Probing Instanton Dynamics in the Pion Vector Form Factor with Gradient Flow

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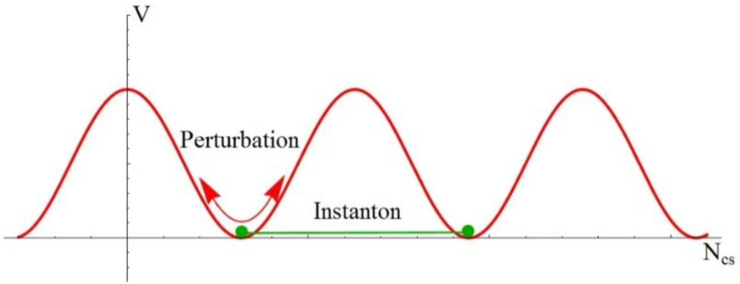
Standard Model parameters and observables from gradient flow, Edinburgh 2026

Introduction

- QCD is a non-Abelian gauge theory based on the SU(3) color symmetry group.
  It describes interactions between quarks and gluons.
- The QCD vacuum is not empty, as it contains topologically non-trivial gluon field configurations.

- Instantons are localized, classical solutions of the Euclidean Yang–Mills equations.
- They describe **tunneling** between distinct classical QCD vacua.



R. Jackiw and C. Rebbi,Phys. Rev. Lett. 37, 172 (1976)

Vainshtein, A. I.; Zakharov, V. I.; Novikov, V. A.; Shifman, M. A. "ABC of Instantons." *Sov. Phys. Usp.* **25** (1982): 195–215.

Wilson Flow (Gradient Flow)

- The QCD vacuum on the lattice is full of UV noise, tiny fluctuations that obscure physical, topological structures.

- Wilson flow is a process that evolves lattice gauge fields $U_\mu(x)$ along a continuous "flow time" ***t*** by solving:

$$\frac{dU_\mu(x,t)}{dt} = -g_0^2 \left\{ \frac{\partial S_g[U(t)]}{\partial U_\mu(x,t)} \right\} U_\mu(x,t), \qquad U_\mu(x,0) = U_\mu(x),$$

- The gauge fields become smoother as ***t*** increases, with small-scale noise diffused away.

- Applications in Lattice Field Theory:
  - Topological Charge and Susceptibility
  - Scale Setting
  - Improve Signal to noise

M. Luscher, "Properties and uses of the Wilson flow in lattice QCD", 23 June 2010, arXiv:1006.4518v1 [hep-lat];
Mattia Bruno, Tomasz Korzec, Stefan Schaefer Phys.Rev.D 95 (2017) 074504 arXiv: 1608.08900 [hep-lat]

Wilson Flow and Instantons

- Each instanton carries a topological charge corresponding to tunneling between degenerate QCD vacua.

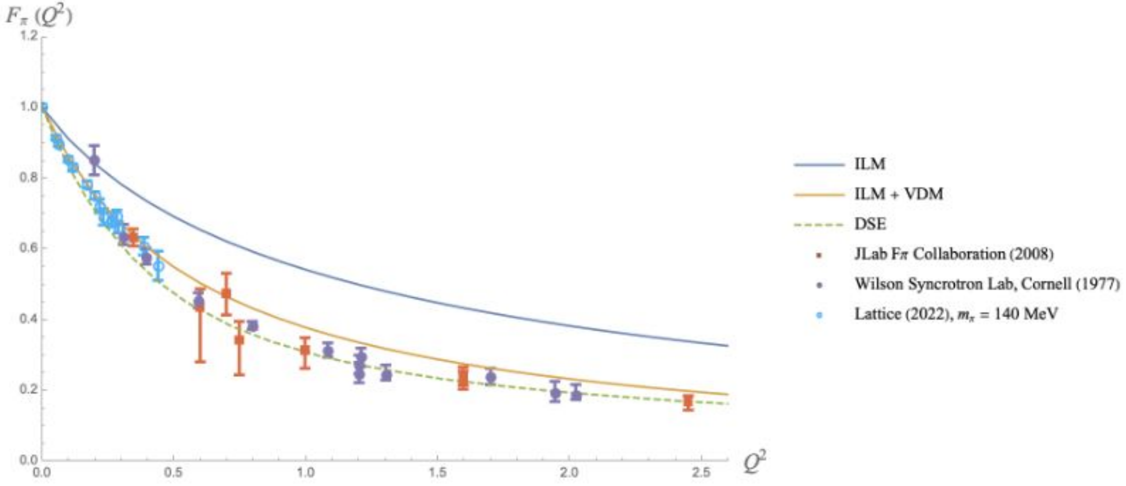
$$Q = \frac{1}{32\pi^2} \int d^4x \, G^a_{\mu\nu} \tilde{G}^a_{\mu\nu}$$

- At **small flow times**, UV fluctuations are suppressed, but physical long-distance structures remain intact.
- At **moderate flow**, instanton-like objects appear as **coherent lumps** in the topological charge.
- If the flow time is too long, instantons may shrink and annihilate (over-smoothing), so an **optimal range of flow time** is chosen.
- In this work, we study hadronic matrix elements directly at finite Wilson flow time $t > 0$, and understand how they change as the flow time is varied.

Andreas Athinodorou, Philippe Boucaud, Feliciano de Soto, José Rodríguez-Quintero and Savvas Zafeiropoulos EPJ Web Conf., 175 (2018) 12011

Instantons and Pion Form Factors

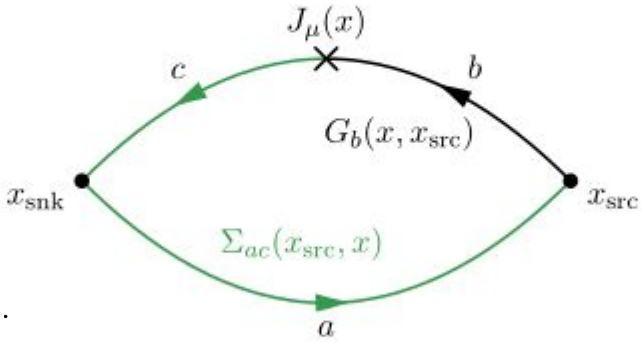
- The **Instanton Liquid Model (ILM)** provides a mathematical framework to describe this complex vacuum by modeling it as an ensemble of interacting instantons and anti-instantons.
- One particularly insightful observable is the **pion electromagnetic form factor**, which probes the internal structure and dynamics of the pion as a bound state of quarks.
- The Pure Instanton Liquid Model gives a higher value of the pion form factor and does not agree with lattice qcd and experiments .
- We study the pion form factor for different values for ***t***.



Wei-Yang Liu, Edward Shuryak, and Ismail Zahed, Phys. Rev. D 109, 074029 (2024),doi:10.1103/PhysRevD.109.074029

Pion Form factors from Lattice

$$D_{\text{clover}} = D_{\text{Wilson}}(\kappa) + c_{\text{SW}}(\text{clover term}).$$



$$C_\pi^{2\text{pt}}(t,\boldsymbol{p}) = \sum_{\boldsymbol{x}} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \langle 0|O_\pi(\boldsymbol{x},t)O_\pi^\dagger(0,0)|0\rangle.$$

$$C_\pi^{3\text{pt}}(t,t_{\text{sep}},\boldsymbol{p}_f,\boldsymbol{p}_i) = \sum_{\boldsymbol{x},\boldsymbol{y}} e^{-i\boldsymbol{p}_f\cdot(\boldsymbol{x}-\boldsymbol{y})} e^{-i\boldsymbol{p}_i\cdot\boldsymbol{y}} \langle 0|O_\pi(\boldsymbol{x},t_{\text{sep}})J_\mu(\boldsymbol{y},t)O_\pi^\dagger(0,0)|0\rangle.$$

- Fix the sink in the middle of the lattice and in the frame where final pion is at rest.
- Use sequential propagator to get the 3-point correlators.
- To get the matrix elements, we use ratios of 3-point and 2-point correlation functions of pions.

$$R(t_{\text{sink}},\tau,\vec{p}',\vec{p}) = \frac{C_{3pt}^{\mathcal{O}}(\tau,\vec{p}',\vec{p})}{C_{2pt}(t_{\text{sink}},\vec{p}')}\sqrt{\frac{C_{2pt}(t_{\text{sink}}-\tau,\vec{p})C_{2pt}(\tau,\vec{p}')C_{2pt}(t_{\text{sink}},\vec{p}')}{C_{2pt}(t_{\text{sink}}-\tau,\vec{p}')C_{2pt}(\tau,\vec{p})C_{2pt}(t_{\text{sink}},\vec{p})}}.$$

$$2\sqrt{E_{\vec{p}}E_{\vec{p}'}} \times Z_V \times \langle \pi(\vec{p}')|\,\mathcal{O}_V^\mu(0)|\pi(\vec{p})\rangle = (p_f + p_i)_\mu \times f_{\pi\pi}(-q^2)$$

D. Brömmel et al. The Pion form-factor from lattice QCD with two dynamical flavours. Eur.Phys.J., C51:335–345, 2007.

Lattice Setup

- The study use publicly available PACS-CS gauge fields ensemble.
- Fermionic part o(a)-improved wilson action with Nf = 2 + 1 dynamical quarks and gauge part is Iwasaki gauge action.
- Bare parameters for the ensemble

β	a [fm]	k_l	L/a	T/a	c_{sw}	N_G	M_π	Z_v
1.96	0.0907(13)	0.13754	32	64	1.715	200	409.7(7)	0.7354(37)

PACS-CS, S. Aoki et al., Phys. Rev. D79, 034503 (2009), 0807.1661.

Procedure

Lattice ensemble

↓

Flow the gauge field to time t

↓

Tune the κ at $M_\pi \simeq 400$ MeV for c_{sw}

↓

Calculate the 2 point and 3 point function.

↓

Extract the matrix elements

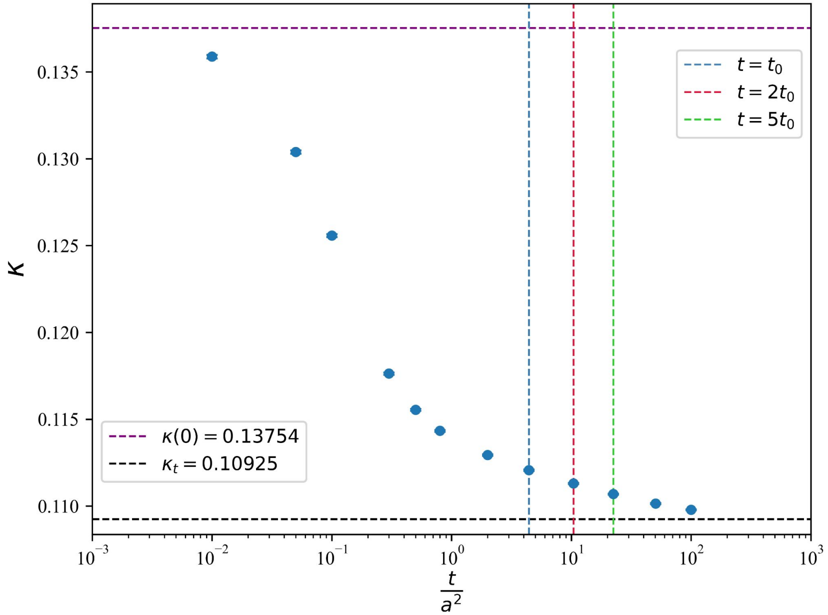
↓

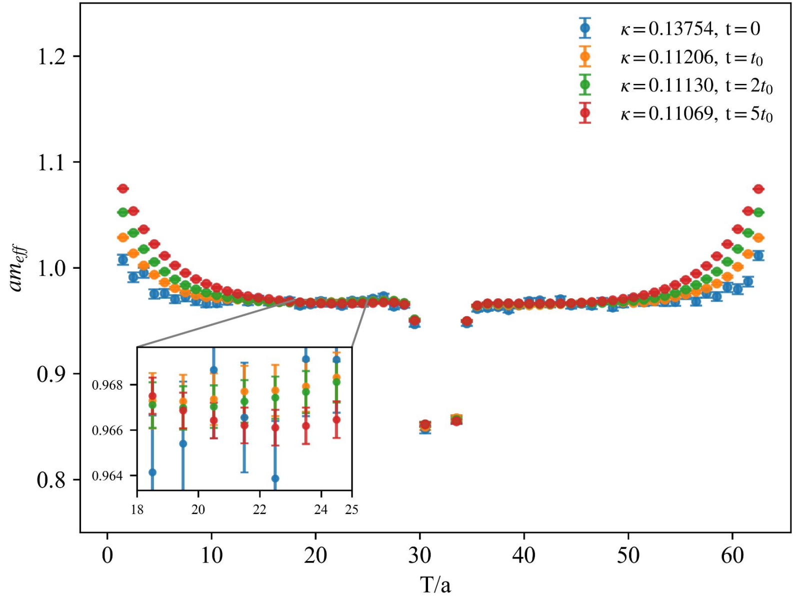
Calculate the pion form factors at t.

Setting up the bare parameters

- Preliminary: set c_{sw} = 1.0. A more precise analysis requires tuning c_{SW} and c_A for example using PCAC relations.

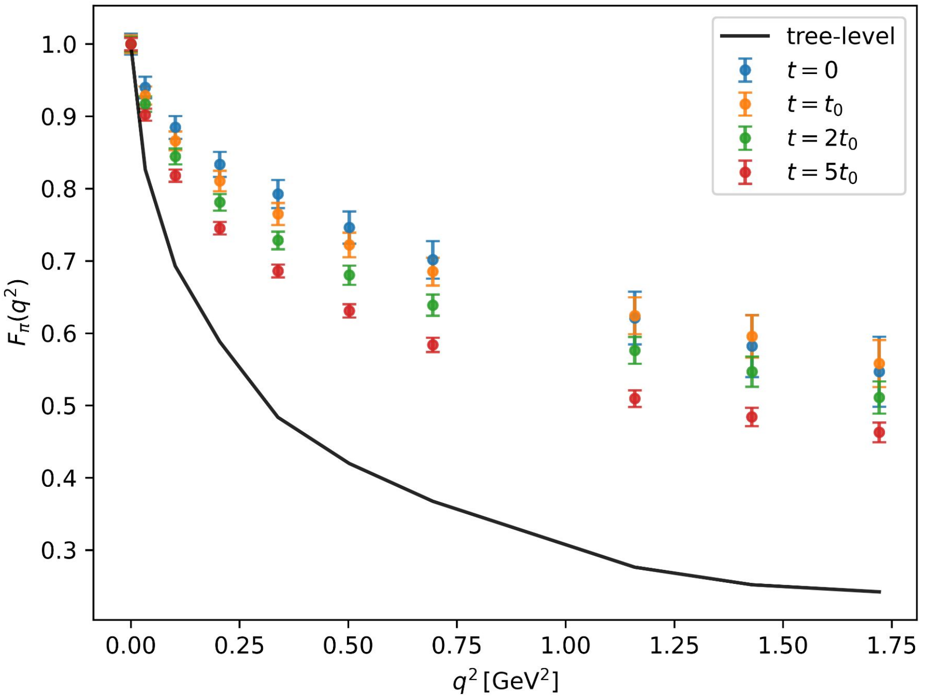
- Rather than performing ADDITIONAL simulations at vanishing quark mass, we attempt to introduce a massive renormalization and improvement scheme, extracting the relevant parameters directly from our available ensemble at $M_\pi \simeq 400$ MeV.





Setting up the bare parameters

We see a significant difference between the form factors at tree-level and flow time = $5t_0$



Setting up the bare parameters

- PCAC relation

$$\langle \partial_\mu A_\mu(x)\mathcal{O}(y)\rangle = 2m\langle P(x)\mathcal{O}(y)\rangle$$ where, $A_\mu(x) = \overline{u}(x)\gamma_\mu\gamma_5 d(x)$, $P(x) = \overline{u}(x)\gamma_5 d(x)$

For simplicity, $\mathcal{O}(y) = P^\dagger(y)$

$$\tfrac{1}{2}\left(\partial^*_\mu + \partial_\mu\right)\left\langle A^I_\mu(x)P^\dagger(y)\right\rangle = 2m\left\langle P(x)P^\dagger(y)\right\rangle$$

$$A^I_\mu(x) = A_\mu(x) + c_A \tfrac{1}{2}\left(\partial^*_\mu + \partial_\mu\right)P(x)$$

We set $\mu = 0$

$$\overleftrightarrow{\partial_0}\left\langle\left\{A_0(x) + c_A\overleftrightarrow{\partial_0}\right)P(x)\right\}P^\dagger(y)\right\rangle = 2m\left\langle P(x)P^\dagger(y)\right\rangle$$

- With symmetric derivative $\overleftrightarrow{\partial_0} = \frac{1}{2}\left(\partial^*_0 + \partial_0\right)$

$$\overleftrightarrow{\partial_0}\left\langle A_0(x)P^\dagger(y)\right\rangle + c_A\overleftrightarrow{\partial_0}\overleftrightarrow{\partial_0}\left\langle P(x)P^\dagger(y)\right\rangle = 2m\left\langle P(x)P^\dagger(y)\right\rangle$$

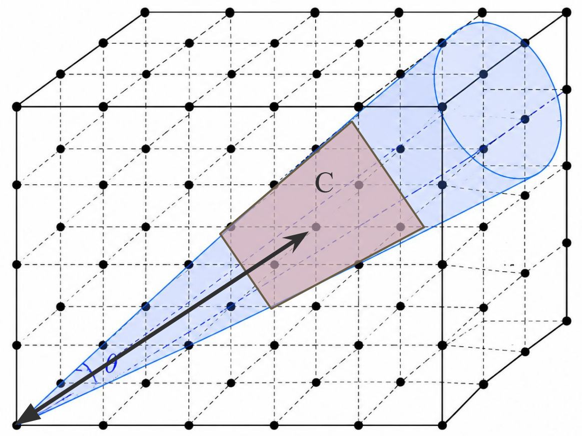
- Now we have ratio

$$m(x,y;c_A,c_{SW},m_\pi) = \frac{\overleftrightarrow{\partial_0}\,A_d\left\langle A_0(x)P^\dagger(y)\right\rangle + c_A\,\overleftrightarrow{\partial_0}\overleftrightarrow{\partial_0}\left\langle P(x)P^\dagger(y)\right\rangle}{\left\langle P(x)P^\dagger(y)\right\rangle}.$$

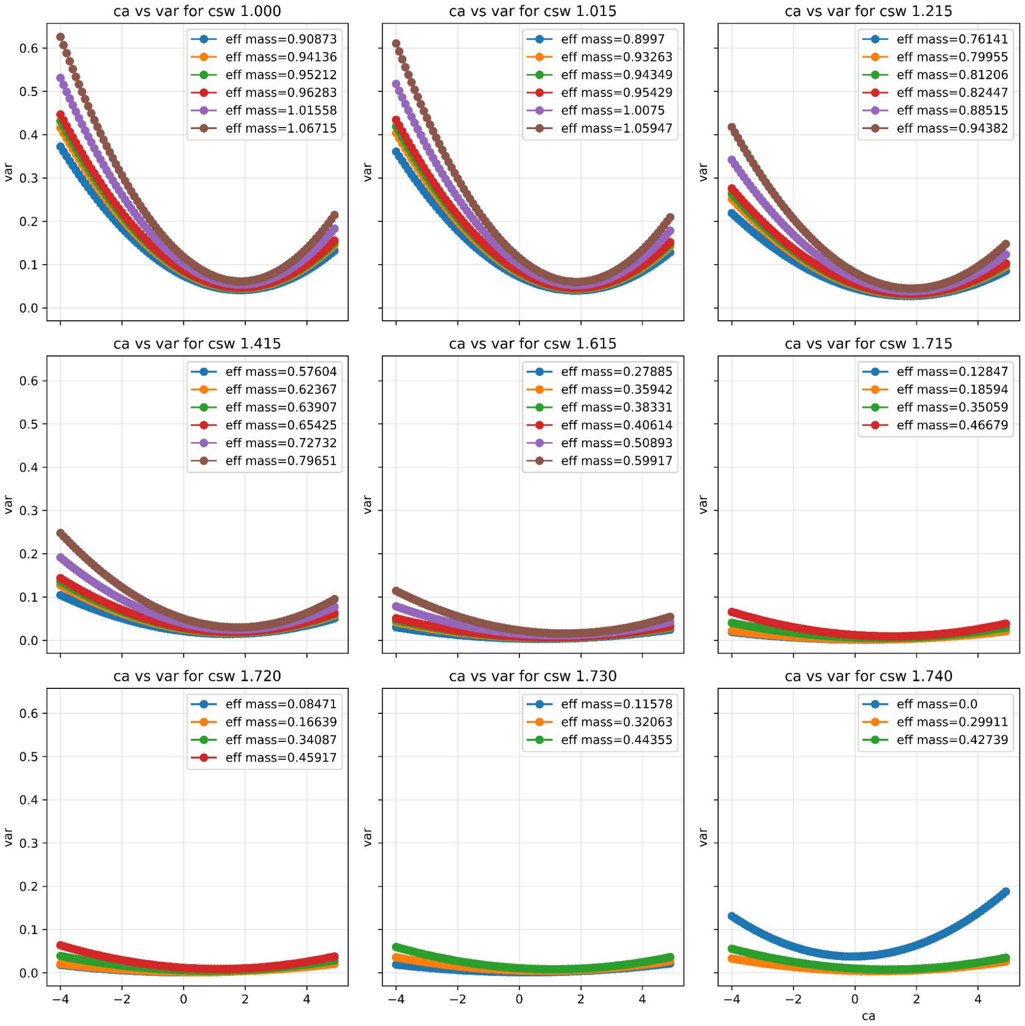
Setting up the bare parameters

$$m(x,y;c_A,c_{SW},m_\pi)=\frac{\overleftarrow{\partial}_0\langle A_0(x)P^\dagger(y)\rangle+c_A\,\overleftarrow{\partial}_0\overrightarrow{\partial}_0\langle P(x)P^\dagger(y)\rangle}{\langle P(x)P^\dagger(y)\rangle}.$$

- The ratio is plotted with c_A
- Choose the points in the cone which lies on the hyper diagonal of the lattice.
- The cone is defined using **physical distances.**
- Then choose the points which lies inside the shaded frustum.
- Larger the angle θ, the more points to consider
- Calculate the variance of $m(y;c_{SW}, c_A)$ using points lies in the frustum various c_{sw} and M_π.



$$\mathrm{Var}(m)=\frac{1}{N-1}\sum_{y\in\mathcal{C}}(m(y)-\overline{m})^2$$


ca vs var for csw 1.000
eff mass=0.90873
eff mass=0.94136
eff mass=0.95212
eff mass=0.96283
eff mass=1.01558
eff mass=1.06715
ca vs var for csw 1.015
eff mass=0.8997
eff mass=0.93263
eff mass=0.94349
eff mass=0.95429
eff mass=1.0075
eff mass=1.05947
ca vs var for csw 1.215
eff mass=0.76141
eff mass=0.79955
eff mass=0.81206
eff mass=0.82447
eff mass=0.88515
eff mass=0.94382
ca vs var for csw 1.415
eff mass=0.57604
eff mass=0.62367
eff mass=0.63907
eff mass=0.65425
eff mass=0.72732
eff mass=0.79651
ca vs var for csw 1.615
eff mass=0.27885
eff mass=0.35942
eff mass=0.38331
eff mass=0.40614
eff mass=0.50893
eff mass=0.59917
ca vs var for csw 1.715
eff mass=0.12847
eff mass=0.18594
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ca vs var for csw 1.720
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eff mass=0.45917
ca vs var for csw 1.730
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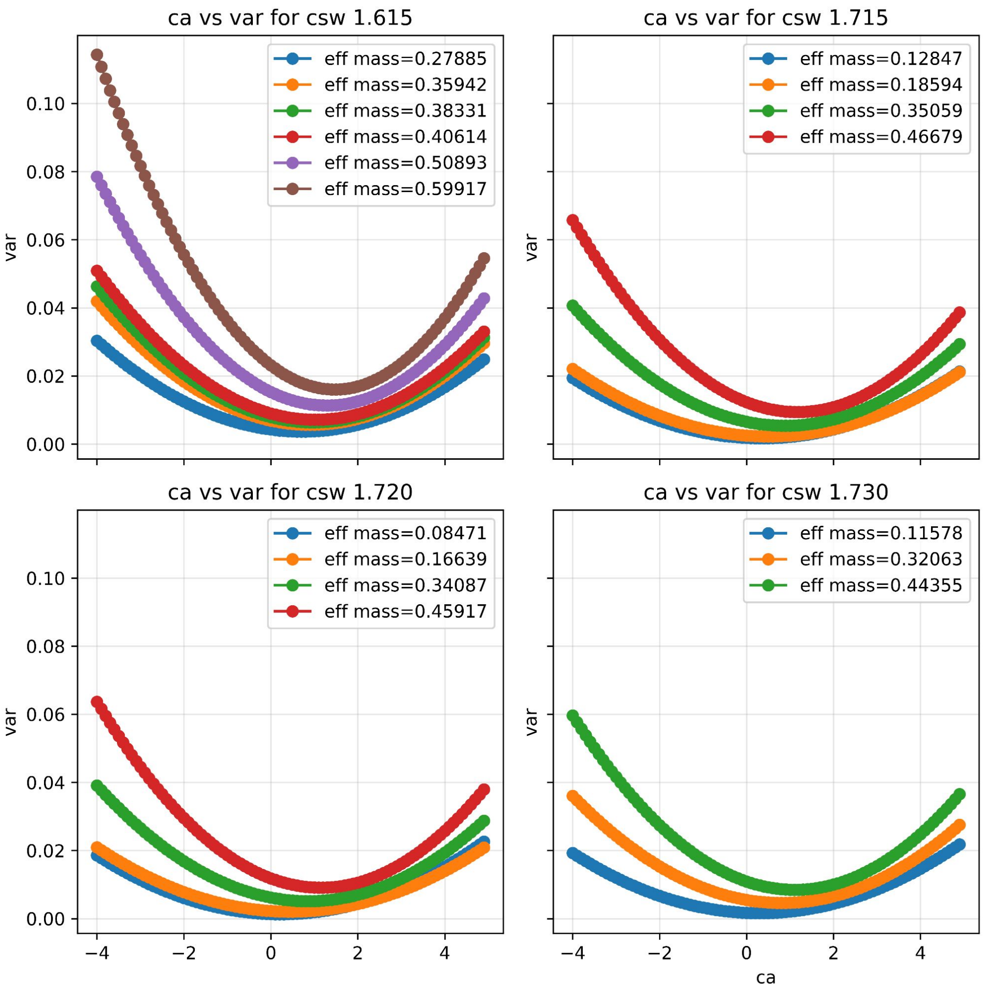
var
ca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eff mass=0.32063
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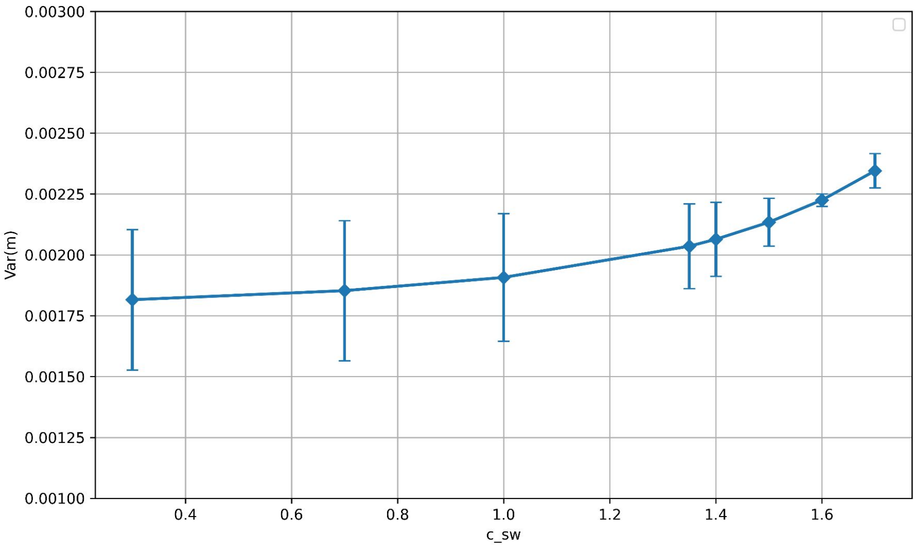
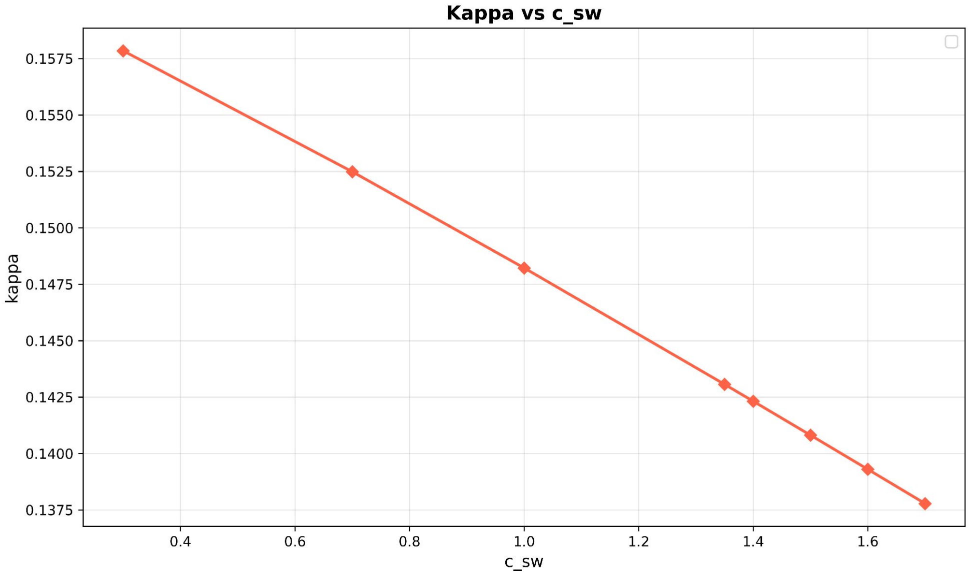
var
ca

Setting up the bare parameters

- Once we fix c_sw we will know all necessary parameters and could estimate the form factor.
- The algorithm can be applied at t>0.

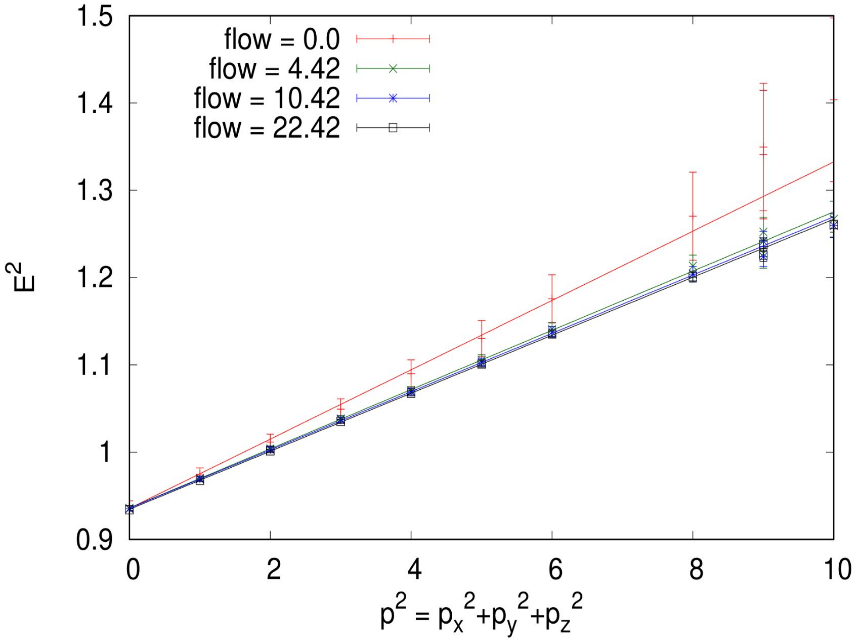


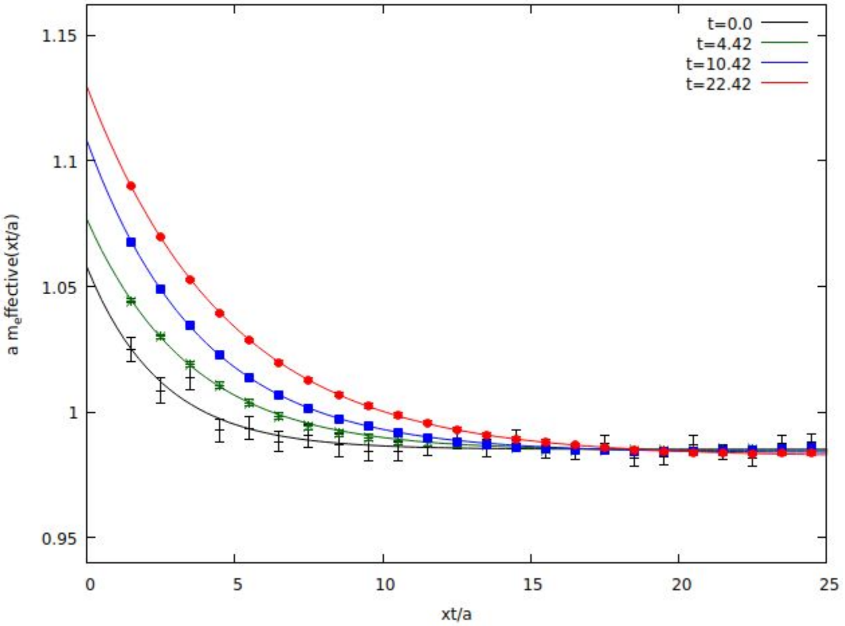


Preliminary results for parameter tuning at t = 0, for 20 gauge configurations.

Setting up the bare parameters

- Can matrix elements at large flow time be extracted using the same procedure as at t = 0?
- Model calculations suggest that the standard extraction procedure remains valid at finite flow time.
- After tuning κ for each flow time, the pion mass is consistent across flows. However, excited-state contributions are enhanced at larger flow times.





Preliminary model calculations done in collaboration with Dr. Y. Liu... work in progress

Summary and Outlook

- Tune the parameters c_{sw}, κ
- In future, form factor will be calculated with twisted boundary conditions to get access to more momenta between the integer values.
- After tuning the parameters, calculation of form factors with higher values of flow time will be used.
- Extrapolation to $a \longrightarrow 0$
- Use more lattice ensembles for broder result.

Thank You