Scale setting in finite-temperature QCD via gradient-flow rescaling

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Outline

1 **Introduction:** finite-temperature QCD and scale-setting

2 **Flow rescaling:** idea and implementation to set the scale

3 **Pure-gauge tests:** T=0 checks, discretization and volume effects

4 **Preliminary results:** tuning of bare coupling for 2+1+1 QCD

5 **Conclusions:** summary and plans

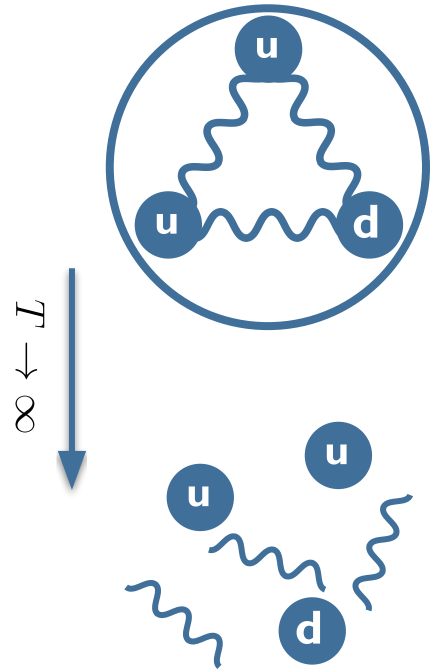
Introduction – Finite-T QCD and phenomenology

Finite-temperature regime of QCD from lattice simulations:

- Deconfinement transition
- Equation of state
- Topological features

Phenomenological implications for collider physics and cosmology:

- Quark-gluon plasma in neutron stars and heavy-ion collision
- Axion mass and abundance from topological susceptibility



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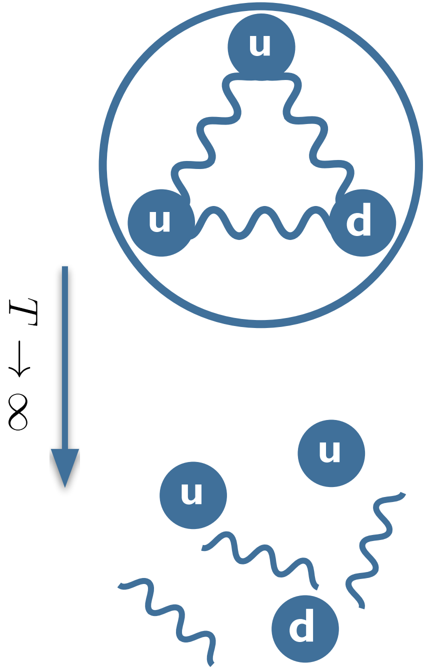


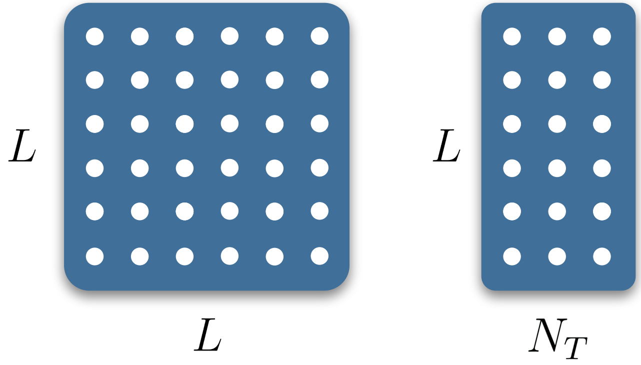
Precise determinations of temperature-dependence require precise scale-setting over broad range of temperatures:

$$\chi = \frac{\langle Q^2 \rangle}{V}\,, \qquad \chi(T) \propto T^{-8} \quad \Longrightarrow \quad \frac{\delta[a^4\chi(T)]}{a^4\chi(T)} \sim 12\frac{\delta a}{a}$$

Introduction – Finite-T scale setting

Direct scale setting:

- Tune bare masses $\hat{m}_q$ and set β at $T = 0$,
  $m_\pi \ell \gtrsim 3$ to control finite-volume effects
- Repeat simulation at finite $T^{-1} = a(\beta) N_T$,
  $N_T \geq 8$ to control lattice artifacts
- To reach $T \simeq 1\ \mathrm{GeV}$ scale:
  $a(\beta) \lesssim 0.025\ \mathrm{fm}\,,\ L = \ell/a \gtrsim 170$



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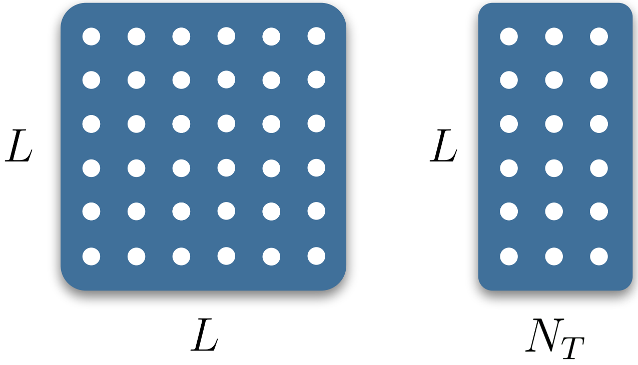
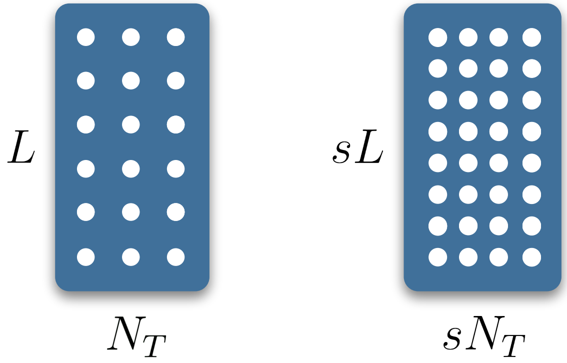


Relative scale setting:

- Take already set $\beta^{(\mathrm{ref})}$, $\hat{m}_q^{(\mathrm{ref})}$ as reference
- Determine β, $\hat{m}_q$ such that $s = a(\beta^{(\mathrm{ref})})/a(\beta) > 1$ by tuning same $g^2(\mu = T)$ and screening masses $M_s(T)/T$
- Step-scaling (Lüscher et al., 1991) to connect different T, no need for increasingly large lattices



Introduction – Step-scaling with gradient flow

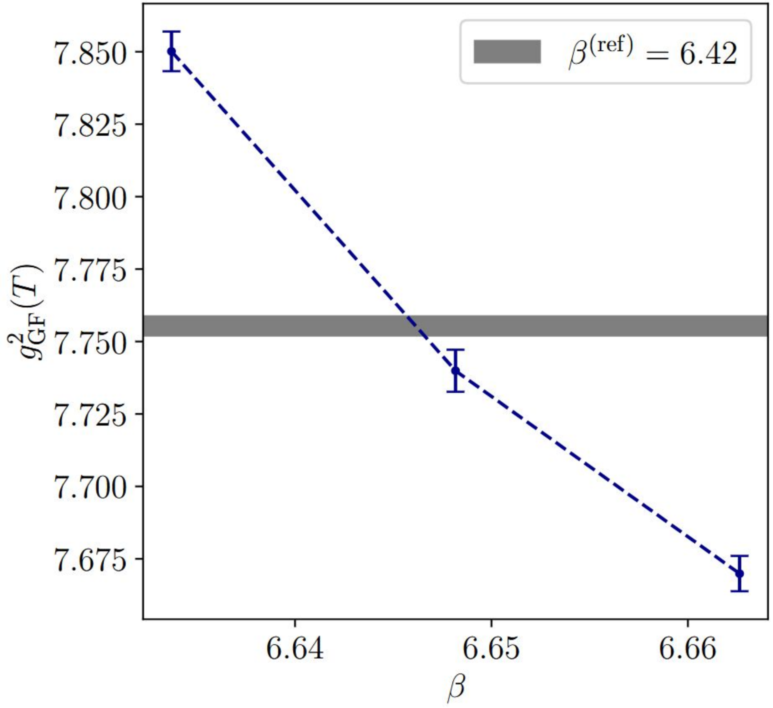
Gradient-flow coupling in finite-size ℓ scheme:

$$g^2_{\rm GF}(\mu) = \frac{16\pi^2}{3}\,\langle t^2 E(t)\rangle\Big|_{\sqrt{8t_c}=\mu^{-1}} = g^2_{\overline{MS}}(\mu) + O(g^4_{\overline{MS}})$$

Choice of $\sqrt{8t_c} = c\ell$ defines different schemes, trade-off between cutoff effects and statistical precision (Fritzsch, Ramos, 2013)

With thermal boundary conditions:

$$\sqrt{8t_c} = T^{-1} \implies \mu = T$$



Example in pure-gauge theory. Possible tuning: PT-inspired fit and interpolation

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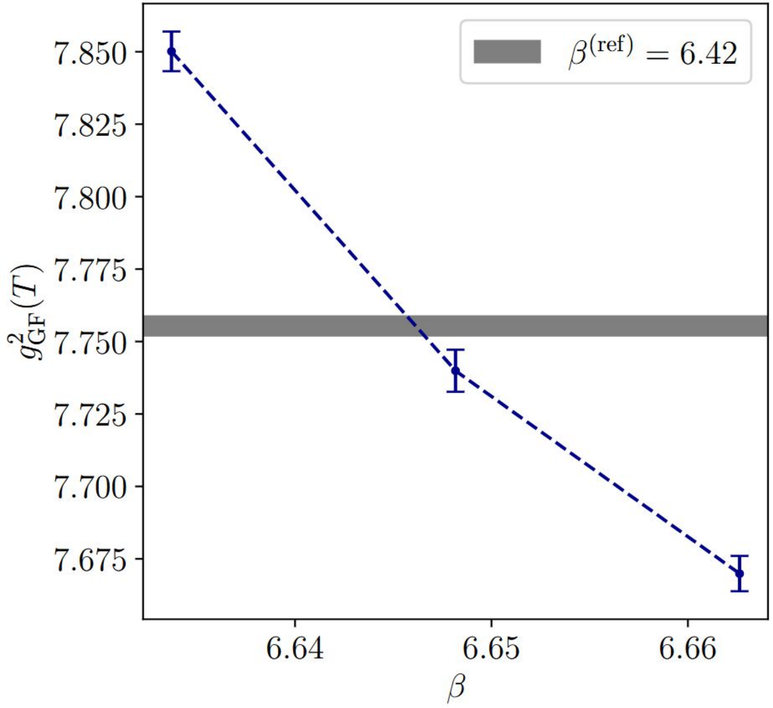
On the lattice, integrate up to and measure at:

$$\hat{t}_c \equiv t_c/a^2 = N_T^2/8$$

Idea: use information from multiple flow times to control systematics of the choice of $\hat{t}_c$



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Flow rescaling – General idea

Continuum flow at finite volume $\ell_0 = 1/T$, ℓ_i (omitting m_q):

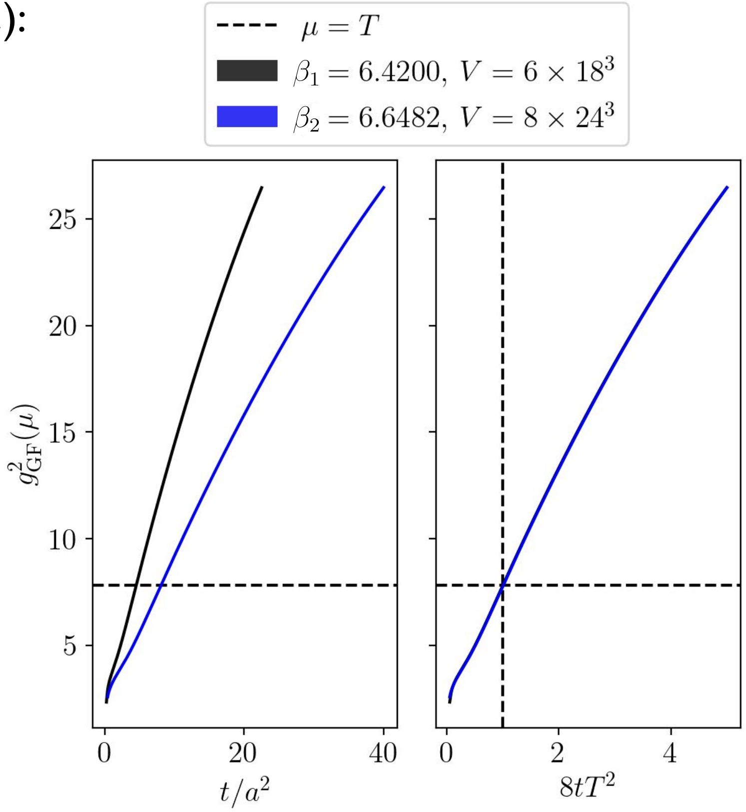
$\varphi(t,\ell) \equiv \langle t^2 E(t,\ell)\rangle \propto g^2_{\rm GF}$

Lattice flows at different $a(\beta)$ but same ℓ differ for (up to lattice artifacts)

rescaling factor $s(\beta_1,\beta_2) \equiv \dfrac{a(\beta_1)}{a(\beta_2)}$

Curves collapse if $\hat{t}_2 \longrightarrow \hat{t}_2/s^2$:

$\hat{\varphi}_1(\hat{t},\ell) \simeq \hat{\varphi}_2(s^{-2}\hat{t},\ell)$



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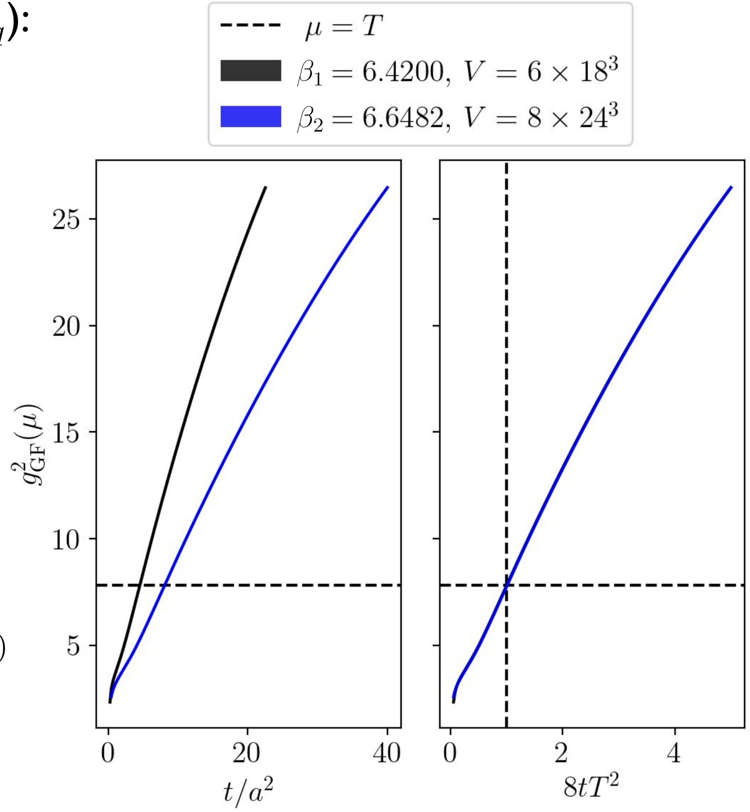
Equal-volume (and masses) lattices require scale setting, but mismatch effects can be iteratively reduced:

1. Guess β_2 (PT, extrapolation) for desired $\bar{s} = N_T^{(2)}/N_T^{(1)}$
2. Determine $s(\beta_1,\beta_2)$ (up to systematics)
3. Refine β_2 until $s(\beta_1,\beta_2) \simeq \bar{s}$



Flow rescaling – Rescaling via best-fit

Original approach (Junnarkar, Moore, and Chaumet, 2023):

Find $s(\beta_1, \beta_2)$ optimizing superposition of spline interpolations of $\hat{\varphi}_1(\hat{t})$, $\hat{\varphi}_2(s^{-2}\hat{t})$
Fit parameters: shared spline coefficients and s

Tested in pure-gauge and applied to 2+1+1 HISQ QCD, scale setting for $T \simeq 341,\ 435,\ 511\ \mathrm{MeV}$ down to $a \simeq 0.014\ \mathrm{fm}$

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- Measures at close flow times are strongly correlated: weights in χ^2 multiplied by $\Delta\hat{t}/\hat{t}$ to compensate
- Fit must be restricted to appropriate window: lattice artifacts at small $\hat{t}$, volume mismatch effects at large $\hat{t}$

Idea: avoid need for best-fit by working with inverse function $\hat{t}(\hat{\varphi})$

Flow rescaling – Rescaling via inverse flow

Finite-volume $\ell = a(\beta)L$ flow on the lattice:

$$\hat{\varphi}(\hat{t}, \beta, L) = \varphi(t, \ell)\left[1 + \frac{a^2(\beta)}{8t} f(\varphi, \ell) + \dots\right]$$

$$\varphi(t, \ell) \equiv \langle t^2 E(t, \ell)\rangle \,, \quad \psi(t, \ell) \equiv t \frac{\partial}{\partial t} \varphi(t, \ell)$$

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Inverse function (at intermediate flow times):

$$\hat{t}(\hat{\varphi}, \beta, L) = a^{-2}(\beta) t(\varphi, \ell)\left[1 + \frac{\varphi}{\psi(t, \ell)}\frac{1}{8\hat{t}} f(\varphi, \ell) + \dots\right]$$

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Two lattices tuned to have $a(\beta_2) \simeq a(\beta_1)/\bar{s}$:

$$L_2 = \bar{s} L_1 \quad \delta \equiv \ell_2/\ell_1 - 1 = \bar{s}/s(\beta_1, \beta_2) - 1$$

Flow rescaling – Rescaling via inverse flow

Ratio of lattice flow times $\hat{t}$ at equal flow $\hat{\varphi}$:

$$\frac{\hat{t}(\hat{\varphi},\beta_2,L_2)}{\hat{t}(\hat{\varphi},\beta_1,L_1)} = s^2(\beta_1,\beta_2)\left[1+\sum_\mu g_\mu(\varphi,\ell)\delta + \frac{1-1/\bar{s}^2}{8\hat{t}(\varphi,\beta_1,L_1)}h(\varphi,\ell)+\dots\right]$$

Actual ratio of lattice spacings

$$s(\beta_1,\beta_2) \equiv \frac{a(\beta_1)}{a(\beta_2)}$$

IR effects suppressed by tuning:

$$g_\mu(\varphi,\ell) = \frac{\partial \log[t(\varphi,\ell)]}{\partial \log \ell_\mu}$$

$$\delta = \bar{s}/s(\beta_1,\beta_2) - 1$$

Realistically $\delta \sim O(10^{-2})$

Includes m_q-derivatives in QCD

Dominates at large $\hat{t}(\hat{\varphi})$

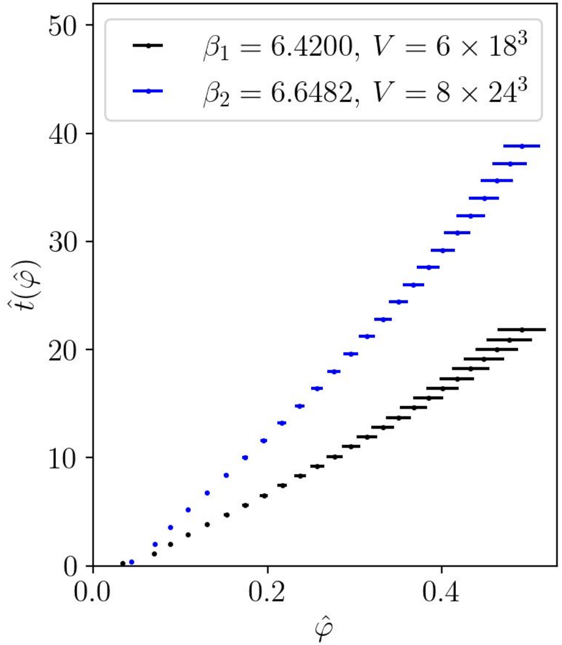
Lattice artifacts:

$$h(\varphi,\ell) = -\frac{\varphi}{\psi(\varphi,\ell)}f(\varphi,\ell)$$

Partial cancellation

Dominates at small $\hat{t}(\hat{\varphi})$

(error x20, 1/20 points)



Lattice measures of $\hat{\varphi}(\hat{t})$ at discrete times

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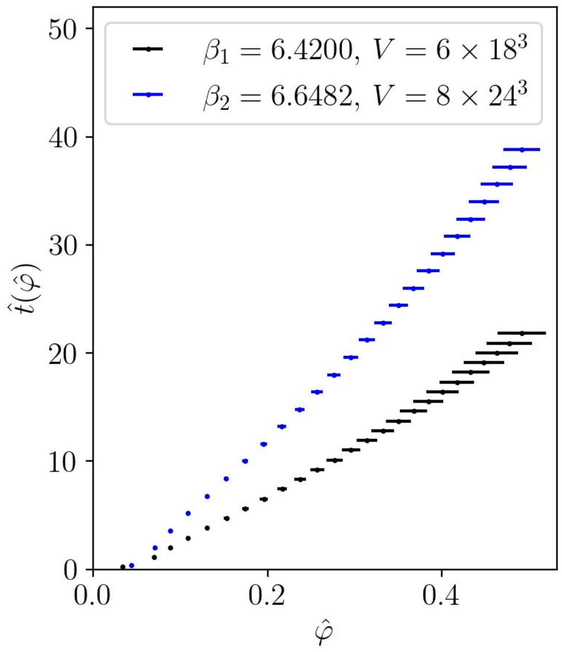

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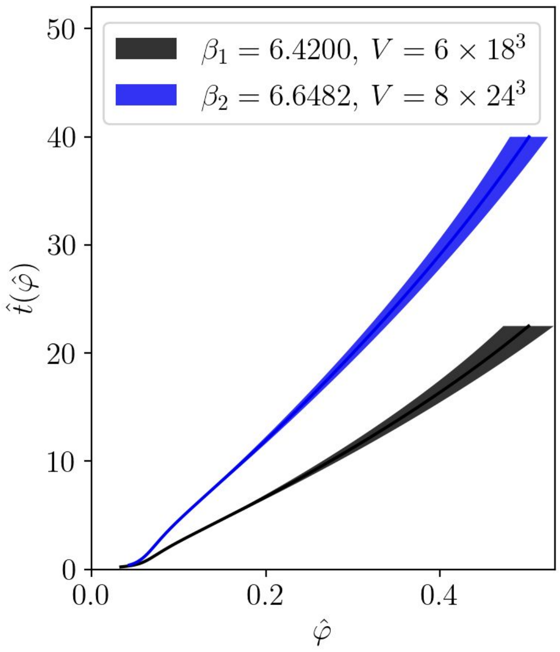

Lattice measures of $\hat{\varphi}(\hat{t})$ at discrete times → Spline interpolation to evaluate at any point

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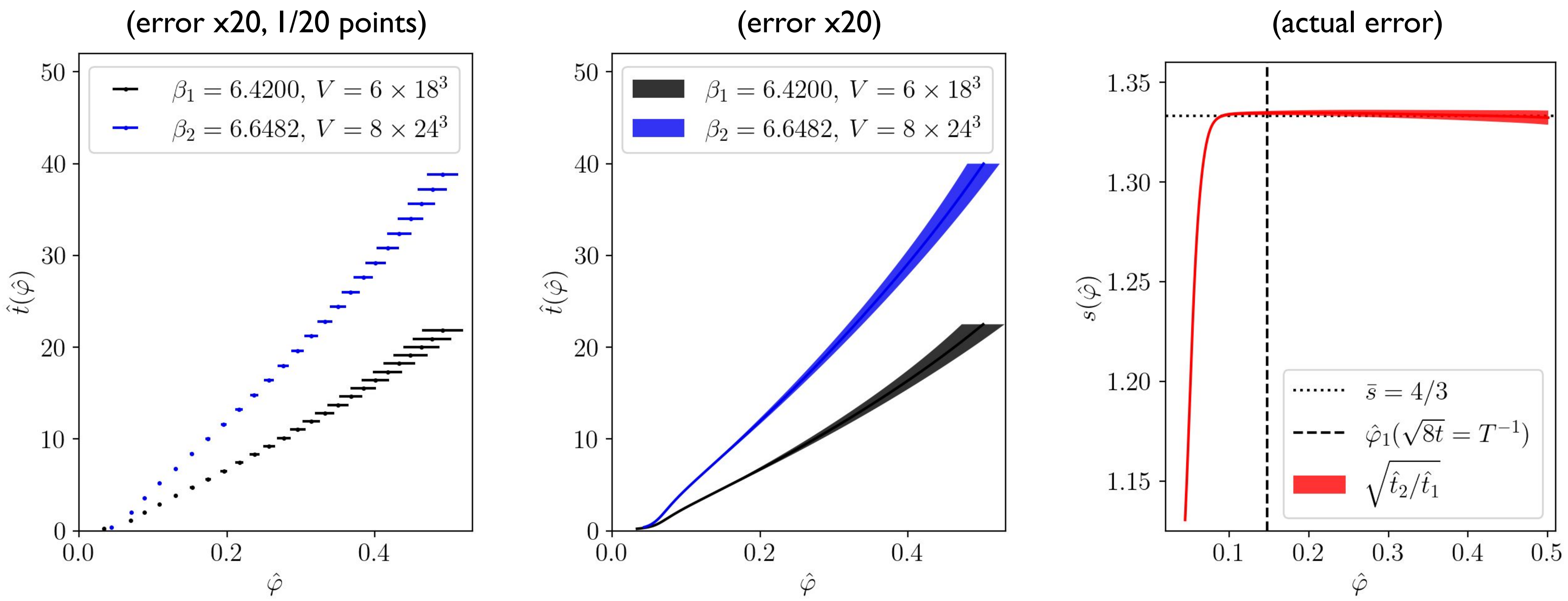
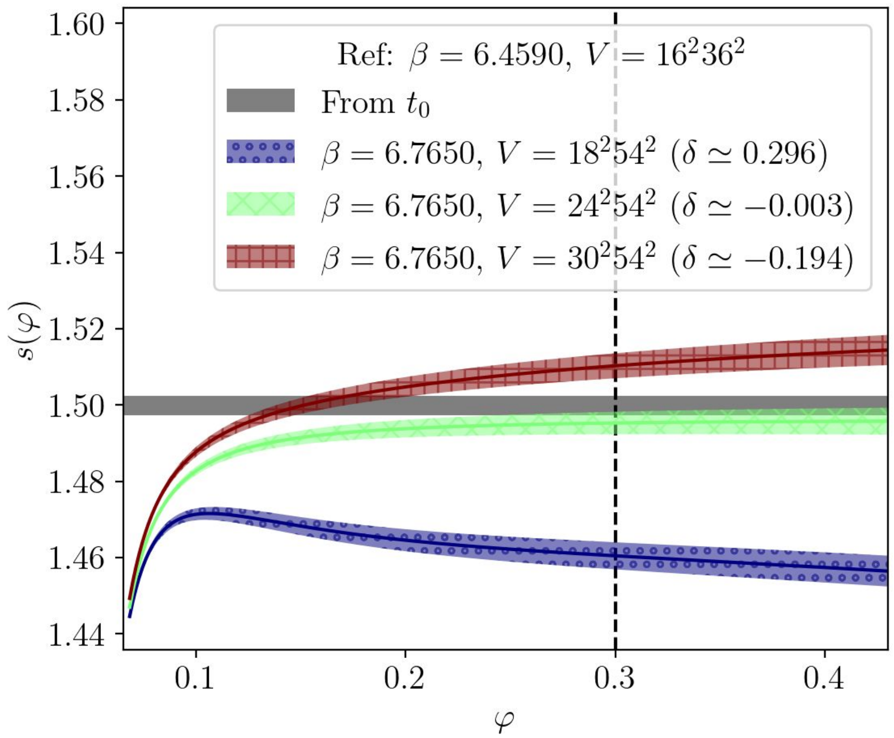


Lattice measures of $\hat{\varphi}(\hat{t})$ at discrete times → Spline interpolation to evaluate at any point → Ratio of times at equal $\hat{\varphi}(\hat{t})$, s taken inside plateau

Pure-gauge tests – T=0 checks

Comparison with standard scale-setting for $V \to \infty$ (data from Bonanno et al., 2026)

Flow rescaling gives $s^2(\beta_1, \beta_2)$ compatible with $\hat{t}_0(\beta_2)/\hat{t}_0(\beta_1)$ (Giusti and Lüsher, 2019) using lattice pairs tuned by coincidence



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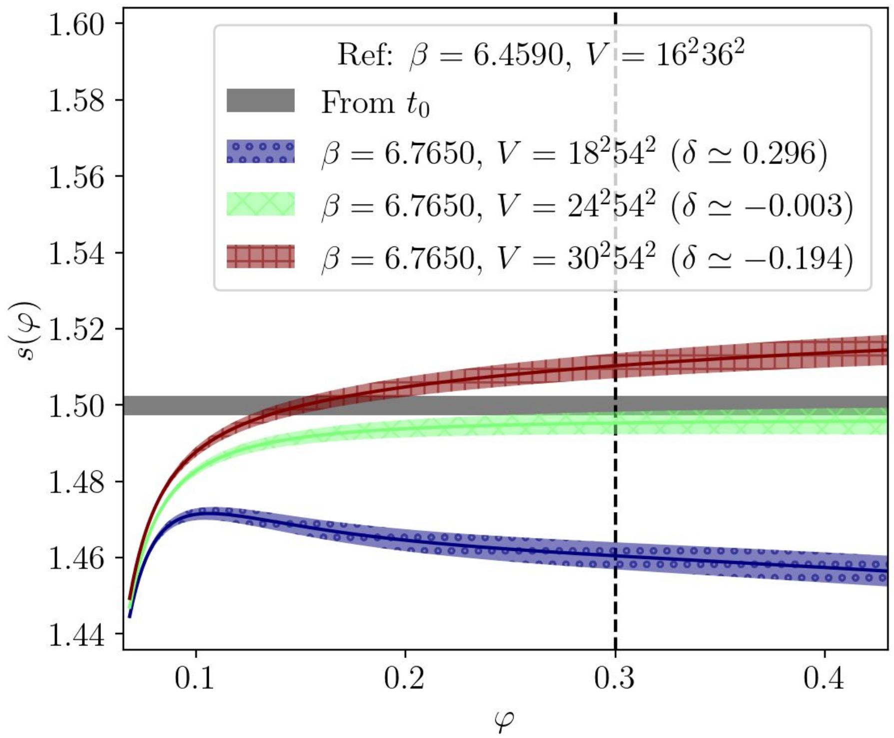
Not tuned lattices have small systematics anyway: $\ell \to \infty \implies g_\mu(\varphi, \ell) \to 0$

So $g_\mu(\varphi, \ell)\delta$ is small even if $\delta \sim 0.2$
At finite T, $g_0 \propto T\partial_T \varphi(t, T)$ dominates

Volume effects also small at small flow times: $\varphi \to 0 \implies g_\mu(\varphi, \ell) \to 0$



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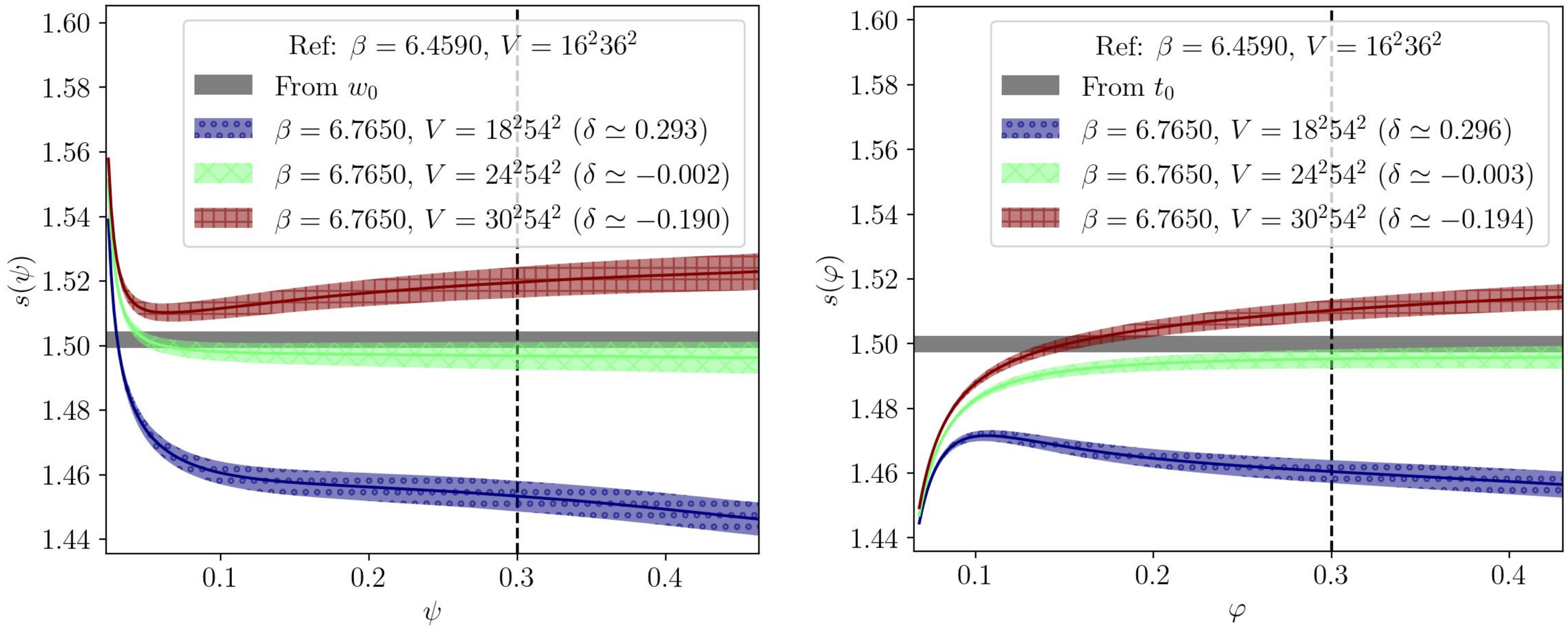


The method can be applied to $\psi(t,\ell) = t\partial_t \varphi(t,\ell)$ without changes

Pure-gauge tests – Effect of topology

Possible bias from **topological freezing**:

rescaling requires lattices with same IR physics, including topology

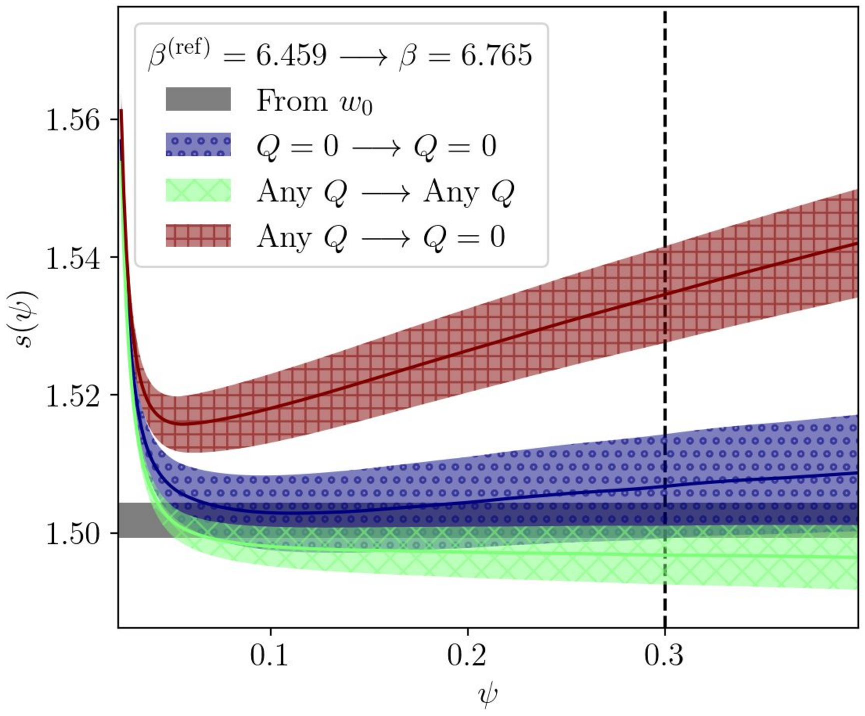
Example: well-sampled topology at coarse spacing, fine spacing frozen in $Q = 0$ sector

↓

different flows even at exact tuning

↓

volume systematics not suppressed by δ in ratio of flow times



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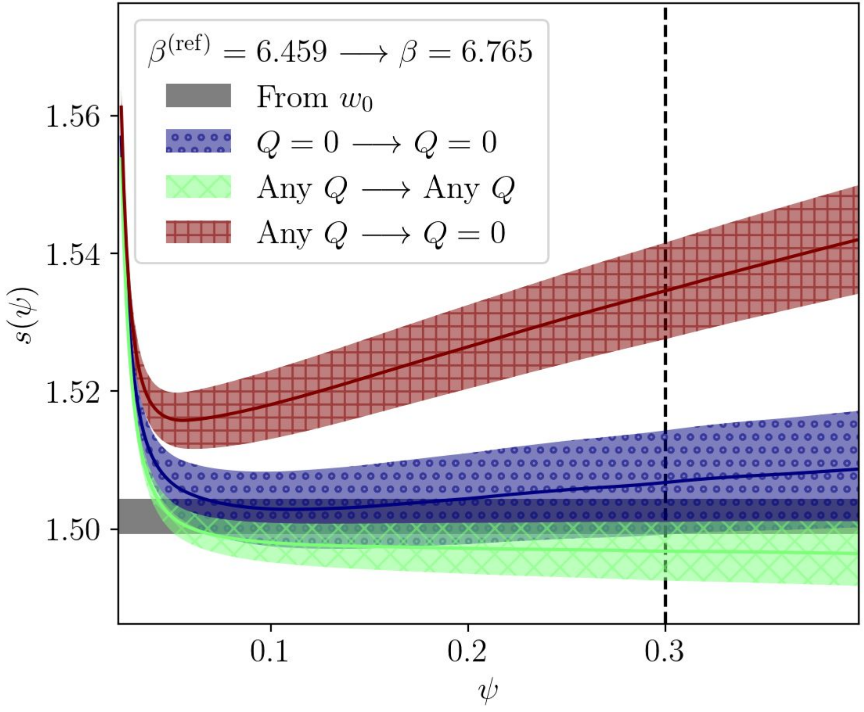
Workaround: projection to $Q = 0$ of both ensembles if topology can't be sampled correctly.
Prescription already tested for step-scaling determinations of Λ (Bonanno et al., 2024)

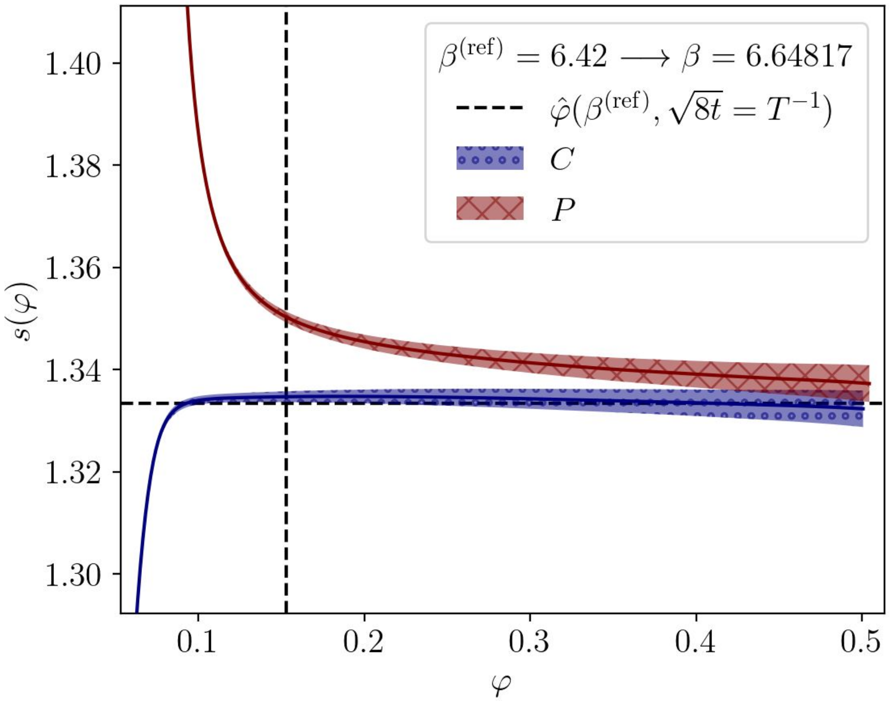


Reference lattice:

$\beta_1 = 6.42\,,\ V_1 = 6\times 18^3$
$a \simeq 0.05\ \mathrm{fm}\,,\ T \simeq 655\ \mathrm{MeV}$

Chosen spacings ratio: $\bar{s} = 4/3 \implies V_2 = 8\times 24^3$
2-loop PT guess: $\beta_2 = 6.64817$

Test of different discretizations of
(with Wilson action and Wilson flow):
$C:\ \mathrm{Clover}\,,\ P:\ \mathrm{Plaquette}$



Pure-gauge tests – Finite-T lattice artifacts

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Test of different discretizations of
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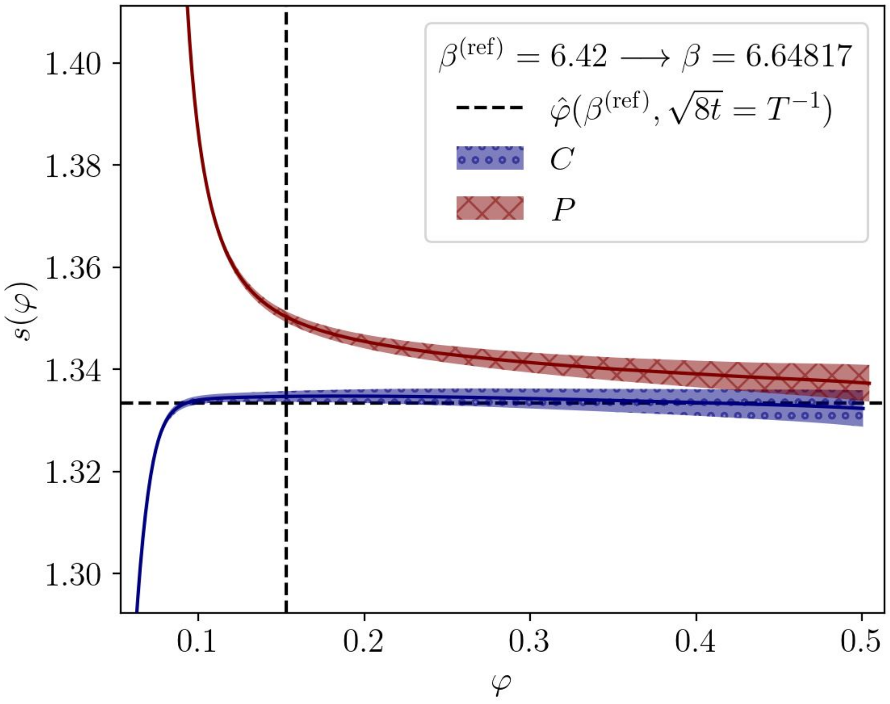
- Lattice artifacts at $\sqrt{8t} = T^{-1}$ around 1% for P, 0.1% for C
- Convergence to same plateau
- PT guess for β gives precise tuning

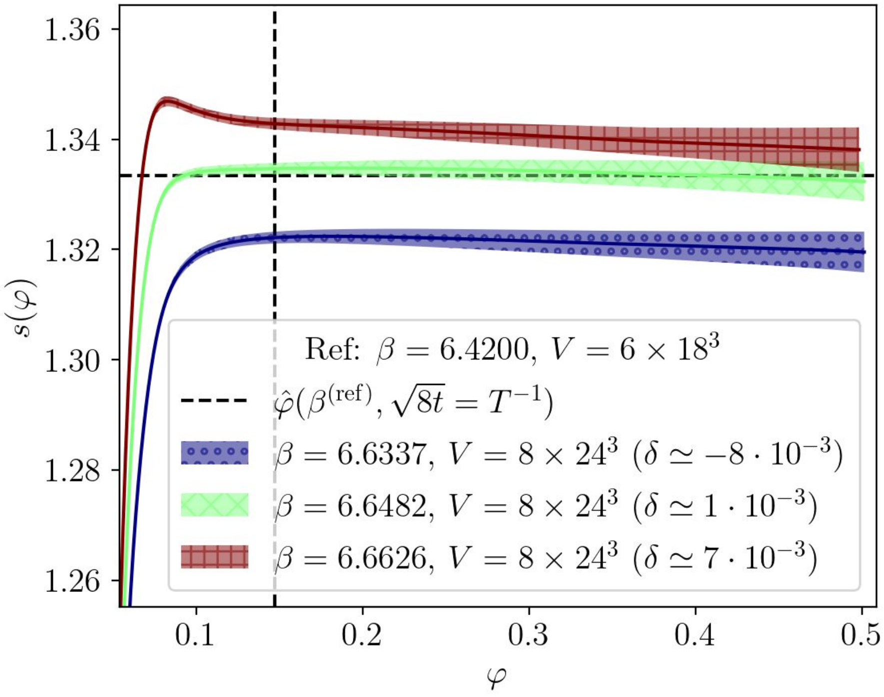


Pure-gauge tests – Finite-T scale setting

Tuning of β (with C discretization at $\sqrt{8t} = T^{-1}$):

1. Simulations around PT guess:
   $\beta_{\mathrm{PT}} = f_{\mathrm{PT}}(\bar{s}, \beta_1)\,,\ \beta_{\mathrm{PT}} \pm 0.2\%$
2. Fit $s(\beta_1, \beta)$ vs β and interpolate at $\bar{s}$
3. Repeat with simulations around tuned β_2 if $s(\varphi)$-dependence is significant

Iterate with $\beta_1,\ V_1 \to \beta_2,\ V_2$ to reach finer spacings



Tuning result: $\beta_2 = 6.6483(9)$

Preliminary results – Tuning for 2+1+1 QCD

Tuning of β for QCD with 2+1+1 4-stout SQ
(Symanzik action, Wilson flow, P discretization of energy)

Reference lattice:

$$\beta_1 = 3.8726\,,\ V_1 = 6 \times 24^3\,,\ \ \hat{m}_c^{(1)} = 0.43453$$
$$m_c/m_s = 11.85\,,\ m_s/m_l = 26.908$$
$$a \simeq 0.0853 \text{ fm}\,,\ T \simeq 385 \text{ MeV}$$

Chosen spacings ratio:

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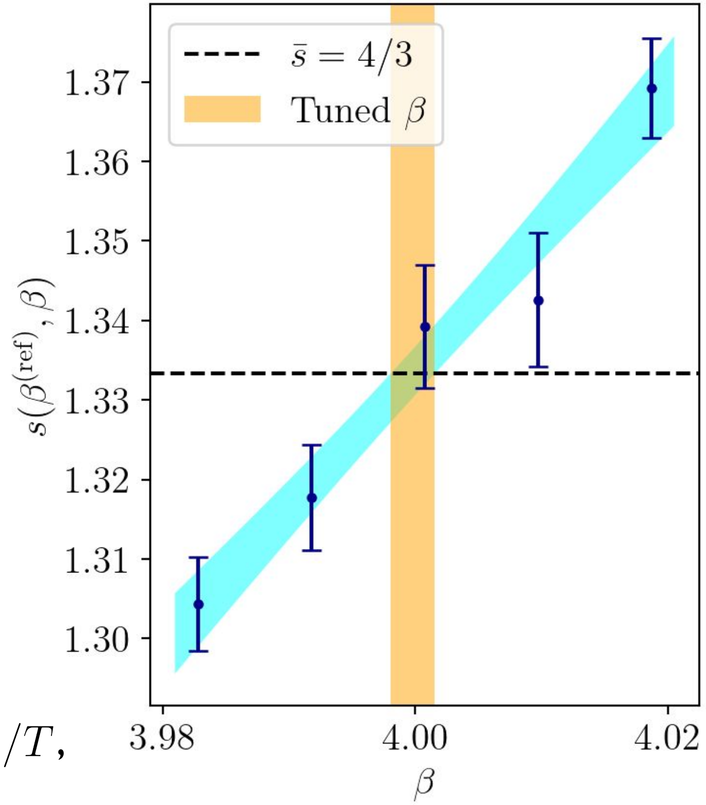
PT guess: $\beta_2 = 4.0457 \implies s = 1.422(9)\ (\delta \simeq 0.07)$
Tuning: $\beta_2 = 3.9998(17)$



Quark masses to be tuned by matching screening masses $M_s(T)/T$, then re-tuning of β

Conclusions

Results

Rescaling of gradient-flow curves allows to set the scale of finite-T QCD

- Method to tune the lattice parameters to achieve a desired ratio of lattice spacings, avoiding the need of large volumes to reach high temperatures
- Dependence of the rescaling factor as ratio of flow times at equal flow value to identify discretization and finite-size effects

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To be done

- Set LCP for 2+1+1 QCD and check consistency with T=0 scale setting in the continuum limit
- Use step-scaling to set LCPs at higher temperatures, to be used for determinations of the topological susceptibility

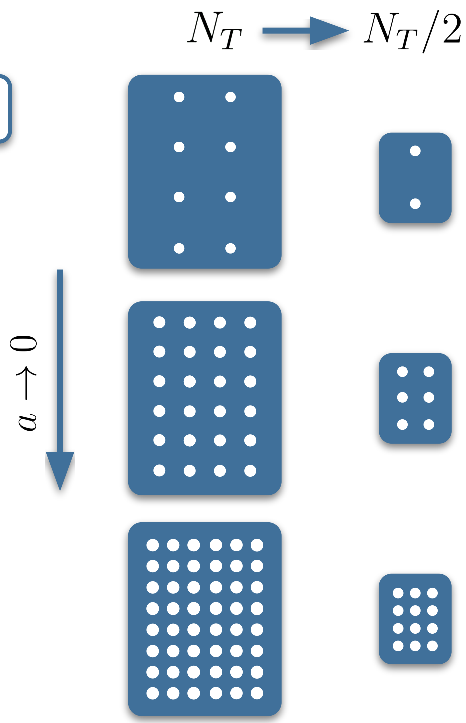
Backup – Step-scaling

- Finite-T renormalized coupling $g^2(\mu = T)$ → Proxy for T
- Mesonic screening masses $M_s(T)$ → One for each free $\hat{m}_q$

Step A: Tune β, $\hat{m}_q$ for different N_T (fixed aspect ratio) to get same $g^2(T)$, $M_s(T)/T$
→ Line of Constant Physics at T

Step B: Repeat measures for $N_T \to N_T/2$ and extrapolate $a \to 0$ to get $g^2(2T)$, $M_s(2T)/(2T)$

Step C: Re-tune β, $\hat{m}_q$ to get $g^2(2T)$, $M_s(2T)/(2T)$ of **B**
→ Line of Constant Physics at $2T$



Iteration allows to connect hadronic and perturbative scales.
Established method to determine the Λ-parameter in SF scheme (FLAG, 2024)

Backup – Scale setting with gradient flow

Energy density of gauge fields at gradient-flow time t:

$$E(t) = \frac{1}{2}\mathrm{Tr}\left[F_{\mu\nu}(t)F_{\mu\nu}(t)\right]$$

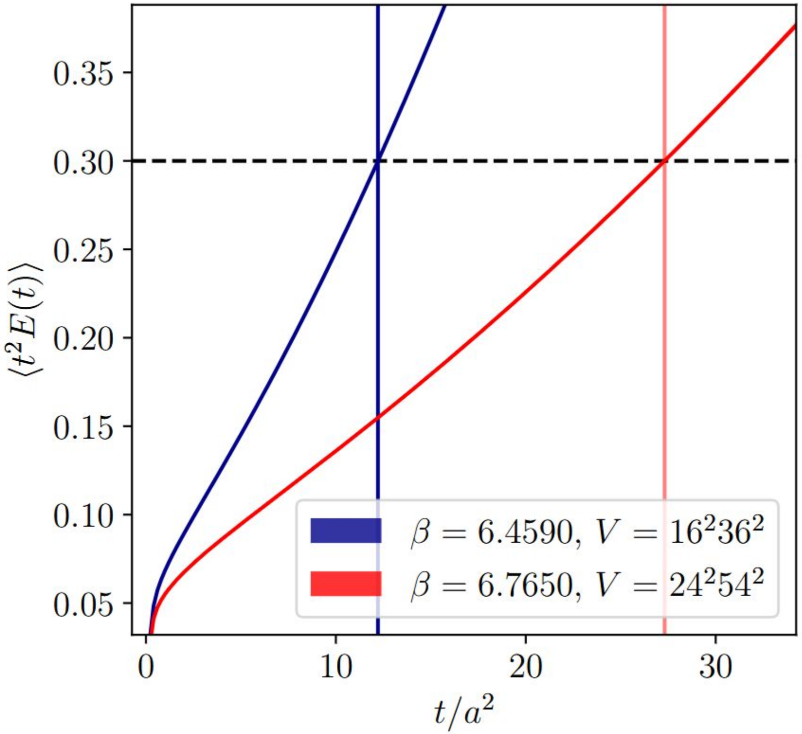
Gradient-flow scales at $T = 0$:
(Lüscher, 2010, Borsányi et al., 2012)

$$\langle t^2 E(t)\rangle\big|_{t=t_0} = 0.3 \quad \Longrightarrow \quad \sqrt{t_0} \simeq 0.14 \text{ fm}$$

$$t\frac{d}{dt}\langle t^2 E(t)\rangle\big|_{t=w_0^2} = 0.3 \quad \Longrightarrow \quad w_0 \simeq 0.17 \text{ fm}$$

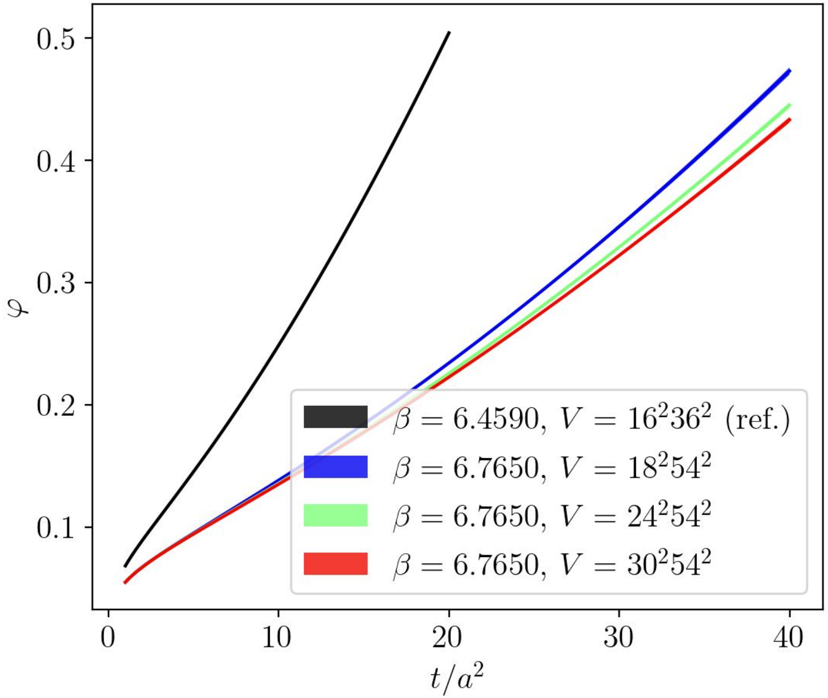
Relatively easy to control finite-volume effects:
in pure-gauge theory,

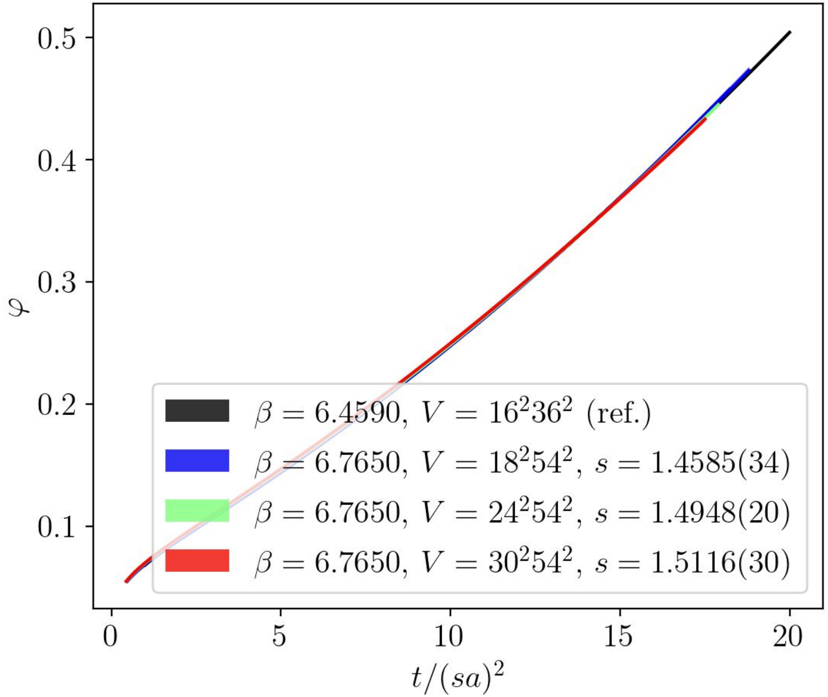
$$\ell = aL \gtrsim 3\sqrt{8t_0} \simeq 1.2 \text{ fm}$$



Examples of t_0 in pure-gauge theory
(Bonanno et al., 2026)

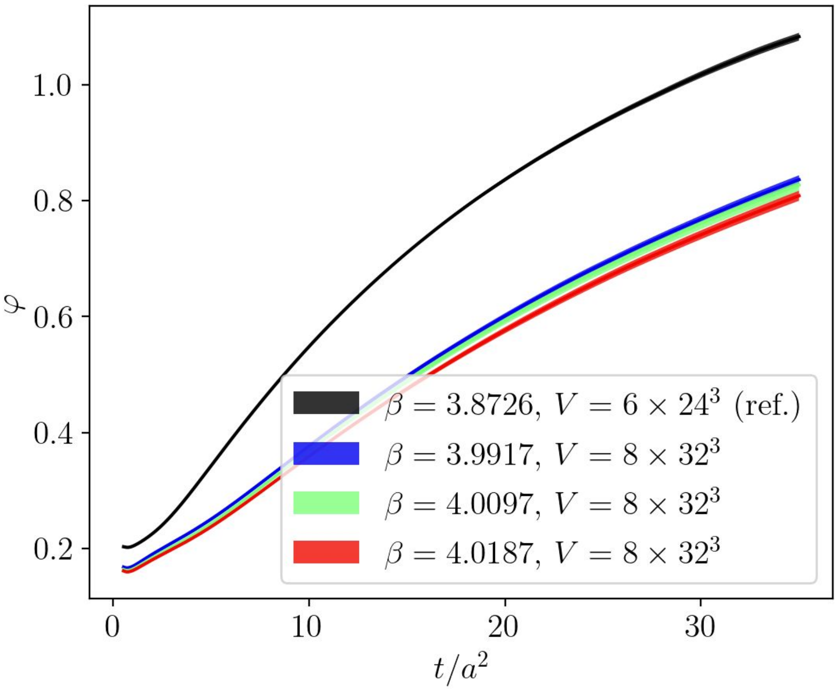
Backup – Pure-gauge tests

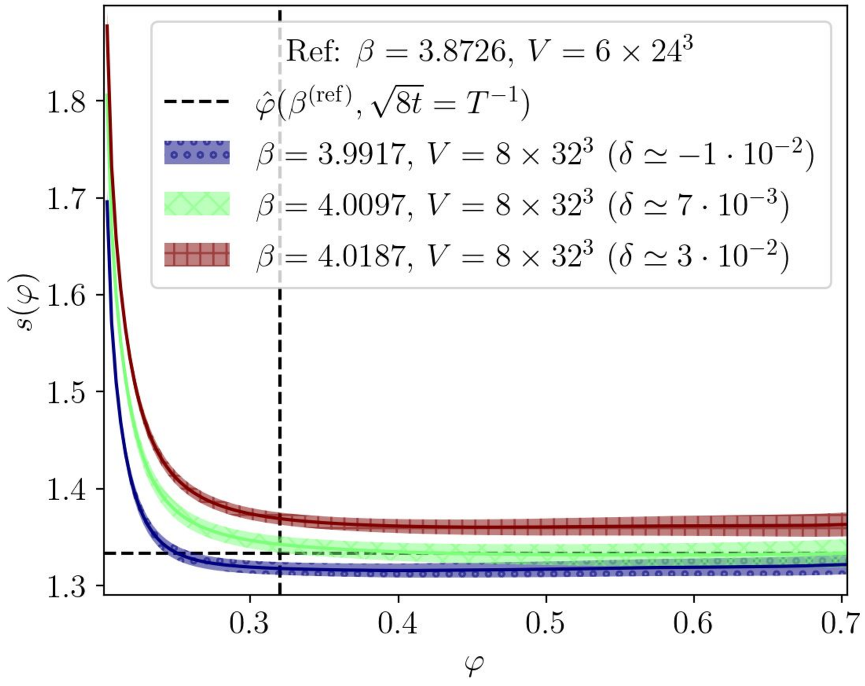




Collapse of gradient-flow curves after rescaling

Backup – Flow rescaling in 2+1+1 QCD





Rescaling factor at selected bare couplings