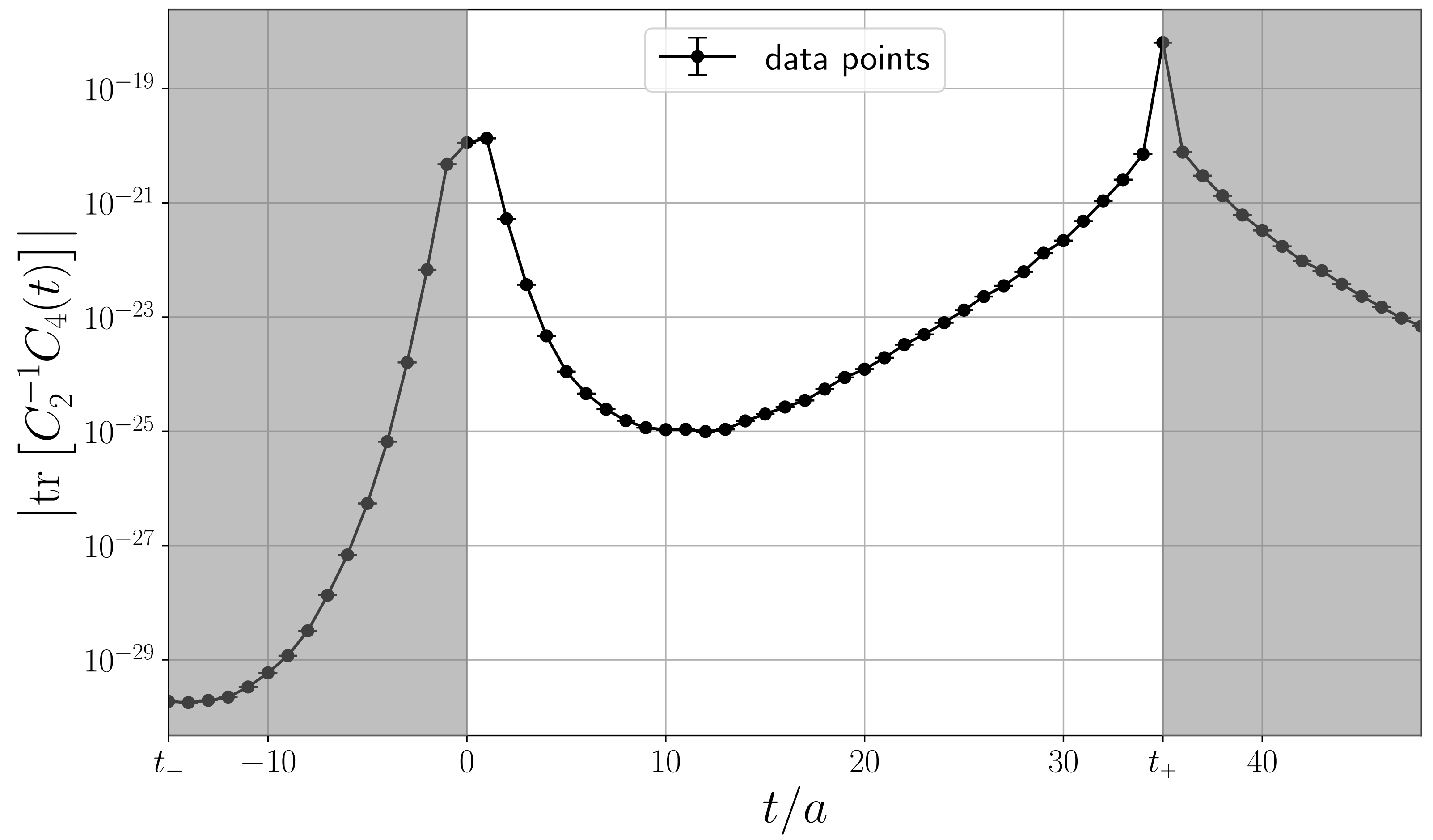
$$\begin{aligned} C_4 &= \langle 0|\tau_\alpha(t_+)\mathcal{H}(t)\mathcal{H}(0)\bar{\tau}_\beta(t_-)|0\rangle \\ &= \sum_{r,s} \frac{Z_\tau}{(2m_\tau)^2} e^{-m_\tau(t_+ + |t_-|)} \langle \tau^r(\underline{0})|\mathcal{H}(t)\mathcal{H}(0)|\tau^s(\underline{0})\rangle u^r(\underline{0})\bar{u}^s(\underline{0}) + \dots \end{aligned}$$

To get the right polarization sums we then consider

MATRIX ELEMENT

$$\mathrm{tr}\left[C_2(t_+ + |t_-|)^{-1} C_4(t_+, t, t_-)\right] = \sum_r \langle \tau^r(\underline{0})|\mathcal{H}(t)\mathcal{H}(0)|\tau^r(\underline{0})\rangle + \dots$$

MATRIX ELEMENT DATA



§6 · SECTION FIVE OF FIVE

Outlook & to-do list

What we still have to do

Outlook & to-do list

SPECTRAL RECONSTRUCTION

Extract $\Gamma(\tau \to X\nu)$ using the HLT

CONTINUUM LIMIT

Perform the continuum $a \to 0$ limit

INFINITE-VOLUME EXTRAPOLATION

Analyse finite volume effects (more severe for QED)

RENORMALIZATION CONSTANTS

Define suitable renormalization conditions and measure the renormalization constants

Plan: use flowed probes
$\to$ Lukas Holan 11:30

MATCHING TO THE SM

Run to high energies to match to perturbation theory

GOING TO THE PHYSICAL POINT

Current ensembles are far away from the physical point
$m_\pi \approx 380\,\mathrm{MeV} \gg 135\,\mathrm{MeV}$

SECTION SIX OF FIVE

Appendix

Some more details

Constraints from the anomalous chiral symmetry

ANOMALY IN THE MASSLESS THEORY

In the massless theory the contributions from $Q \neq 0$ sectors automatically vanish by the index theorem

$$Z_\theta = \int \mathcal{D}[A,\psi,\bar{\psi}]\, e^{-S_G[A]-\bar{\psi}D\psi-i\theta Q} = \sum_{n=0} \int_{Q[A]=n} \mathcal{D}A\, e^{-S_G[A]-i\theta n} \det D = \int_{Q[A]=0} \mathcal{D}A\, e^{-S_G[A]} \det D = Z_0$$

Since $\mathcal{H}$ does not mix with lower dimensional operators, the renormalization constants can be chosen to be mass independent. So we can calculate them in the massless theory and here

$$\begin{aligned} Z[J] &= \int \mathcal{D}[A,\psi',\bar{\psi}']\, e^{-S_G[A]-\bar{\psi}'D\psi'+JO[\psi',\bar{\psi}']} \\ &= \int \mathcal{D}[A,\psi,\bar{\psi}] e^{-i\theta Q[A]}\, e^{-S_G[A]-\bar{\psi}D\psi+J\left(\rho(\alpha)O[\psi,\bar{\psi}]\right)} \\ &= \int \mathcal{D}[A,\psi,\bar{\psi}]\, e^{-S_G[A]-\bar{\psi}D\psi+(J\rho(\alpha))O[\psi,\bar{\psi}]} \\ &= Z[J\rho(\alpha)] \end{aligned} \qquad (\psi' = e^{i\alpha\gamma_5}\psi \quad \bar{\psi}' = \bar{\psi}e^{i\alpha\gamma_5})$$

The same symmetry constraint then holds for the (renormalized) vertex functional $\Gamma[J] = \Gamma[J\rho(\alpha)]$. Suppose now that O transforms in some (1-dimensional) irrep ρ_q of $\mathrm{U}(1)_A$ and O' in a different inequivalent irrep $\rho_{q'}$ then

$$JZO' \overset{!}{=} J\rho_q(\alpha)Z\rho_{q'}(-\alpha)O' = e^{i(q-q')\alpha}JZO'$$

which cannot be true for all α when $q \neq q'$. This implies that JO' cannot appear in Γ_R.

About the polarization sums...

About the disconnected diagram...

doesn't contribute for kinematical reasons...

can be represented by introducing a second, identical τ species ...

in the Wilson averages....