Determining m_{cr} and c_{fl} via Finite Axial Ward Identities of Flowed Fermion Fields

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Long-Term Motivation

Twisted Mass Fermions in the Continuum

Lattice Version

$\mathcal{O}(a)$-effects

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$$H_{\mathrm{ew}} = Z\left\{ O_1 + \sum_{j=2}^{5} r_j O_j \right\} \qquad O_1 = \overline{d}_L \gamma_\mu u_L \overline{\nu}_L \gamma_\mu \tau_L \;, \quad O_2 = \ldots \;, \quad \ldots$$

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RTG 2575: Rethinking Quantum Field Theory

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- On the lattice true up to $\mathcal{O}(a)$-effects $\implies$ Use the 4 equations as a condition to determine the r_j
- Due to multiplicative renormalization of flowed composite operators this leads to a system of equations that are linear in the r_j

Current Presentation

- In this talk we will not present anything on H_{ew}
- Instead I will present very preliminary data on the determination of m_{cr} and c_{fl}
- Uses the same ingredients:
  - Finite spurionic axial symmetries $U(1)_A$
  - Probes at positive flow time

Twisted Mass Fermions in the Continuum

Continuums Twisted Mass Fermions

- The action of twisted mass fermions $\overline{\chi}$ and χ is given by

$$S_F = \int dx^4 \, \overline{\chi}(\not{D} + m_0 + i\gamma_5\mu_0)\chi$$

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- With $\cot\alpha = \frac{m_0}{\mu_0}$, α the twist angle and $M_0 = \sqrt{m_0^2 + \mu_0^2}$

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- With $\cot\alpha = \frac{m_0}{\mu_0}$, α the twist angle and $M_0 = \sqrt{m_0^2 + \mu_0^2}$
- In a partially quenched setup we add the action of the bosonic ghosts $\overline{c}$ and c with

$$S_{\text{gh}} = \int dx^4 \, \overline{c}(\not{D} + M_0 e^{-i\alpha\gamma_5})c$$

- When integrating over $\overline{c}$, c, $\overline{\chi}$ and χ the fermionic determinant cancels

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$$S_F = \int dx^4 \, \overline{\chi}(\not{D} + M_0 e^{i\gamma_5 \alpha})\chi = \int dx^4 \, \overline{\psi}(\not{D} + M_0)\psi$$

- In a quenched continuum theory equivalent to standard fermions with a mass M_0

Rethinking Quantum Field Theory

- More precisely assuming some regularization that preserves the axial spurionic symmetry one has $\cot\alpha = \frac{m_R}{\mu_R}$ with m_R and μ_R the renormalized masses

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- The flowed fermion fields $\overline{\chi}_t$ and χ_t transform the same under the spurionic axial symmetry as $\overline{\chi}$ and χ

- For example the scalar $\langle \chi_t \chi_t \rangle$ and pseudo scalar $\langle \chi_t i\gamma_5 \chi_t \rangle$ transform into each other

$$\begin{pmatrix} \cos(\omega) & \sin(\omega) \\ -\sin(\omega) & \cos(\omega) \end{pmatrix} \begin{pmatrix} \langle \overline{\chi}_t \chi_t \rangle_{(M_R,\alpha)} \\ \langle \overline{\chi}_t i\gamma_5 \chi_t \rangle_{(M_R,\alpha)} \end{pmatrix} = \begin{pmatrix} \langle \overline{\chi}_t \chi_t \rangle_{(M_R,\alpha-\omega)} \\ \langle \overline{\chi}_t i\gamma_5 \chi_t \rangle_{(M_R,\alpha-\omega)} \end{pmatrix}$$

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$$\frac{m_R}{\mu_R} = \cot\alpha = \frac{\langle \overline{\chi}_t \chi_t \rangle_{(M_R,\alpha)}}{\langle \overline{\chi}_t i\gamma_5 \chi_t \rangle_{(M_R,\alpha)}}$$

- This equation is not satisfied on the lattice so we plan to used it to extract the critical mass m_{cr} and the improvement coefficient c_{fl}

Lattice Version

Lattice Twisted Mass Fermions

$$S_F = \sum_x a^4 \overline{\chi}(D_{0,\mathrm{SW}} + m_0 + i\gamma_5\mu_0)\chi \quad \text{and} \quad S_{\mathrm{gh}} = \sum_x a^4 \overline{c}(D_{0,\mathrm{SW}} + m_0 + i\gamma_5\mu_0)c$$

$$D_{0,SW} = \gamma_\mu \nabla^f_\mu - \frac{a}{2}\nabla^f\nabla^b + \frac{a}{2} c_{sw}\sigma_{\mu\nu}\hat{F}_{\mu\nu}$$

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- The lattice regularization has an additional explicit breaking of the spurionic axial symmetry via the Wilson term
- The condition holds only up to $\mathcal{O}(a)$-effects

$$\mathcal{O}(a) + \frac{Z_m}{Z_\mu}\frac{m_0 - \frac{1}{a}\hat{m}_{cr}}{\mu_0} = \frac{m_R}{\mu_R} = \cot\alpha = \frac{\langle \overline{\chi}_t \chi_t \rangle_{(m_0,\mu_0)}}{\langle \overline{\chi}_t i\gamma_5 \chi_t \rangle_{(m_0,\mu_0)}} + \mathcal{O}(a)$$

Extraction of m_{cr}

$$\frac{Z_m}{Z_\mu}\frac{m_0 - \frac{1}{a}\hat{m}_{cr}}{\mu_0} = \frac{\langle \overline{\chi}_t \chi_t \rangle_{(m_0,\mu_0)}}{\langle \overline{\chi}_t i\gamma_5 \chi_t \rangle_{(m_0,\mu_0)}}$$

- Measure the righthand side for different masses (m_0, μ_0)
- Extract $\frac{Z_m}{Z_\mu}$ and $\hat{m}_{\mathrm{cr}}$ from a fit up to relative $\mathcal{O}(a)$-effects
- Even with c_{sw} tuned there are $\mathcal{O}(a)$-effects coming from the flow and the renormalized masses

$\mathcal{O}(a)$-effects

- The original equation that determines α up to $\mathcal{O}(a)$-effects

$$Z_\chi \Big[-\sin(\alpha)\langle\overline{\chi}_t\chi_t\rangle_{(m_0,\mu_0)} + \cos(\alpha)\langle\overline{\chi}_t i\gamma_5\chi_t\rangle_{(m_0,\mu_0)}\Big] = \mathcal{O}(a)$$

[1] Andrea Shindler. In: *Nuclear Physics B* 881 (Apr. 2014).

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- The c_{fl} and b_2 are improvement coefficients.
- As a condition to determine m_{cr} and c_{fl} we can impose that the term in the brackets vanishes.

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$\mathcal{O}(a)$-effects

$$\cot(\tilde{\alpha})\langle\overline{\chi}_t i\gamma_5\chi_t\rangle_{(m_0,\mu_0)} + ac_{fl}\langle\overline{\eta}_t\eta_t\rangle = \langle\overline{\chi}_t\chi_t\rangle_{(m_0,\mu_0)} \qquad \tilde{\alpha} = \alpha + ab_2\mu$$

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- This determines $\cot\tilde{\alpha}$ and c_{fl} as a function of m_0, μ_0, t_1 and t_2

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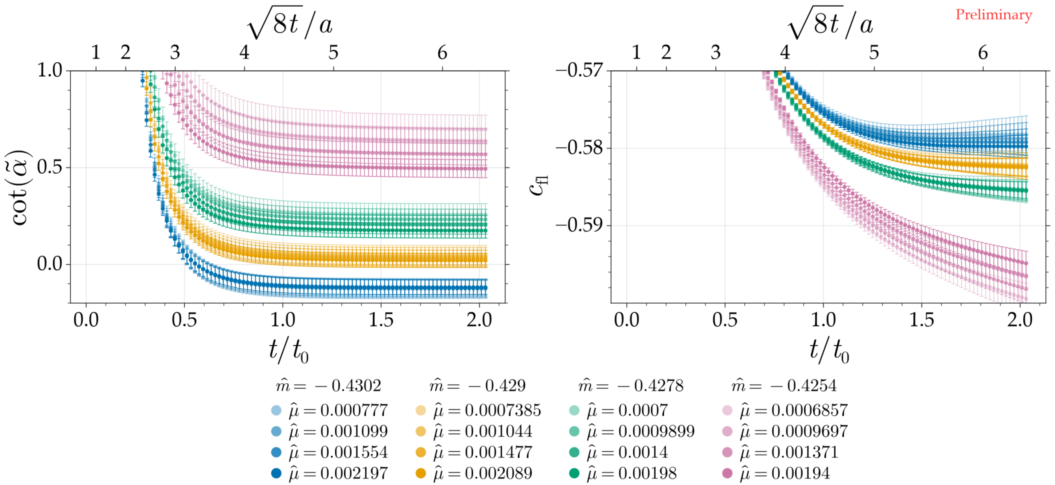
- This determines $\cot\tilde{\alpha}$ and c_{fl} as a function of m_0, μ_0, t_1 and t_2
- We set $t_1 = t$ and $t_2 = t + \delta t$ and plot $\cot\tilde{\alpha}$ and c_{fl} as function of t

$\cot(\tilde{\alpha})$ and c_{fl}



ETMC Ensembles

$N_T \times N_L^3$	a	M_π	LM_π	N_{cnfg} used
96×48^3	0.09 fm	174 Mev	3.8	47

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Determination of $m_{\rm cr}$

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- Determining $\hat{m}_{\rm cr}$ from $\tilde{\alpha} = \alpha + ab_2\mu$ with $\cot(\alpha) = \frac{m_R}{\mu_R}$
- One needs to include the $\mathcal{O}(a)$-effects in the renormalized masses, so with $\delta m = m_0 - \frac{1}{a}\hat{m}_{\rm cr}$ we impose

$$\tilde{\alpha} = \mathrm{acot}\left(\frac{Z_m}{Z_\mu}\frac{\delta m + N_1 a\delta m^2 + N_2 a\mu^2}{\mu + D_1 a\delta m\mu}\right) + b_2 a\mu$$

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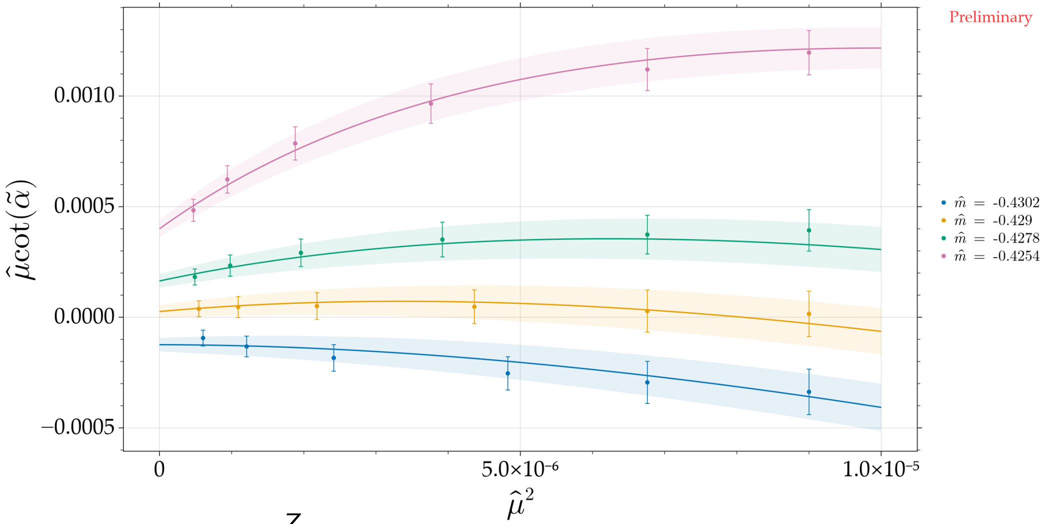
- Performe a non-linear fit to extract $\hat{m}_{\mathrm{cr}}$ and $\frac{Z_m}{Z_\mu}$

Non-Linear Fit



$$\frac{Z_m}{Z_\mu} = 0.122(5) \qquad \hat{m}_{cr} = -0.42924(24)$$

Outlook

- Next step is to calculate Z_A the renormalisation current of the axial current
- Then calculate the relative renormalization constants of

$$H_{ew} = Z\left\{ O_1 + \sum_{j=2}^{5} r_j O_j \right\}$$

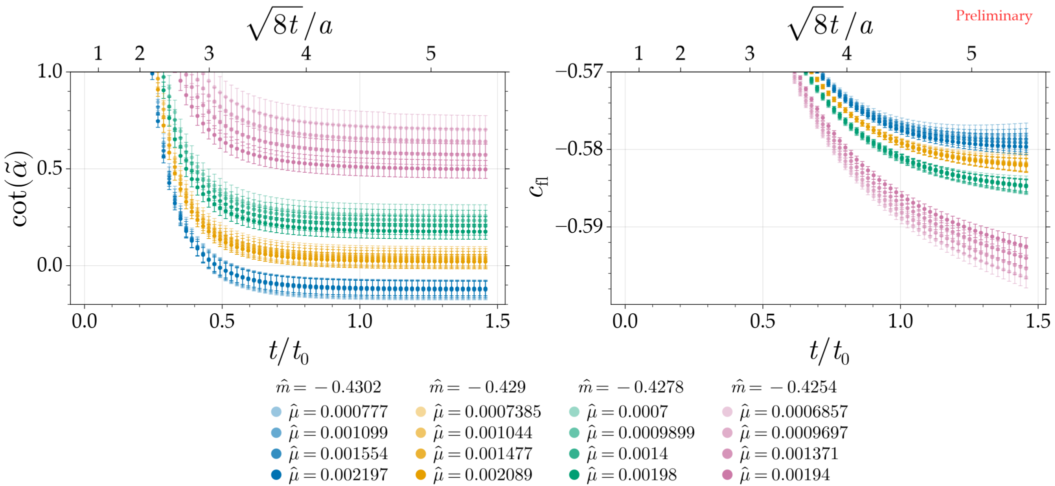
by imposing

$$\langle H_{\mathrm{ew}} O(t)_j^{\dagger} \rangle = 0 \qquad \text{for} \qquad j = 2, 3, 4, 5$$

Full Fit Results

$$\hat{m}_{cr} = -0.42924(24) \qquad Z = 0.122(5)$$
$$N_1 = -147(10) \qquad N_2 = 1996(118)$$
$$D_1 = -140(4) \qquad b_2 = 220(7)$$

$t_1 = t$ and $t_2 = \sqrt{2}t$



$$S = S_G + S_{G,\text{fl}} + S_{\text{sea}} + S_{\text{val}} + S_{\text{val,fl}} + S_{\text{gh}} + S_{\text{ms}}$$

- The ETMC setup for the gluon is the mean-field improved Iwasaki action

$$S_G = \frac{\beta}{3}\sum_x \sum_{\mu<\nu}\left(b_0\left\{1 - \mathrm{ReTr}[U_{\mu\nu}(x)^{1\times 1}]\right\} + b_1\left\{1 - \mathrm{ReTr}[U_{\mu\nu}(x)^{1\times 2}]\right\}\right)$$

- $b_1 = 0.311$ and $b_0 = 1 - 8b_1$
- The sea quarks are also twisted mass fermions consisting of a light-quark doublet $\chi_l^T = \begin{pmatrix} u & d \end{pmatrix}$ and a heavy doublet $\chi_h^T = \begin{pmatrix} c & s \end{pmatrix}$

$$S_{\text{sea}} = \sum_x \overline{\chi}_{\text{sea}}\left\{D_{0,\text{sw}}[U] + M_{\text{sea}} i\gamma_5 + M_{5,\text{sea}}\right\}\chi_{\text{sea}}$$

$$\chi_{\text{sea}}^T = \begin{pmatrix} \chi_l^T & \chi_h^T \end{pmatrix} \quad M_{\text{sea}} = m_{\text{cr}}^{\text{sim}} I_4 + \begin{pmatrix} 0 & 0 \\ 0 & \sigma_3 \end{pmatrix} \quad M_{\text{sea}} = \begin{pmatrix} \sigma_3 \mu_l^{\text{sim}} & 0 \\ 0 & \sigma_1 \mu_\sigma^{\text{sim}} \end{pmatrix}$$

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- Flowed valance action

$$S_{\mathrm{val,fl}} = \epsilon \sum_{t=0}^{T-\epsilon} \sum_{x} \left[\overline{\lambda}_t(x)(\partial_t - \Delta)\chi_t(x) + \overline{\chi}_t(x)(\overleftarrow{\partial}_t - \overleftarrow{\Delta})\lambda_t(x) \right]$$

- $\chi_t(x)\big|_{t=0} = \chi(x)$ and $\overline{\chi}_t(x)\big|_{t=0} = \overline{\chi}(x)$
- Flowed gluon action

$$S_{G,\mathrm{fl}} = -2 \sum_{t=0}^{T-\epsilon} \sum_{x,\mu} \mathrm{Tr}\left\{ L_t(x,\mu)\mathcal{P}\left(V_{t+\epsilon}(x,\mu) V_t(x,\mu)^{-1} - e^{-\epsilon g_0^2 (\partial S_W)(V,x,\mu)} \right) \right\}$$

- Projection to algebra $\mathcal{P}(M) = \frac{1}{2}(M - M^\dagger) - \frac{1}{6}\mathrm{Tr}(M - M^\dagger)$

Symanzik Improvement

- The improvements were characterized for Wilson by M. Lüscher and for twisted mass by A. Shindler

$$\int dx^4 c_2 \overline{\lambda}_0(x)\lambda_0(x)$$

$$\int dx^4 \Big\{ \overline{\lambda}_0(x)(c_3 M_{q,\mathrm{val}} + c_4 \mathrm{Tr} M_{q,\mathrm{val}} + c_5 \mathrm{Tr} M_{q,\mathrm{sea}})\chi(x) + \\ \chi(x)(c_3 M_{q,\mathrm{val}} + c_4 \mathrm{Tr} M_{q,\mathrm{val}} + c_5 \mathrm{Tr} M_{q,\mathrm{sea}})\lambda_0(x) \Big\}$$

$$\int dx^4 \Big\{ \overline{\lambda}_0(x)(c_6 M_{5,\mathrm{val}} + c_7 \mathrm{Tr} M_{5,\mathrm{val}}) i\gamma_5 \chi(x) + \chi(x)((c_6 M_{5,\mathrm{val}} + c_7 \mathrm{Tr} M_{5,\mathrm{val}}) i\gamma_5 \lambda_0(x) \Big\}$$

- With $M_q = M - m_{\mathrm{cr}} I_{N_f}$

- $\mathcal{O}(a)$-Counterterms due to the flow are given by

$$\delta S = \sum_x a \Big\{ c_{\mathrm{fl}} \overline{\lambda}_0 \lambda_0 + \check{c}_1 \left(\overline{\lambda}_0 M_{5,\mathrm{val}} i\gamma_5 \chi + \overline{\chi} M_{5,\mathrm{val}} i\gamma_5 \lambda_0 \right) + \check{c}_1 \left(\overline{\lambda}_0 \mathrm{Tr}(M_{5,\mathrm{val}}) i\gamma_5 \chi + \overline{\chi} \mathrm{Tr}(M_{5,\mathrm{val}}) i\gamma_5 \lambda_0 \right) \Big\}$$

$$\chi_{R,f,t}(x) = \left\{ Z_\chi (1 + b_\chi a m_{q,f} + \overline{b}_\chi a \mathrm{Tr}(M_{q,\mathrm{val}}) + \overline{b}_{\mathrm{Tr}}^{\mathrm{sea}} (M_{q,\mathrm{sea}})) \right\}^{\frac{1}{2}} \chi_{f,t}(x)$$

- with the subtracted mass for a specific flavor $m_{q,f} = m_{0,f} - m_{\mathrm{cr}}$
- Terms in δS that are linear in $\overline{\lambda}_0$ or λ_0 can be understood as initial conditions of the flow

- Introducing auxiliary fields $\overline{\eta}$ and η

$$\int [\mathcal{D}\overline{\eta}\mathcal{D}\eta] \exp\left(-\sum_{x} \left\{ -\overline{\eta}\eta + (ac_{\mathrm{fl}})^{\frac{1}{2}}\overline{\lambda}_0\eta + (a_c\mathrm{fl})^{\frac{1}{2}}\overline{\eta}\lambda_0 \right\} \right)$$

$$\propto \exp\left\{ -\sum_{x} ac_{\mathrm{fl}}\overline{\lambda}_0\lambda_0 \right\}$$

- Now all terms can be written as initial conditions

- Write the solution of the flow equation as $\chi_{f,t}(x) = K_t[U]\chi_f(x)$ and $\overline{\chi}_{f,t}(x) = \overline{\chi}_f(x)K_t^\dagger[U]$

$$\chi_{R,f,t}(x) = \left\{Z_\chi(1 + b_\chi a m_{q,f} + \overline{b}_\chi a\mathrm{Tr}(M_{q,\mathrm{val}}) + \overline{b}_{\mathrm{Tr}}^{\mathrm{sea}}(M_{q,\mathrm{sea}}))\right\}^{\frac{1}{2}}$$
$$\times\, e^{i\frac{\gamma_5}{2}(a\check{b}_\chi^{(1)}M_{5,\mathrm{val}}+a\check{b}_\chi^{(2)}\mathrm{Tr}(M_{5,\mathrm{val}}))}K_t[U](\chi_f(x) + \sqrt{ac_{\mathrm{fl}}}\eta(x))$$
$$\overline{\chi}_{R,f,t}(x) = \left\{Z_\chi(1 + b_\chi a m_{q,f} + \overline{b}_\chi a\mathrm{Tr}(M_{q,\mathrm{val}}) + \overline{b}_{\mathrm{Tr}}^{\mathrm{sea}}(M_{q,\mathrm{sea}}))\right\}^{\frac{1}{2}}$$
$$\times\, (\overline{\chi}_f(x) + \sqrt{ac_{\mathrm{fl}}}\overline{\eta}(x))K_t^\dagger[U]e^{i\frac{\gamma_5}{2}(a\check{b}_\chi^{(1)}M_{5,\mathrm{val}}+a\check{b}_\chi^{(2)}\mathrm{Tr}(M_{5,\mathrm{val}}))}$$

Symanzik Improvement

- In the physical basis $\psi_{f,t}(x)$, with the twist angle $\cot\alpha_f = \frac{m_{R,f}}{\mu_{R,f}}$

$$\psi_{R,f,t}(x) = \left\{ Z_\chi (1 + b_\chi a m_{q,f} + \overline{b}_\chi a \mathrm{Tr}(M_{q,\mathrm{val}}) + \overline{b}^{\mathrm{sea}}_{\mathrm{Tr}} (M_{q,\mathrm{sea}})) \right\}^{\frac{1}{2}}$$
$$\times\, e^{i\frac{\gamma_5}{2}(\alpha_f + a\check{b}^{(1)}_\chi M_{5,\mathrm{val}} + a\check{b}^{(2)}_\chi \mathrm{Tr}(M_{5,\mathrm{val}}))} K_t[U](\chi_f(x) + \sqrt{a c_{\mathrm{fl}}}\,\eta(x))$$

$$\overline{\psi}_{R,f,t}(x) = \left\{ Z_\chi (1 + b_\chi a m_{q,f} + \overline{b}_\chi a \mathrm{Tr}(M_{q,\mathrm{val}}) + \overline{b}^{\mathrm{sea}}_{\mathrm{Tr}} (M_{q,\mathrm{sea}})) \right\}^{\frac{1}{2}}$$
$$\times\, (\overline{\chi}_f(x) + \sqrt{a c_{\mathrm{fl}}}\,\overline{\eta}(x)) K^\dagger_t[U] e^{i\frac{\gamma_5}{2}(\alpha_f + a\check{b}^{(1)}_\chi M_{5,\mathrm{val}} + a\check{b}^{(2)}_\chi \mathrm{Tr}(M_{5,\mathrm{val}}))}$$

Ensemble Overview

Ensemble	$N_T \times N_L^3$	β	a	M_π	LM_π	κ_{cr}	c_{sw}
A48	96×48^3	1.726	0.09 fm	174 Mev	3.8	0.140065	1.74

- Ensemble of the ETMC collaboration[2]
- $\frac{t_0}{a^2} = 2.438$ and we flow to $2t_0$ with a step size of $\epsilon = 0.01$.
- The one point functions $\langle \overline{\chi}_t \chi_t \rangle_{(M,\alpha)}$, $\langle \overline{\chi}_t i\gamma_5 \chi_t \rangle_{(M,\alpha)}$ and $\langle \overline{\eta}\eta \rangle$ are measured every 5 steps.
- We solve the system for different values of $\hat{m}$ and $\hat{\mu}$ for $t_1 = t$ and $t_2 = t + \delta t$ with $\delta t = 5\epsilon$

[2] C. Alexandrou et al. 2024.