Gradient flow scale setting with the pion decay constant in QCD+QED

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on behalf of the Budapest-Marseille-Wuppertal collaboration

Based on *Hybrid calculation of hadronic vacuum polarization in muon $g-2$ to 0.48%*

Standard Model parameters and observables from gradient flow,
Edinburgh, May 14th 2026

Motivations

Lattice QCD simulations give access to dimensionless quantities:

$$aM_P, af_P, t_0/a^2, w_0/a, \ldots$$

One quantity is used to define the lattice spacing:

$$a = \frac{[a\mathcal{O}]^{\text{latt.}}}{[\mathcal{O}]^{\text{exp.}}}.$$

The quantity $\mathcal{O}$ needs to be:

- Easy to determine precisely numerically;
- Determined experimentally with high precision.

Example of the impact of scale setting on $\left(a_\mu^{\text{hvp}}\right)^{ud}$ [Della Morte et al., 1705.01775]

$$\frac{\delta a}{a} \approx \pm 1\% \rightarrow \frac{\delta \left(a_\mu^{\text{hvp}}\right)^{ud}}{\left(a_\mu^{\text{hvp}}\right)^{ud}} \approx \pm 1.8\%.$$

Gradient flow

The Wilson flow [Lüscher, 1006.4518]:

$$\frac{d}{dt} V_\mu(t,x) = \left\{ -g_0^2 \partial_\mu^x S_{\mathrm{W}}[V] \right\} V_\mu(t,x), \quad V_\mu(0,x) := U_\mu(x).$$

The Zeuthen flow [Ramos and Sint, 1508.05552]:

$$\frac{d}{dt} V_\mu(t,x) = \left\{ -\left(1 + \frac{1}{12} \nabla_\mu^* \nabla_\mu\right) g_0^2 \partial_\mu^x S_{\mathrm{LW}}[V] \right\} V_\mu(t,x)$$

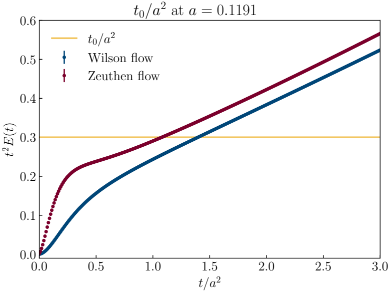
The scale is derived from the action density:

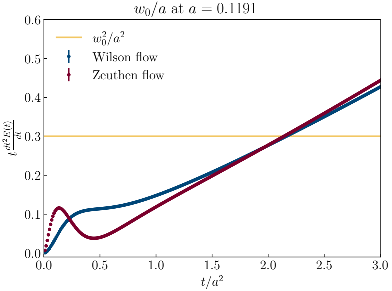
$$E^{\mathrm{cl.}}(t): \text{ clover definition of the action density,}$$
$$E^{\mathrm{im.}}(t): \text{ improved definition of the action density.}$$

Gradient flow scale setting

The scales are t_0/a^2 [Lüscher, 1006.4518] and w_0/a [Borsányi et al., 1203.4469]:

$$\left[t^2\langle E(t)\rangle\right]\Big|_{t=t_0} = 0.3, \qquad t\frac{d\left[t^2\langle E(t)\rangle\right]}{dt}\Bigg|_{t=w_0^2} = 0.3.$$





The two scales are precisely determined from simulations. A physical observable is required to measure the "physical" value of the scale.

Pion decay constant in QCD

In QCD it is computed from the axial current:

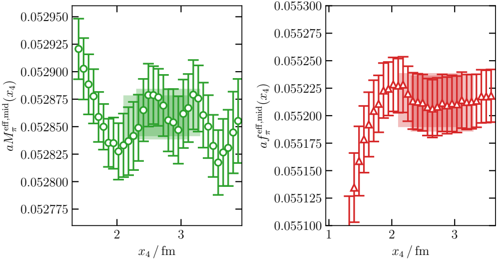
$$\langle 0|\, \bar{u}\gamma_4\gamma_5 d(x)\, |\pi(\vec{p})\rangle = e^{ipx} M_\pi f_\pi\,,$$

computed from:

$$\langle \mathcal{A}_4^{ud}(t) P^{ud\dagger}(0)\rangle \underset{\substack{T-t\to\infty \\ t\to\infty}}{\sim} e^{-tM_\pi}\, \langle 0|\, \mathcal{A}_4^{ud}(t)\, |\pi(\vec{0})\rangle\,.$$

If chiral symmetry is preserved we use the WTI:

$$\frac{\partial_\mu \mathcal{A}_\mu^{ud}(x)}{(m_u + m_d)} = P^{ud}(x) \to f_\pi^{\mathrm{eff}}(x_4) = \sqrt{\frac{8m_l^2 G^P(x_4)}{M_\pi^{\mathrm{eff}}(x_4)^3}\, \frac{e^{M_\pi^{\mathrm{eff}}(x_4)\frac{T}{2}}\left(1 - e^{-M_\pi^{\mathrm{eff}}(x_4)T}\right)}{\cosh\left(M_\pi^{\mathrm{eff}}(x_4)(x_4 - \frac{T}{2})\right)}}\,,$$



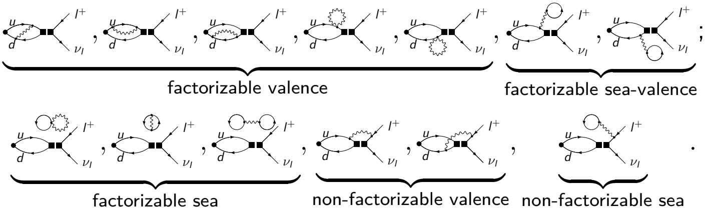
Pion decay constant in QCD+QED

In QCD+QED it is defined following the PDG parametrization:

$$F_\pi^2 = \frac{\Gamma(\pi \to \mu \bar{\nu}_\mu [\gamma])}{\frac{G_F^2}{8\pi} |V_{ud}|^2 M_{\pi^\pm} m_\mu^2 \left(1 - \frac{m_\mu^2}{M_{\pi^\pm}^2}\right)^2} = f_\pi^2 \left[1 + \delta R_\pi\right].$$

Challenges:

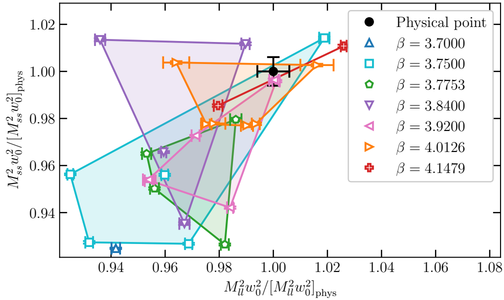
- Renormalization of electroweak hamiltonian;
- Intermediate infrared divergences;
- Power-law finite volume effects;
- Many diagrams to compute:



Only one computation available in literature [Di Carlo et al., 1904.08731]. More results to the IBE to f_K/f_π [Giusti et al., 1711.06537] [Boyle et al., 2211.12865] [Christ et al., 2510.26993].

Simulation details

- Tree-level improved Lüscher-Weisz gauge action;
- $N_f = 2 + 1 + 1$ staggered fermions;
- 4 levels of stout smearing $\rho = 0.125$;
- Physical Pion mass;
- $L \approx 6$ fm;



Analysis details

- Statistical errors from jackknife;
- $w_0 F_\pi$ parametrized by:

$$
\begin{aligned}
w_0 F_\pi = & A + B F_\pi^{-2} M_{\pi^\pm}^2 + C F_\pi^{-2} \left(M_{K^\pm}^2 + M_{K^0}^2 - M_{\pi^\pm}^2 \right) /2 \\
& + E e_v^2 + F e_v e_s + G e_s^2
\end{aligned}
$$

e_v : valence electric charge, e_s : sea electric charge,
$B,\ C$: mass derivatives
$E,\ F,\ G$: electromagnetic derivatives

- $A(a)$ polinomial in a^{2n} up to $n = 3$, $a^2 \alpha_s^\gamma(a)$ with $\gamma \in \{0, 0.5, 1.0, 1.5, 2.0, 2.5\}$;
- $B(a)$ and $C(a)$ constant or linear in a^2;
- Modified Akaike Information Criterion, fit function weighted by [Borsanyi et al., 2002.12347]:

$$
w = \exp\left[-\frac{1}{2} \left(\chi^2 + 2 n_{\text{par}} - n_{\text{data}} \right) \right].
$$

Iso QCD determination in FLAG scheme

Continuum Extrapolation

FLAG scheme [Aoki et al., 2111.09849]:

$$[M_\pi]_{\mathrm{qcd,FLAG}} = 135.0 \text{ MeV},$$
$$\left[\frac{1}{2}(M_{K^\pm} + M_{K^0})\right]_{\mathrm{qcd,FLAG}} = 494.6 \text{ MeV},$$
$$[f_\pi]_{\mathrm{qcd,FLAG}} = 130.5 \text{ MeV}.$$

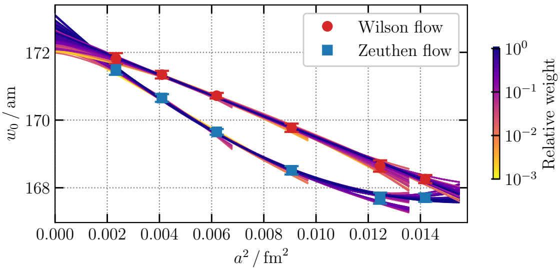
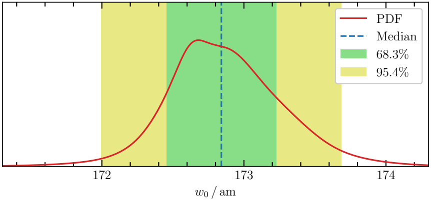


Figure: Continuum limit extrapolation of the iso QCD value of w_0 in physicsl units.

Iso QCD determination in FLAG scheme

Error Budget

Number of fits	12672	
Fits with $P > 0.1$	74%	
Median	172.84	
Total error	0.42	0.24%
Statistical error	0.34	0.20%
Systematic error	0.25	0.14%
Pseudoscalar fits	0.15	0.09%
Finite volume ChiPT	0.03	0.02%
Type of flow (Ze/Wi)	0.14	0.08%
Lattice spacing cut	0.04	0.02%
Fit polynomial order	0.11	0.06%
Log corrections γ	0.07	0.04%



Current mixed determination

Once A, B, C, G, F are computed we can separate:

$$\underbrace{[w_0 F_\pi]_{\text{QCD+QED}}}_{\text{scheme independent}} = \underbrace{[w_0 F_\pi]_{\text{QCD+seaQED}}}_{\text{scheme dependent}} + \underbrace{[w_0 F_\pi]_{\text{valQED}}}_{\text{scheme dependent}}$$

w_0 has no valence quark isospin-breaking effects. The QCD from RM123S is defined in the GRS scheme:

$$\begin{aligned} [M_\pi]^{\text{GRS}} &= 135.0(2)\ \text{MeV} \\ \left[\frac{1}{2}\left(M_{K^\pm} + M_{K^0}\right)\right]^{\text{GRS}} &= 494.6(1)\ \text{MeV} \\ [f_\pi]^{\text{GRS}} &= 130.65(12)\ \text{MeV} \\ [\delta R_\pi]^{\text{GRS}} &= 0.0153(19). \end{aligned}$$

The quantities in the GRS scheme are computed as:

$$[*]^{\text{GRS}} = [*]^{\text{phys}}_{\text{QCD+QED}} - [*]^{\text{GRS}}_{\text{valQED}} = [*]^{\text{GRS}}_{\text{QCD+seaQED}}\,.$$

We can combine the BMW and RM123 results using the parametrisation:

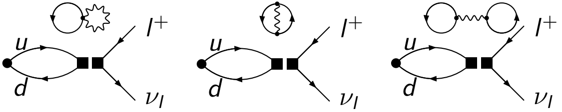
$$[w_0 F_\pi]_{\text{QCD+QED}} = [w_0 F_\pi]^{\text{GRS}}_{\text{QCD+seaQED}} \sqrt{1 + [\delta R_\pi]^{\text{GRS}}}.$$

Sea-sea contribution

The renormalized sea-quark electromagnetic derivative is computed from:

$$G = \left[w_0 F_\pi - B F_\pi^{-2} M_{\pi^\pm}^2 - C F_\pi^{-2} \left(M_{K^\pm}^2 + M_{K^0}^2 - M_{\pi^\pm}^2\right)/2\right]_{\text{sea-sea}} .$$

The sea-sea diagrams are:



.

They have been estimated using the following ensembles and QED_L.

action	β	a [fm]	$L/a \times T/a$	tag	am_s	m_s/m_l	#meas
4stout	3.7000	0.1315	24×48	volume/24	0.057291	27.899	716
			48×64	volume/48	0.057291	27.899	300
	3.7753	0.1116	28×56	dir00	0.047615	27.843	887
	3.8400	0.0952	32×64	dir00	0.043194	28.500	1110
				dir02	0.043194	30.205	1072
				dir04	0.040750	28.007	1036
				dir05	0.039130	26.893	1035
4hex	0.7300	0.1120	56×84	phys2/56	0.06061	33.728	1305

Table: Set of ensembles used for the computation of the sea quark isospin-breaking effects.

Sea-sea contribution

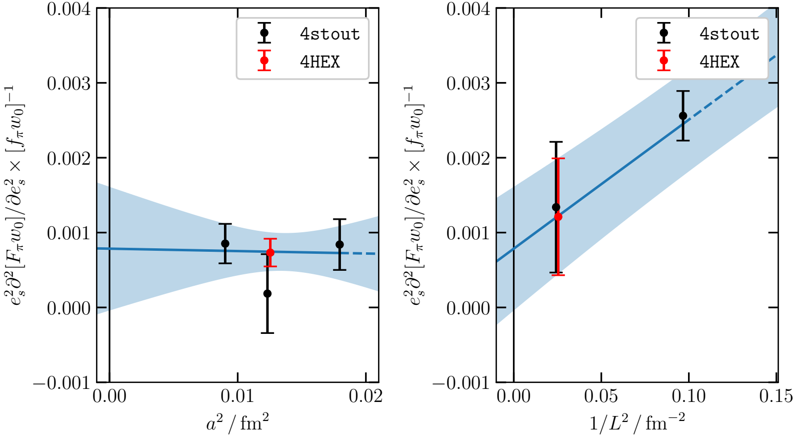


Figure: Renormalized sea quark electromagnetic derivative of $w_0 F_{ud}$ divided by its QCD value.

Total error budget

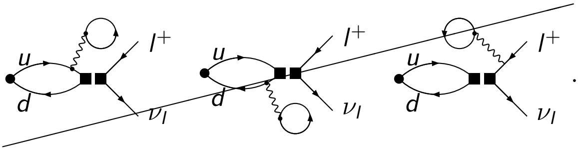
The QCD+QED value of w_0:

$$[w_0]_{\mathrm{QCD+QED}} = 0.17264(36)_{\mathrm{stat}}(33)_{\mathrm{sys}}[48]\ \mathrm{fm}$$

The systematic error is:

$$(33) = \underbrace{(27)}_{\text{isoQCD}}\ \underbrace{(18)}_{\substack{\text{valence}\\ \text{QED}}}\ \underbrace{(9)}_{\text{sea QED}}\ \underbrace{(6)}_{\substack{\text{experimental}\\ \text{value of } V_{ud}}}\ \underbrace{(3)}_{\substack{\text{other experimental}\\ \text{inputs}}}$$

Sea-valence derivative has been estimated to be 20% of the sea-sea derivative.



- room for improvement of the QED valence determination;
- independent determination of δR_π and check FVE [Di Carlo et al., 2109.05002];

Comparison with existing values

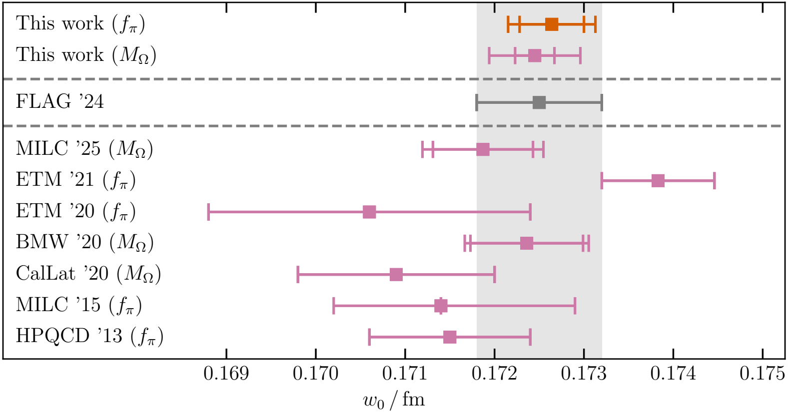


Figure: Comparison between the state of the art determinations of w_0 from different inputs.

Conclusions and outlook

- The determination of w_0 from the pion decay constant is:

$$[w_0]_{\mathrm{QCD+QED}} = 0.17264(36)(33)[48]\ \mathrm{fm};$$

- The resulting value is compatible with the determination based on the M_Ω and with the FLAG average;
- We are working on improving the continuum extrapolation with a finer ensemble;
- We are working on a fully independent determination of the isospin-breaking effects including the sea-valence diagrams;
- We plan a detailed study of the finite volume effects with different volumes (up to $L = 10.8$ fm);

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Thank you for your attention!

Backup: set of ensembles for iso QCD f_π

β	a [fm]	$L/a \times T/a$	tag	am_s	m_s/m_l	#confs
3.7000	0.1315	48×64	dir00	0.057291	27.899	904
3.7500	0.1191	56×96	dir00	0.049593	28.038	315
			dir01	0.049593	26.939	516
			dir02	0.051617	29.183	504
			dir03	0.051617	28.038	522
			dir05	0.055666	28.083	215
3.7753	0.1116	56×84	dir00	0.047615	27.843	510
			dir01	0.048567	28.400	505
			dir02	0.046186	26.469	507
			dir03	0.049520	27.852	385
3.8400	0.0952	64×96	dir00	0.043194	28.500	510
			dir02b	0.043194	30.205	436
			dir04	0.040750	28.007	1503
			dir05	0.039130	26.893	500

Table: List of the ensembles used in this work, with gauge coupling, lattice spacing, lattice size, ensemble tag, strange-quark mass, mass ratio of strange and light quarks and number of configurations.

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β	a [fm]	$L/a \times T/a$	tag	am_s	m_s/m_l	#confs
3.9200	0.0787	80×128	dir02	0.032440	27.679	506
			dir04	0.034240	27.502	512
			dir01b	0.032000	26.512	1001
			dir02b	0.032440	27.679	327
			dir03b	0.033286	27.738	1450
			dir04b	0.034240	27.502	500
4.0126	0.0640	96×144	phys1	0.026500	27.634	446
			phys2	0.026500	27.124	551
			phys1b	0.026500	27.634	2248
			phys2b	0.026500	27.124	1000
			phys3	0.027318	27.263	985
			phys4	0.027318	28.695	1750
4.1479	0.0483	128×192	phys1	0.019370	27.630	2792
			phys2	0.019951	27.104	2225

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