Gradient Flow in Perturbation Theory

Robert Harlander

RWTH Aachen University

13 May 2026

Standard Model parameters and observables from gradient flow
Edinburgh, 12-15 May 2026

CRC TRR 257

supported by:
Deutsche Forschungsgemeinschaft

	Greetings	*Matthew Black et al.*	*Lecture theatre B, 3rd Floor, James Clerk Maxwell Building*	08:45 - 09:00
09:00	**Gradient flow in perturbation theory**	*Robert Harlander*	*Lecture theatre B, 3rd Floor, James Clerk Maxwell Building*	09:00 - 10:00
10:00	**Gradient-flowed operator product expansion of the Adler function without IR renormalons**	*Martin Beneke*	*Lecture theatre B, 3rd Floor, James Clerk Maxwell Building*	10:00 - 10:30
	Coffee Break		*Lecture theatre B, 3rd Floor, James Clerk Maxwell Building*	10:30 - 11:00
11:00	**Quark mass effects in flowed action density**	*Robert Mason*	*Lecture theatre B, 3rd Floor, James Clerk Maxwell Building*	11:00 - 11:30
	The SFTX of LEFT	*Oscar Lara Crosas*	*Lecture theatre B, 3rd Floor, James Clerk Maxwell Building*	11:30 - 12:00
12:00	**The perturbative matching of four-quark operators between gradient flow and MSbar**	*Jonas Kohnen*	*Lecture theatre B, 3rd Floor, James Clerk Maxwell Building*	12:00 - 12:30
	Lattice determination of bag parameters using gradient flow	*Antonio Rago*	*Lecture theatre B, 3rd Floor, James Clerk Maxwell Building*	12:30 - 13:00
13:00	**Lunch**			

Motivation

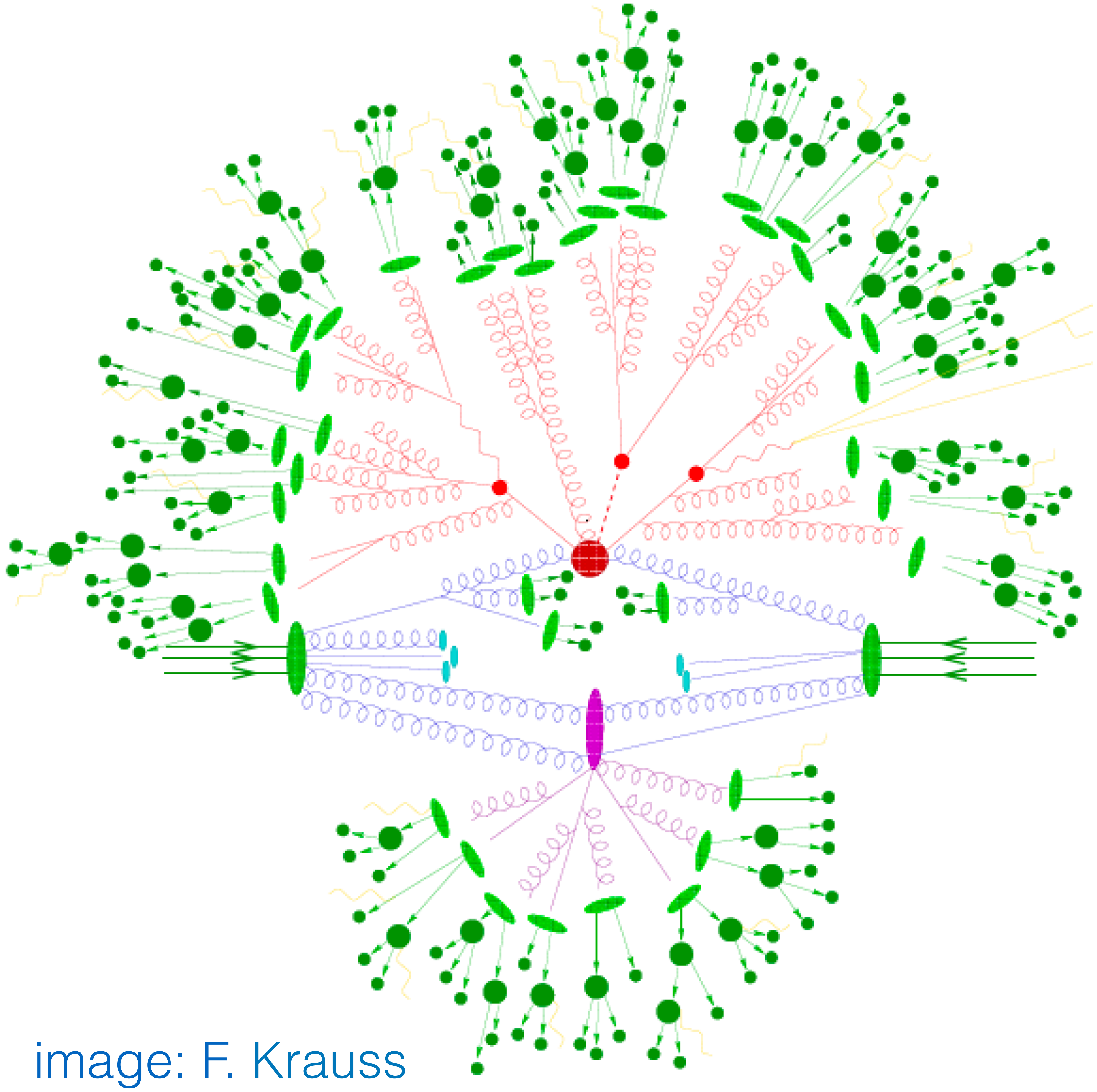


image: F. Krauss

Motivation

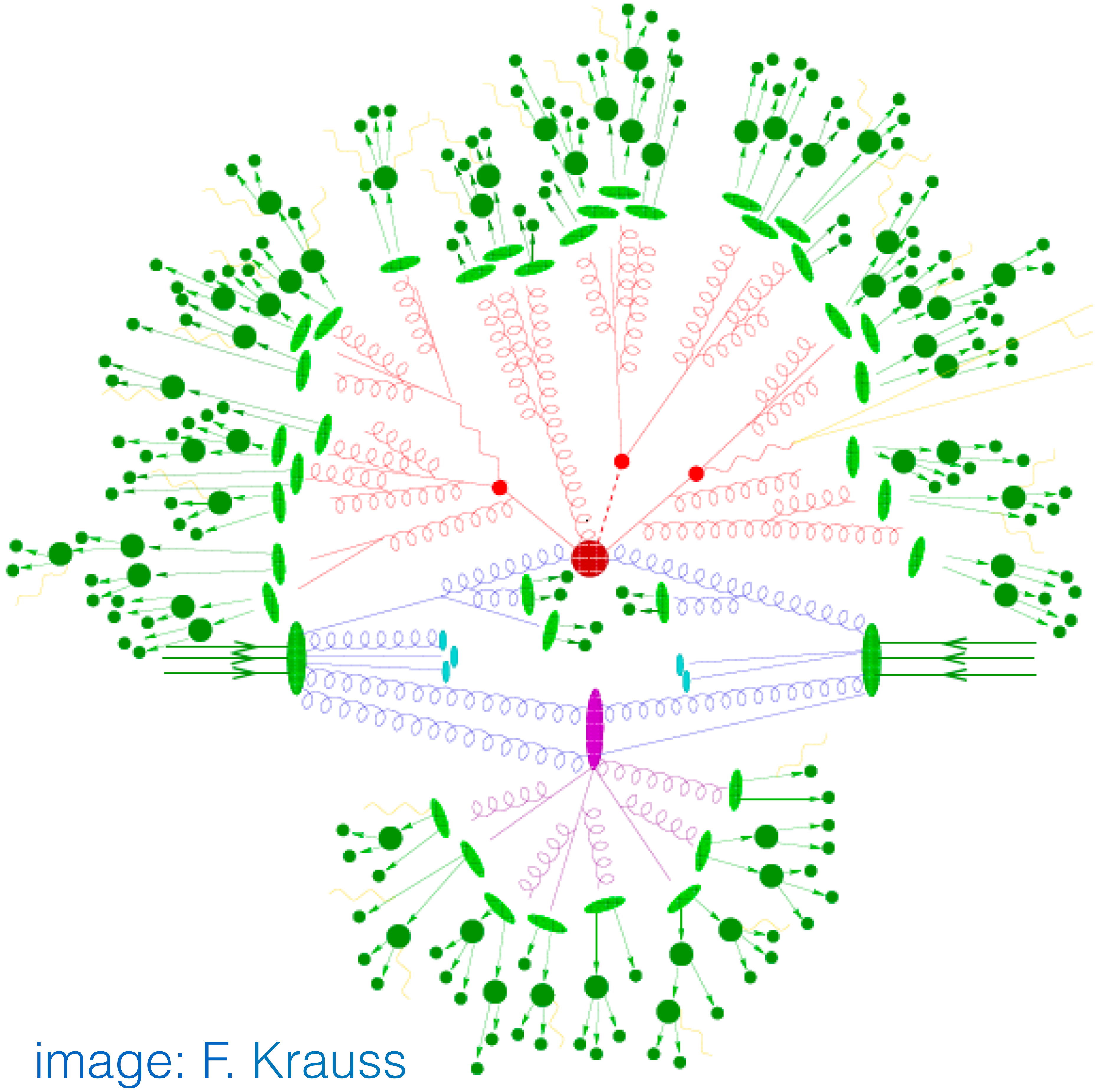


image: F. Krauss

perturbative contributions:
first principles

Motivation

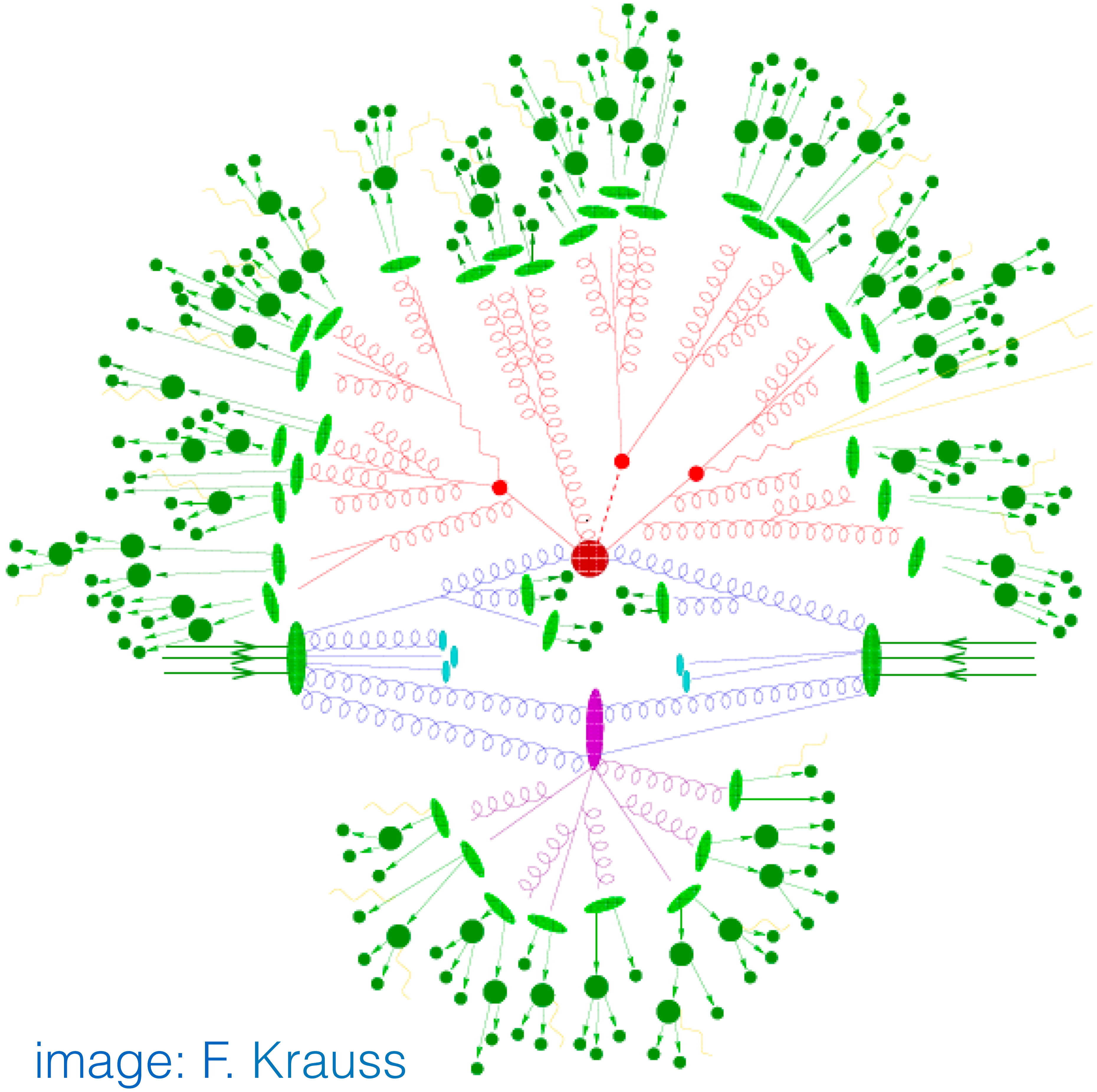


image: F. Krauss

perturbative contributions:
first principles

non-perturbative contributions:
phenomenological fits

Motivation

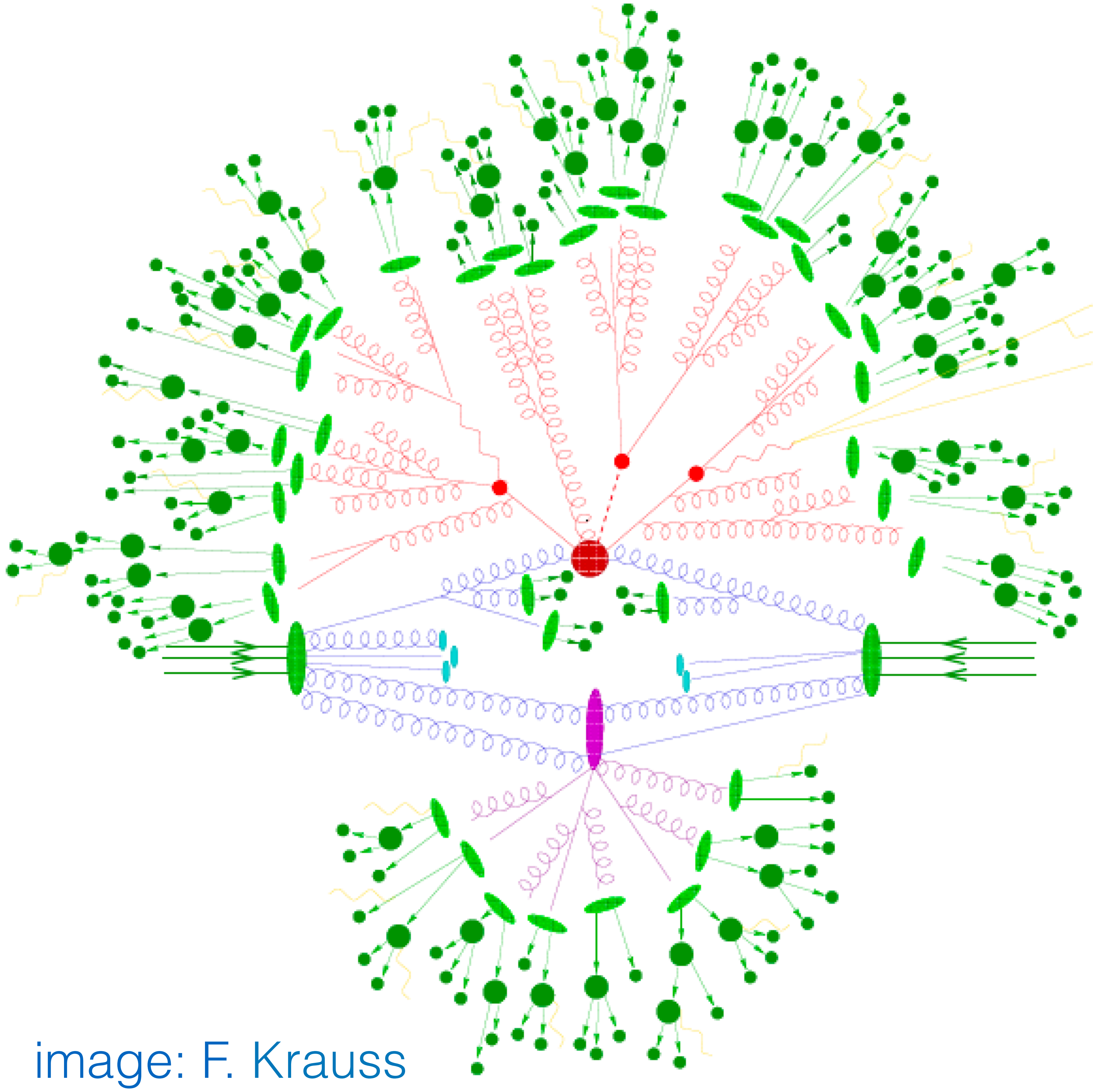


image: F. Krauss

perturbative contributions:
first principles

non-perturbative contributions:
phenomenological fits

lattice?

Motivation

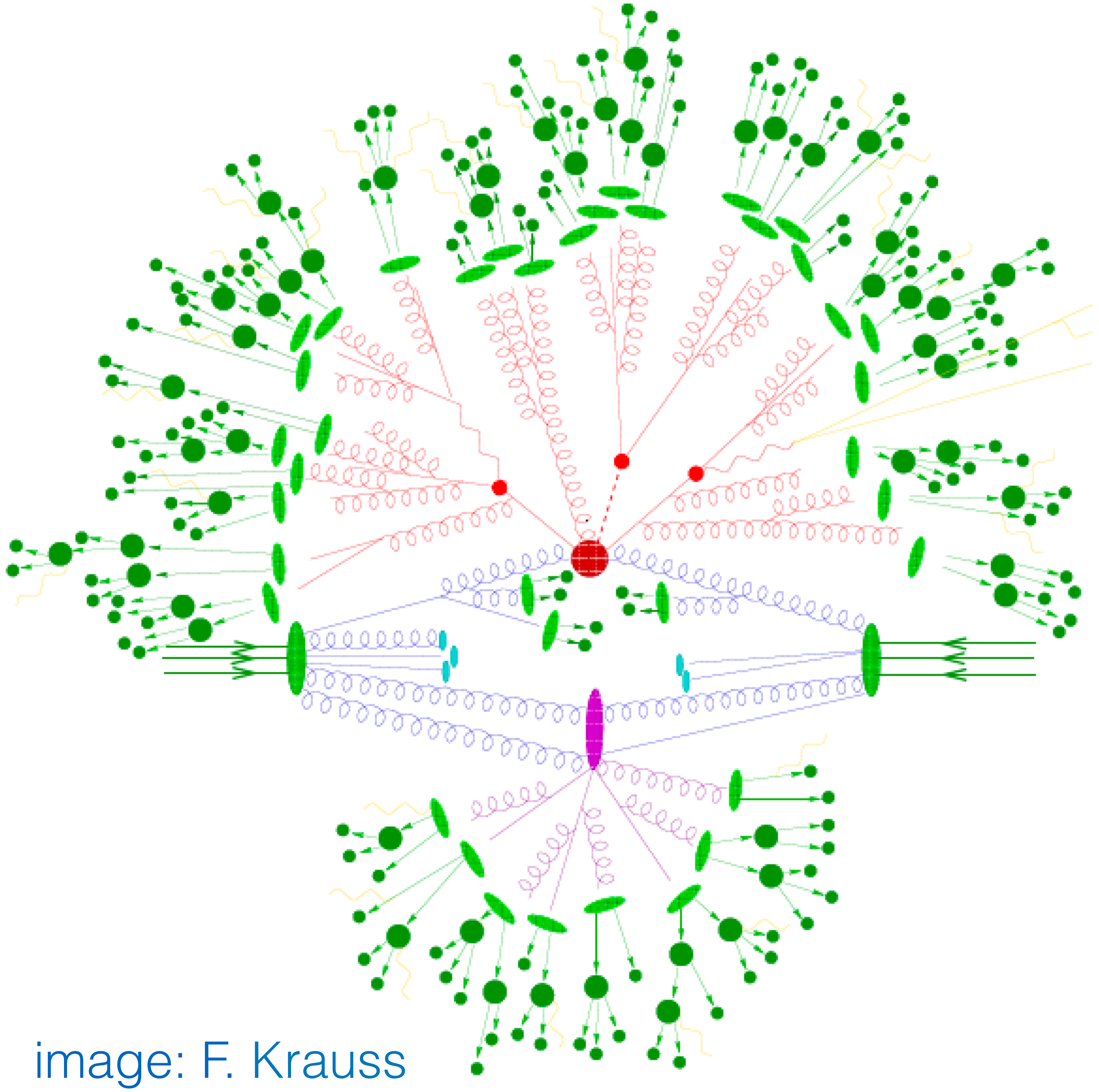


image: F. Krauss

perturbative contributions:
first principles

non-perturbative contributions:
phenomenological fits

lattice?

PDFs defined via light-cone correlators,
lattice formulated in Euclidean space

Motivation

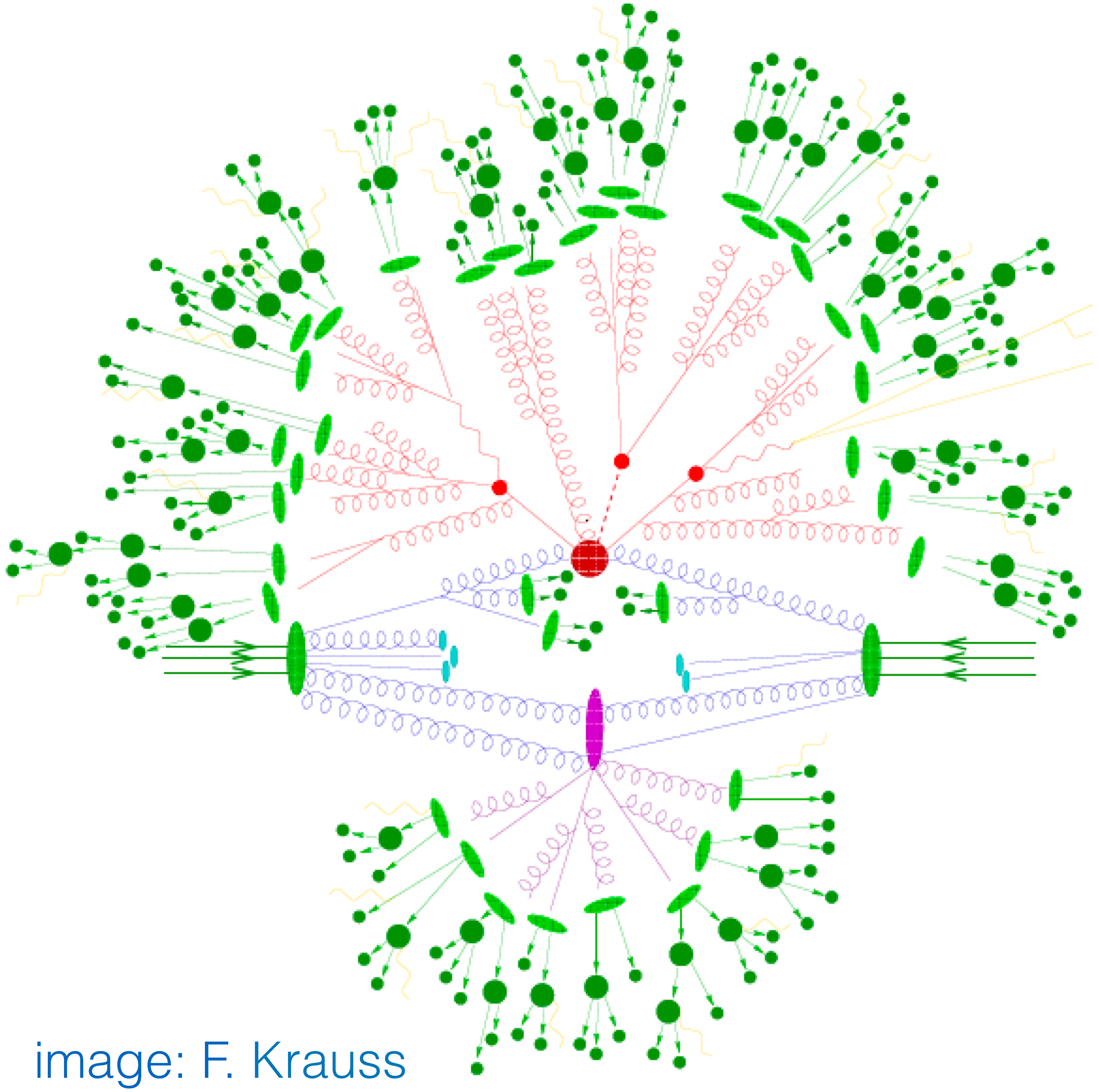


image: F. Krauss

perturbative contributions:
first principles

non-perturbative contributions:
phenomenological fits

lattice?

PDFs defined via light-cone correlators,
lattice formulated in Euclidean space

alternative: moments

$$\langle x^n \rangle_q = \int_0^1 dx\, x^n\, q(x)$$

Moments of parton densities

Moments of parton densities

$$\langle h(p) | \mathcal{O}_{\{\mu_1 \cdots \mu_n\}} | h(p) \rangle \sim p_{\mu_1} \cdots p_{\mu_n} \langle x^{n-1} \rangle$$

Moments of parton densities

$$\langle h(p) | \mathcal{O}_{\{\mu_1\cdots\mu_n\}} | h(p) \rangle \sim p_{\mu_1}\cdots\, p_{\mu_n} \langle x^{n-1} \rangle$$

$$\mathcal{O}_{\mu_1\cdots\mu_n} \sim \int d^4x\, \bar{\psi}(x)\, \gamma_{\mu_1} \overleftrightarrow{D}_{\mu_2} \cdots \overleftrightarrow{D}_{\mu_n}\, \psi(x)$$

$$D_\mu = \partial_\mu + igA_\mu$$

Moments of parton densities

$$\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}\,|\,h(p)\rangle \quad \sim \quad p_{\mu_1}\cdots\; p_{\mu_n}\langle x^{n-1}\rangle$$

$$\mathcal{O}_{\mu_1\cdots\mu_n} \sim \int d^4x\,\bar\psi(x)\,\gamma_{\mu_1}\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n}\,\psi(x)$$

$$D_\mu = \partial_\mu + igA_\mu$$

problem: Lattice breaks Lorentz symmetry

Moments of parton densities

$$\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}\,|\,h(p)\rangle \;\sim\; p_{\mu_1}\cdots\; p_{\mu_n}\langle x^{n-1}\rangle$$

$$\mathcal{O}_{\mu_1\cdots\mu_n} \sim \int d^4x\,\bar{\psi}(x)\,\gamma_{\mu_1}\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n}\,\psi(x)$$

$$D_\mu = \partial_\mu + igA_\mu$$

problem: Lattice breaks Lorentz symmetry

→ mixing with lower-dimensional operators

Institute for Theoretical Particle Physics and Cosmology | RWTH AACHEN UNIVERSITY

Moments of parton densities

$$\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}\,|\,h(p)\rangle \sim p_{\mu_1}\cdots\, p_{\mu_n}\langle x^{n-1}\rangle$$

$$\mathcal{O}_{\mu_1\cdots\mu_n} \sim \int d^4x\,\bar{\psi}(x)\,\gamma_{\mu_1}\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n}\psi(x)$$

$$D_\mu = \partial_\mu + igA_\mu$$

problem: Lattice breaks Lorentz symmetry

→ mixing with lower-dimensional operators

→ $a \to 0$ limit challenging

Moments of parton densities

$$\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}\,|\,h(p)\rangle \quad\sim\quad p_{\mu_1}\cdots\; p_{\mu_n}\,\langle x^{n-1}\rangle$$

$$\mathcal{O}_{\mu_1\cdots\mu_n}\sim\int d^4x\,\bar\psi(x)\,\gamma_{\mu_1}\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n}\,\psi(x)$$

$$D_\mu = \partial_\mu + igA_\mu$$

problem: Lattice breaks Lorentz symmetry

→ mixing with lower-dimensional operators

→ $a \to 0$ limit challenging

→ only results up to $\langle x^3\rangle$ available

Martinelli, Sachrajda ’87, ‘88
Alexandrou et al. ’20, ‘21

Gradient flow as cutoff

$$\langle h(p) | \mathcal{O}_{\{\mu_1 \cdots \mu_n\}} | h(p) \rangle \sim p_{\mu_1} \cdots p_{\mu_n} \langle x^{n-1} \rangle$$

$$\mathcal{O}_{\mu_1 \cdots \mu_n} \sim \int d^4x \, \bar{\psi}(x) \, \gamma_{\mu_1} \overleftrightarrow{D}_{\mu_2} \cdots \overleftrightarrow{D}_{\mu_n} \psi(x)$$

$$D_\mu = \partial_\mu + igA_\mu$$

Gradient flow as cutoff

$$\langle h(p) | \mathcal{O}_{\{\mu_1 \cdots \mu_n\}} | h(p) \rangle \sim p_{\mu_1} \cdots \, p_{\mu_n} \langle x^{n-1} \rangle$$

$$\mathcal{O}_{\mu_1 \cdots \mu_n} \sim \int d^4x \, \bar{\psi}(x) \, \gamma_{\mu_1} \overleftrightarrow{D}_{\mu_2} \cdots \overleftrightarrow{D}_{\mu_n} \psi(x)$$

$$D_\mu = \partial_\mu + igA_\mu$$

In $\mathcal{O}_{\mu_1 \cdots \mu_n}$, remove high-momentum modes $p^2 \gtrsim 1/t$

$$\psi(x) \to \chi(x, t)$$

$$A_\mu(x) \to B_\mu(x, t)$$

in a gauge and Lorentz invariant way

Gradient flow as cutoff

$$\langle h(p) | \mathcal{O}_{\{\mu_1 \cdots \mu_n\}} | h(p) \rangle \sim p_{\mu_1} \cdots p_{\mu_n} \langle x^{n-1} \rangle$$

$$\mathcal{O}_{\mu_1 \cdots \mu_n} \sim \int d^4x \, \bar{\psi}(x) \, \gamma_{\mu_1} \overleftrightarrow{D}_{\mu_2} \cdots \overleftrightarrow{D}_{\mu_n} \psi(x)$$

$$D_\mu = \partial_\mu + igA_\mu$$

In $\mathcal{O}_{\mu_1 \cdots \mu_n}$, remove high-momentum modes $p^2 \gtrsim 1/t$

$$\psi(x) \to \chi(x, t)$$

$$A_\mu(x) \to B_\mu(x, t)$$

in a gauge and Lorentz invariant way

→ continuum limit can be taken, result depends on UV cutoff t

Gradient flow as cutoff

Gradient flow as cutoff

$$B_\mu(x, t) = \int^{1/\sqrt{t}} d^4k\, e^{ikx}\, \tilde{A}_\mu(k)$$

Gradient flow as cutoff

$$B_\mu(x, t) = \int^{1/\sqrt{t}} d^4k\, e^{ikx}\, \tilde{A}_\mu(k)$$

→ not Lorentz invariant!

Institute for Theoretical Particle Physics and Cosmology
RWTH AACHEN UNIVERSITY

Gradient flow as cutoff

$$B_\mu(x, t) = \int^{1/\sqrt{t}} d^4k\, e^{ikx}\, \tilde{A}_\mu(k)$$

→ not Lorentz invariant!

$$B_\mu(x, t) = \int d^4k\, e^{ikx}\, e^{-tk^2} \tilde{A}_\mu(k)$$

Gradient flow as cutoff

$$B_\mu(x, t) = \int^{1/\sqrt{t}} d^4k\, e^{ikx}\, \tilde{A}_\mu(k)$$ → not Lorentz invariant!

$$B_\mu(x, t) = \int d^4k\, e^{ikx}\, e^{-tk^2} \tilde{A}_\mu(k)$$ → not gauge invariant!

Gradient flow as cutoff

$$B_\mu(x,t) = \int^{1/\sqrt{t}} d^4k\, e^{ikx}\, \tilde{A}_\mu(k)$$ → not Lorentz invariant!

$$B_\mu(x,t) = \int d^4k\, e^{ikx}\, e^{-tk^2} \tilde{A}_\mu(k)$$ → not gauge invariant!

$$\Updownarrow$$

$$\partial_t B_\mu(x,t) = \Box\, B_\mu(x,t)$$

Gradient flow as cutoff

$$B_\mu(x,t) = \int^{1/\sqrt{t}} d^4k\, e^{ikx}\, \tilde{A}_\mu(k)$$ → not Lorentz invariant!

$$B_\mu(x,t) = \int d^4k\, e^{ikx}\, e^{-tk^2} \tilde{A}_\mu(k)$$ → not gauge invariant!

$$\Updownarrow$$

$$\partial_t B_\mu(x,t) = \Box\, B_\mu(x,t) \quad \rightarrow \quad \partial_t B_\mu(x,t) = D_\nu(t)\, G_{\nu\mu}(x,t)$$

Gradient flow as cutoff

$$B_\mu(x,t) = \int^{1/\sqrt{t}} d^4k\, e^{ikx}\, \tilde{A}_\mu(k)$$ → not Lorentz invariant!

$$B_\mu(x,t) = \int d^4k\, e^{ikx}\, e^{-tk^2} \tilde{A}_\mu(k)$$ → not gauge invariant!

$$\Updownarrow$$

$$\partial_t B_\mu(x,t) = \Box\, B_\mu(x,t) \quad \rightarrow \quad \partial_t B_\mu(x,t) = D_\nu(t)\, G_{\nu\mu}(x,t)$$

$$D_\mu(t) = \partial_\mu + ig[B_\mu(x,t), \cdot\,]$$

$$G_{\mu\nu}(x,t) \sim [D_\mu(t), D_\nu(t)]$$

Institute for Theoretical Particle Physics and Cosmology | RWTH AACHEN UNIVERSITY

Gradient flow as cutoff

$$B_\mu(x,t) = \int^{1/\sqrt{t}} d^4k\, e^{ikx}\, \tilde{A}_\mu(k)$$ → not Lorentz invariant!

$$B_\mu(x,t) = \int d^4k\, e^{ikx}\, e^{-tk^2} \tilde{A}_\mu(k)$$ → not gauge invariant!

$$\Updownarrow$$

$$\partial_t B_\mu(x,t) = \Box\, B_\mu(x,t) \quad \to \quad \partial_t B_\mu(x,t) = D_\nu(t)\, G_{\nu\mu}(x,t)$$

$$B_\mu(x,0) = A_\mu(x)$$

$$D_\mu(t) = \partial_\mu + ig[B_\mu(x,t), \cdot\,]$$

$$G_{\mu\nu}(x,t) \sim [D_\mu(t), D_\nu(t)]$$

Gradient flow as cutoff

$$B_\mu(x,t) = \int^{1/\sqrt{t}} d^4k\, e^{ikx}\, \tilde{A}_\mu(k)$$ → not Lorentz invariant!

$$B_\mu(x,t) = \int d^4k\, e^{ikx}\, e^{-tk^2} \tilde{A}_\mu(k)$$ → not gauge invariant!

$$\Updownarrow$$

$$\partial_t B_\mu(x,t) = \Box B_\mu(x,t) \quad \to$$

$$\partial_t B_\mu(x,t) = D_\nu(t)\, G_{\nu\mu}(x,t)$$
$$B_\mu(x,0) = A_\mu(x)$$

$$D_\mu(t) = \partial_\mu + ig[B_\mu(x,t),\, \cdot\,]$$

$$G_{\mu\nu}(x,t) \sim [D_\mu(t), D_\nu(t)]$$

Gradient flow as cutoff

$$B_\mu(x,t) = \int^{1/\sqrt{t}} d^4k\, e^{ikx}\, \tilde{A}_\mu(k)$$ → not Lorentz invariant!

$$B_\mu(x,t) = \int d^4k\, e^{ikx}\, e^{-tk^2} \tilde{A}_\mu(k)$$ → not gauge invariant!

$$\Updownarrow$$

$$\partial_t B_\mu(x,t) = \Box B_\mu(x,t) \quad \rightarrow \quad \boxed{\begin{aligned} \partial_t B_\mu(x,t) &= D_\nu(t)\, G_{\nu\mu}(x,t) \\ B_\mu(x,0) &= A_\mu(x) \end{aligned}}$$

$$D_\mu(t) = \partial_\mu + ig[B_\mu(x,t), \cdot\,]$$

$$G_{\mu\nu}(x,t) \sim [D_\mu(t), D_\nu(t)]$$

analogously:

$$\boxed{\begin{aligned} \partial_t \chi(x,t) &= D^2 \chi(x,t) \\ \chi(x,0) &= \psi(x) \end{aligned}}$$

TTK Institute for Theoretical Particle Physics and Cosmology | RWTH AACHEN UNIVERSITY

Moments of parton densities

$$\langle h(p) | \mathcal{O}_{\{\mu_1 \cdots \mu_n\}} | h(p) \rangle \quad \sim \quad p_{\mu_1} \cdots \; p_{\mu_n} \langle x^{n-1} \rangle$$

Moments of parton densities

$$\langle h(p) | \mathcal{O}_{\{\mu_1 \cdots \mu_n\}} | h(p) \rangle \sim p_{\mu_1} \cdots p_{\mu_n} \langle x^{n-1} \rangle$$

Shindler '24

$$\langle h(p) | \mathcal{O}_{\{\mu_1 \cdots \mu_n\}}(t) | h(p) \rangle$$

Moments of parton densities

$$\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}\,|\,h(p)\rangle \quad \sim \quad p_{\mu_1}\cdots\; p_{\mu_n}\langle x^{n-1}\rangle$$

Shindler '24

$$\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}(t)\,|\,h(p)\rangle$$

$$\mathcal{O}_{\mu_1\cdots\mu_n}(t) \sim \int d^4x\,\bar{\psi}(x,t)\,\gamma_{\mu_1}\overleftrightarrow{D}_{\mu_2}(t)\cdots\overleftrightarrow{D}_{\mu_n}(t)\,\psi(x,t)$$

Moments of parton densities

$$\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}\,|\,h(p)\rangle \quad\sim\quad p_{\mu_1}\cdots\; p_{\mu_n}\langle x^{n-1}\rangle$$

↓ Shindler '24

$$\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}(t)\,|\,h(p)\rangle$$

$$\mathcal{O}_{\mu_1\cdots\mu_n}(t)\sim\int d^4x\,\bar{\psi}(x,t)\,\gamma_{\mu_1}\overleftrightarrow{D}_{\mu_2}(t)\cdots\overleftrightarrow{D}_{\mu_n}(t)\,\psi(x,t)$$

Andrea Shindler (Fri 09:30)

Moments of parton densities

$$\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}\,|\,h(p)\rangle \quad\sim\quad p_{\mu_1}\cdots\; p_{\mu_n}\,\langle x^{n-1}\rangle$$

↓ Shindler '24

Andrea Shindler (Fri 09:30)

$$\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}(t)\,|\,h(p)\rangle$$

$$\mathcal{O}_{\mu_1\cdots\mu_n}(t) \sim \int d^4x\,\bar{\psi}(x,t)\,\gamma_{\mu_1}\overleftrightarrow{D}_{\mu_2}(t)\cdots\overleftrightarrow{D}_{\mu_n}(t)\,\psi(x,t)$$

Continuum limit exists.

Moments of parton densities

$$\langle h(p) | \mathcal{O}_{\{\mu_1\cdots\mu_n\}} | h(p)\rangle \sim p_{\mu_1}\cdots p_{\mu_n}\langle x^{n-1}\rangle$$

Shindler '24

Andrea Shindler (Fri 09:30)

$$\langle h(p) | \mathcal{O}_{\{\mu_1\cdots\mu_n\}}(t) | h(p)\rangle$$

$$\mathcal{O}_{\mu_1\cdots\mu_n}(t) \sim \int d^4x\, \bar{\psi}(x,t)\,\gamma_{\mu_1}\overleftrightarrow{D}_{\mu_2}(t)\cdots\overleftrightarrow{D}_{\mu_n}(t)\,\psi(x,t)$$

Continuum limit exists.

Lorentz and gauge invariance preserved.

Moments of parton densities

$$\langle h(p) | \mathcal{O}_{\{\mu_1 \cdots \mu_n\}} | h(p) \rangle \sim p_{\mu_1} \cdots p_{\mu_n} \langle x^{n-1} \rangle$$

Shindler '24

$$\langle h(p) | \mathcal{O}_{\{\mu_1 \cdots \mu_n\}}(t) | h(p) \rangle$$

Andrea Shindler (Fri 09:30)

$$\mathcal{O}_{\mu_1 \cdots \mu_n}(t) \sim \int d^4x \, \bar{\psi}(x,t) \gamma_{\mu_1} \overleftrightarrow{D}_{\mu_2}(t) \cdots \overleftrightarrow{D}_{\mu_n}(t) \psi(x,t)$$

Continuum limit exists.

Lorentz and gauge invariance preserved.

No mixing with other operators.

Moments of parton densities

$$\langle h(p) | \mathcal{O}_{\{\mu_1 \cdots \mu_n\}} | h(p) \rangle \sim p_{\mu_1} \cdots p_{\mu_n} \langle x^{n-1} \rangle$$

Shindler '24

$$\langle h(p) | \mathcal{O}_{\{\mu_1 \cdots \mu_n\}}(t) | h(p) \rangle$$

Andrea Shindler (Fri 09:30)

$$\mathcal{O}_{\mu_1 \cdots \mu_n}(t) \sim \int d^4x\, \bar{\psi}(x,t)\, \gamma_{\mu_1} \overleftrightarrow{D}_{\mu_2}(t) \cdots \overleftrightarrow{D}_{\mu_n}(t)\, \psi(x,t)$$

Continuum limit exists.

Lorentz and gauge invariance preserved.

No mixing with other operators.

$t \to 0$?

Short flow time expansion

Short flow time expansion

$$\mathcal{O}_n(t) = \sum_m \zeta_{nm}(t)\,\mathcal{O}_m \; + \; t\,(\;\cdots\;)$$

Lüscher '14
Suzuki '15

Short flow time expansion

$$\mathcal{O}_n(t) = \sum_m \zeta_{nm}(t)\,\mathcal{O}_m \;+\; t\,(\;\cdots\;)$$

Lüscher '14
Suzuki '15

higher-dimensional operators → neglect

Short flow time expansion

$$\mathcal{O}_n(t) = \sum_m \zeta_{nm}(t)\,\mathcal{O}_m \;+\; t\,(\;\cdots\;)$$

Lüscher '14
Suzuki '15

higher-dimensional operators → neglect

$$\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}\,|\,h(p)\rangle = \sum_m \zeta^{-1}_{mn}(t)\,\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}(t)\,|\,h(p)\rangle$$

Short flow time expansion

$$\mathcal{O}_n(t) = \sum_m \zeta_{nm}(t)\,\mathcal{O}_m \; + \; t\,(\;\cdots\;)$$

Lüscher '14
Suzuki '15

higher-dimensional operators → neglect

$$\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}\,|\,h(p)\rangle = \sum_m \zeta^{-1}_{mn}(t)\,\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}(t)\,|\,h(p)\rangle$$

lattice

TTK Institute for Theoretical Particle Physics and Cosmology | RWTH AACHEN UNIVERSITY

Short flow time expansion

$$\mathcal{O}_n(t) = \sum_m \zeta_{nm}(t)\, \mathcal{O}_m \; + \; t\,(\;\cdots\;)$$

Lüscher '14
Suzuki '15

higher-dimensional operators → neglect

$$\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}\,|\,h(p)\rangle = \sum_m \zeta^{-1}_{mn}(t)\,\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}(t)\,|\,h(p)\rangle$$

perturbation theory

lattice

Short flow time expansion

$$\mathcal{O}_n(t) = \sum_m \zeta_{nm}(t)\,\mathcal{O}_m \;+\; t\,(\;\cdots\;)$$

Lüscher '14
Suzuki '15

$\cdots$ ← higher-dimensional operators → neglect

$$\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}\,|\,h(p)\rangle = \sum_m \zeta^{-1}_{mn}(t)\,\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}(t)\,|\,h(p)\rangle$$

$\zeta^{-1}_{mn}(t)$ ← perturbation theory

$\langle h(p)|\mathcal{O}_{\{\mu_1\cdots\mu_n\}}(t)|h(p)\rangle$ ← lattice

ACCREDITED
Loops and Legs

Short flow time expansion

$$\mathcal{O}_n(t) = \sum_m \zeta_{nm}(t)\,\mathcal{O}_m \;+\; t\,(\;\cdots\;)$$

Lüscher '14
Suzuki '15

higher-dimensional operators → neglect

$$\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}\,|\,h(p)\rangle = \sum_m \zeta^{-1}_{mn}(t)\,\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}(t)\,|\,h(p)\rangle$$

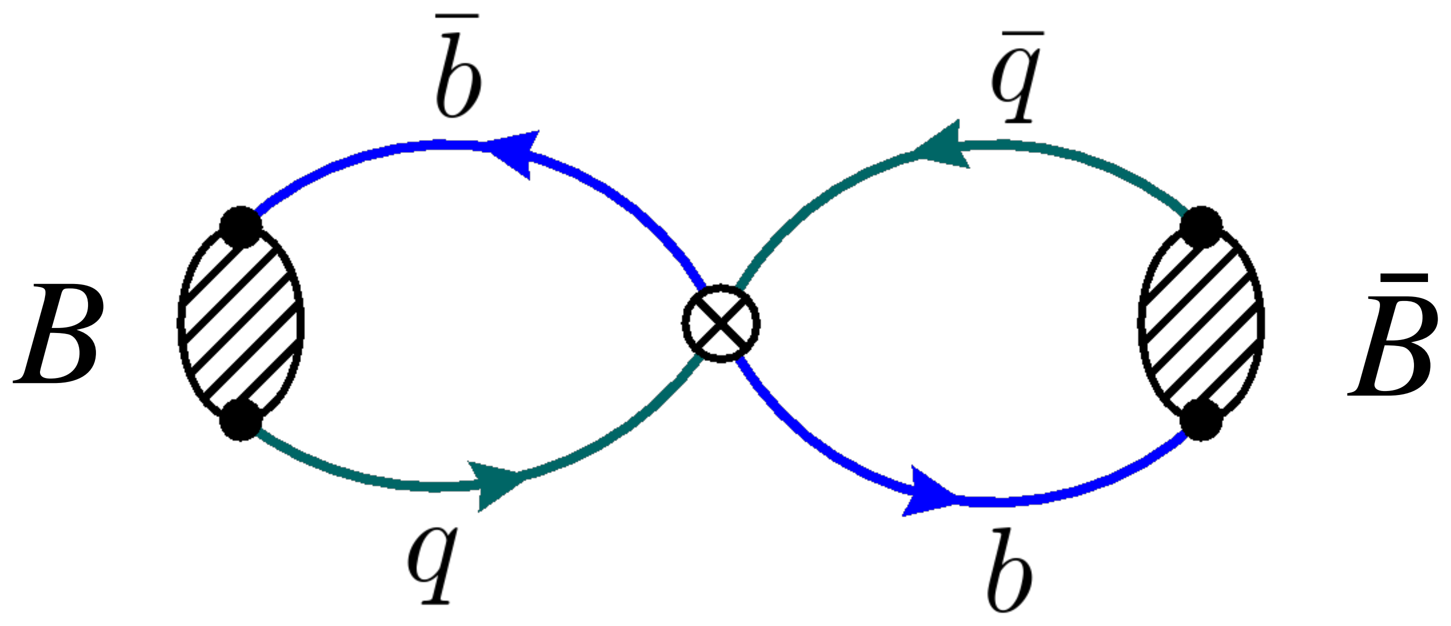
perturbation theory

lattice

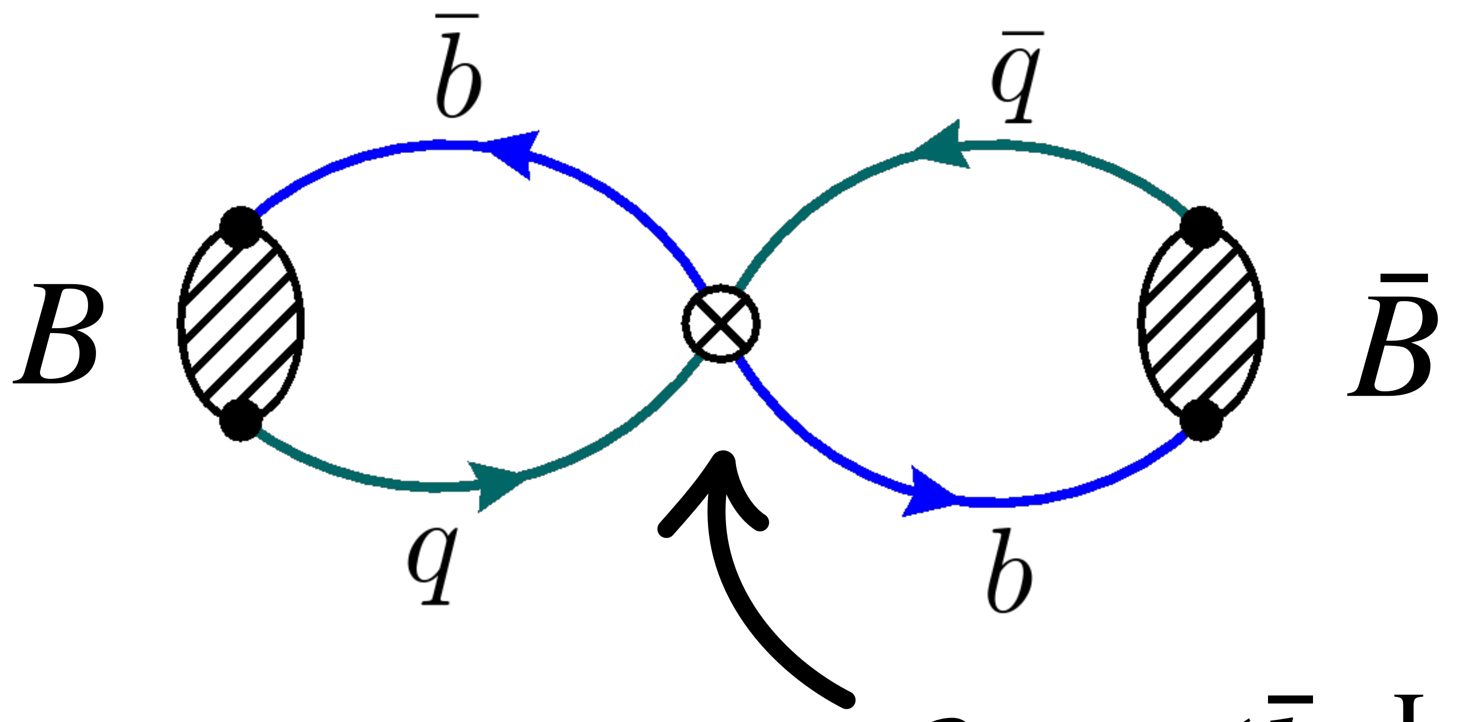
actually:

$$\zeta^{\rm R}_{nm}(t,\mu) = \sum_k \zeta_{nk}(t) Z^{-1}_{km}$$

Meson mixing

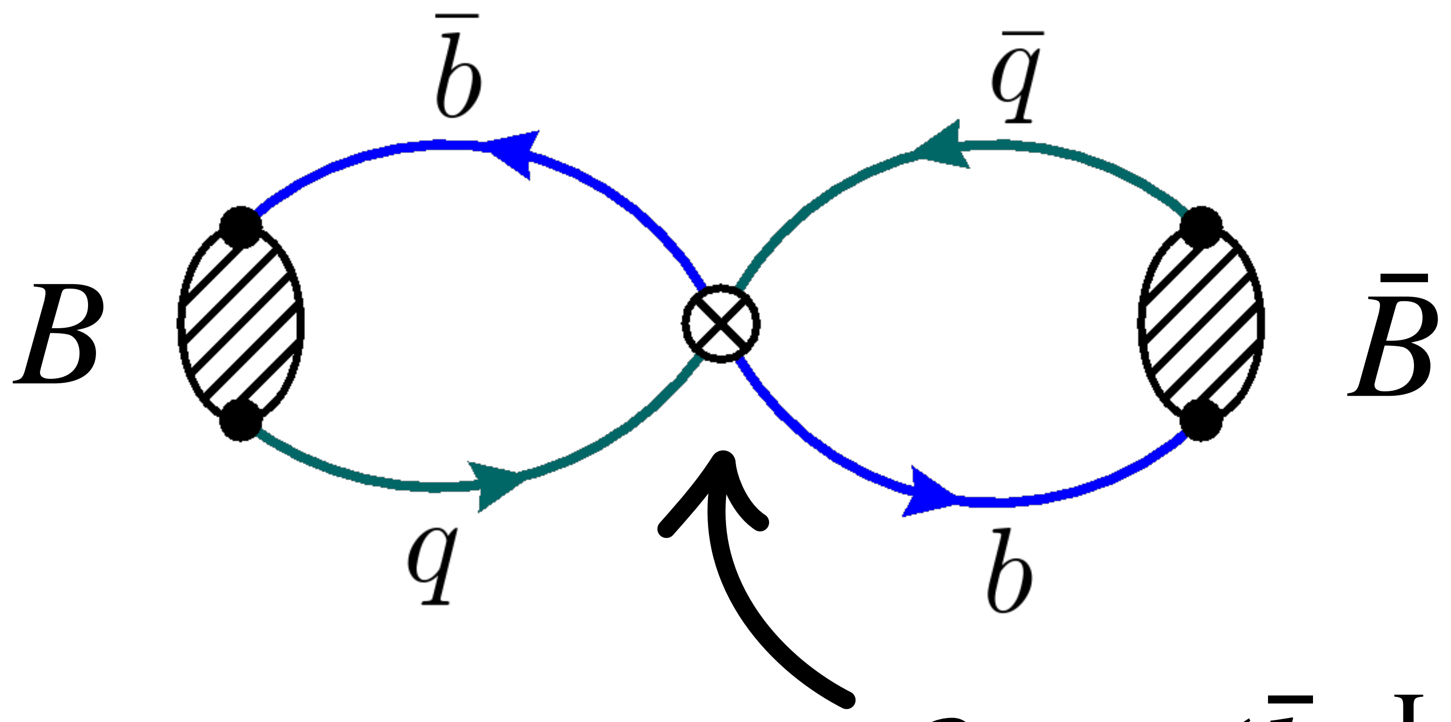


Meson mixing



$$\mathcal{O}_1 = (\bar{b}\gamma_\mu^{\mathrm{L}} q)(\bar{b}\gamma_\mu^{\mathrm{L}} q)$$

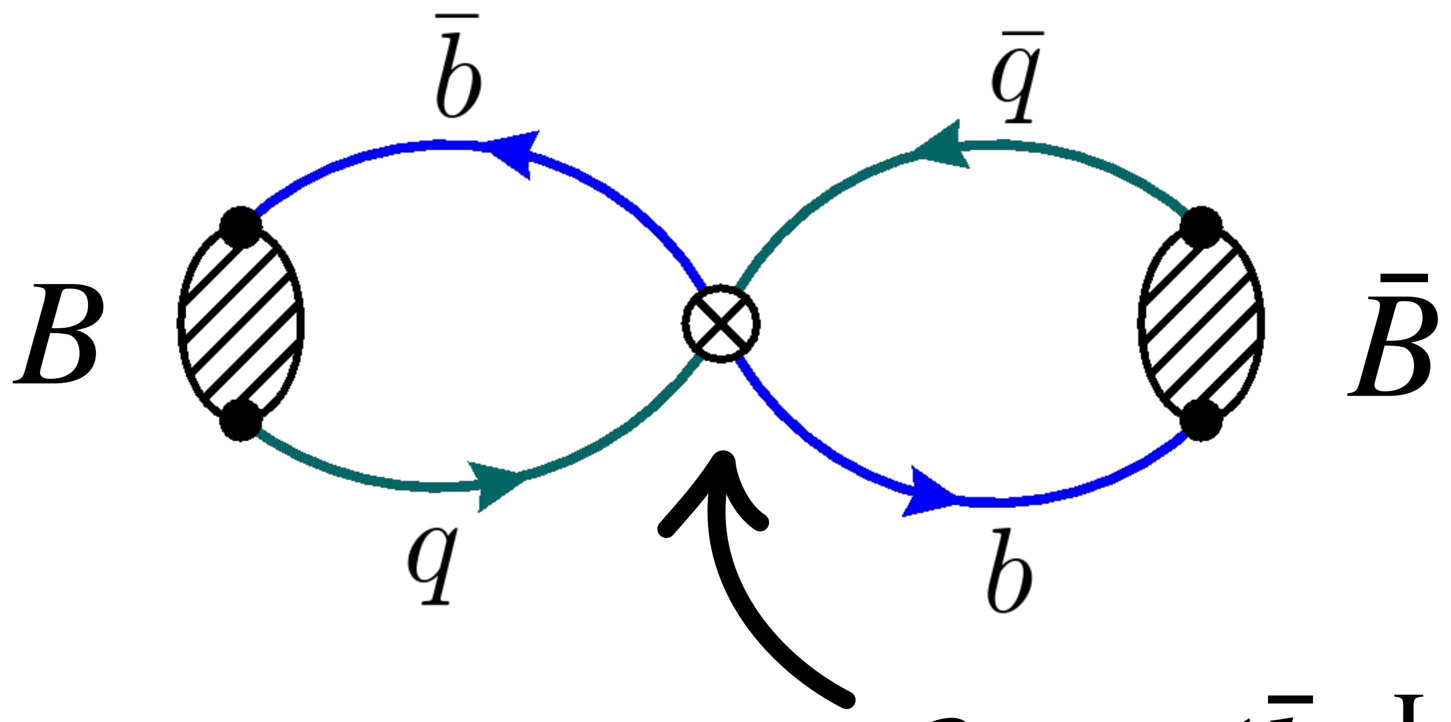
Meson mixing



$$\mathcal{O}_1 = (\bar{b}\gamma_\mu^{\mathrm{L}} q)(\bar{b}\gamma_\mu^{\mathrm{L}} q)$$

$\longrightarrow$ $B_1 \sim \langle B \,|\, \mathcal{O}_1 \,|\, B \rangle$ bag parameter

Meson mixing

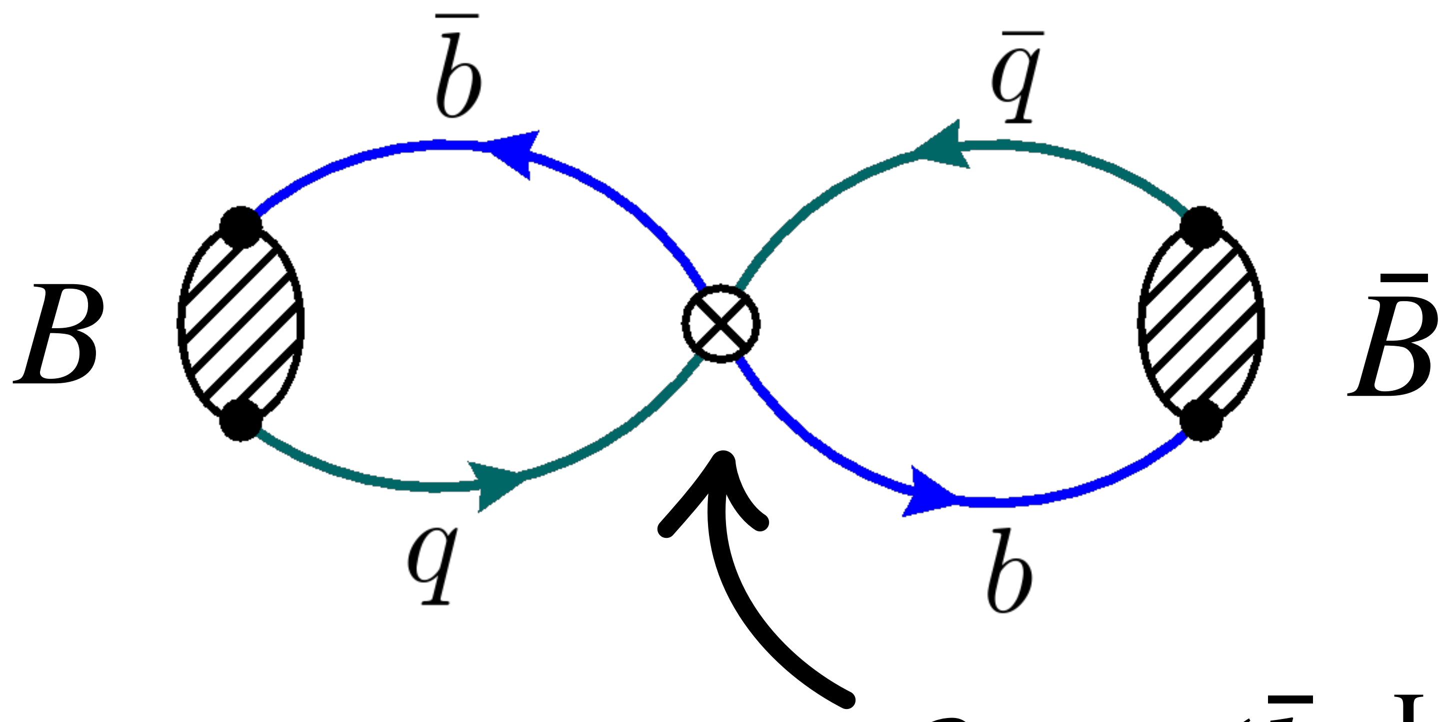


$$\mathcal{O}_1 = (\bar{b}\gamma_\mu^{\mathrm{L}} q)(\bar{b}\gamma_\mu^{\mathrm{L}} q)$$

$\longrightarrow$ $B_1 \sim \langle B | \mathcal{O}_1 | B \rangle$ bag parameter

$$= \zeta^{-1}(t)\langle B | \mathcal{O}_1(t) | B \rangle$$

Meson mixing



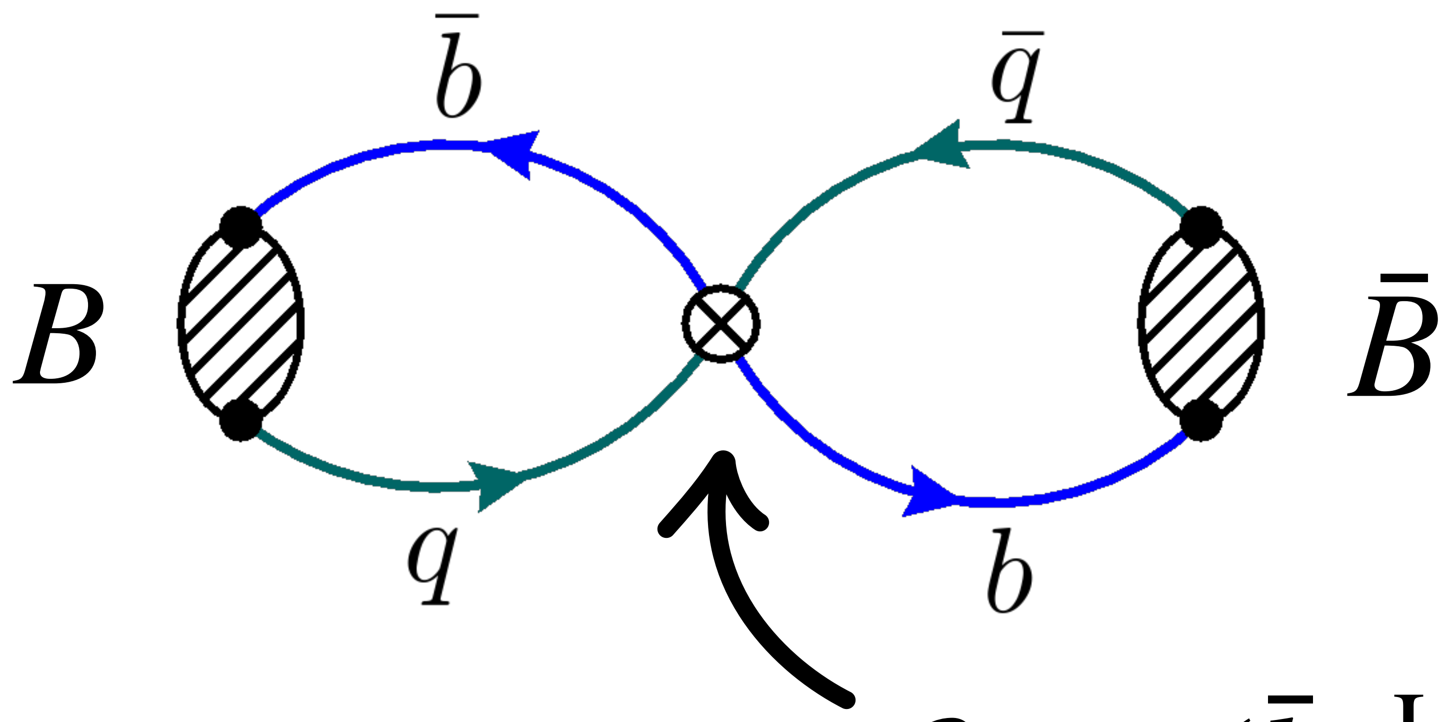
$$\mathcal{O}_1 = (\bar{b}\gamma_\mu^{\mathrm{L}} q)(\bar{b}\gamma_\mu^{\mathrm{L}} q)$$

$\longrightarrow \quad B_1 \sim \langle B \,|\, \mathcal{O}_1 \,|\, B \rangle$ bag parameter

$$= \zeta^{-1}(t)\langle B \,|\, \mathcal{O}_1(t) \,|\, B \rangle$$

perturbative

Meson mixing



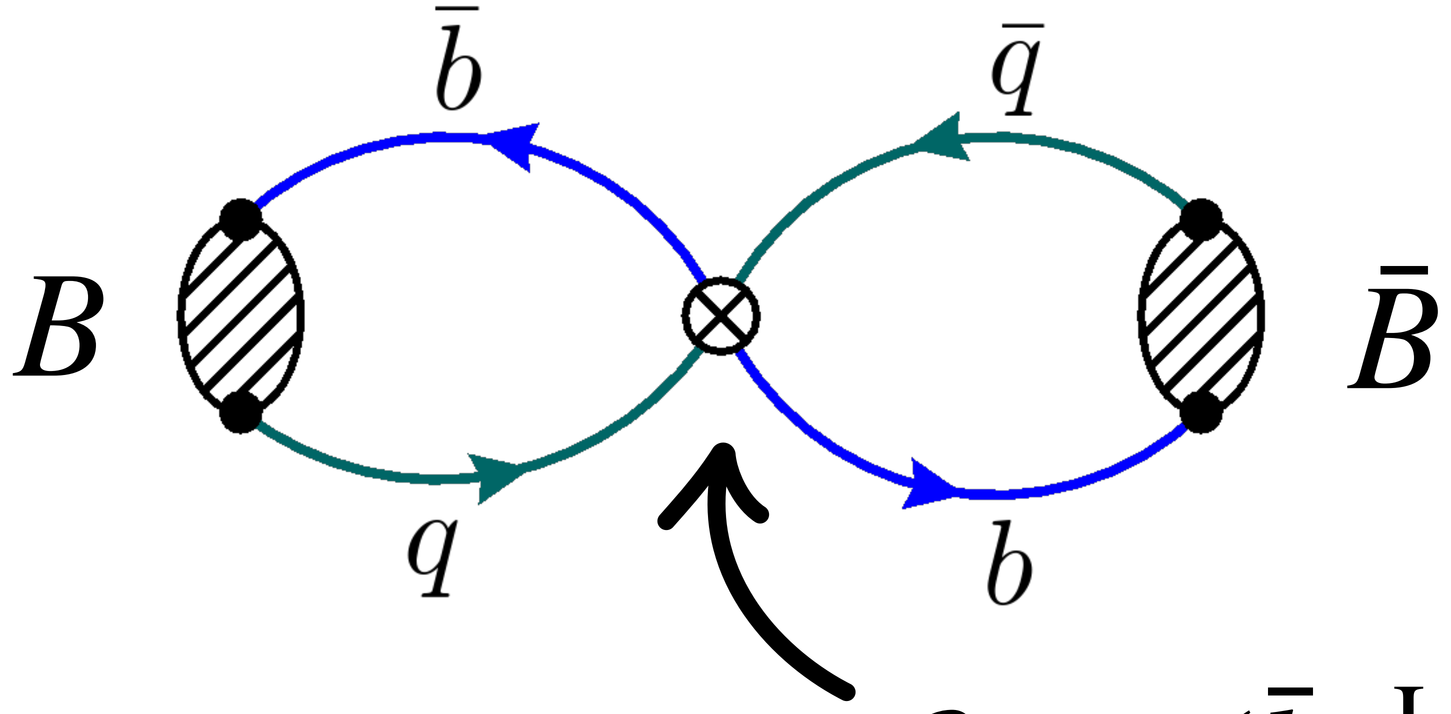
$$\mathcal{O}_1 = (\bar{b}\gamma_\mu^{\mathrm{L}} q)(\bar{b}\gamma_\mu^{\mathrm{L}} q)$$

$\longrightarrow \quad B_1 \sim \langle B | \mathcal{O}_1 | B \rangle$ bag parameter

$$= \zeta^{-1}(t)\langle B | \mathcal{O}_1(t) | B \rangle$$

perturbative ↗ ↖ lattice

Meson mixing



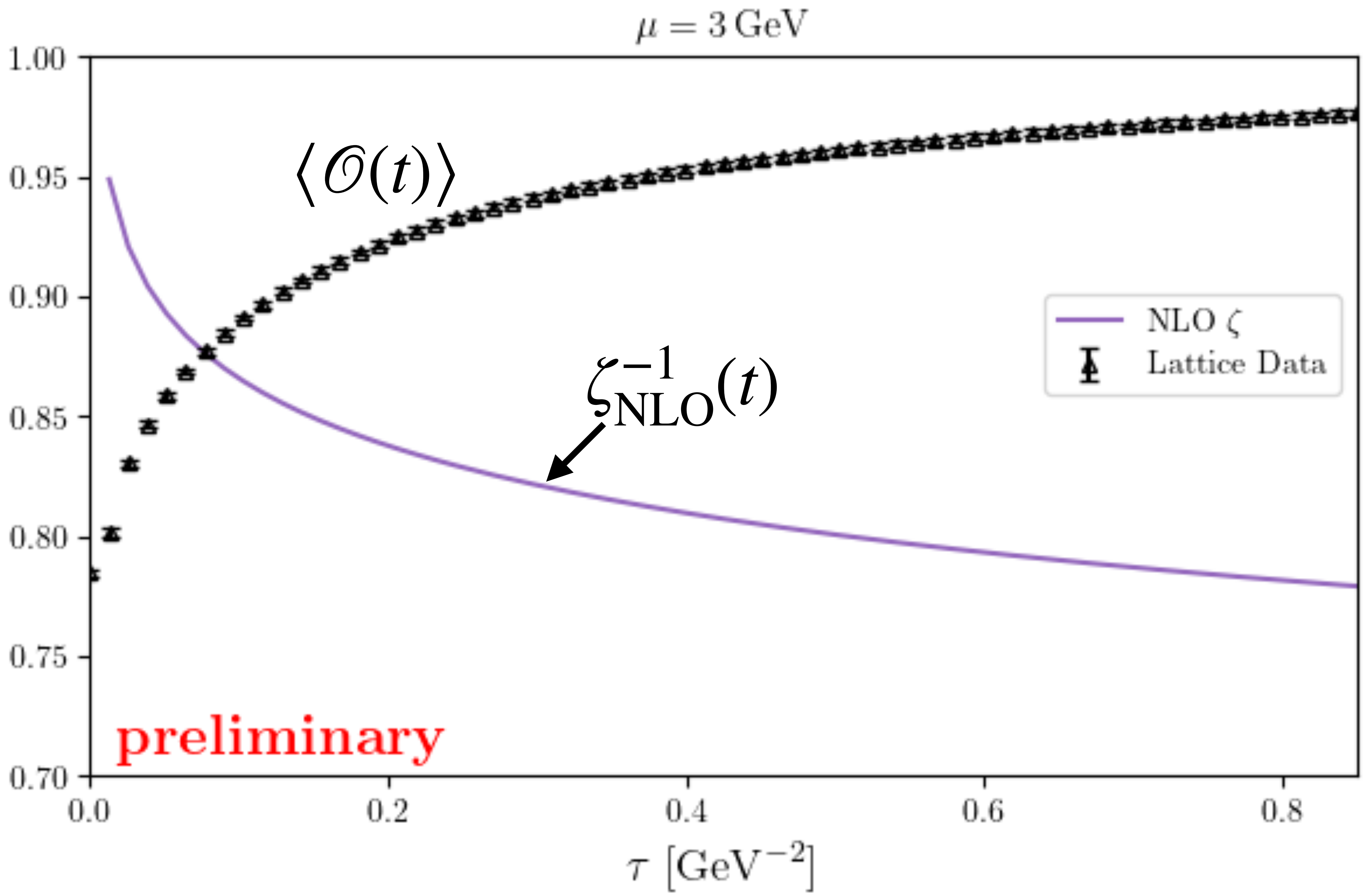
$\mathcal{O}_1 = (\bar{b}\gamma_\mu^{\mathrm{L}} q)(\bar{b}\gamma_\mu^{\mathrm{L}} q)$

$\longrightarrow \quad B_1 \sim \langle B \,|\, \mathcal{O}_1 \,|\, B \rangle$ bag parameter

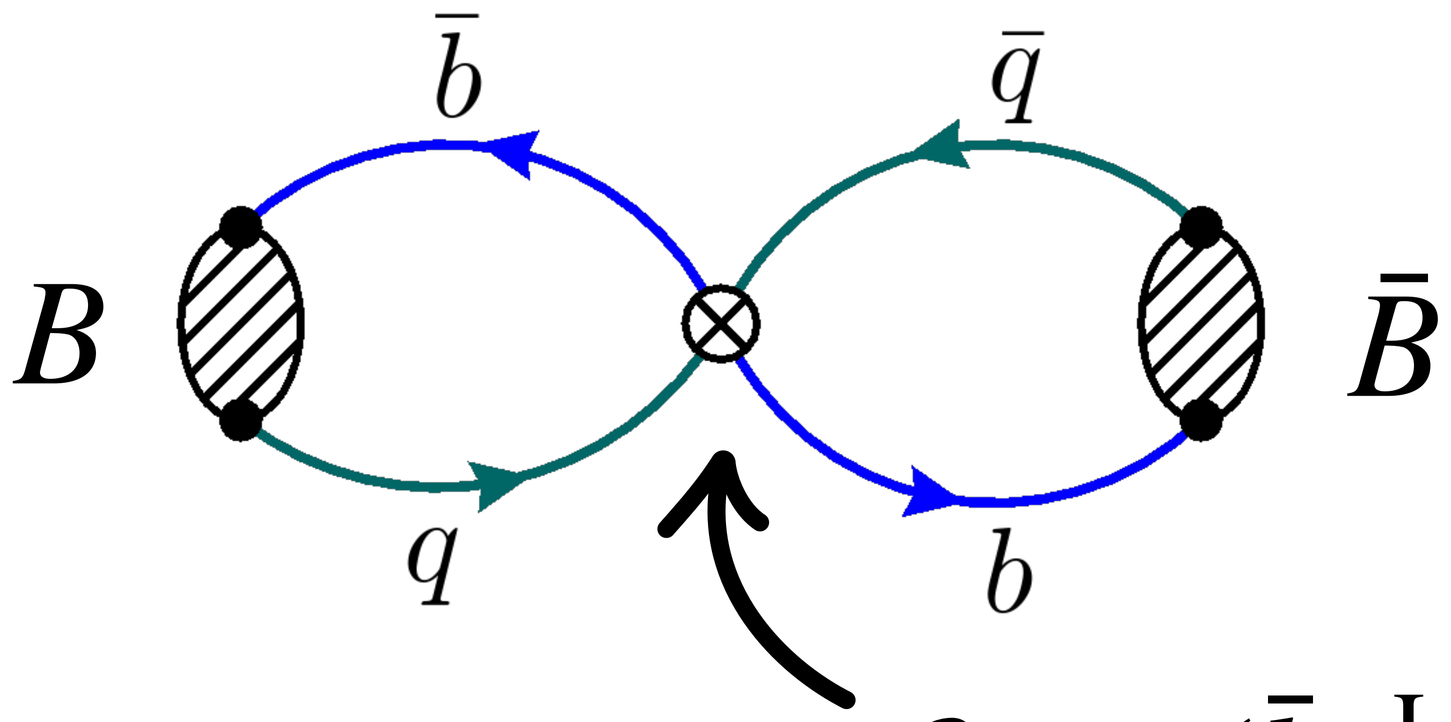
$= \zeta^{-1}(t)\langle B \,|\, \mathcal{O}_1(t) \,|\, B \rangle$

perturbative

lattice



Meson mixing

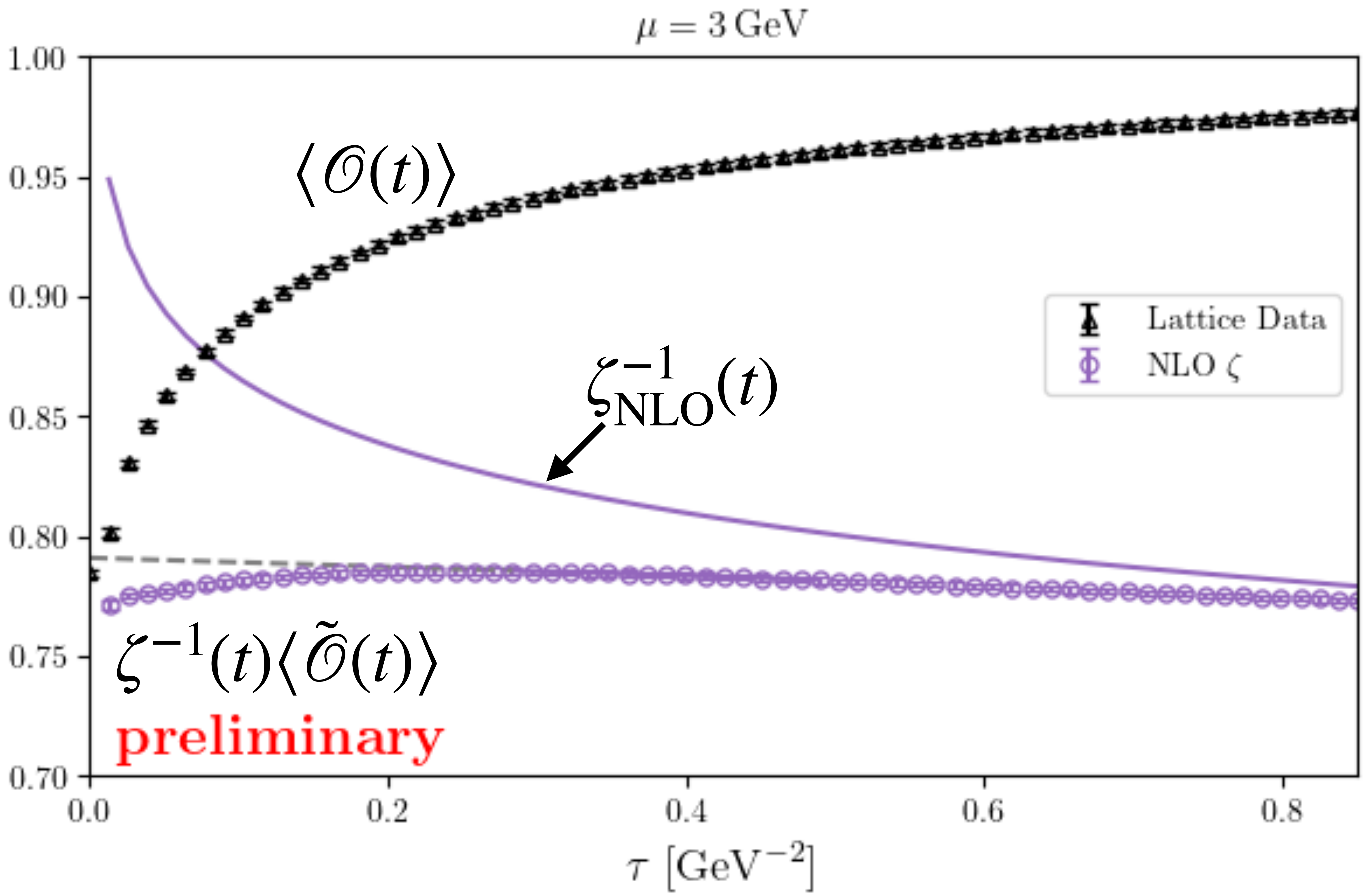


$\mathcal{O}_1 = (\bar{b}\gamma_\mu^{\mathrm{L}} q)(\bar{b}\gamma_\mu^{\mathrm{L}} q)$

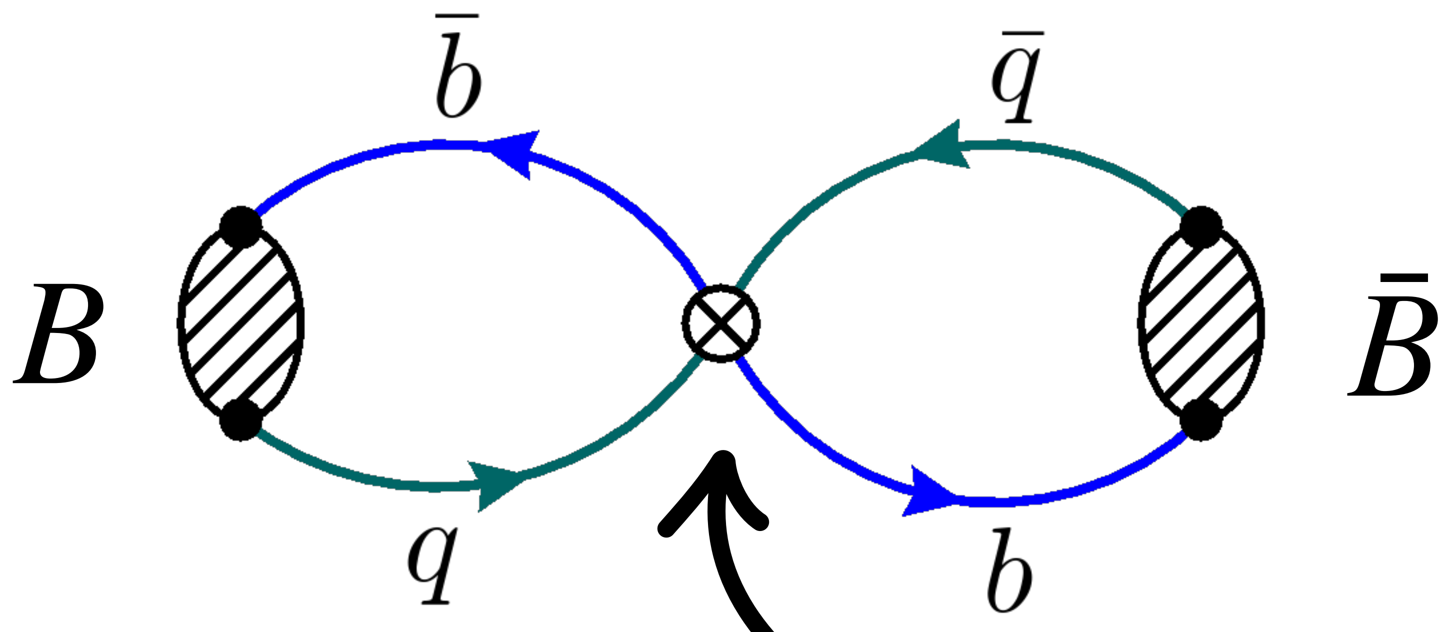
$\rightarrow \quad B_1 \sim \langle B | \mathcal{O}_1 | B \rangle$ bag parameter

$= \zeta^{-1}(t)\langle B | \mathcal{O}_1(t) | B \rangle$

perturbative ↗ ↖ lattice



Meson mixing

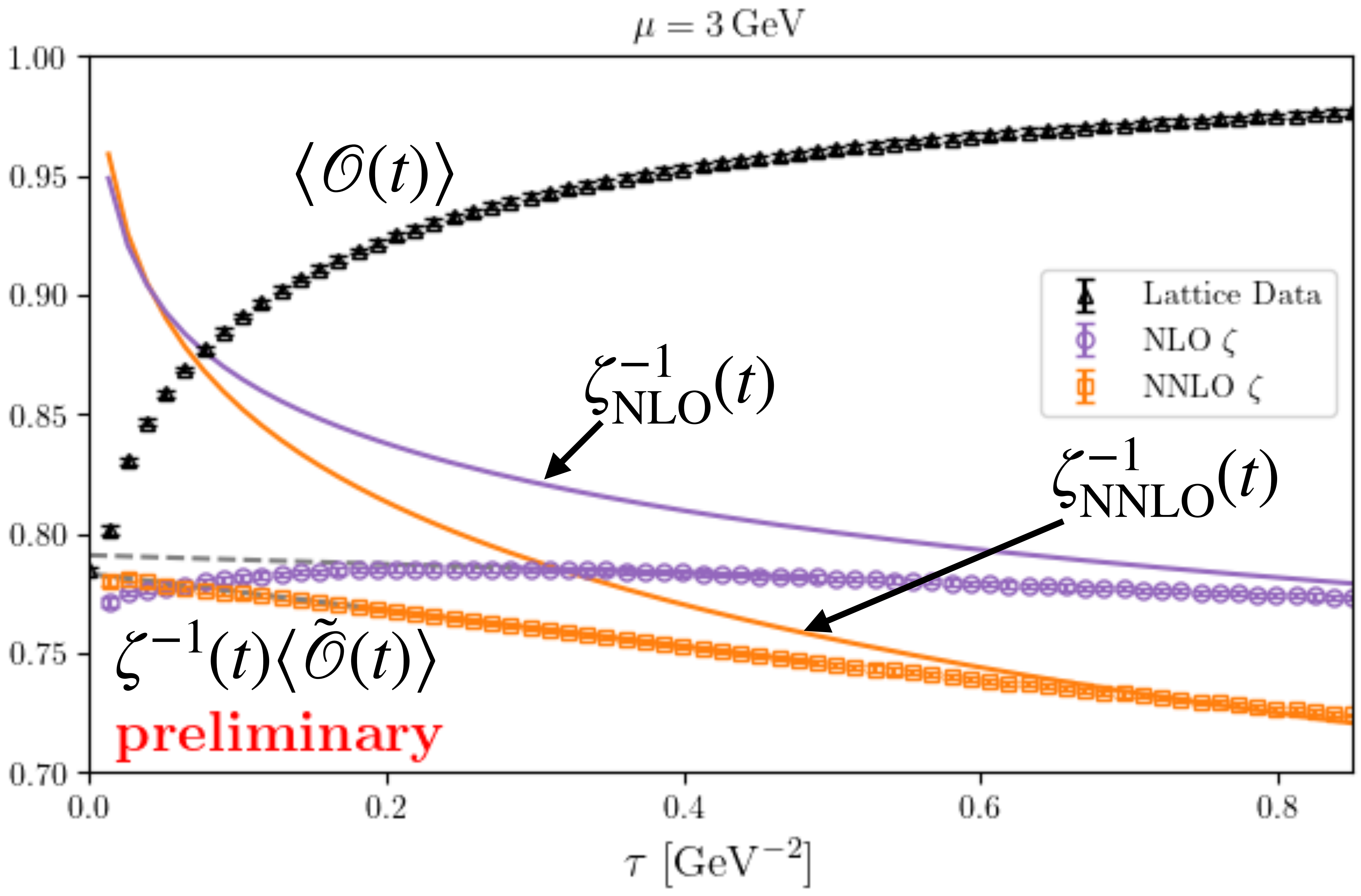


$$\mathcal{O}_1 = (\bar{b}\gamma_\mu^{\mathrm{L}} q)(\bar{b}\gamma_\mu^{\mathrm{L}} q)$$

$\longrightarrow \quad B_1 \sim \langle B | \mathcal{O}_1 | B \rangle$ bag parameter

$$= \zeta^{-1}(t) \langle B | \mathcal{O}_1(t) | B \rangle$$

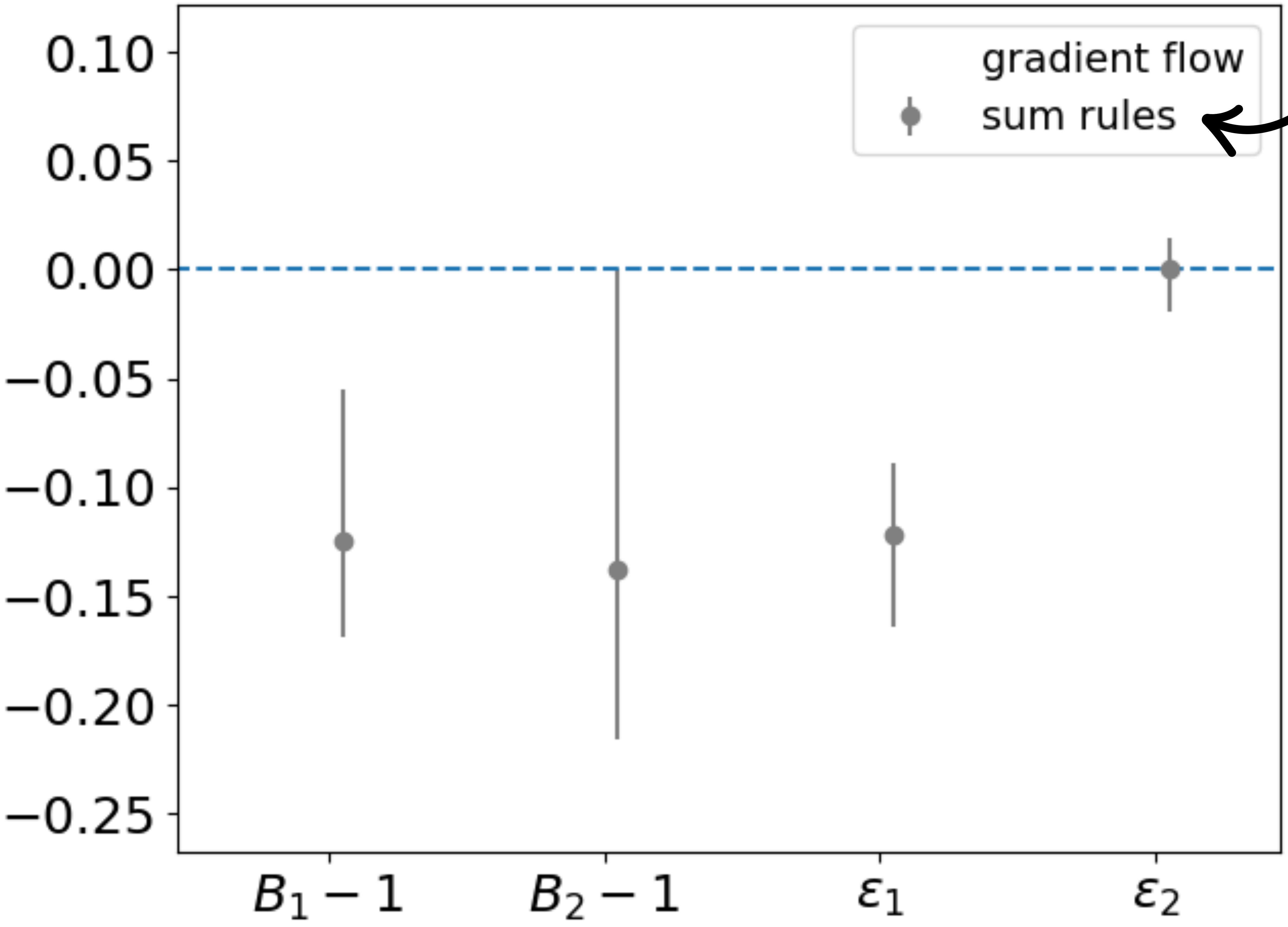
perturbative ↗ ↖ lattice



Black, RH, Lange, Rago, Shindler, Witzel ’26

CRC TRR 257

D meson lifetime bag parameters



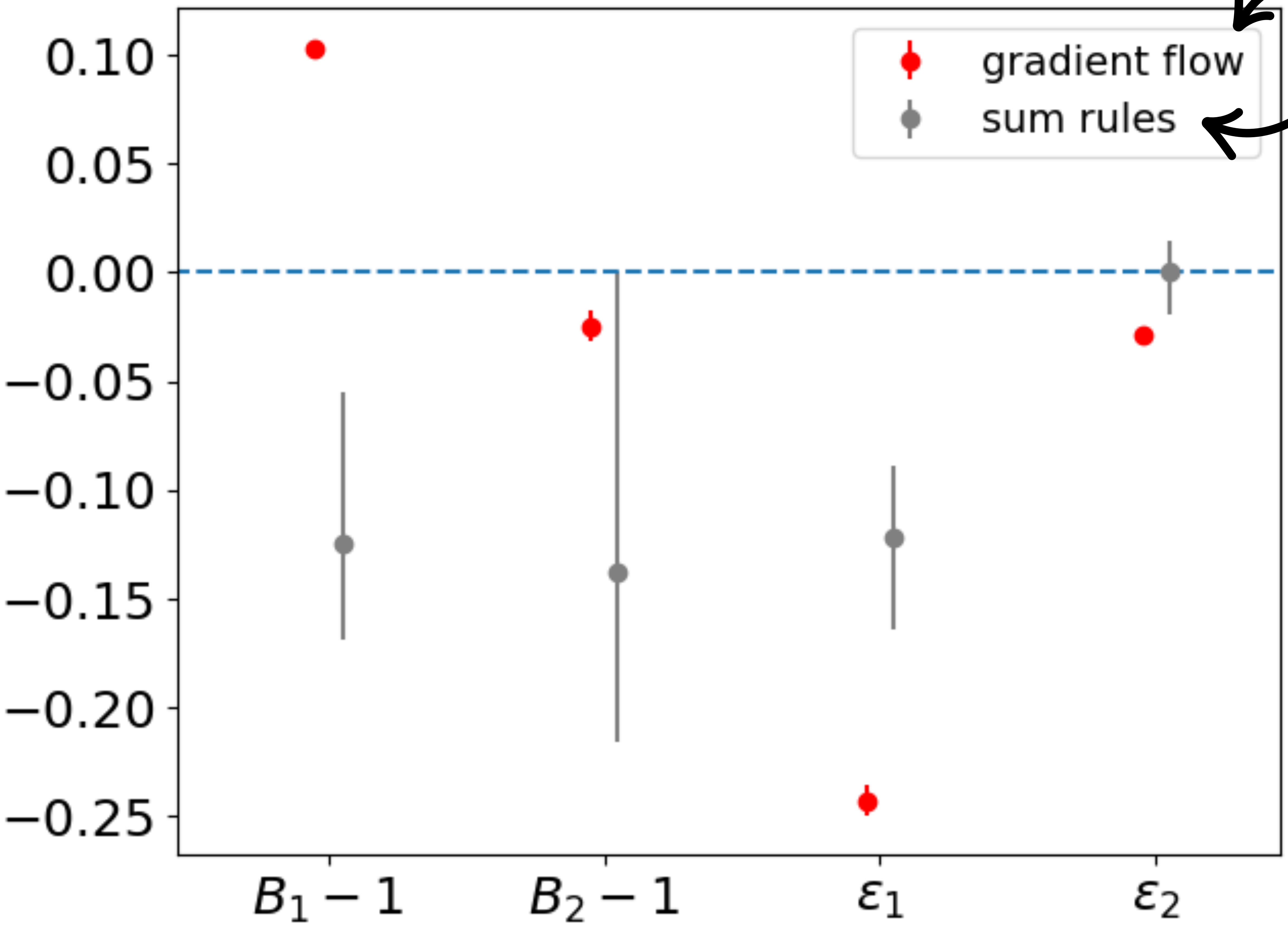
Black, Lang, Lenz, Wüthrich '25

$$\mathcal{O}_1 = (\bar{Q}\gamma_\mu(1-\gamma_5)q)(\bar{q}\gamma^\mu(1-\gamma_5)Q),$$
$$T_1 = (\bar{Q}\gamma_\mu(1-\gamma_5)T^A q)(\bar{q}\gamma^\mu(1-\gamma_5)T^A Q),$$
$$\mathcal{O}_2 = (\bar{Q}(1-\gamma_5)q)(\bar{q}(1+\gamma_5)Q),$$
$$T_2 = (\bar{Q}(1-\gamma_5)T^A q)(\bar{q}(1+\gamma_5)T^A Q),$$

D meson lifetime bag parameters

Black, RH, Lange, Rago, Shindler, Witzel '26

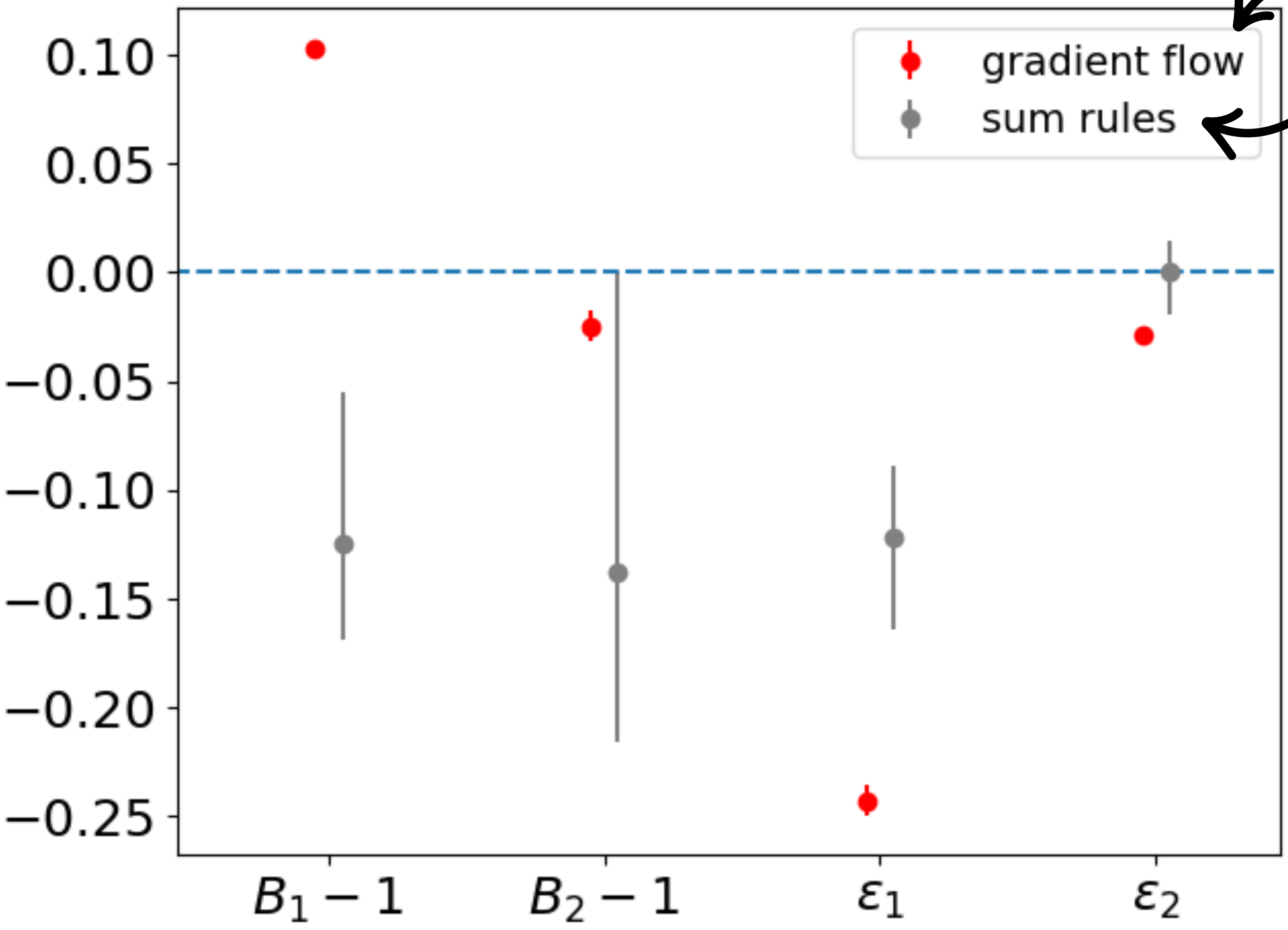
Black, Lang, Lenz, Wüthrich '25



$$\mathcal{O}_1 = (\bar{Q}\gamma_\mu(1-\gamma_5)q)(\bar{q}\gamma^\mu(1-\gamma_5)Q),$$
$$T_1 = (\bar{Q}\gamma_\mu(1-\gamma_5)T^A q)(\bar{q}\gamma^\mu(1-\gamma_5)T^A Q),$$
$$\mathcal{O}_2 = (\bar{Q}(1-\gamma_5)q)(\bar{q}(1+\gamma_5)Q),$$
$$T_2 = (\bar{Q}(1-\gamma_5)T^A q)(\bar{q}(1+\gamma_5)T^A Q),$$

D meson lifetime bag parameters

CRC TRR 257



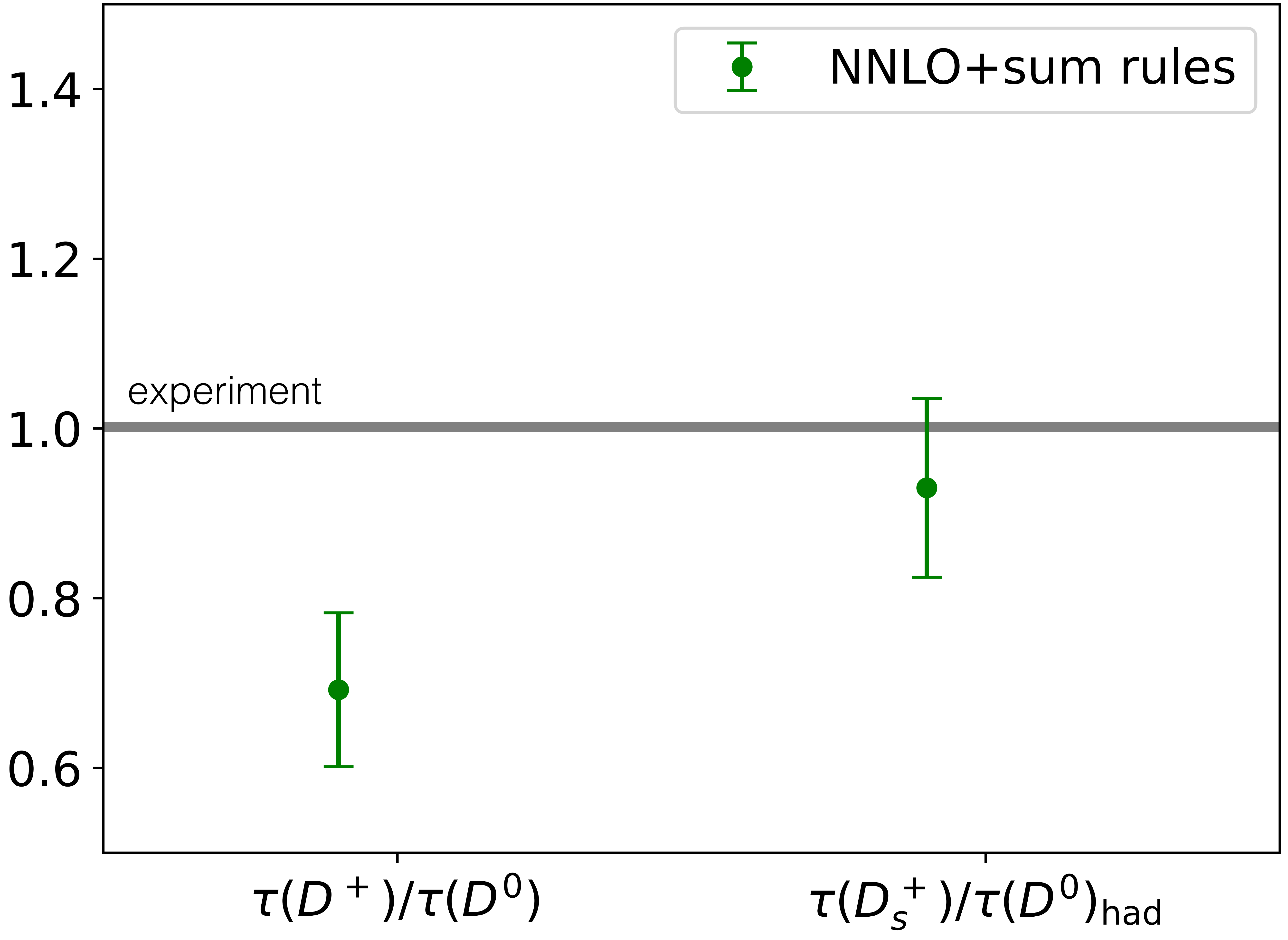
Black, RH, Lange, Rago, Shindler, Witzel ‘26

Black, Lang, Lenz, Wüthrich ‘25

$$\mathcal{O}_1 = (\bar{Q}\gamma_\mu(1-\gamma_5)q)(\bar{q}\gamma^\mu(1-\gamma_5)Q),$$
$$T_1 = (\bar{Q}\gamma_\mu(1-\gamma_5)T^A q)(\bar{q}\gamma^\mu(1-\gamma_5)T^A Q),$$
$$\mathcal{O}_2 = (\bar{Q}(1-\gamma_5)q)(\bar{q}(1+\gamma_5)Q),$$
$$T_2 = (\bar{Q}(1-\gamma_5)T^A q)(\bar{q}(1+\gamma_5)T^A Q),$$

but: comparison not fully consistent

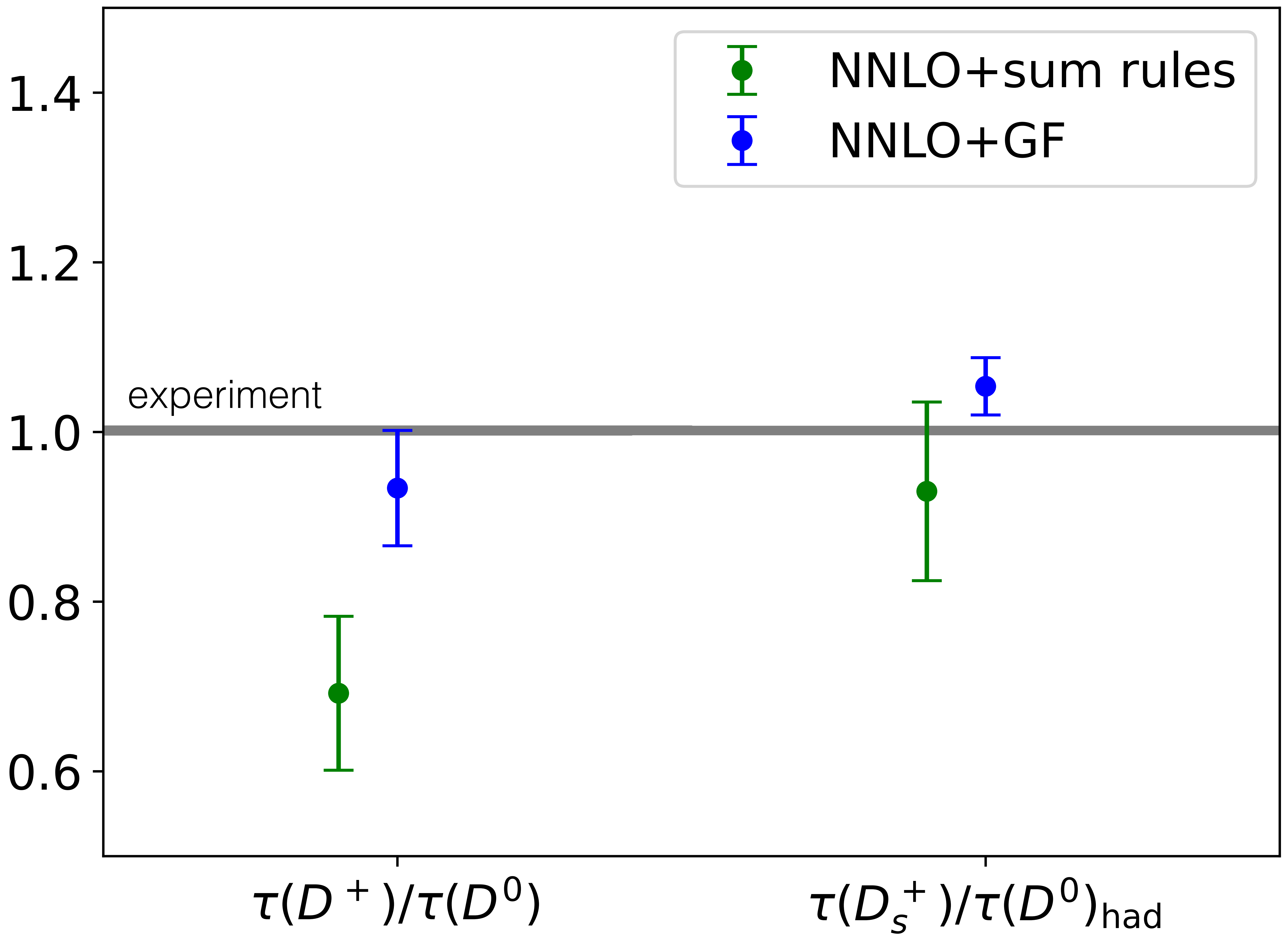
Lifetime ratios



Combining with NNLO Wilson coefficients

Moretti, Nierste, Reeck, Steinhauser '26

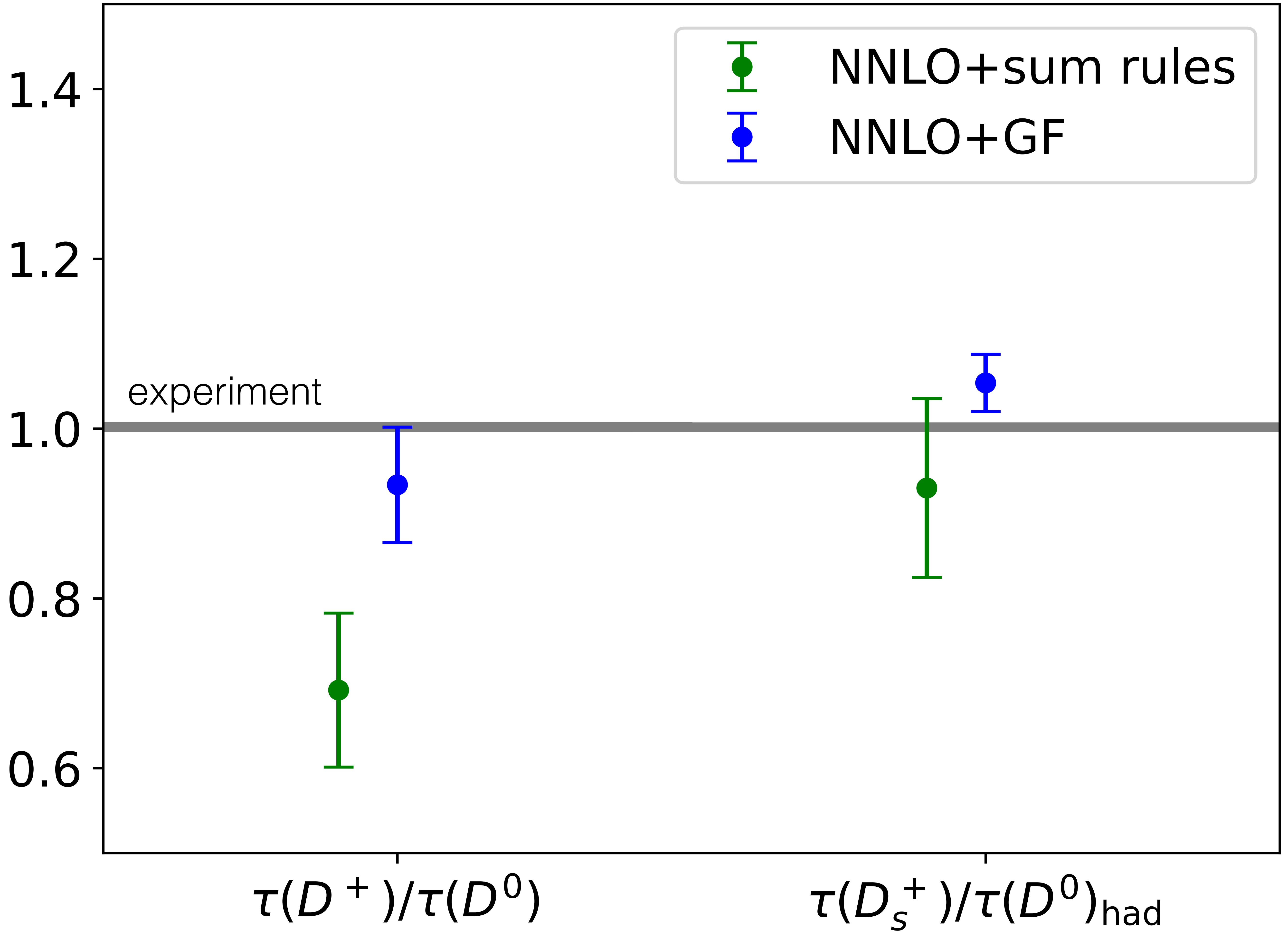
Lifetime ratios



Combining with NNLO Wilson coefficients

Moretti, Nierste, Reeck, Steinhauser ‘26

Lifetime ratios

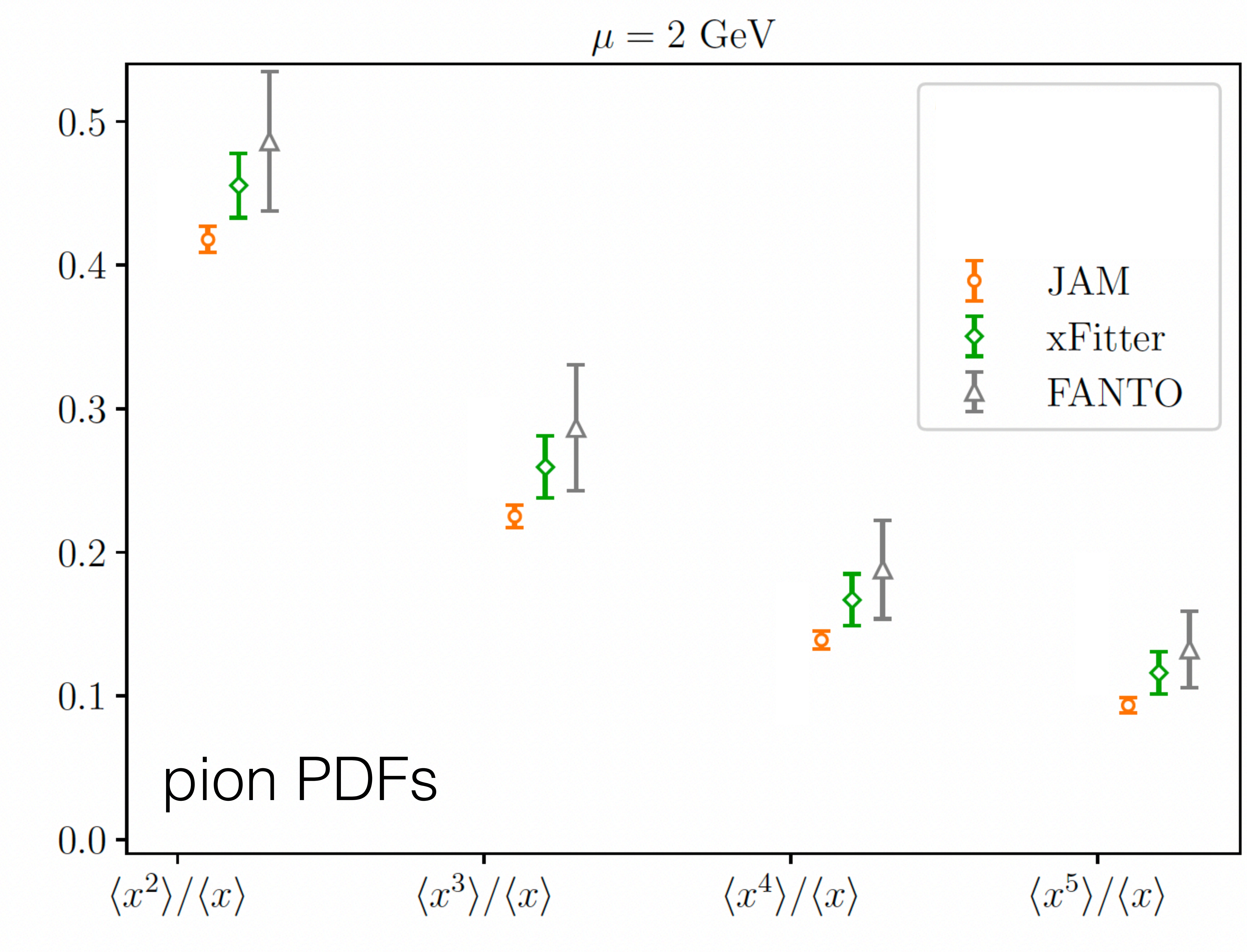


Combining with NNLO Wilson coefficients

Moretti, Nierste, Reeck, Steinhauser '26

Jonas Kohnen (Tue 12:00)
Antonio Rago (Tue 12:30)

Moments of parton densities



$$\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}\,|\,h(p)\rangle =$$

$$= \sum_m \zeta^{-1}_{mn}(t)\,\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}(t)\,|\,h(p)\rangle$$

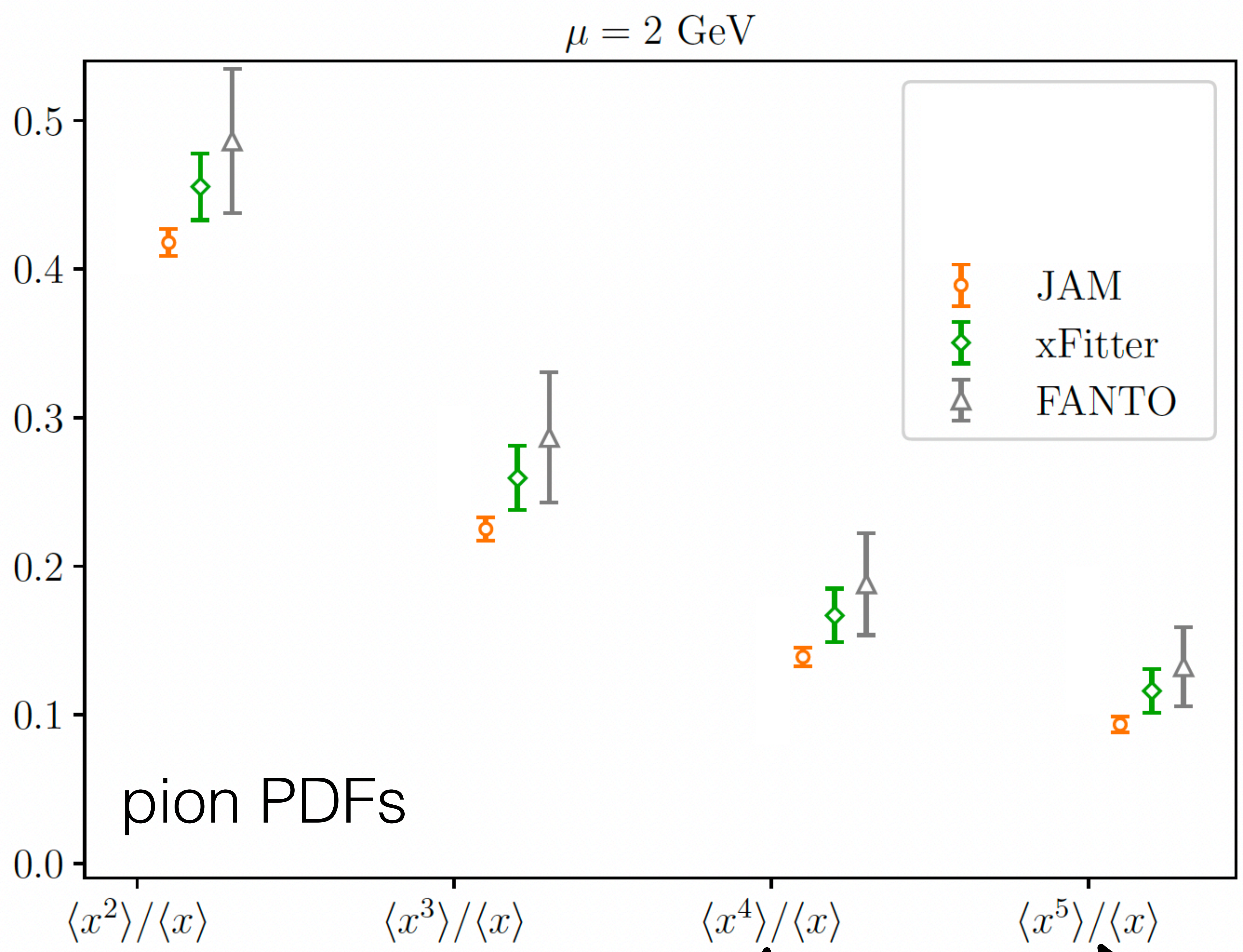
Francis, …, Shindler, … '25

Francis, …, RH, Kohnen, Shindler, … '25

RH, Kohnen, Shindler '25

see also Edwards et al. '26

Moments of parton densities



unreachable w/o gradient flow

$$\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}\,|\,h(p)\rangle =$$
$$= \sum_m \zeta^{-1}_{mn}(t)\,\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}(t)\,|\,h(p)\rangle$$

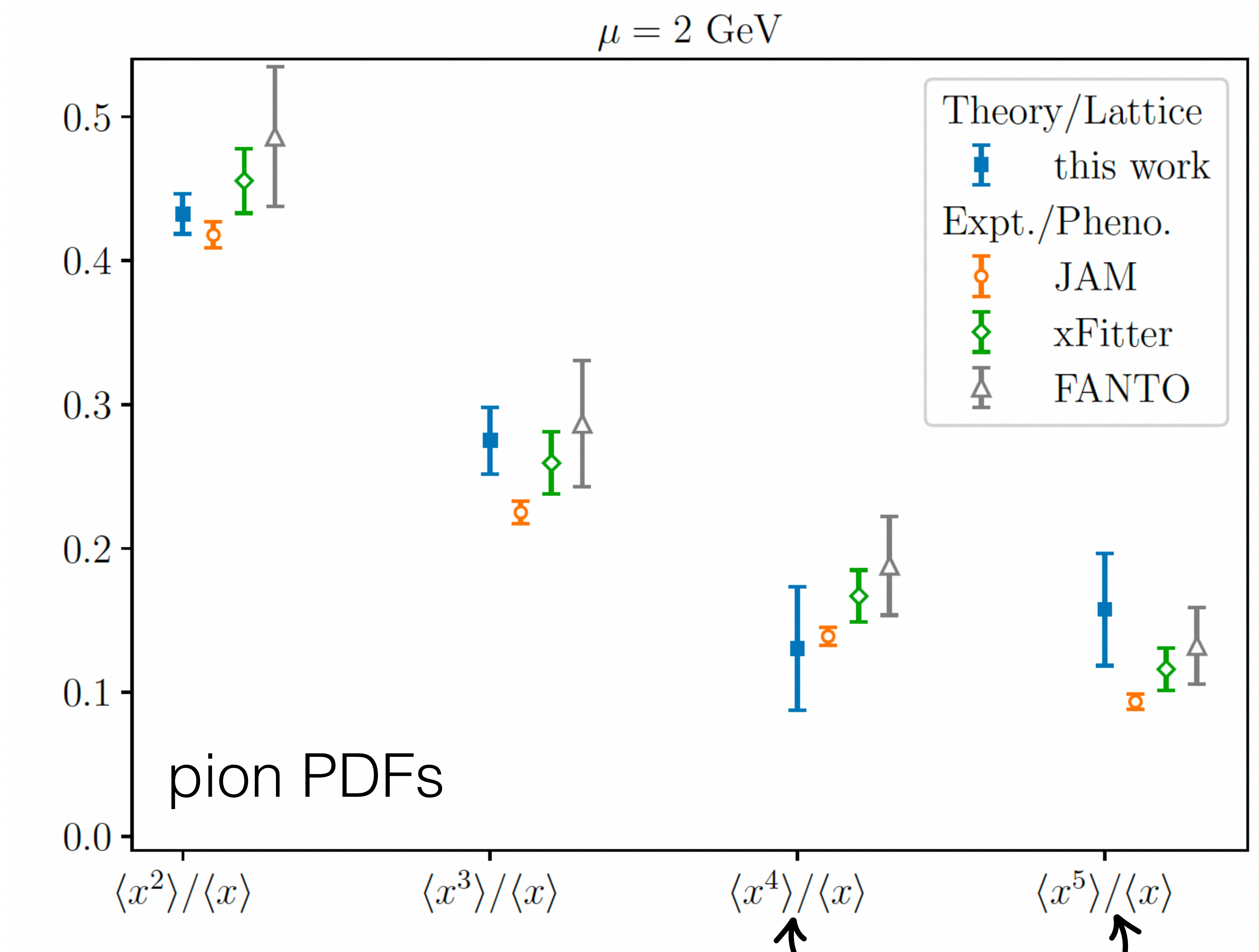
Francis, …, Shindler, … ‘25

Francis, …, RH, Kohnen, Shindler, … ‘25

RH, Kohnen, Shindler ‘25

see also Edwards et al. ‘26

Moments of parton densities



unreachable w/o gradient flow

$$\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}\,|\,h(p)\rangle =$$
$$= \sum_m \zeta^{-1}_{mn}(t)\,\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}(t)\,|\,h(p)\rangle$$

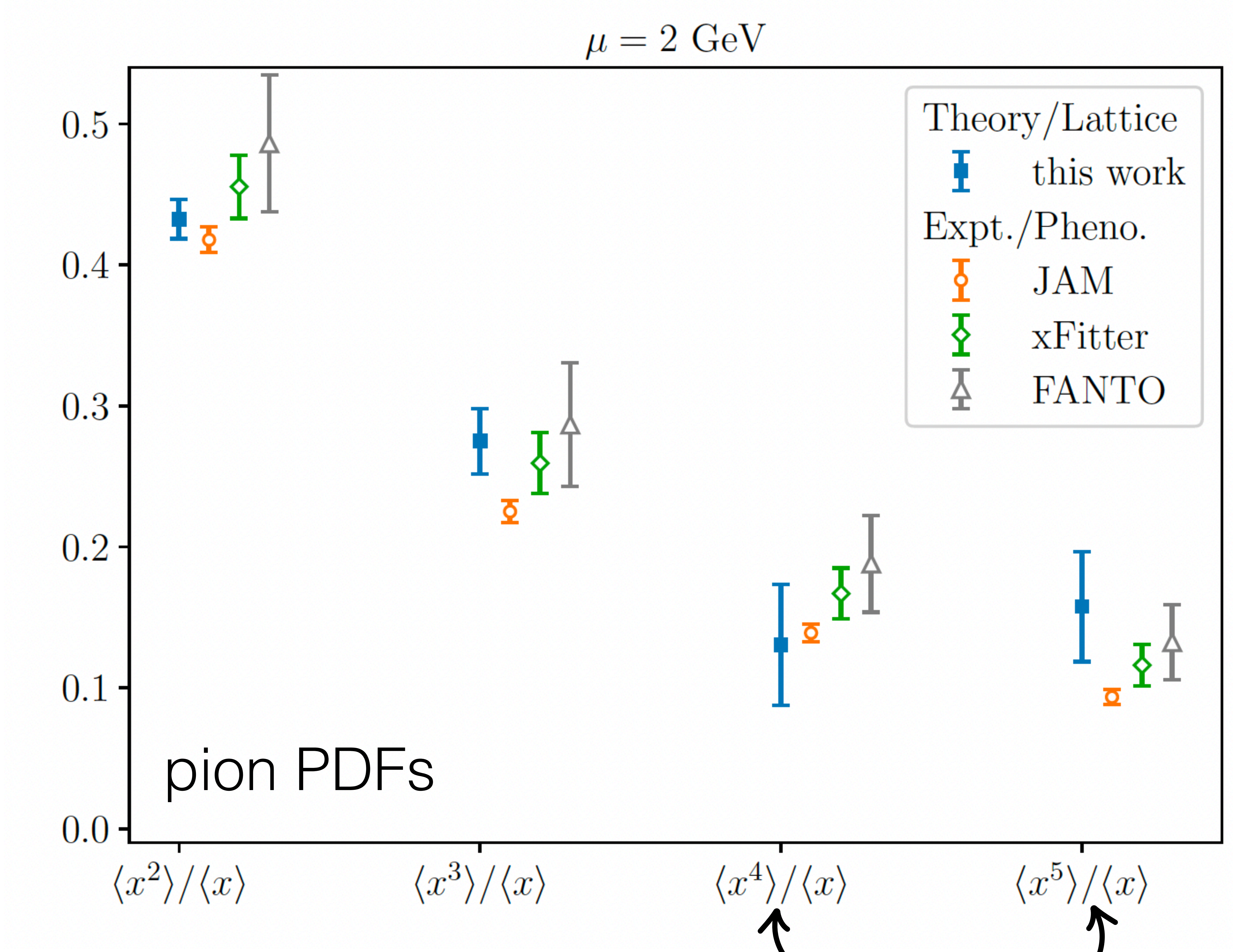
Francis, …, Shindler, … '25

Francis, …, RH, Kohnen, Shindler, … '25

RH, Kohnen, Shindler '25

see also Edwards et al. '26

Moments of parton densities



unreachable w/o gradient flow

$$\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}\,|\,h(p)\rangle =$$
$$= \sum_m \zeta^{-1}_{mn}(t)\,\langle h(p)\,|\,\mathcal{O}_{\{\mu_1\cdots\mu_n\}}(t)\,|\,h(p)\rangle$$

Andrea Shindler (Fri 09:30)
Chris Monahan (Fri 09:00)

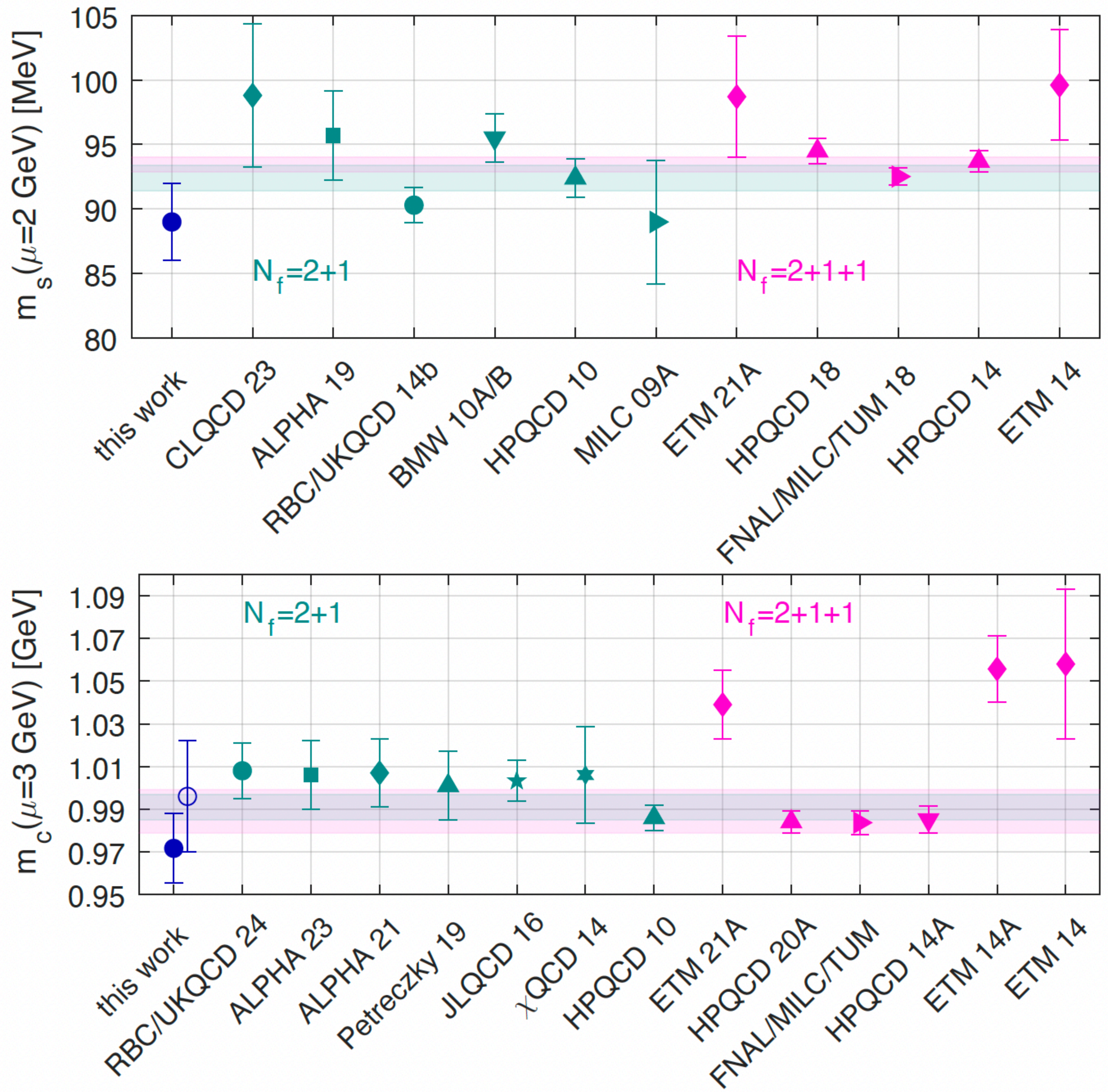
Francis, …, Shindler, … ‘25

Francis, …, RH, Kohnen, Shindler, … ‘25

RH, Kohnen, Shindler ‘25

see also Edwards et al. ‘26

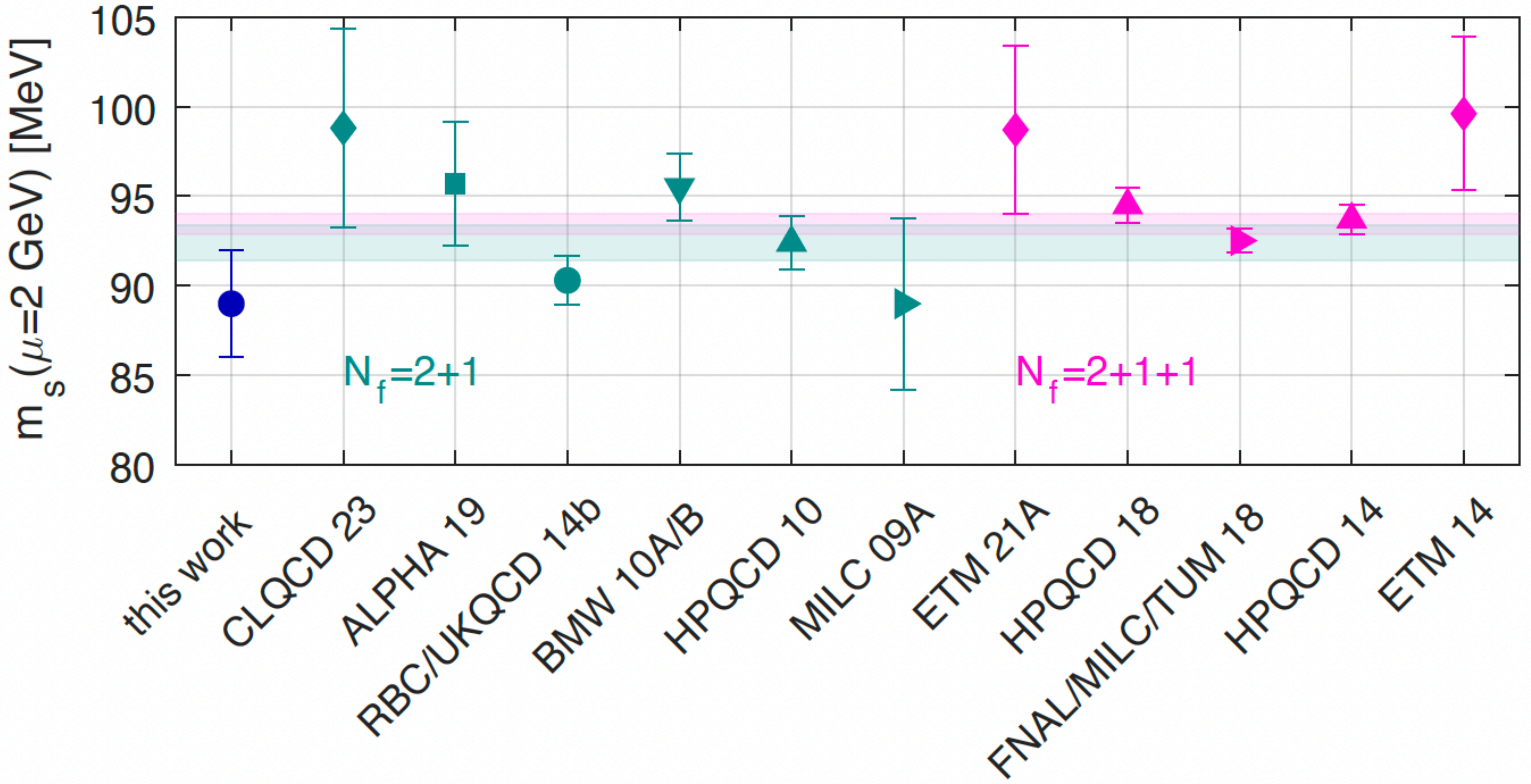
Quark masses

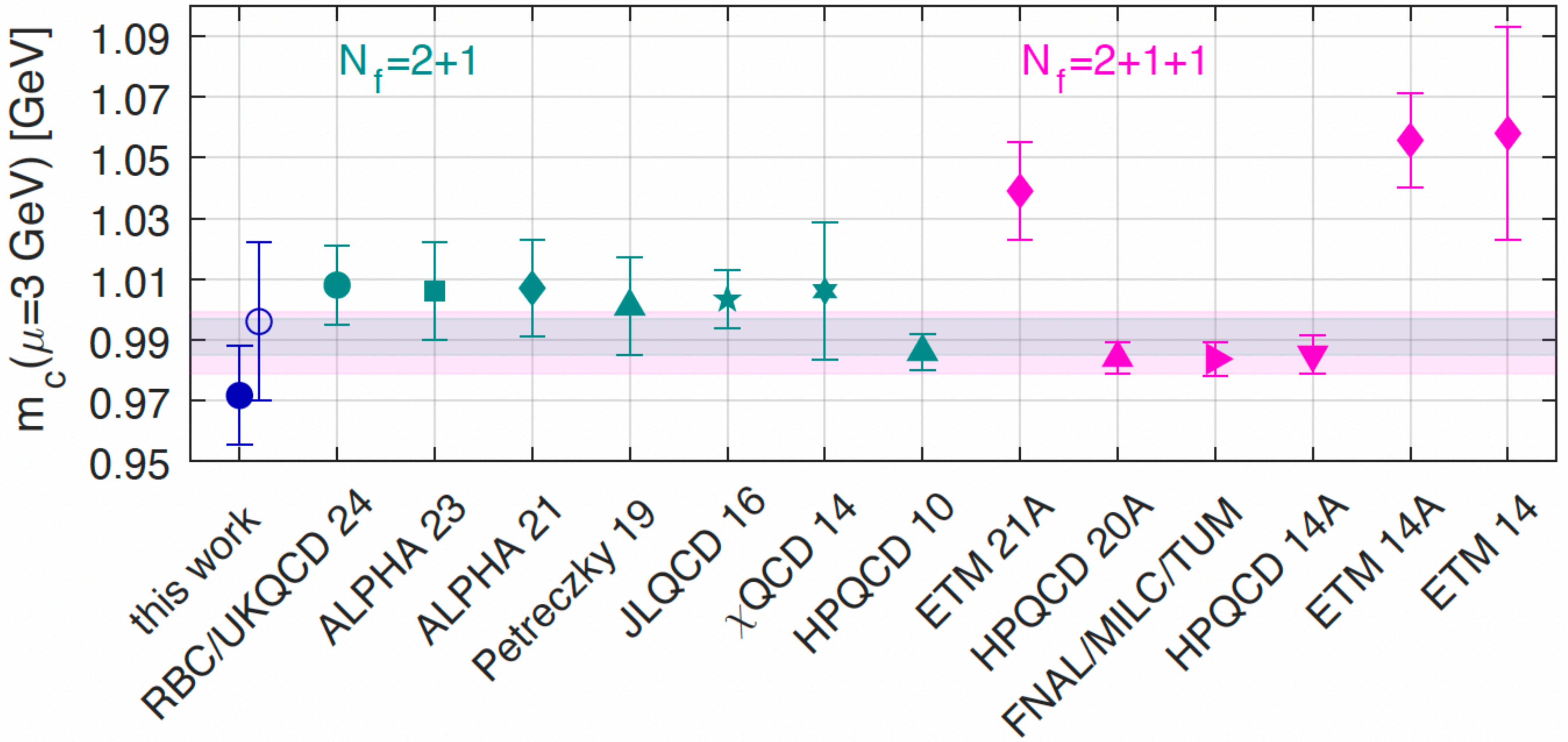


Quark masses

Black, RH, Hasenfratz, Rago, Witzel ‘25

CRC TRR 257





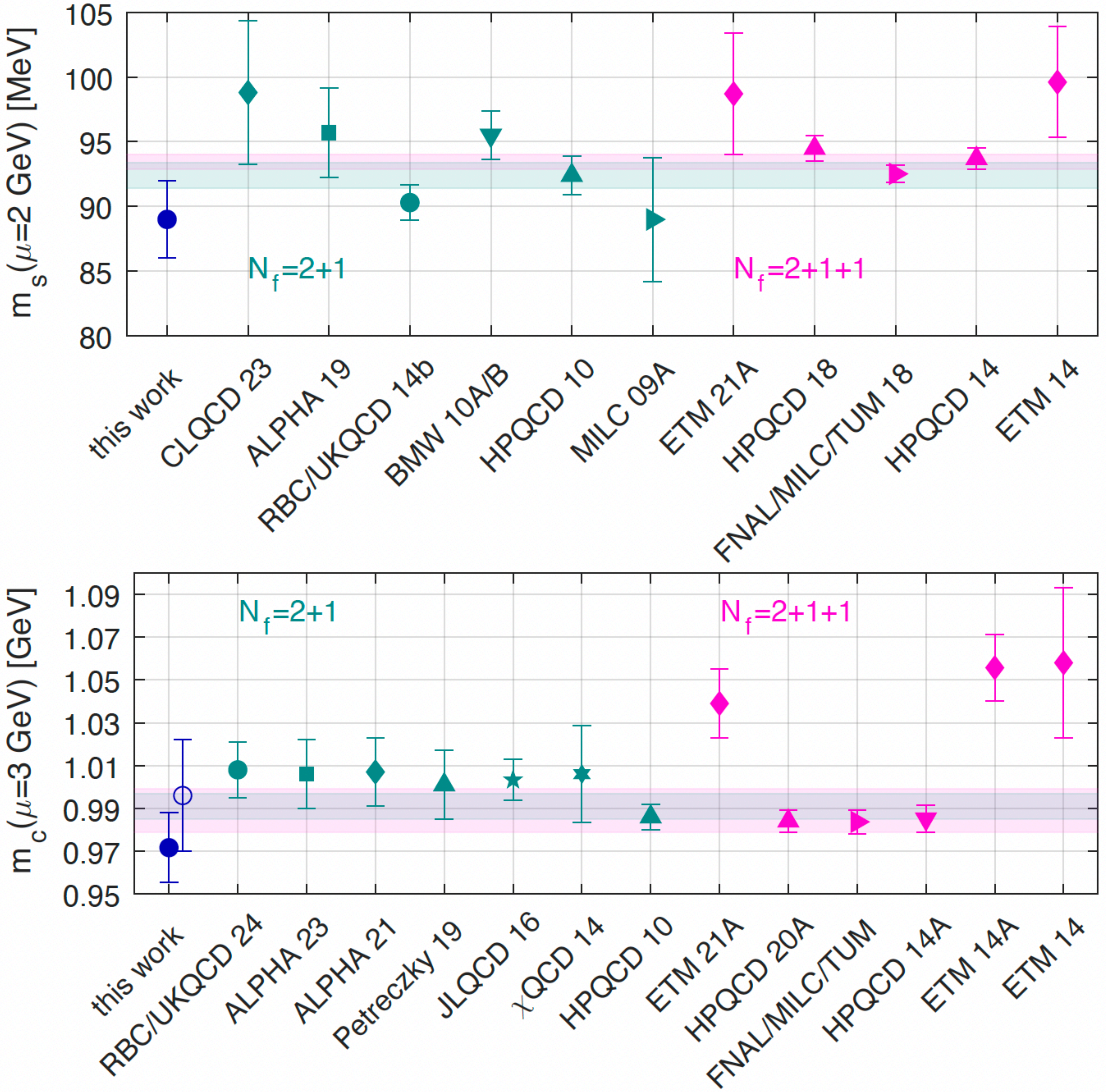
Quark masses

Black, RH, Hasenfratz, Rago, Witzel '25

P H
CRC TRR 257

PCAC:

$$R(t, \tau) = -\frac{\langle j_A^0(t, \tau) j_P(t=0) \rangle}{\langle j_P(t, \tau) j_P(t=0) \rangle}$$



Quark masses

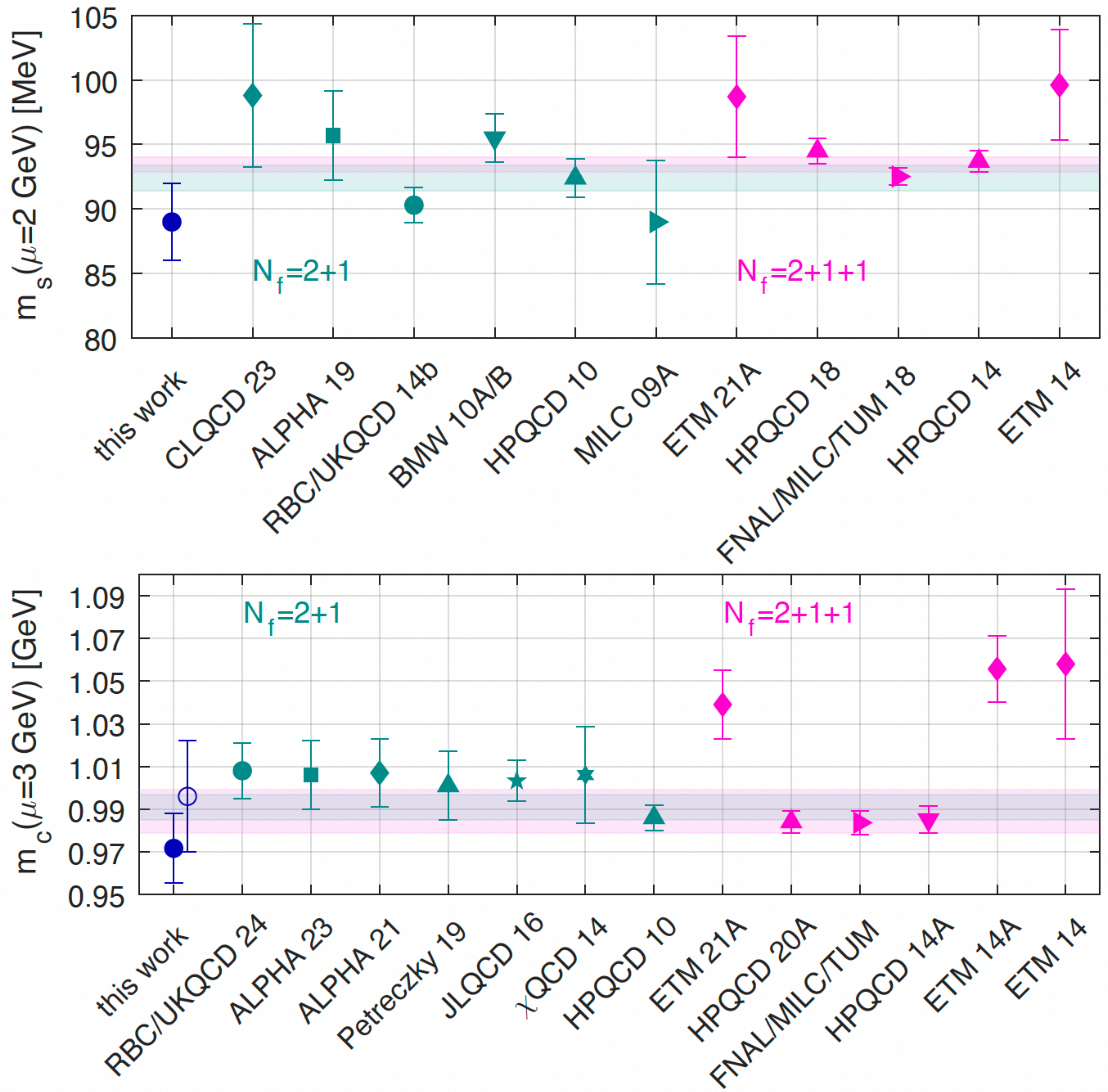
Black, RH, Hasenfratz, Rago, Witzel ‘25

P H
CRC TRR 257

PCAC:

$$R(t, \tau) = -\frac{\langle j_A^0(t, \tau) j_P(t=0)\rangle}{\langle j_P(t, \tau) j_P(t=0)\rangle}$$

$$m(\mu) = \lim_{\tau\to 0}\lim_{t\to\infty} \zeta^{-1}(\tau, \mu)\, M_\pi\, R(t, \tau)$$



Quark masses

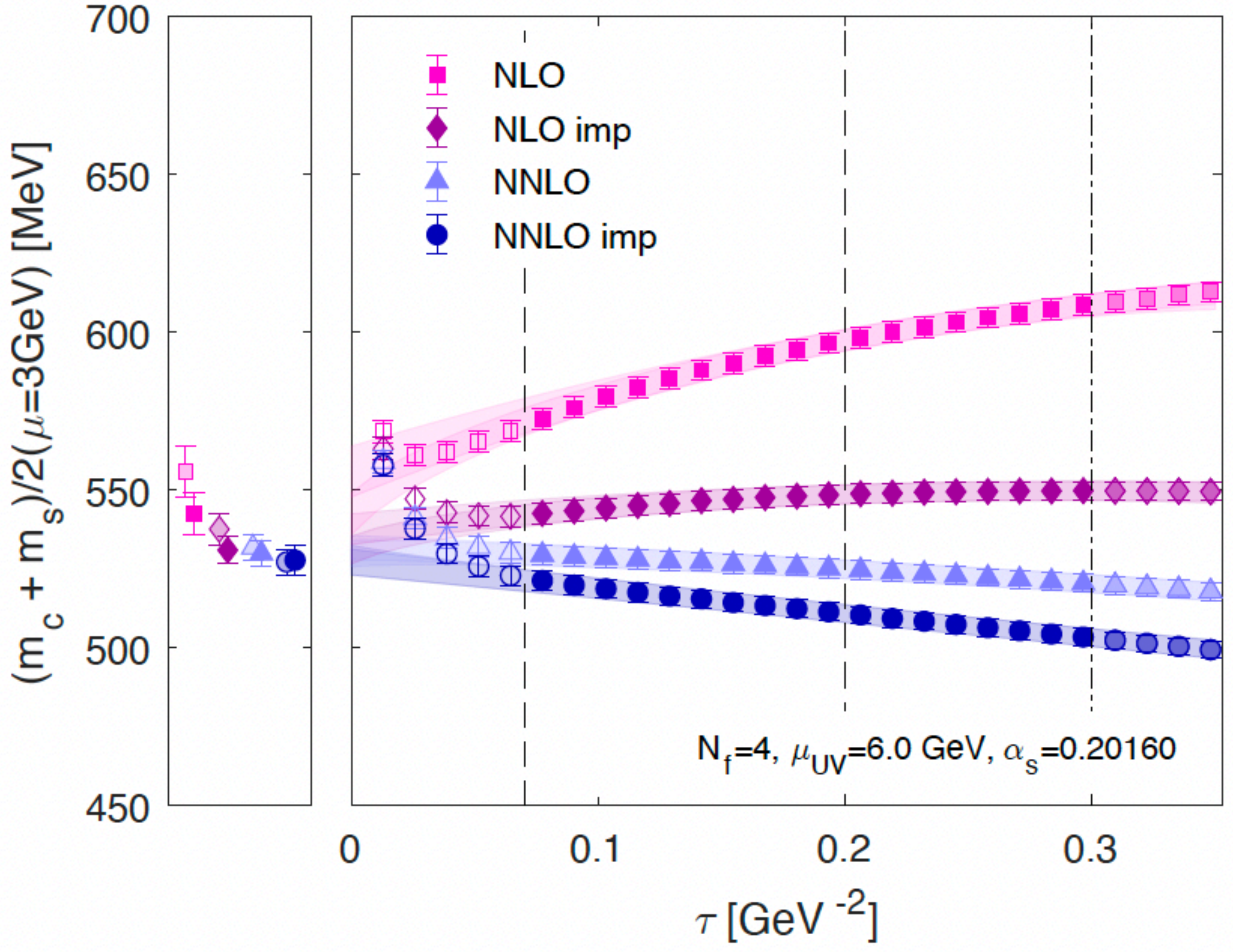
Black, RH, Hasenfratz, Rago, Witzel '25

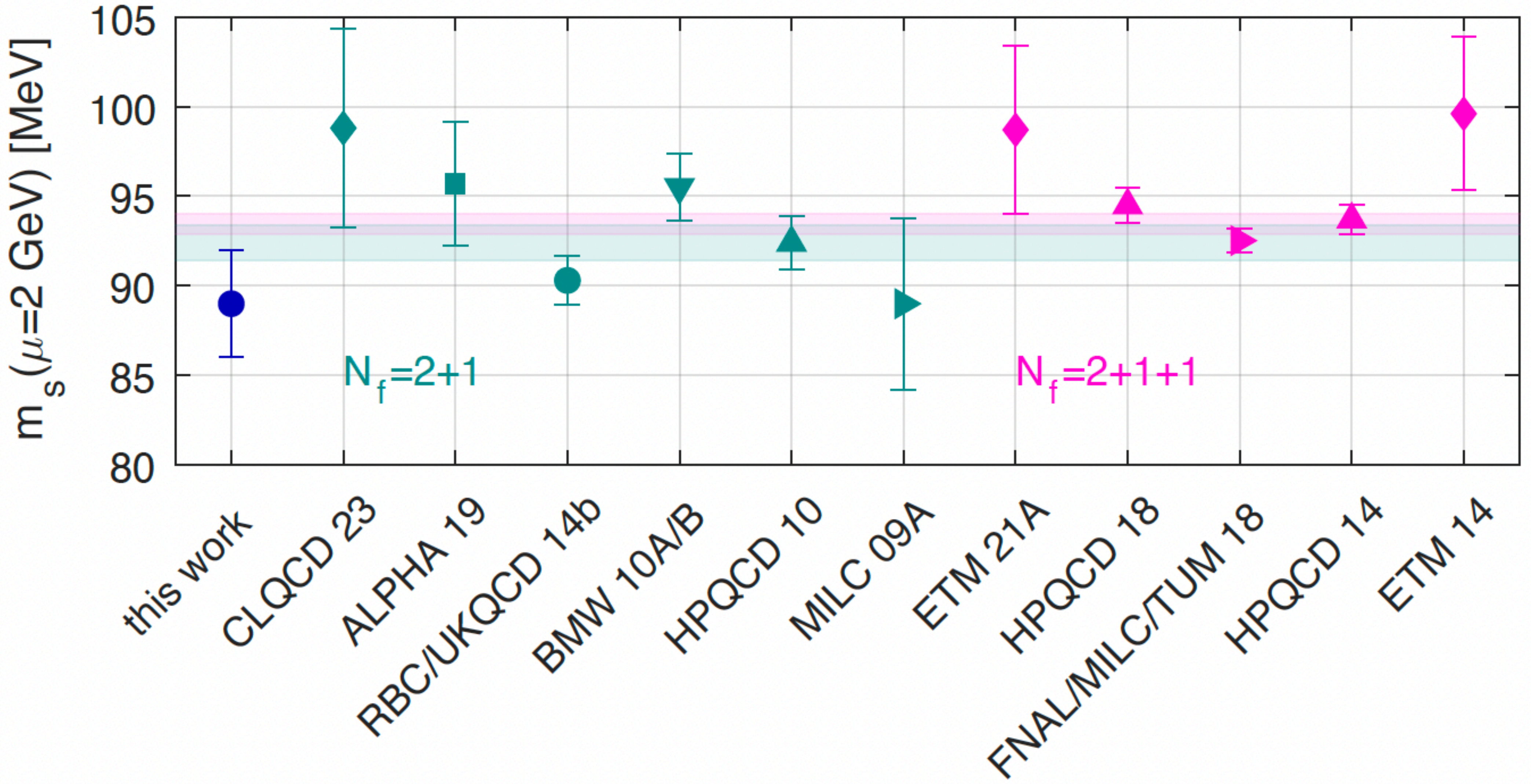
CRC TRR 257

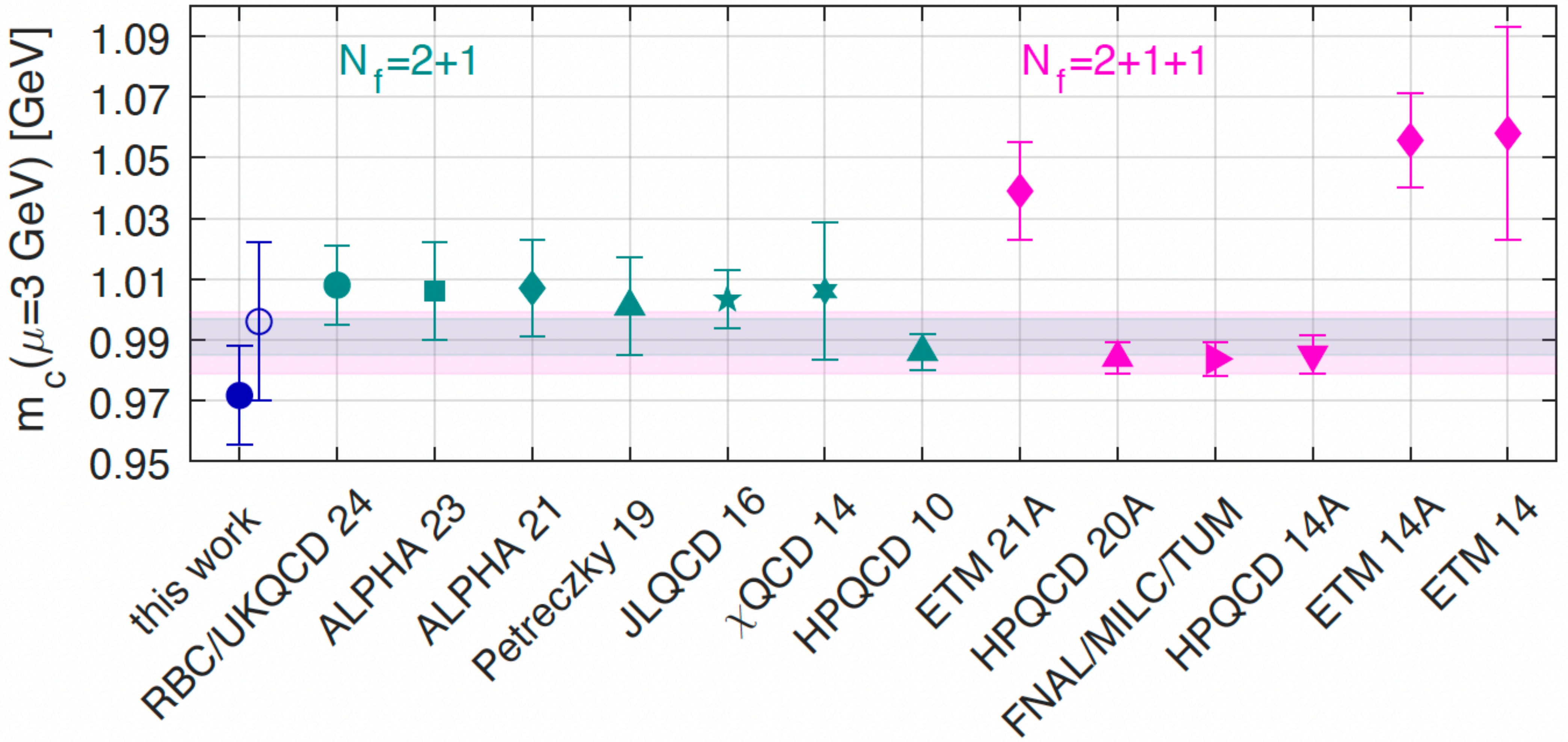
PCAC:

$$R(t,\tau) = -\frac{\langle j_A^0(t,\tau) j_P(t=0)\rangle}{\langle j_P(t,\tau) j_P(t=0)\rangle}$$

$$m(\mu) = \lim_{\tau\to 0}\lim_{t\to\infty} \zeta^{-1}(\tau,\mu)\, M_\pi\, R(t,\tau)$$







Quark masses

Black, RH, Hasenfratz, Rago, Witzel '25

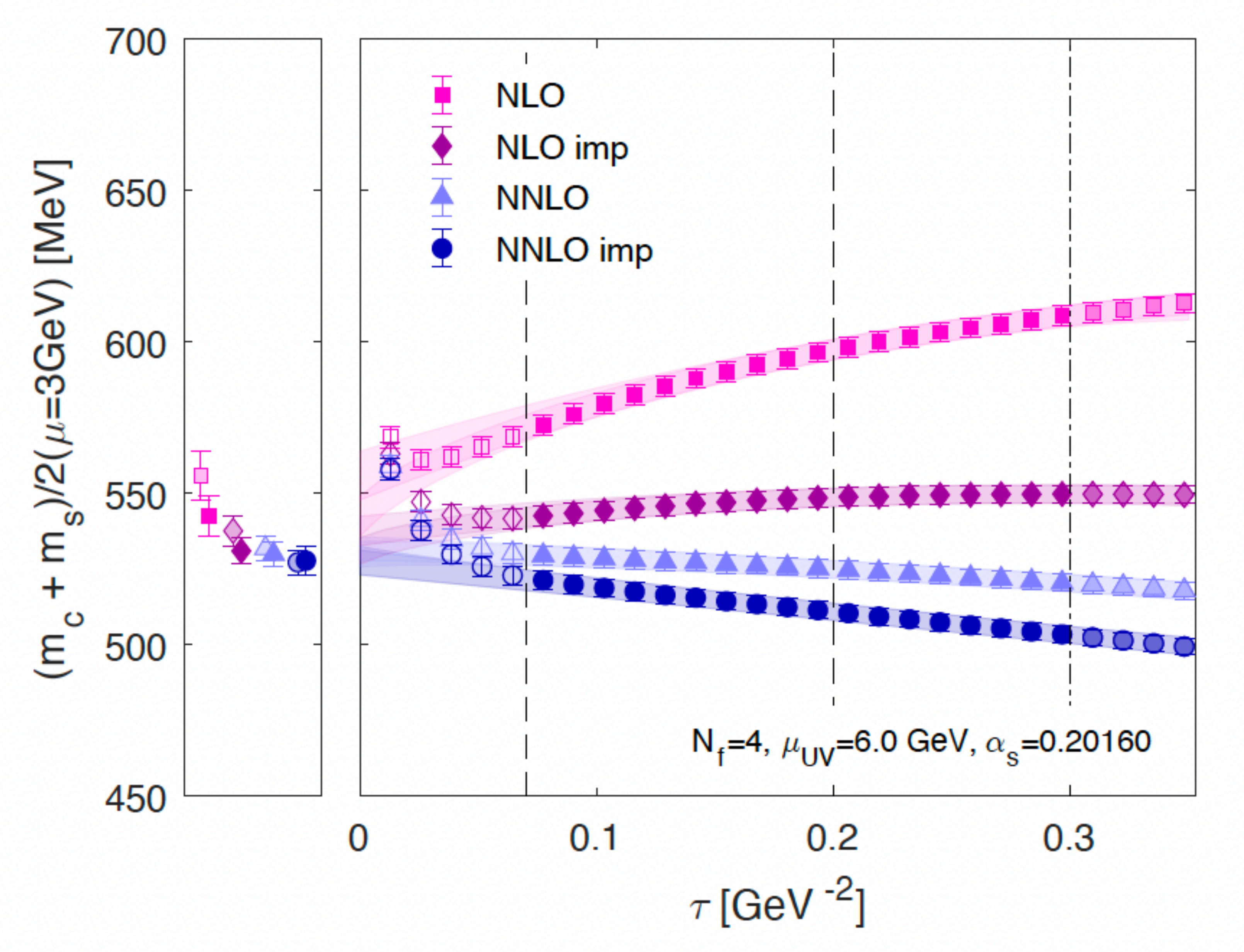
CRC TRR 257

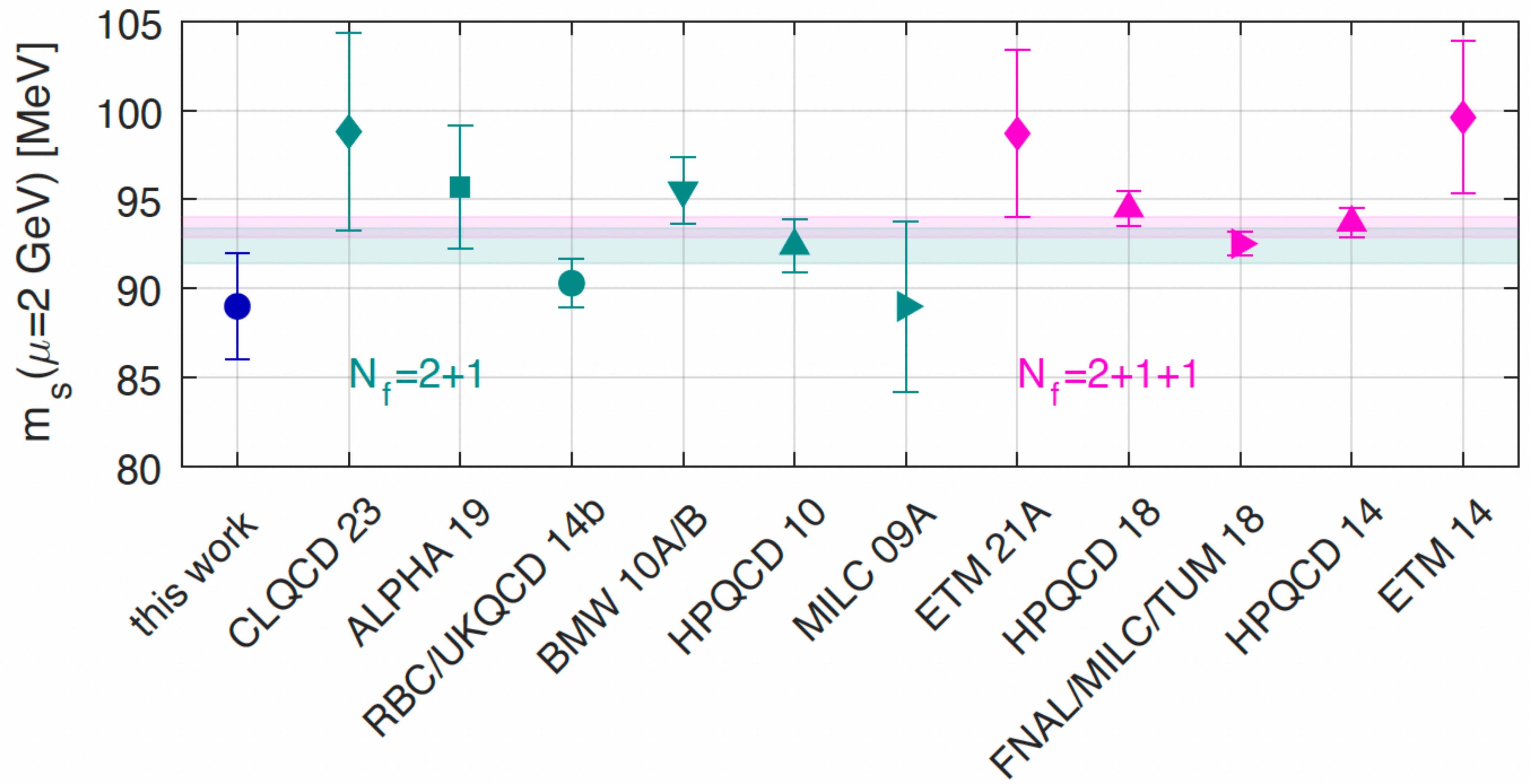
Oliver Witzel (Tue 10:00)

PCAC:

$$R(t,\tau) = -\frac{\langle j_A^0(t,\tau) j_P(t=0)\rangle}{\langle j_P(t,\tau) j_P(t=0)\rangle}$$

$$m(\mu) = \lim_{\tau\to 0}\lim_{t\to\infty} \zeta^{-1}(\tau,\mu)\, M_\pi\, R(t,\tau)$$





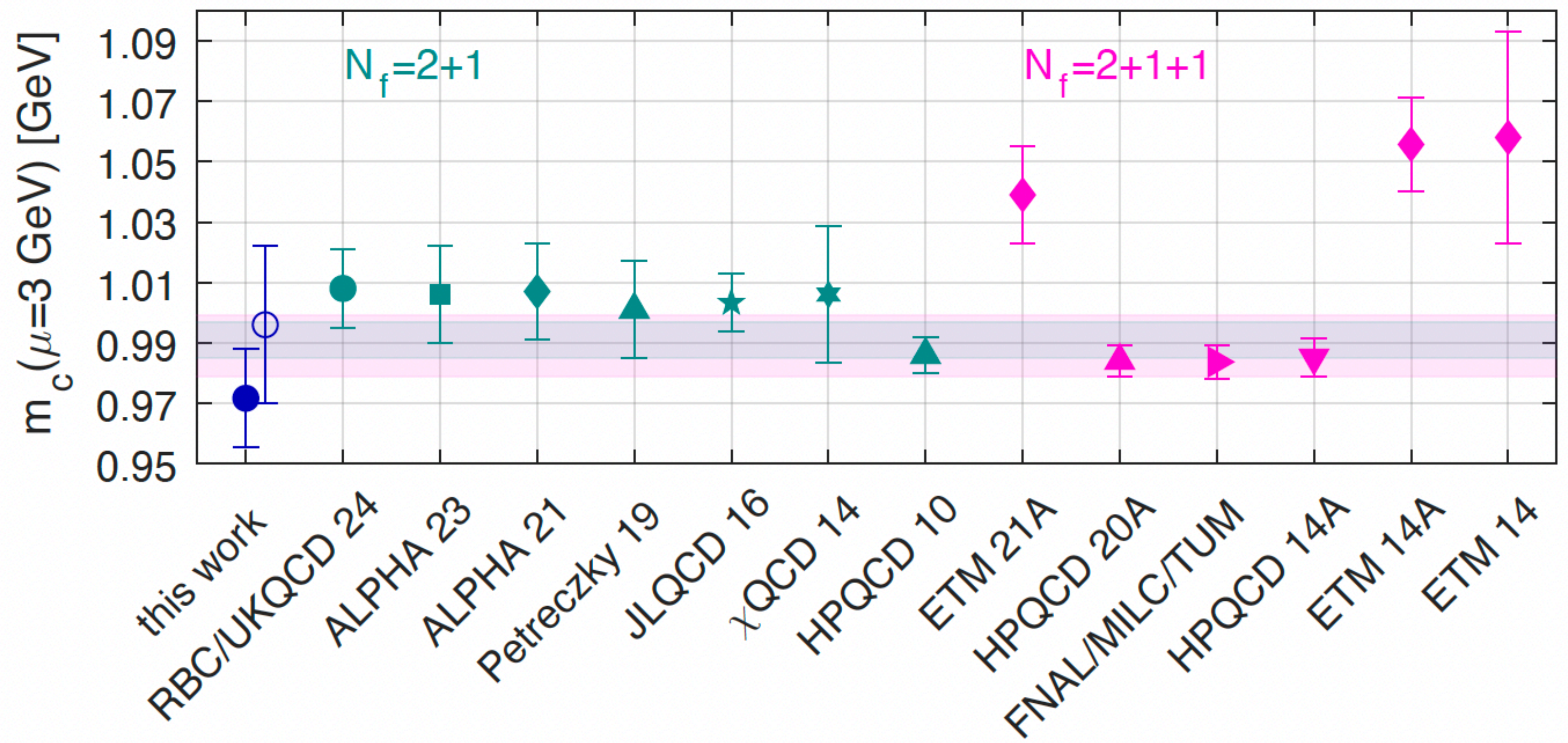


Table of contents

Table of contents

Table of contents

Table of contents

Institute for Theoretical Particle Physics and Cosmology
RWTH AACHEN UNIVERSITY

Table of contents

1. **Motivation** (LHC → PDFs → SFTX → Flavor physics → PDFs)
2. **Introduction**
3. ~~**Main part**~~
4. **Conclusions**

Institute for Theoretical Particle Physics and Cosmology
RWTH AACHEN UNIVERSITY

Gradient flow Lagrangian

flowed gauge field:

$$\frac{\partial}{\partial t} B_\mu(t,x) = \mathscr{D}_\nu G_{\nu\mu}(t,x)$$

$$B_\mu(t=0,x) = A_\mu(x)$$

flowed quark field:

$$\frac{\partial}{\partial t} \chi(t,x) = \mathscr{D}^2 \chi(t,x)$$

$$\chi(t=0,x) = \psi(x)$$

Gradient flow Lagrangian

flowed gauge field:

$$\frac{\partial}{\partial t} B_\mu(t,x) = \mathscr{D}_\nu G_{\nu\mu}(t,x)$$
$$B_\mu(t=0,x) = A_\mu(x)$$

flowed quark field:

$$\frac{\partial}{\partial t} \chi(t,x) = \mathscr{D}^2 \chi(t,x)$$
$$\chi(t=0,x) = \psi(x)$$

$$\mathscr{L} = \mathscr{L}_{\mathrm{QCD}} + \mathscr{L}_B + \mathscr{L}_\chi$$

$$\mathscr{L}_B \sim \int_0^\infty dt\, L_\mu \left(\partial_t B_\mu - \mathscr{D}_\nu G_{\nu\mu} \right)$$

$$\mathscr{L}_\chi \sim \int_0^\infty dt\, \bar{\lambda} \left(\partial_t - \mathscr{D}^2 \right) \chi + \mathrm{h.c.}$$

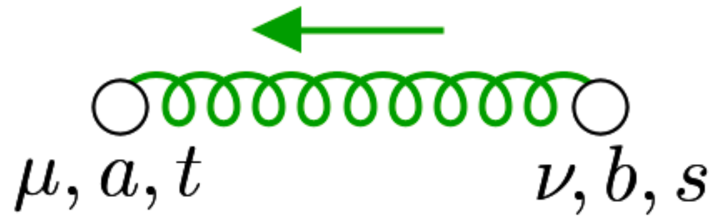
Lüscher, Weisz 2011
Lüscher 2013

Perturbative approach

$$\mathscr{L}_B \sim \int_0^\infty dt\, L_\mu \left(\partial_t B_\mu - \mathscr{D}_\nu G_{\nu\mu}\right)$$

Perturbative approach

$$\mathscr{L}_B \sim \int_0^\infty dt\, L_\mu \left(\partial_t B_\mu - \mathscr{D}_\nu G_{\nu\mu}\right)$$



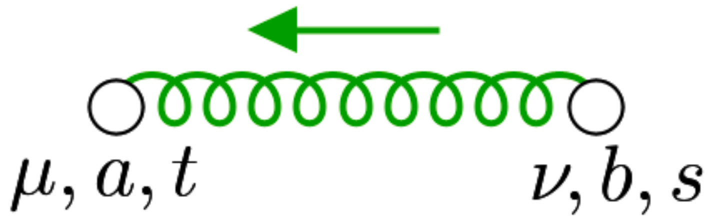
$$\delta_{ab}\delta_{\mu\nu}\,\theta(t-s)\,e^{-(t-s)p^2}$$

"gluon flow line"

$$\sim \langle 0 | T\, L_\mu^a(t,x)\, B_\nu^b(s,0) | 0 \rangle$$

Perturbative approach

$$\mathscr{L}_B \sim \int_0^\infty dt\, L_\mu \left(\partial_t B_\mu - \mathscr{D}_\nu G_{\nu\mu} \right)$$



$$\delta_{ab}\delta_{\mu\nu}\,\theta(t-s)\,e^{-(t-s)p^2}$$

“gluon flow line”

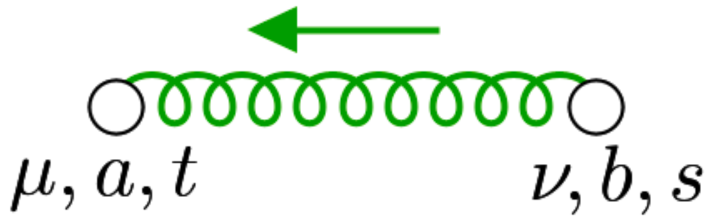
$$\sim \langle 0 | T\, L_\mu^a(t,x)\, B_\nu^b(s,0) | 0 \rangle$$

$$\frac{\delta^{ab}}{p^2}\left(\delta_{\mu\nu} - \xi \frac{p_\mu p_\nu}{p^2}\right) e^{-(t+s)p^2}$$

$$\sim \langle 0 | T\, B_\mu^a(t,x)\, B_\nu^b(s,0) | 0 \rangle$$

Perturbative approach

$$\mathscr{L}_B \sim \int_0^\infty dt\, L_\mu \left(\partial_t B_\mu - \mathscr{D}_\nu G_{\nu\mu}\right)$$



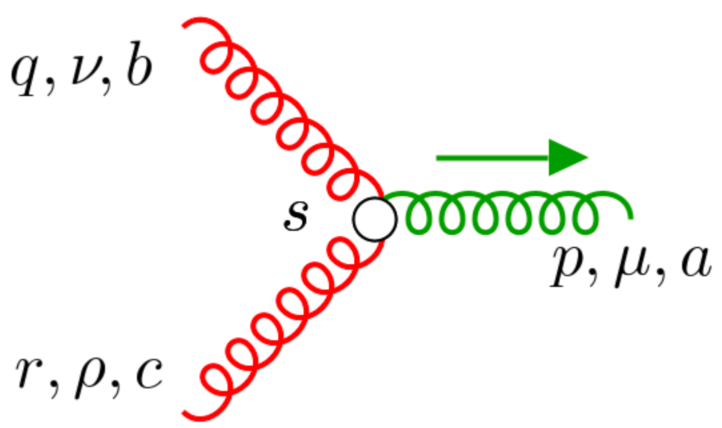
$$\delta_{ab}\delta_{\mu\nu}\,\theta(t-s)\,e^{-(t-s)p^2}$$

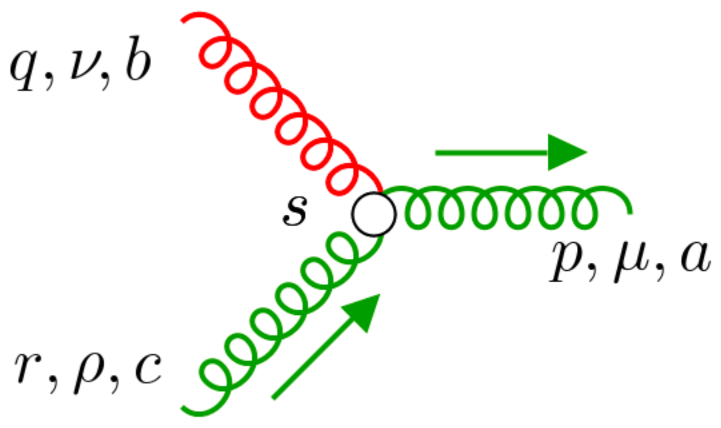
"gluon flow line"

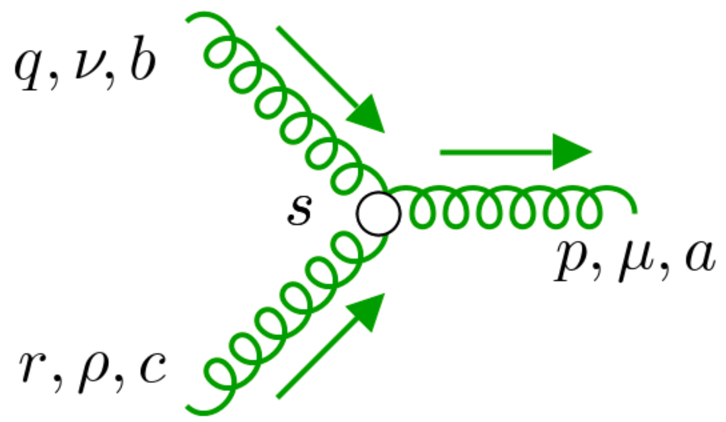
$$\sim \langle 0 | T\, L_\mu^a(t,x)\, B_\nu^b(s,0) | 0 \rangle$$

$$\frac{\delta^{ab}}{p^2}\left(\delta_{\mu\nu} - \xi\frac{p_\mu p_\nu}{p^2}\right) e^{-(t+s)p^2}$$

$$\sim \langle 0 | T\, B_\mu^a(t,x)\, B_\nu^b(s,0) | 0 \rangle$$







$$-\mathrm{i}gf^{abc}\int_0^\infty \mathrm{d}s\,\big(\delta_{\nu\rho}(r-q)_\mu + 2\delta_{\mu\nu}q_\rho - 2\delta_{\mu\rho}r_\nu + (\kappa-1)(\delta_{\mu\rho}q_\nu - \delta_{\mu\nu}r_\rho)\big)$$

+ 4-gluon vertex

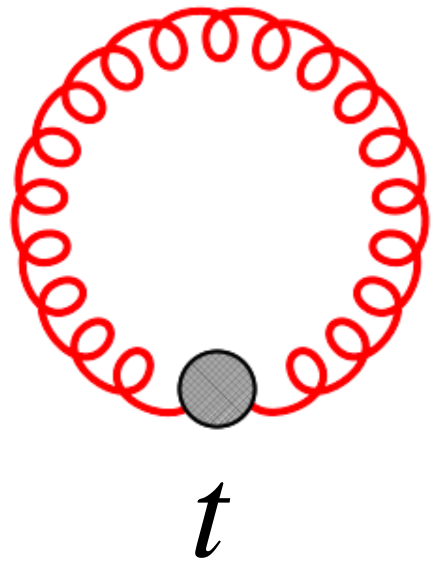
Let's calculate

$$E(t) \equiv \frac{1}{4}\langle G^a_{\mu\nu}(t)G^{a,\mu\nu}(t)\rangle$$

Let's calculate

$$E(t) \equiv \frac{1}{4}\langle G^a_{\mu\nu}(t) G^{a,\mu\nu}(t)\rangle$$

LO:

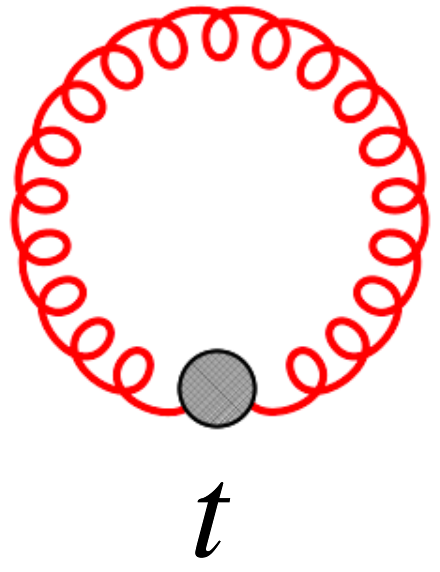


Let's calculate

$$E(t) \equiv \frac{1}{4}\langle G^a_{\mu\nu}(t) G^{a,\mu\nu}(t)\rangle$$

$$\frac{\delta^{ab}}{p^2}\left(\delta_{\mu\nu} - \xi\frac{p_\mu p_\nu}{p^2}\right) e^{-(t+s)p^2}$$

LO:

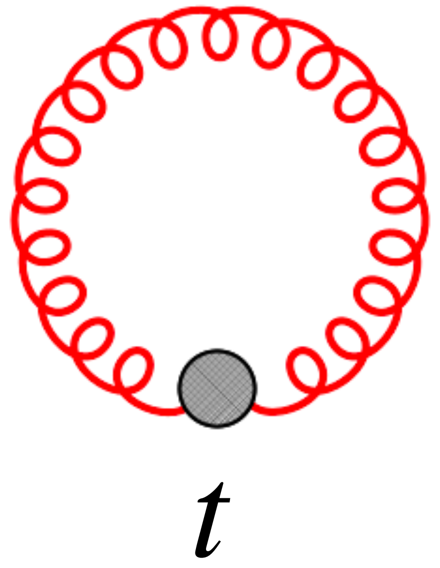


Let's calculate

$$E(t) \equiv \frac{1}{4}\langle G^a_{\mu\nu}(t) G^{a,\mu\nu}(t)\rangle$$

$$\frac{\delta^{ab}}{p^2}\left(\delta_{\mu\nu} - \xi\frac{p_\mu p_\nu}{p^2}\right) e^{-(t+s)p^2}$$

LO:



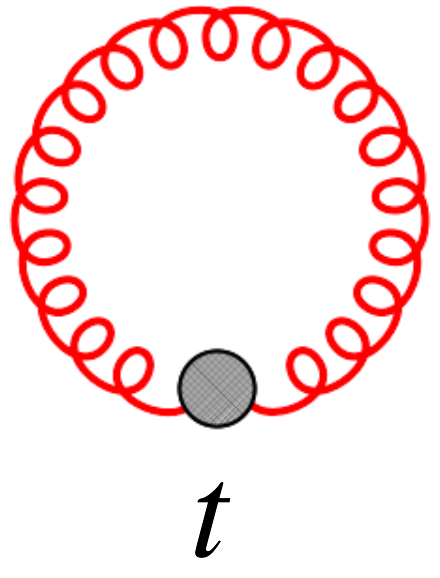
$$\sim \int d^D p\, e^{-2tp^2} \sim t^{-2+\epsilon} \neq 0$$

Let's calculate

$$E(t) \equiv \frac{1}{4}\langle G^a_{\mu\nu}(t) G^{a,\mu\nu}(t)\rangle$$

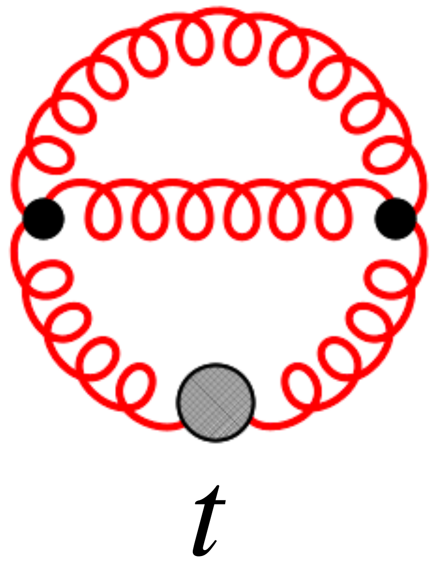
$$\frac{\delta^{ab}}{p^2}\left(\delta_{\mu\nu} - \xi\frac{p_\mu p_\nu}{p^2}\right) e^{-(t+s)p^2}$$

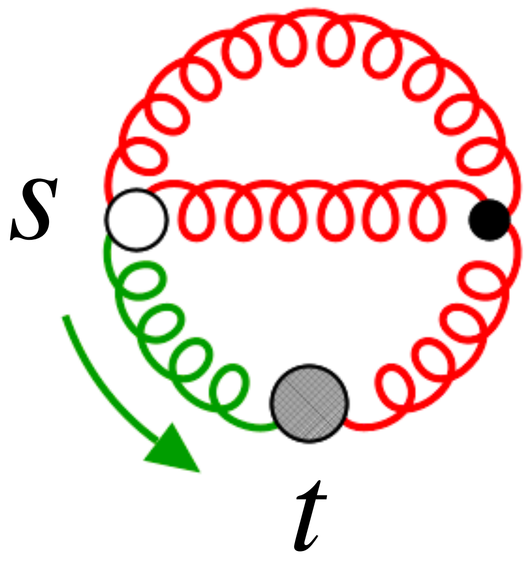
LO:



$$\sim \int d^D p\, e^{-2tp^2} \sim t^{-2+\epsilon} \neq 0$$

NLO:





$$\sim \int_p \int_k \frac{e^{-2\,t\,p^2}}{p^4\, k^2\, (p-k)^2}$$

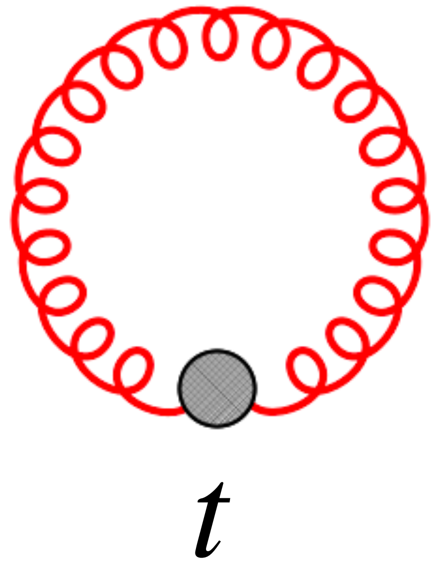
$$\int_0^t ds \int_p \int_k \frac{e^{-(2t-s)p^2}}{p^2\, k^2\, (p-k)^2}$$

Let's calculate

$$E(t) \equiv \frac{1}{4}\langle G^a_{\mu\nu}(t) G^{a,\mu\nu}(t)\rangle$$

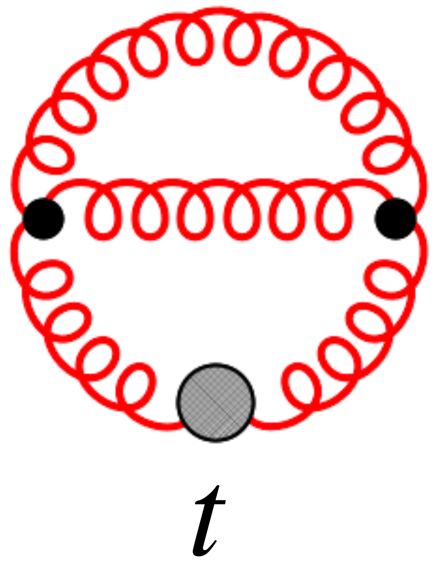
$$\frac{\delta^{ab}}{p^2}\left(\delta_{\mu\nu} - \xi \frac{p_\mu p_\nu}{p^2}\right) e^{-(t+s)p^2}$$

LO:

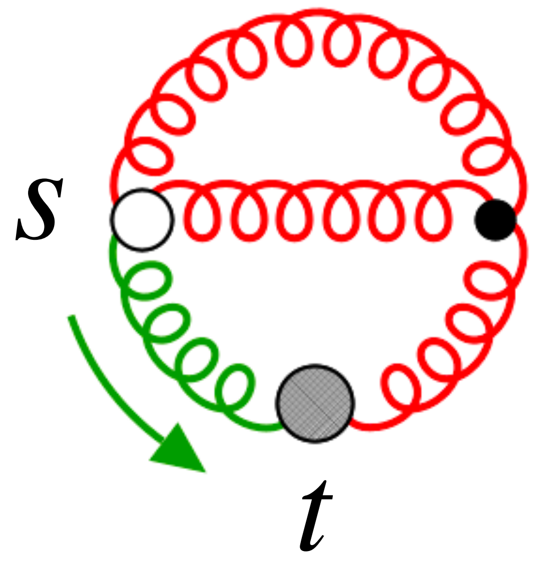


$$\sim \int d^D p\, e^{-2tp^2} \sim t^{-2+\epsilon} \neq 0$$

NLO:

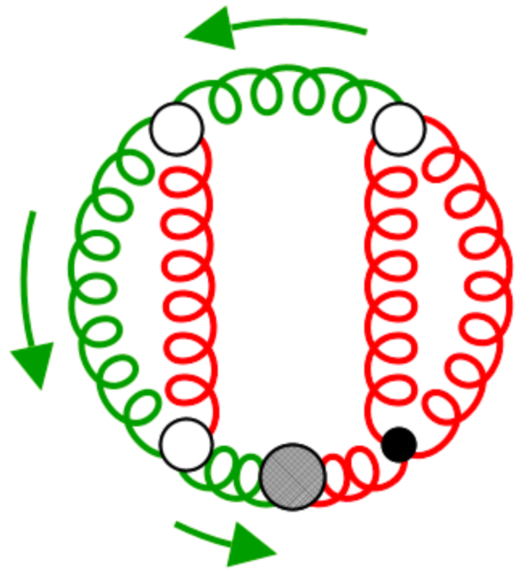


$$\sim \int_p \int_k \frac{e^{-2\,t\,p^2}}{p^4\, k^2\, (p-k)^2}$$



$$\int_0^t ds \int_p \int_k \frac{e^{-(2t-s)p^2}}{p^2\, k^2\, (p-k)^2}$$

NNLO:



$$\int_0^t ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3 \int_p \int_k \int_l \cdots$$

The perturbative toolbox

[For details, see: Artz, RH, Lange, Neumann, Prausa 2019]

diagram generation:	qgraf	Nogueira 1993
diagram analyzation:	q2e/exp	RH, Seidensticker, Steinhauser 1997
	→ tapir/exp	Gerlach, Herren, Lang 2022
algebraic manipulations:	FORM	Vermaseren 2000, …
reduction to masters:	Kira ⊗ FireFly	Usovitsch, Uwer, Maierhöfer 2017 ⊗ Klappert, Klein, Lange 2019

Chetyrkin, Tkachov 1981
Laporta 2000

The perturbative toolbox

[For details, see: Artz, RH, Lange, Neumann, Prausa 2019]

diagram generation: qgraf — Nogueira 1993

diagram analyzation: q2e/exp — RH, Seidensticker, Steinhauser 1997

$\longrightarrow$ tapir/exp — Gerlach, Herren, Lang 2022

algebraic manipulations: FORM — Vermaseren 2000, …

reduction to masters: Kira $\otimes$ FireFly — Usovitsch, Uwer, Maierhöfer 2017 $\otimes$ Klappert, Klein, Lange 2019

Chetyrkin, Tkachov 1981
Laporta 2000

sector decomposition:

Binoth, Heinrich 2002

$$\int d^D k \int d^D p \int_0^t ds \frac{e^{-tp^2 - s(k-p)^2}}{k^2 p^2 (k-p^2)} = \frac{A}{\epsilon^2} + \frac{B}{\epsilon} + C + \ldots$$

The perturbative toolbox

[For details, see: Artz, RH, Lange, Neumann, Prausa 2019]

diagram generation: qgraf Nogueira 1993

diagram analyzation: q2e/exp RH, Seidensticker, Steinhauser 1997

$\longrightarrow$ tapir/exp Gerlach, Herren, Lang 2022

algebraic manipulations: FORM Vermaseren 2000, …

reduction to masters: Kira $\otimes$ FireFly Usovitsch, Uwer, Maierhöfer 2017 $\otimes$ Klappert, Klein, Lange 2019

Chetyrkin, Tkachov 1981

Laporta 2000

sector decomposition:

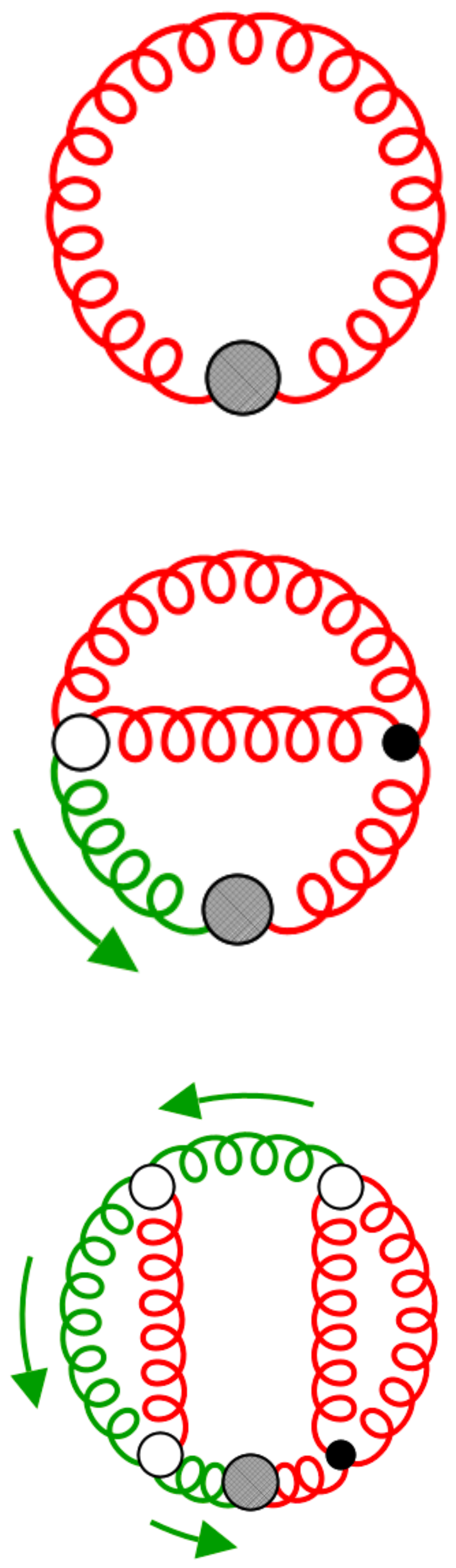
Binoth, Heinrich 2002

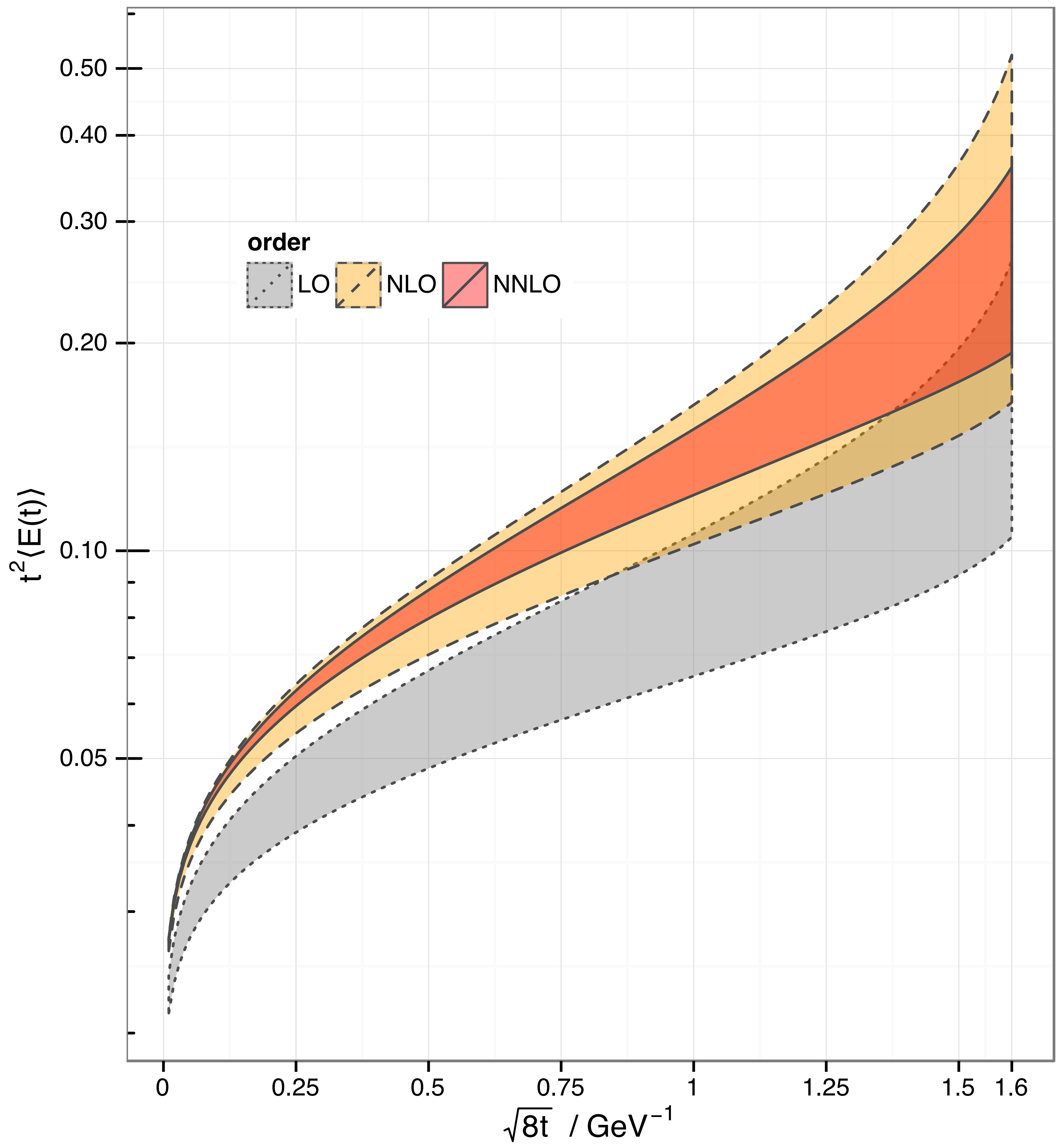
$$\int d^D k \int d^D p \int_0^t ds \frac{e^{-tp^2 - s(k-p)^2}}{k^2 p^2 (k-p^2)} = \frac{A}{\epsilon^2} + \frac{B}{\epsilon} + C + \ldots$$

→ ftint RH, Nellopoulos, Olsson, Wesle '25

$$t^2E(t) = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right] \equiv \frac{3}{4\pi}\alpha_s^{\mathrm{GF}}(t)$$

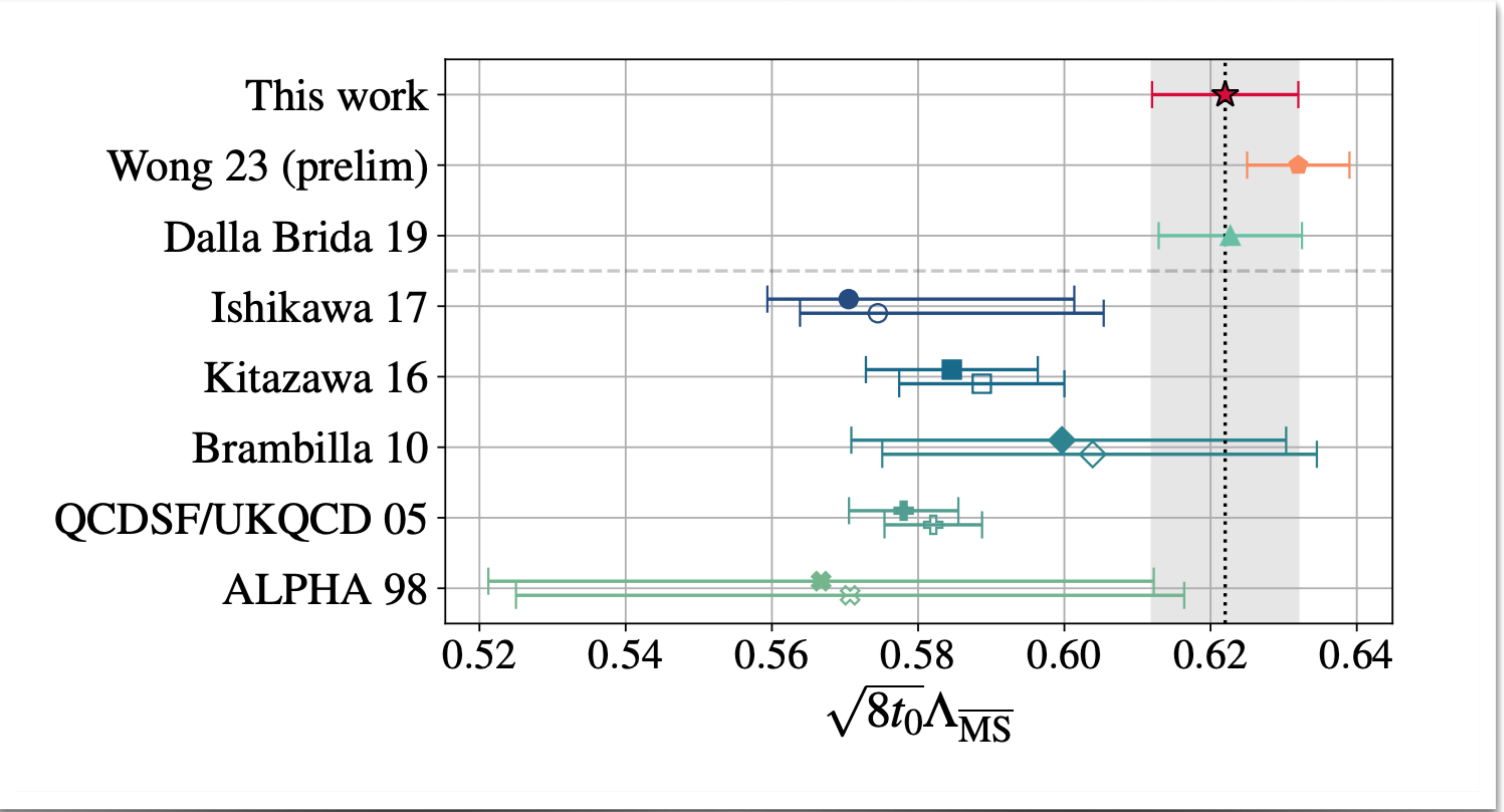
RH, Neumann 2016





Renormalization scheme for α_s compatible with lattice and PT!

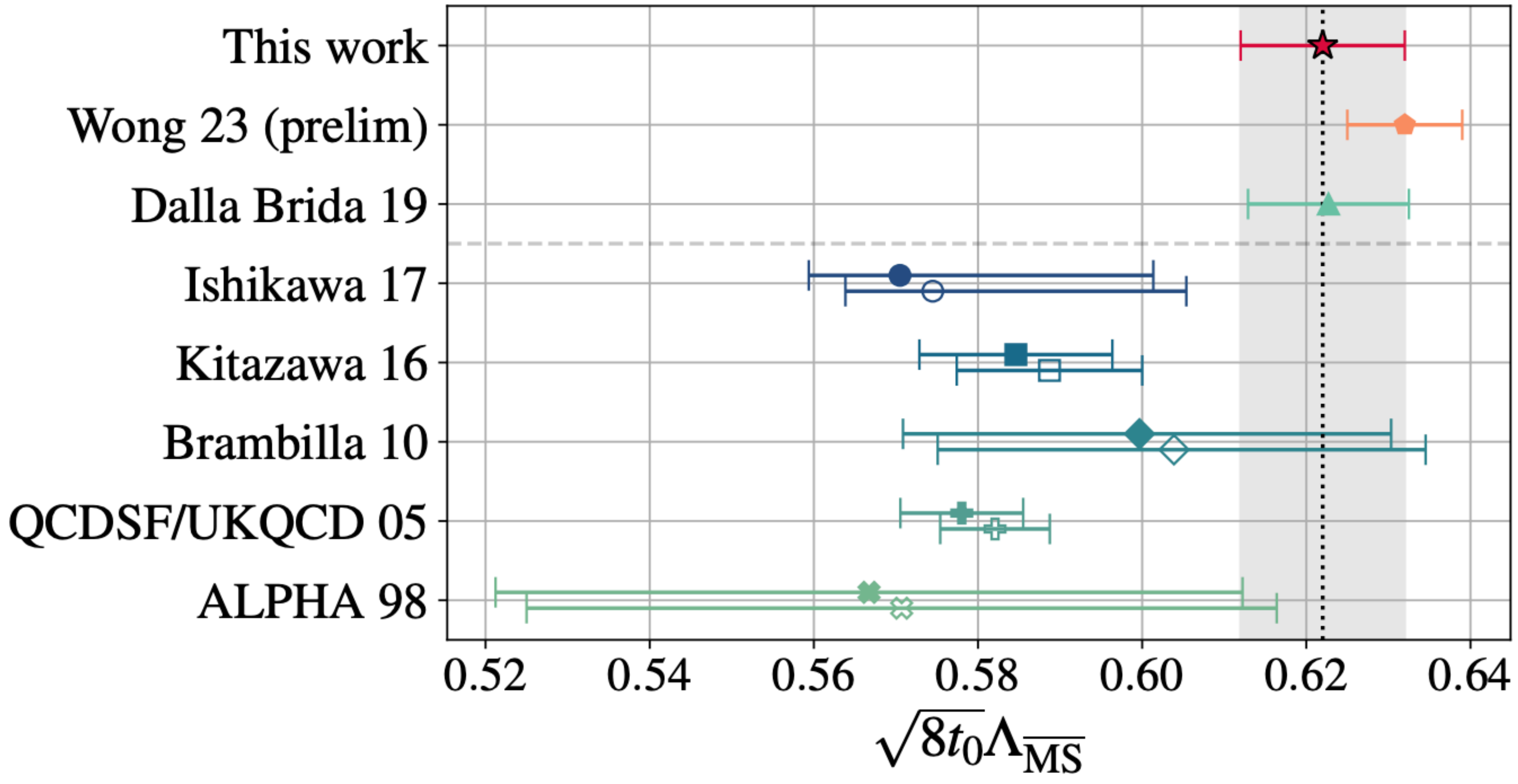
Determine Λ_{QCD}



Hasenfratz, Peterson, van Sickle, Witzel (2023)

see also Wong, Borsanyi, Fodor, Holland, Kuti (2023)

Determine Λ_{QCD}



Hasenfratz (Tue 9:00)

Hasenfratz, Peterson, van Sickle, Witzel (2023)

see also Wong, Borsanyi, Fodor, Holland, Kuti (2023)

Three thoughts on the GF scheme

1

$$t^2 E(t) = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right] \equiv \frac{3}{4\pi}\alpha_s^{\mathrm{GF}}(t)$$

Three thoughts on the GF scheme

1

$$t^2 E(t) = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right] \equiv \frac{3}{4\pi}\alpha_s^{\mathrm{GF}}(t)$$

$$+t^2\tilde{C}_F(t)\,\langle F_{\mu\nu}F^{\mu\nu}\rangle + \cdots$$

non-perturbative cont's

Three thoughts on the GF scheme

1

$$t^2 E(t) = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right] \equiv \frac{3}{4\pi}\alpha_s^{\rm GF}(t)$$

$$+t^2\tilde{C}_F(t)\,\langle F_{\mu\nu}F^{\mu\nu}\rangle + \cdots$$

non-perturbative cont's

perturbative contribution: renormalons!

Renormalon subtraction

Beneke, Takaura ‘25

$$\Pi(q^2) \sim \int d^4x\, e^{iqx} \langle 0 \,|\, T\, j(x) j(0) \,|\, 0 \rangle$$

Adler function: $$D(Q^2) = -\,Q^2 \frac{d}{dQ^2} \Pi(Q^2) = C_1(Q^2) + \frac{1}{Q^4} C_F(Q^2) \langle F_{\mu\nu} F^{\mu\nu} \rangle + \cdots$$

Baikov, Chetyrkin, Kühn ’08

RH ‘98

perturbative

Institute for Theoretical Particle Physics and Cosmology | RWTH AACHEN UNIVERSITY

Renormalon subtraction

Beneke, Takaura '25

$$\Pi(q^2) \sim \int d^4x\, e^{iqx} \langle 0 | T j(x) j(0) | 0 \rangle$$

Adler function: $$D(Q^2) = -Q^2 \frac{d}{dQ^2} \Pi(Q^2) = C_1(Q^2) + \frac{1}{Q^4} C_F(Q^2) \langle F_{\mu\nu} F^{\mu\nu} \rangle + \cdots$$

Baikov, Chetyrkin, Kühn '08

RH '98

perturbative

$$\langle G_{\mu\nu}(t) G^{\mu\nu}(t) \rangle = t^{-2} \tilde{C}_1(t) + \tilde{C}_F(t) \langle F_{\mu\nu} F^{\mu\nu} \rangle + \cdots$$

Renormalon subtraction

Beneke, Takaura '25

$$\Pi(q^2) \sim \int d^4x\, e^{iqx} \langle 0 | T j(x) j(0) | 0 \rangle$$

Adler function: $D(Q^2) = -Q^2 \frac{d}{dQ^2} \Pi(Q^2) = C_1(Q^2) + \frac{1}{Q^4} C_F(Q^2) \langle F_{\mu\nu} F^{\mu\nu} \rangle + \cdots$

Baikov, Chetyrkin, Kühn '08

RH '98

perturbative

RH, Neumann '16

RH, Lange, Neumann '20

$$\langle G_{\mu\nu}(t) G^{\mu\nu}(t) \rangle = t^{-2} \tilde{C}_1(t) + \tilde{C}_F(t) \langle F_{\mu\nu} F^{\mu\nu} \rangle + \cdots$$

Renormalon subtraction

Beneke, Takaura '25

$$\Pi(q^2) \sim \int d^4x\, e^{iqx} \langle 0 \,|\, T j(x) j(0) \,|\, 0 \rangle$$

Adler function: $$D(Q^2) = -Q^2 \frac{d}{dQ^2}\Pi(Q^2) = C_1(Q^2) + \frac{1}{Q^4} C_F(Q^2) \langle F_{\mu\nu} F^{\mu\nu} \rangle + \cdots$$

Baikov, Chetyrkin, Kühn '08

RH '98

perturbative

RH, Neumann '16

RH, Lange, Neumann '20

$$\langle G_{\mu\nu}(t) G^{\mu\nu}(t) \rangle = t^{-2} \tilde{C}_1(t) + \tilde{C}_F(t) \langle F_{\mu\nu} F^{\mu\nu} \rangle + \cdots$$

$$D(Q^2) = C_1(Q^2) - \frac{1}{t^2 Q^4} \frac{C_F(Q^2)}{\tilde{C}_F(t)} \tilde{C}_1(t) + \frac{1}{Q^4} \frac{C_F(Q^2)}{\tilde{C}_F(t)} \langle G_{\mu\nu}(t) G^{\mu\nu}(t) \rangle + \cdots$$

Renormalon subtraction

Beneke, Takaura '25

$$\Pi(q^2) \sim \int d^4x\, e^{iqx} \langle 0 | T j(x) j(0) | 0 \rangle$$

Adler function:

$$D(Q^2) = -Q^2 \frac{d}{dQ^2} \Pi(Q^2) = C_1(Q^2) + \frac{1}{Q^4} C_F(Q^2) \langle F_{\mu\nu} F^{\mu\nu} \rangle + \cdots$$

Baikov, Chetyrkin, Kühn '08

RH '98

perturbative

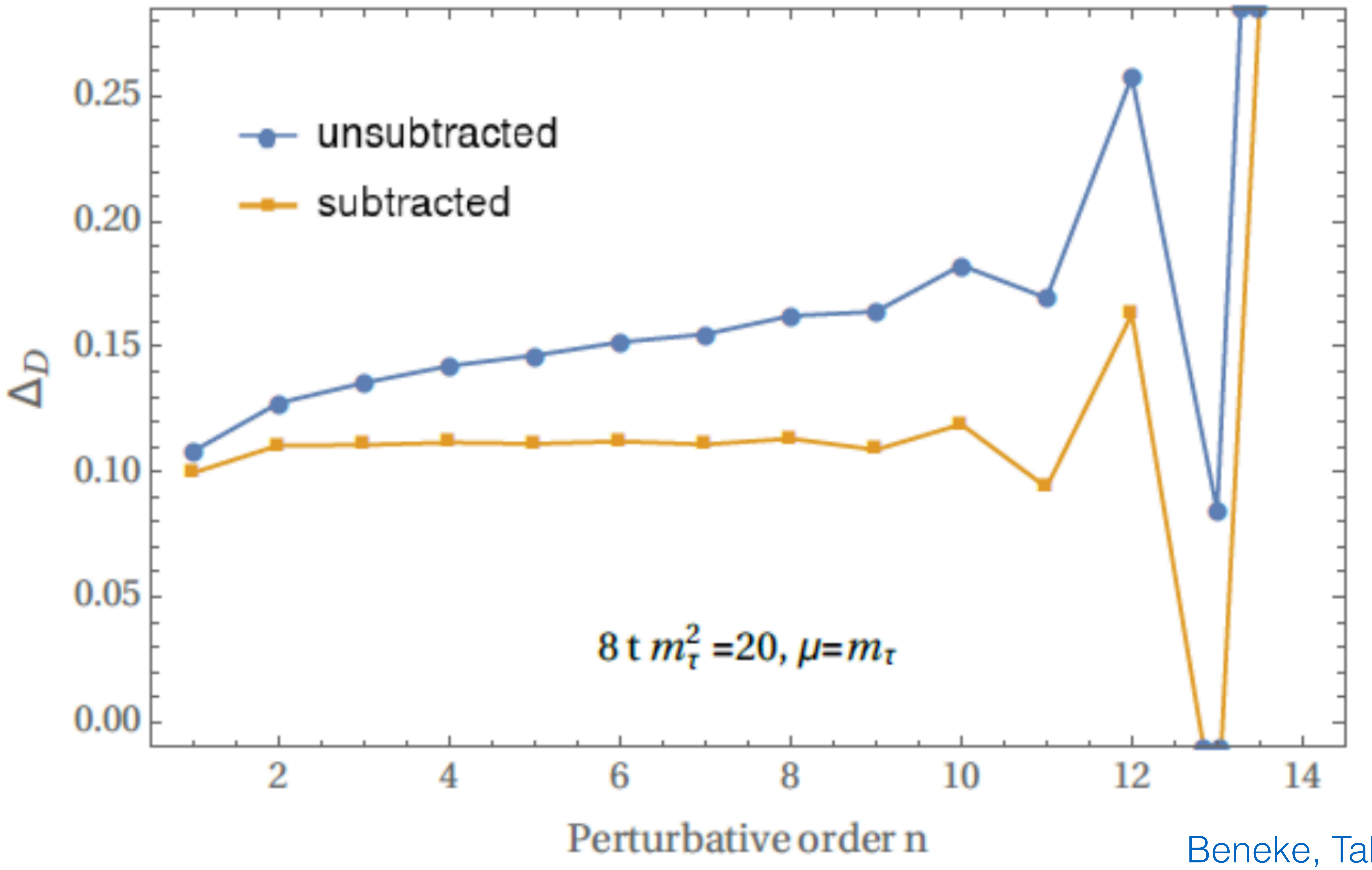
RH, Neumann '16

RH, Lange, Neumann '20

$$\langle G_{\mu\nu}(t) G^{\mu\nu}(t) \rangle = t^{-2} \tilde{C}_1(t) + \tilde{C}_F(t) \langle F_{\mu\nu} F^{\mu\nu} \rangle + \cdots$$

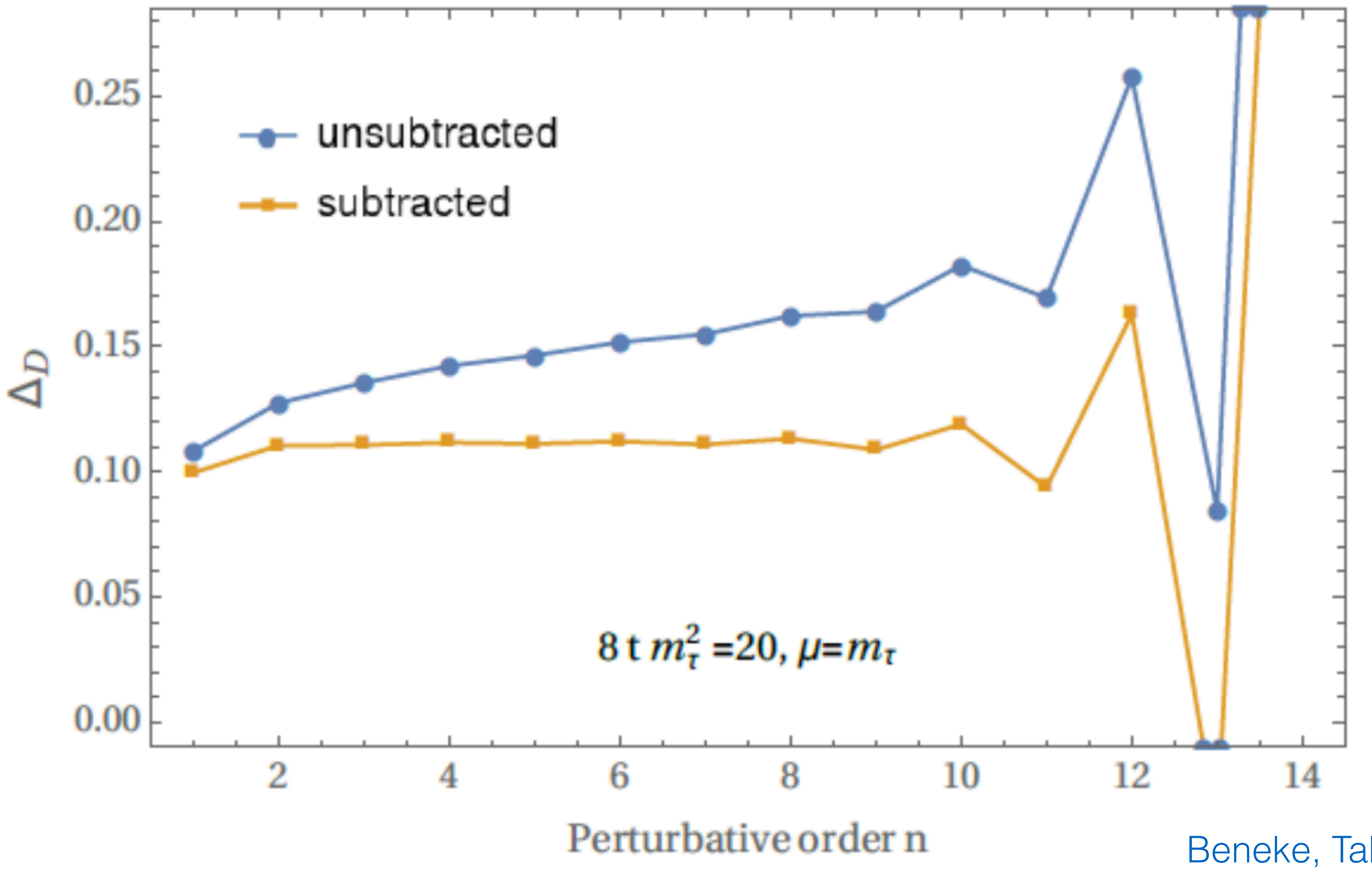
$$D(Q^2) = C_1(Q^2) - \frac{1}{t^2 Q^4} \frac{C_F(Q^2)}{\tilde{C}_F(t)} \tilde{C}_1(t) + \frac{1}{Q^4} \frac{C_F(Q^2)}{\tilde{C}_F(t)} \langle G_{\mu\nu}(t) G^{\mu\nu}(t) \rangle + \cdots$$

renormalon cancels



Beneke, Takaura '25

Martin Beneke (Wed 10:00)



Beneke, Takaura '25

Three thoughts on the GF scheme

2

Three thoughts on the GF scheme

2

Problems with $\overline{\mathrm{MS}}$: mass independent

Institute for Theoretical Particle Physics and Cosmology
RWTH AACHEN UNIVERSITY

Three thoughts on the GF scheme

2

Problems with $\overline{\text{MS}}$: mass independent

decoupling must be implemented "by hand"

Three thoughts on the GF scheme

2

Problems with $\overline{\text{MS}}$: mass independent

decoupling must be implemented "by hand"

e.g.: $\alpha_s^{(n_f=3)}(\mu)$ ok for $\mu \lesssim m_c$

Three thoughts on the GF scheme

2

Problems with $\overline{\text{MS}}$: mass independent

decoupling must be implemented "by hand"

e.g.: $\alpha_s^{(n_f=3)}(\mu)$ ok for $\mu \lesssim m_c$

for $\mu > m_c$, switch to $\alpha_s^{(4)}(\mu)$

Three thoughts on the GF scheme

2

Problems with $\overline{\text{MS}}$: mass independent

decoupling must be implemented "by hand"

e.g.: $\alpha_s^{(n_f=3)}(\mu)$ ok for $\mu \lesssim m_c$

$$\beta^{\overline{\text{MS}}}(\alpha_s) = -\frac{\alpha_s}{4\pi}\left(11 - \frac{2}{3}n_f\right) + \cdots$$

for $\mu > m_c$, switch to $\alpha_s^{(4)}(\mu)$

Three thoughts on the GF scheme

2

Problems with $\overline{\text{MS}}$: mass independent

decoupling must be implemented "by hand"

e.g.: $\alpha_s^{(n_f=3)}(\mu)$ ok for $\mu \lesssim m_c$

$$\beta^{\overline{\text{MS}}}(\alpha_s) = -\frac{\alpha_s}{4\pi}\left(11 - \frac{2}{3}n_f\right) + \cdots$$

for $\mu > m_c$, switch to $\alpha_s^{(4)}(\mu)$

decoupling relation: $$\alpha_s^{(4)}(\mu) = \alpha_s^{(3)}(\mu)\left[1 + c_1(m_c,\mu)\,\alpha_s^{(3)}(\mu) + \cdots\right]$$

Three thoughts on the GF scheme

2

Problems with $\overline{\text{MS}}$: mass independent

decoupling must be implemented "by hand"

e.g.: $\alpha_s^{(n_f=3)}(\mu)$ ok for $\mu \lesssim m_c$

$$\beta^{\overline{\text{MS}}}(\alpha_s) = -\frac{\alpha_s}{4\pi}\left(11 - \frac{2}{3}n_f\right) + \cdots$$

for $\mu > m_c$, switch to $\alpha_s^{(4)}(\mu)$

decoupling relation: $\alpha_s^{(4)}(\mu) = \alpha_s^{(3)}(\mu)\left[1 + c_1(m_c, \mu)\,\alpha_s^{(3)}(\mu) + \cdots\right]$

$\alpha_s(\mu)$ has a kink/step at some "decoupling scale" μ_{thr}

Three thoughts on the GF scheme

2

$$t^2 E(t) = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right] \equiv \frac{3}{4\pi}\alpha_s^{\mathrm{GF}}(t)$$

Three thoughts on the GF scheme

2

$$t^2 E(t) = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right] \equiv \frac{3}{4\pi}\alpha_s^{\mathrm{GF}}(t)$$

$$k_1 = k_1\Big|_{n_f=0} - \sum_q \frac{1}{6\pi}\left(1 \;+\; \Omega_{1q}(m_q,t)\right)$$

Lüscher 2010
RH, Neumann 2016

Three thoughts on the GF scheme

2

$$t^2 E(t) = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right] \equiv \frac{3}{4\pi}\alpha_s^{\mathrm{GF}}(t)$$

$$k_1 = k_1\Big|_{n_f=0} - \sum_q \frac{1}{6\pi}\Big(1 \; + \; \Omega_{1q}(m_q, t)\Big)$$

Lüscher 2010
RH, Neumann 2016

$$\beta^{\overline{\mathrm{MS}}}(\alpha_s) = -\frac{\alpha_s}{4\pi}\left(11 - \frac{2}{3}\sum_q \left(1 \; + \; t\partial_t \sum_q \Omega_{q1}\right)\right)_{t\sim 1/\mu^2} + \cdots$$

Three thoughts on the GF scheme

2

$$t^2 E(t) = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right] \equiv \frac{3}{4\pi}\alpha_s^{\mathrm{GF}}(t)$$

$$k_1 = k_1\Big|_{n_f=0} - \sum_q \frac{1}{6\pi}\Big(1 + \Omega_{1q}(m_q, t)\Big)$$

Lüscher 2010
RH, Neumann 2016

$$\beta^{\overline{\mathrm{MS}}}(\alpha_s) = -\frac{\alpha_s}{4\pi}\left(11 - \frac{2}{3}\sum_q\left(1 + t\partial_t \sum_q \Omega_{q1}\right)\right)\Bigg|_{t\sim 1/\mu^2} + \cdots$$

→ 0 for $m_q \to 0$

Three thoughts on the GF scheme

2

$$t^2E(t) = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right] \equiv \frac{3}{4\pi}\alpha_s^{\rm GF}(t)$$

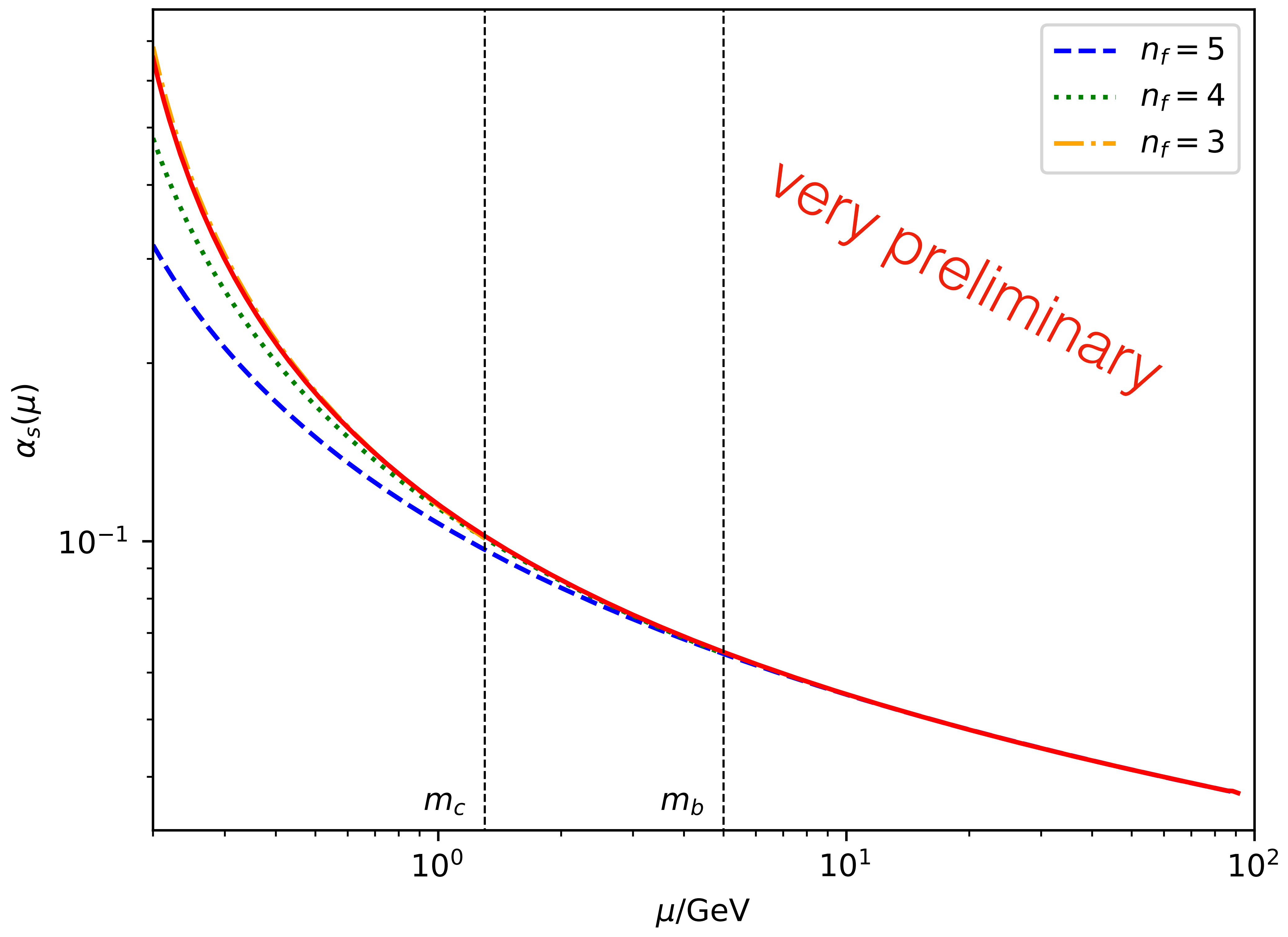
$$k_1 = k_1\Big|_{n_f=0} - \sum_q \frac{1}{6\pi}\left(1 + \Omega_{1q}(m_q,t)\right)$$

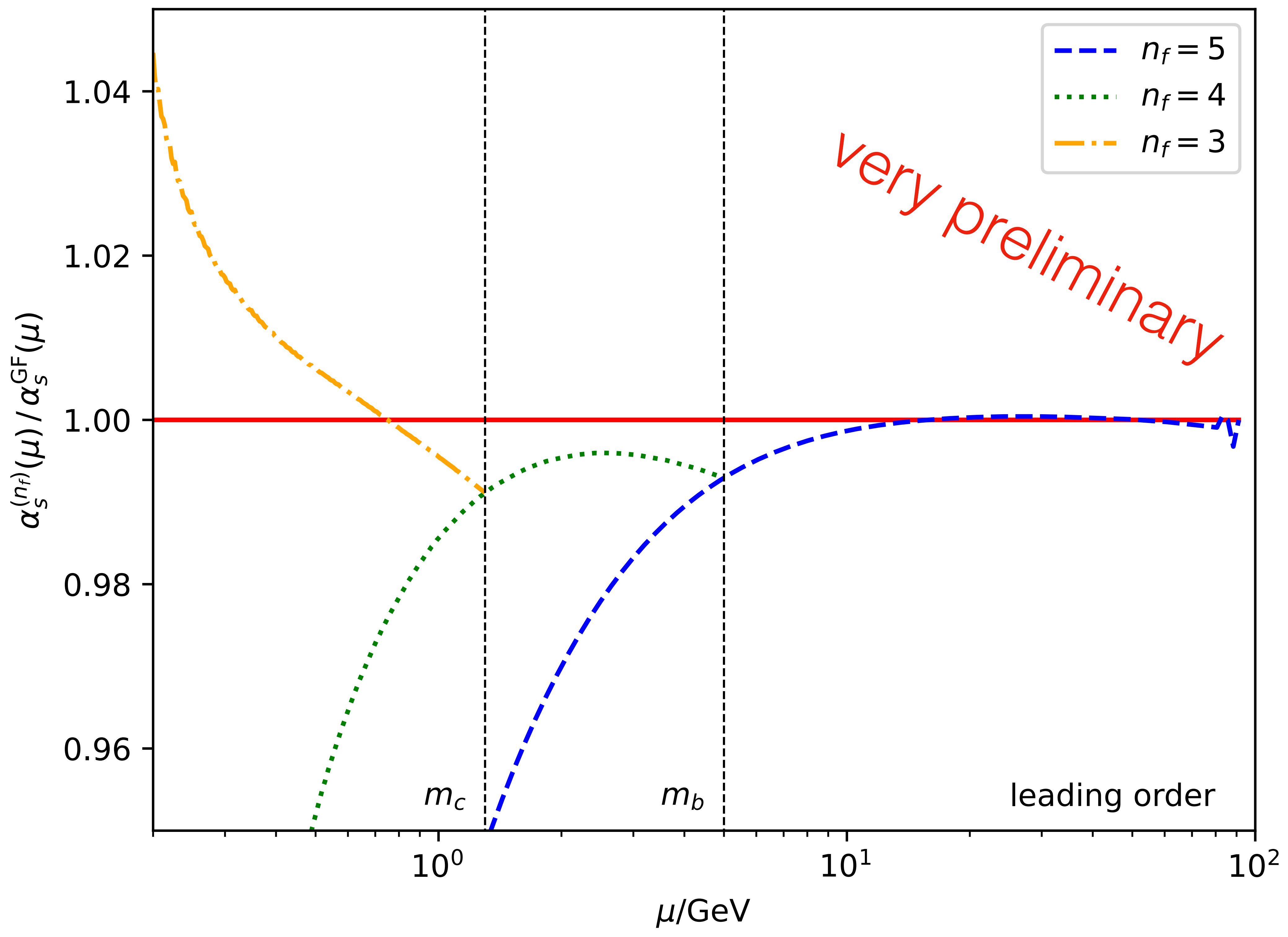
Lüscher 2010
RH, Neumann 2016

$$\beta^{\overline{\rm MS}}(\alpha_s) = -\frac{\alpha_s}{4\pi}\left(11 - \frac{2}{3}\sum_q\left(1 + t\partial_t\sum_q \Omega_{q1}\right)\right)_{t\sim 1/\mu^2} + \cdots$$

-1 for $m_q \to \infty$ 0 for $m_q \to 0$

GF vs $\overline{\text{MS}}$





Three thoughts on the GF scheme

2

$$t^2E(t) = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right] \equiv \frac{3}{4\pi}\alpha_s^{\mathrm{GF}}(t)$$

$$k_1 = k_1\Big|_{n_f=0} - \sum_q \frac{1}{6\pi}\Big(1 + \Omega_{1q}(m_q,t)\Big)$$

Lüscher 2010
RH, Neumann 2016

$$\beta^{\overline{\mathrm{MS}}}(\alpha_s) = -\frac{\alpha_s}{4\pi}\left(11 - \frac{2}{3}\sum_q\left(1 + t\partial_t\sum_q\Omega_{q1}\right)\right)_{t\sim 1/\mu^2} + \cdots$$

-1 for $m_q \to \infty$

0 for $m_q \to 0$

Three thoughts on the GF scheme

2

$$t^2E(t) = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right] \equiv \frac{3}{4\pi}\alpha_s^{\mathrm{GF}}(t)$$

$$k_1 = k_1\Big|_{n_f=0} - \sum_q \frac{1}{6\pi}\Big(1 \;+\; \Omega_{1q}(m_q,t)\Big)$$

Lüscher 2010
RH, Neumann 2016

$$\beta^{\overline{\mathrm{MS}}}(\alpha_s) = -\frac{\alpha_s}{4\pi}\left(11 - \frac{2}{3}\sum_q\left(1 \;+\; t\partial_t \sum_q \Omega_{q1}\right)\right)_{t\sim 1/\mu^2} + \cdots$$

-1 for $m_q \to \infty$

0 for $m_q \to 0$

NNLO: RH, Mason 2025

Three thoughts on the GF scheme

2

$$t^2 E(t) = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right] \equiv \frac{3}{4\pi}\alpha_s^{\mathrm{GF}}(t)$$

$$k_1 = k_1\Big|_{n_f=0} - \sum_q \frac{1}{6\pi}\left(1 \;+\; \Omega_{1q}(m_q,t)\right)$$

Lüscher 2010
RH, Neumann 2016

$$\beta^{\overline{\mathrm{MS}}}(\alpha_s) = -\frac{\alpha_s}{4\pi}\left(11 - \frac{2}{3}\sum_q\left(1 \;+\; t\partial_t \sum_q \Omega_{q1}\right)\right)_{t\sim 1/\mu^2} + \cdots$$

-1 for $m_q \to \infty$ — 0 for $m_q \to 0$

NNLO: RH, Mason 2025

Robert Mason (Wed 11:00)

Three thoughts on the GF scheme

Kluth ’24, ‘26

3

Three thoughts on the GF scheme

Kluth '24, '26

3

Problems with $\overline{\text{MS}}$: insensitive to power divergences

Three thoughts on the GF scheme

Kluth ’24, ‘26

3

Problems with $\overline{\text{MS}}$: insensitive to power divergences

Example:

$$\int \frac{d^D p}{\pi^{D/2}} \frac{1}{p^2 + m^2} = m^{D-2}\,\Gamma(1 - D/2)$$

Three thoughts on the GF scheme

Kluth '24, '26

3

Problems with $\overline{\text{MS}}$: insensitive to power divergences

Example:

$$\int \frac{d^D p}{\pi^{D/2}} \frac{1}{p^2 + m^2} = m^{D-2}\,\Gamma(1 - D/2)$$

$$D = 4 - 2\epsilon \qquad = m^{2-2\epsilon}\,\Gamma(-1+\epsilon) = \frac{m^{2-2\epsilon}}{\epsilon - 1}\,\Gamma(\epsilon) = \frac{m^2}{\epsilon}$$

Three thoughts on the GF scheme

Kluth '24, '26

3

Problems with $\overline{\text{MS}}$: insensitive to power divergences

Example:

$$\int \frac{d^Dp}{\pi^{D/2}} \frac{1}{p^2+m^2} = m^{D-2}\,\Gamma(1-D/2)$$

$$D = 4 - 2\epsilon \qquad = m^{2-2\epsilon}\,\Gamma(-1+\epsilon) = \frac{m^{2-2\epsilon}}{\epsilon-1}\,\Gamma(\epsilon) = \frac{m^2}{\epsilon}$$

cf. $$\int \frac{d^Dp}{\pi^{D/2}} \frac{1}{p^2+m^2} = \int \frac{d^Dp}{\pi^{D/2}} \left[\frac{1}{p^2} - \frac{m^2}{p^4} + \cdots\right]$$

Three thoughts on the GF scheme

Kluth '24, '26

3

Problems with $\overline{\text{MS}}$: insensitive to power divergences

Example:

$$\int \frac{d^D p}{\pi^{D/2}} \frac{1}{p^2 + m^2} = m^{D-2}\,\Gamma(1 - D/2)$$

$$D = 4 - 2\epsilon \qquad = m^{2-2\epsilon}\,\Gamma(-1+\epsilon) = \frac{m^{2-2\epsilon}}{\epsilon - 1}\,\Gamma(\epsilon) = \frac{m^2}{\epsilon}$$

cf. $$\int \frac{d^D p}{\pi^{D/2}} \frac{1}{p^2 + m^2} = \int \frac{d^D p}{\pi^{D/2}} \left[\frac{1}{p^2} - \frac{m^2}{p^4} + \cdots\right]$$

i.e., only logarithmic divergence contributes!

Three thoughts on the GF scheme

Kluth '24, '26

3

Problems with $\overline{\text{MS}}$: insensitive to power divergences

Example:

$$\int \frac{d^D p}{\pi^{D/2}} \frac{1}{p^2 + m^2} = m^{D-2}\,\Gamma(1 - D/2)$$

$$D = 4 - 2\epsilon \qquad = m^{2-2\epsilon}\,\Gamma(-1+\epsilon) = \frac{m^{2-2\epsilon}}{\epsilon - 1}\,\Gamma(\epsilon) = \frac{m^2}{\epsilon}$$

cf. $$\int \frac{d^D p}{\pi^{D/2}} \frac{1}{p^2 + m^2} = \int \frac{d^D p}{\pi^{D/2}} \left[\frac{1}{p^2} - \frac{m^2}{p^4} + \cdots\right]$$

i.e., only logarithmic divergence contributes!

→ Fixed point in gravity is hidden in $\overline{\text{MS}}$!

Three thoughts on the GF scheme

Kluth '24, '26

3

Problems with $\overline{\text{MS}}$: insensitive to power divergences

Example:

$$\int \frac{d^Dp}{\pi^{D/2}} \frac{1}{p^2+m^2} = m^{D-2}\,\Gamma(1-D/2)$$

$$D = 4-2\epsilon \qquad = m^{2-2\epsilon}\,\Gamma(-1+\epsilon) = \frac{m^{2-2\epsilon}}{\epsilon-1}\,\Gamma(\epsilon) = \frac{m^2}{\epsilon}$$

cf. $$\int \frac{d^Dp}{\pi^{D/2}} \frac{1}{p^2+m^2} = \int \frac{d^Dp}{\pi^{D/2}} \left[\frac{1}{p^2} - \frac{m^2}{p^4} + \cdots\right]$$

i.e., only logarithmic divergence contributes!

→ Fixed point in gravity is hidden in $\overline{\text{MS}}$!

Need mass dependent renormalization scheme

Application to gravity

RH, Kluth, Kohnen, Werthenbach '26

flowed gauge field:

$$\frac{\partial}{\partial t} B_\mu(t,x) = \mathscr{D}_\nu G_{\nu\mu}(t,x)$$
$$B_\mu(t=0,x) = A_\mu(x)$$

Application to gravity

RH, Kluth, Kohnen, Werthenbach '26

flowed gauge field:

$$\frac{\partial}{\partial t} B_\mu(t, x) = \mathcal{D}_\nu G_{\nu\mu}(t, x)$$
$$B_\mu(t = 0, x) = A_\mu(x)$$

gravity →

$$\partial_t g_{\mu\nu}(t) = -2R_{\mu\nu}(t)$$

Ricci flow

Application to gravity

RH, Kluth, Kohnen, Werthenbach '26

flowed gauge field:

$$\frac{\partial}{\partial t} B_\mu(t,x) = \mathcal{D}_\nu G_{\nu\mu}(t,x)$$
$$B_\mu(t=0,x) = A_\mu(x)$$

gravity →

$$\partial_t g_{\mu\nu}(t) = -2R_{\mu\nu}(t)$$

Ricci flow

flowed metric requires field renormalization (analogous to Z_χ in QCD)

Application to gravity

RH, Kluth, Kohnen, Werthenbach '26

flowed gauge field:

$$\frac{\partial}{\partial t} B_\mu(t,x) = \mathscr{D}_\nu G_{\nu\mu}(t,x)$$
$$B_\mu(t=0,x) = A_\mu(x)$$

gravity →

$$\partial_t g_{\mu\nu}(t) = -2R_{\mu\nu}(t)$$

Ricci flow

flowed metric requires field renormalization (analogous to Z_χ in QCD)

define "fixed-volume scheme":

$$\langle\sqrt{g(t)}\rangle \equiv -\frac{3G_\mathrm{N}}{4\pi t}$$

Application to gravity

RH, Kluth, Kohnen, Werthenbach '26

flowed gauge field:

$$\frac{\partial}{\partial t} B_\mu(t,x) = \mathscr{D}_\nu G_{\nu\mu}(t,x)$$

$$B_\mu(t=0,x) = A_\mu(x)$$

gravity →

$$\partial_t g_{\mu\nu}(t) = -2R_{\mu\nu}(t)$$

Ricci flow

flowed metric requires field renormalization (analogous to Z_χ in QCD)

define "fixed-volume scheme":

$$\langle \sqrt{g(t)} \rangle \equiv -\frac{3G_\mathrm{N}}{4\pi t}$$

cf.: $\langle \mathring{\bar{\chi}}(t) \overleftrightarrow{\not{D}} \mathring{\chi}(t) \rangle \equiv -\frac{2n_c}{(4\pi t)^2}$

Application to gravity

RH, Kluth, Kohnen, Werthenbach '26

define Newton coupling in Ricci flow scheme:

$$g_{\mathrm{RF}}(\mu) \equiv -\frac{t}{3\nu}\langle\sqrt{g(t)}\,R(t)\rangle\Big|_{t=c/\mu^2}$$

cf. $\alpha_s^{\mathrm{GF}}(\mu) \equiv \frac{4\pi t^2}{3}\langle G_{\mu\nu}(t)G^{\mu\nu}(t)\rangle\Big|_{t=c/\mu^2}$

Application to gravity

RH, Kluth, Kohnen, Werthenbach ‘26

define Newton coupling in Ricci flow scheme:

$$g_{\rm RF}(\mu) \equiv -\frac{t}{3\nu}\langle\sqrt{g(t)}\,R(t)\rangle\Big|_{t=c/\mu^2}$$

$$= \frac{G_{\rm N}}{4\pi t\nu}\left[1 - \frac{5G_{\rm N}}{4\pi t} + \cdots\right]$$

cf. $\alpha_s^{\rm GF}(\mu) \equiv \frac{4\pi t^2}{3}\langle G_{\mu\nu}(t)G^{\mu\nu}(t)\rangle\Big|_{t=c/\mu^2}$

Application to gravity

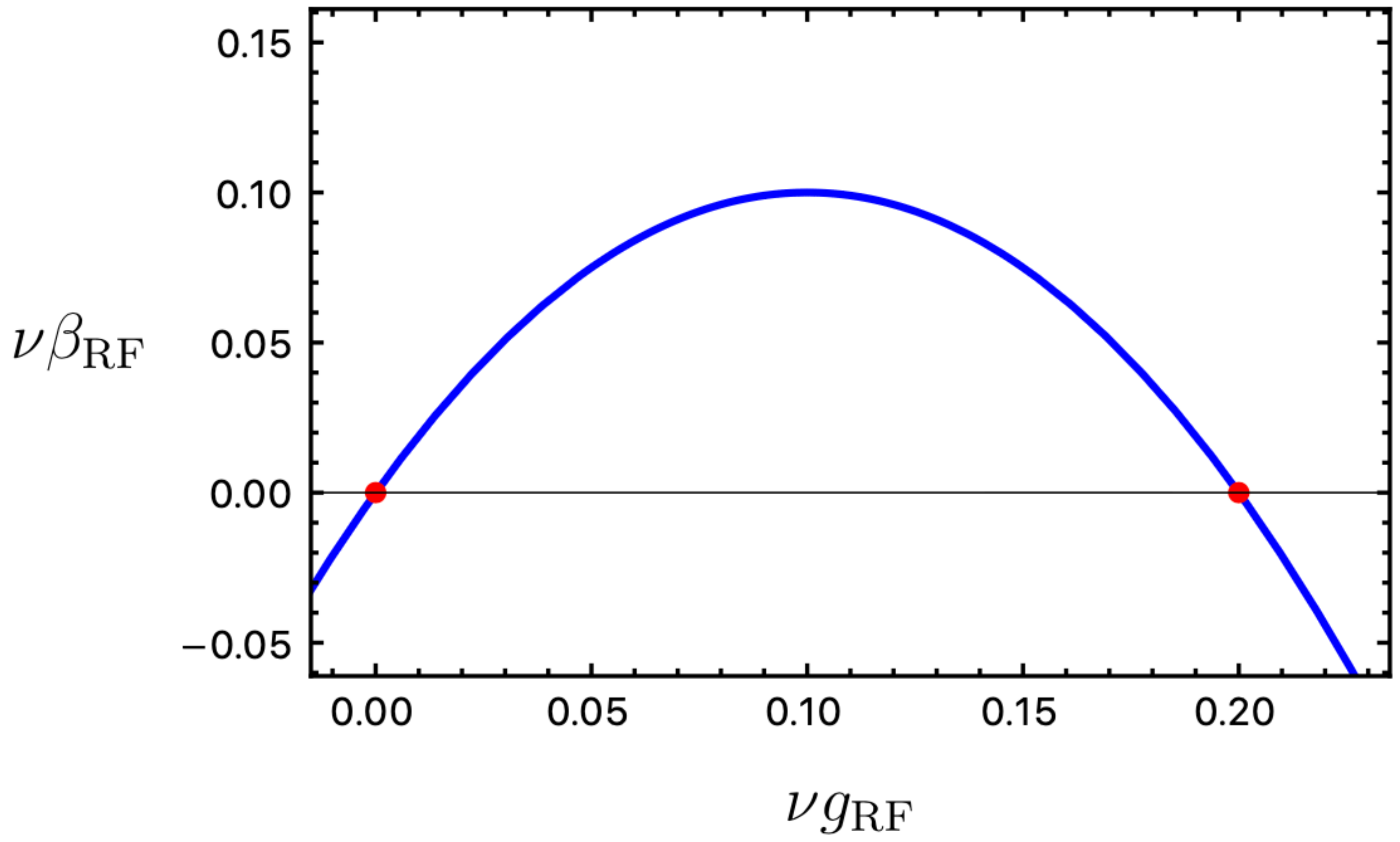
RH, Kluth, Kohnen, Werthenbach ‘26

define Newton coupling in Ricci flow scheme:

$$g_{\mathrm{RF}}(\mu) \equiv -\frac{t}{3\nu}\langle\sqrt{g(t)}\,R(t)\rangle\Big|_{t=c/\mu^2}$$

$$= \frac{G_{\mathrm{N}}}{4\pi t\nu}\left[1-\frac{5G_{\mathrm{N}}}{4\pi t}+\cdots\right]$$

cf. $\alpha_s^{\mathrm{GF}}(\mu) \equiv \frac{4\pi t^2}{3}\langle G_{\mu\nu}(t)G^{\mu\nu}(t)\rangle\Big|_{t=c/\mu^2}$



Application to gravity

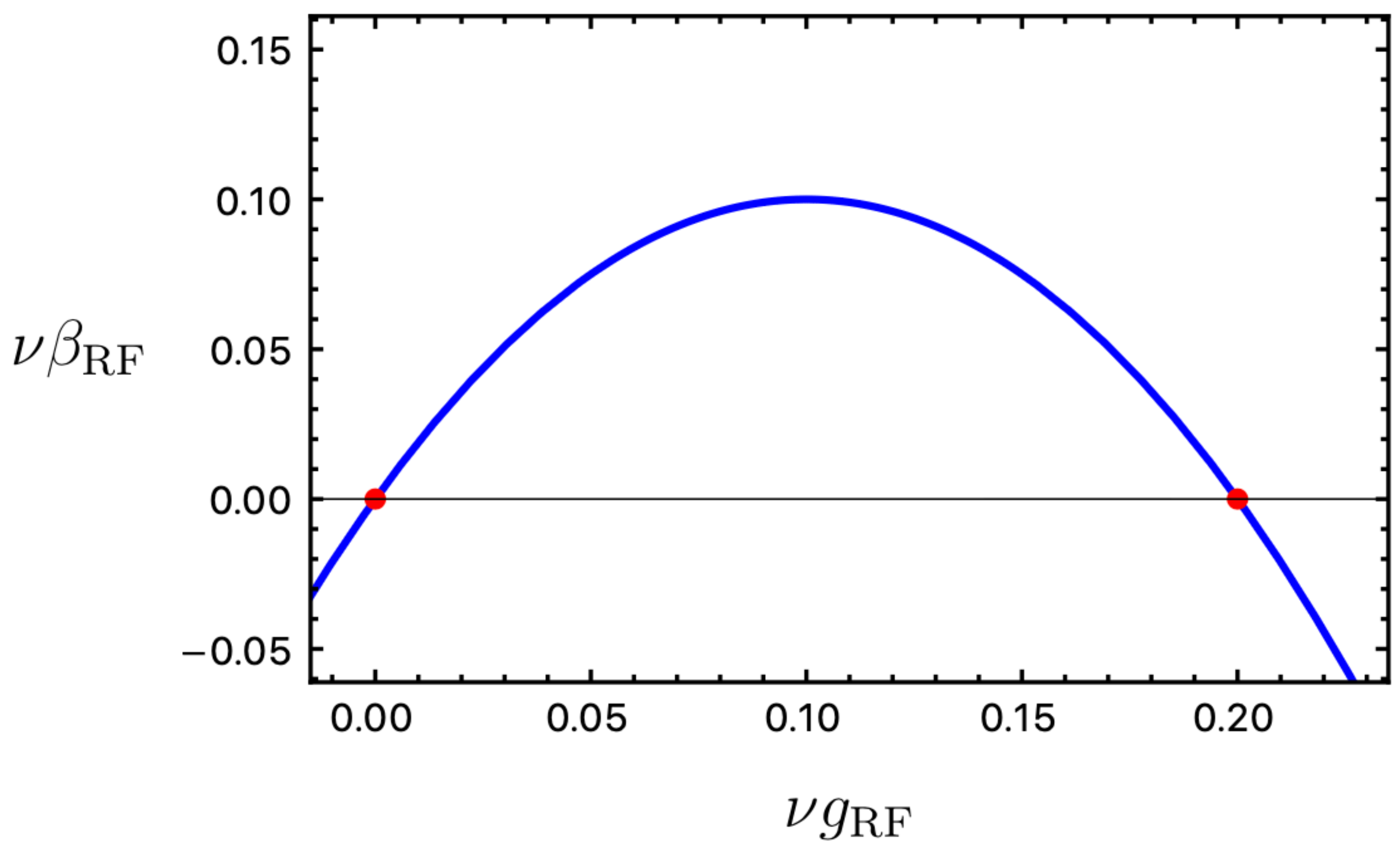
RH, Kluth, Kohnen, Werthenbach '26

define Newton coupling in Ricci flow scheme:

$$g_{\rm RF}(\mu) \equiv -\frac{t}{3\nu}\langle\sqrt{g(t)}\,R(t)\rangle\Big|_{t=c/\mu^2}$$

cf. $\alpha_s^{\rm GF}(\mu) \equiv \frac{4\pi t^2}{3}\langle G_{\mu\nu}(t)G^{\mu\nu}(t)\rangle\Big|_{t=c/\mu^2}$

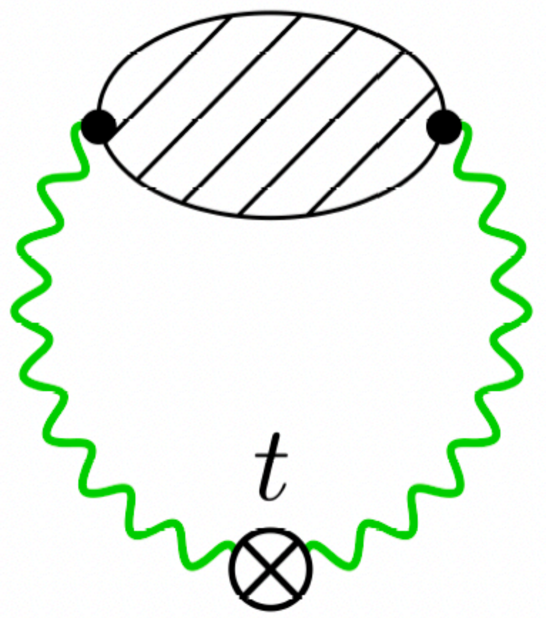
$$= \frac{G_{\rm N}}{4\pi t\nu}\left[1 - \frac{5G_{\rm N}}{4\pi t} + \cdots\right]$$



Henry Werthenbach (Tue 16:00)

Application to QED_3

$E(t) =$

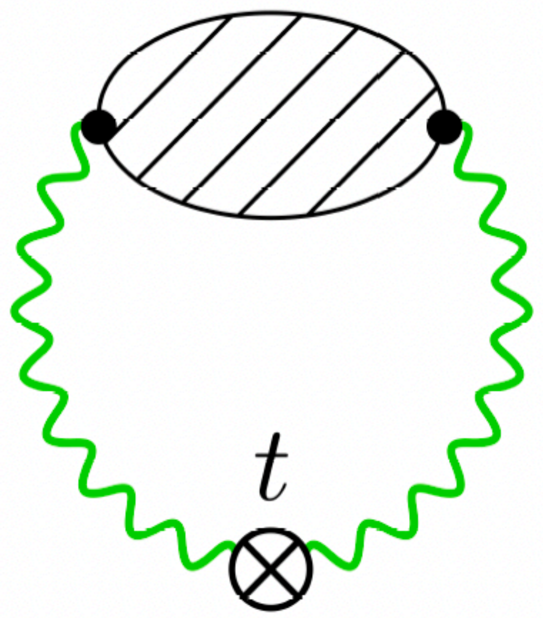


$= \langle F_{\mu\nu}(t) F_{\mu\nu}(t) \rangle$

Application to QED$_3$

in D dimensions:

Georg, RH, Mason '26

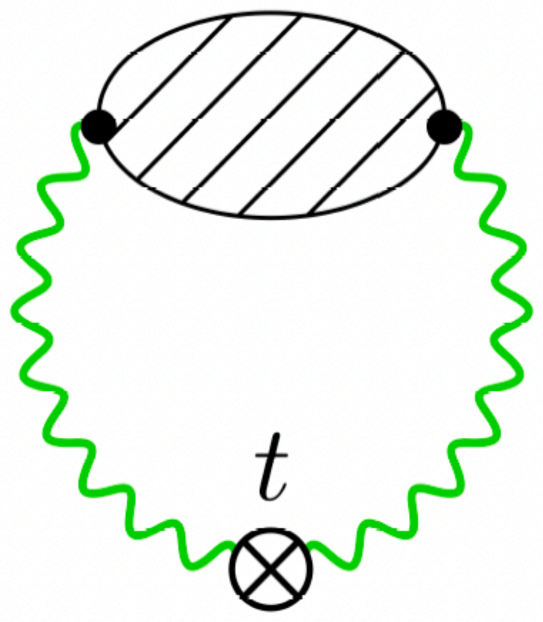
$E(t) =$ 

$$= \langle F_{\mu\nu}(t) F_{\mu\nu}(t) \rangle = \frac{e^2}{2}(D-1)\int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{e^{-2tp^2}}{1+\Pi_{\mathrm{R}}(p)}$$

Application to QED$_3$

in D dimensions:

Georg, RH, Mason '26

$$E(t) =$$ 

$$= \langle F_{\mu\nu}(t) F_{\mu\nu}(t) \rangle = \frac{e^2}{2}(D-1) \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{e^{-2tp^2}}{1 + \Pi_\mathrm{R}(p)}$$

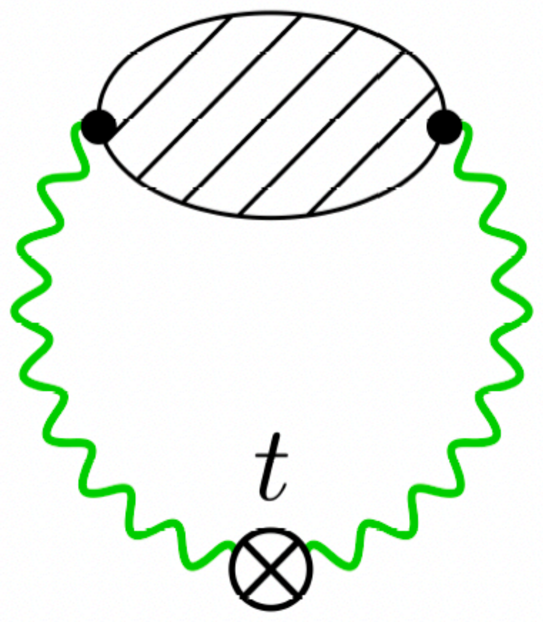
regular photon polarization function

Application to QED$_3$

in D dimensions:

Georg, RH, Mason '26

$E(t) =$



$$= \langle F_{\mu\nu}(t)F_{\mu\nu}(t)\rangle = \frac{e^2}{2}(D-1)\int \frac{\mathrm{d}^D p}{(2\pi)^D}\frac{e^{-2tp^2}}{1-\Pi_{\mathrm{R}}(p)}$$

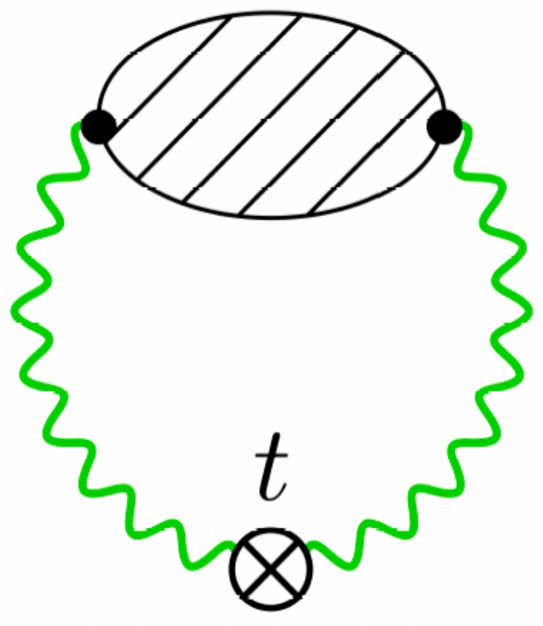
regular photon polarization function

D=3:

Application to QED_3

in D dimensions:

Georg, RH, Mason ‘26

$E(t) =$ 

$$= \langle F_{\mu\nu}(t) F_{\mu\nu}(t) \rangle = \frac{e^2}{2}(D-1) \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{e^{-2tp^2}}{1 - \Pi_{\mathrm{R}}(p)}$$

regular photon polarization function

D=3: α mass dimension = 1

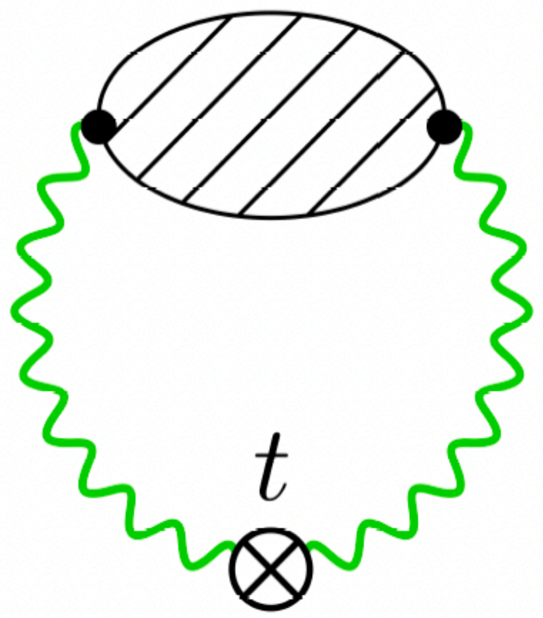
$$\hat{\alpha} \sim \alpha / p$$

Appelquist, Pisarski ‘81
Pisarski ‘84

Application to QED$_3$

in D dimensions:

Georg, RH, Mason ‘26

$E(t) =$ 

$$= \langle F_{\mu\nu}(t) F_{\mu\nu}(t) \rangle = \frac{e^2}{2}(D-1)\int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{e^{-2tp^2}}{1 - \Pi_{\mathrm{R}}(p)}$$

regular photon polarization function

D=3: α mass dimension = 1

super-renormalizable

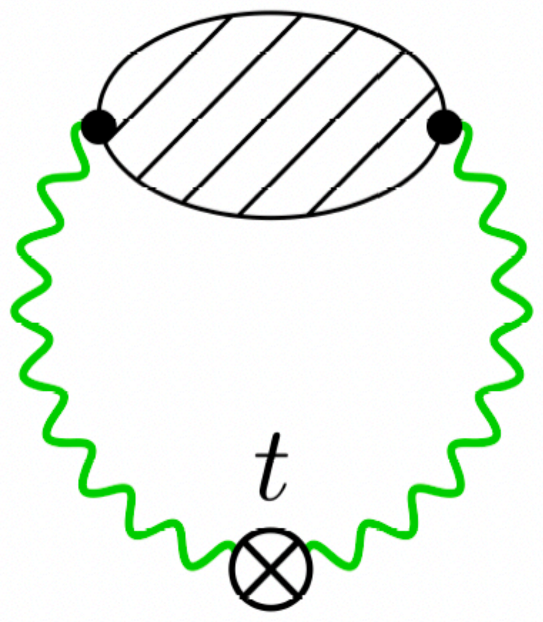
$$\hat{\alpha} \sim \alpha/p$$

Appelquist, Pisarski ‘81
Pisarski ‘84

Application to QED$_3$

in D dimensions:

Georg, RH, Mason ‘26

$E(t) =$ 

$= \langle F_{\mu\nu}(t) F_{\mu\nu}(t) \rangle = \frac{e^2}{2}(D-1)\int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{e^{-2tp^2}}{1 + \Pi_{\mathrm{R}}(p)}$

regular photon polarization function

D=3: α mass dimension = 1

super-renormalizable

strongly interacting in IR

$\hat{\alpha} \sim \alpha/p$

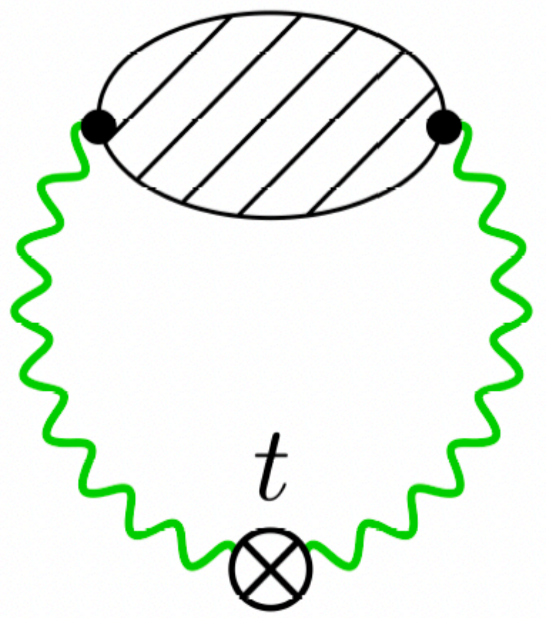
Appelquist, Pisarski ‘81
Pisarski ‘84

Application to QED$_3$

in D dimensions:

Georg, RH, Mason '26

$E(t) =$



$$= \langle F_{\mu\nu}(t)F_{\mu\nu}(t)\rangle = \frac{e^2}{2}(D-1)\int \frac{\mathrm{d}^D p}{(2\pi)^D}\frac{e^{-2tp^2}}{1-\Pi_{\mathrm{R}}(p)}$$

regular photon polarization function

D=3: α mass dimension = 1

super-renormalizable

strongly interacting in IR

IR fixed point at large n_f

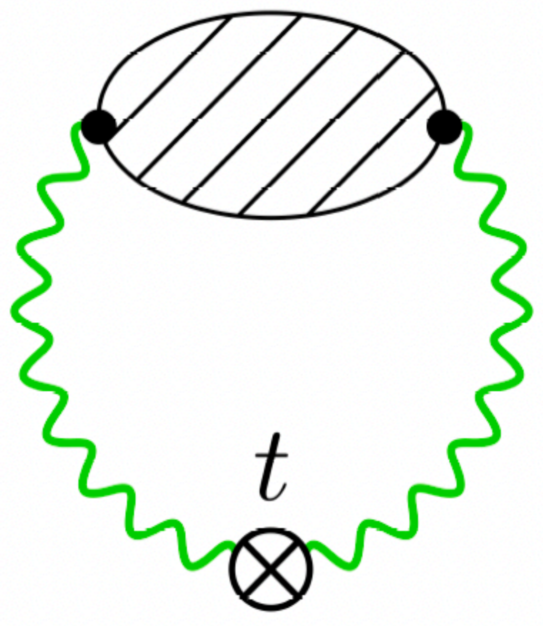
$\hat{\alpha} \sim \alpha/p$

Appelquist, Pisarski '81
Pisarski '84

Application to QED_3

in D dimensions:

Georg, RH, Mason '26

$E(t) =$ 

$= \langle F_{\mu\nu}(t) F_{\mu\nu}(t) \rangle = \frac{e^2}{2}(D-1) \int \frac{\mathrm{d}^D p}{(2\pi)^D} \frac{e^{-2tp^2}}{1 - \Pi_{\mathrm{R}}(p)}$

regular photon polarization function

D=3: α mass dimension = 1 $\hat{\alpha} \sim \alpha/p$

super-renormalizable

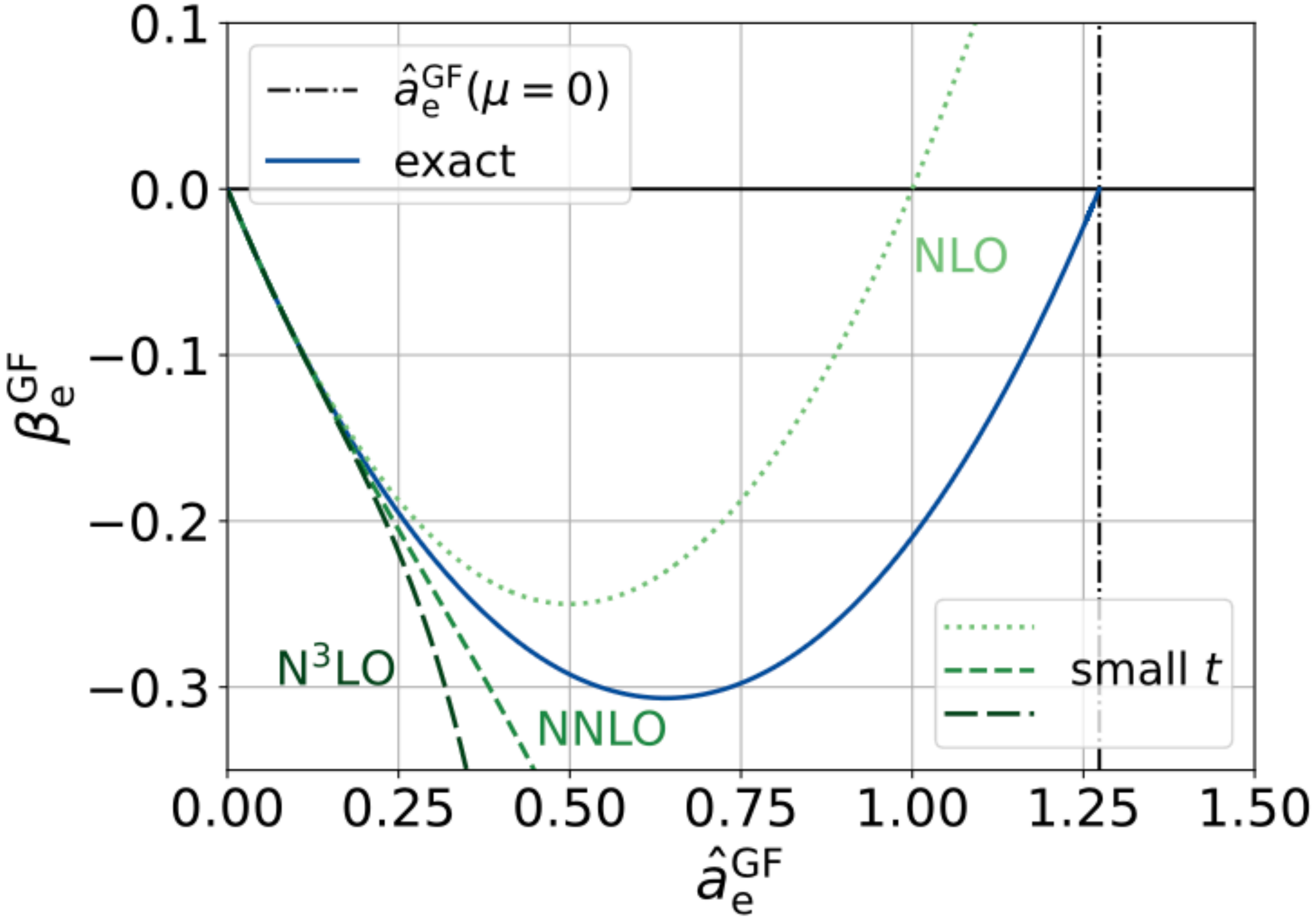
Appelquist, Pisarski '81
Pisarski '84

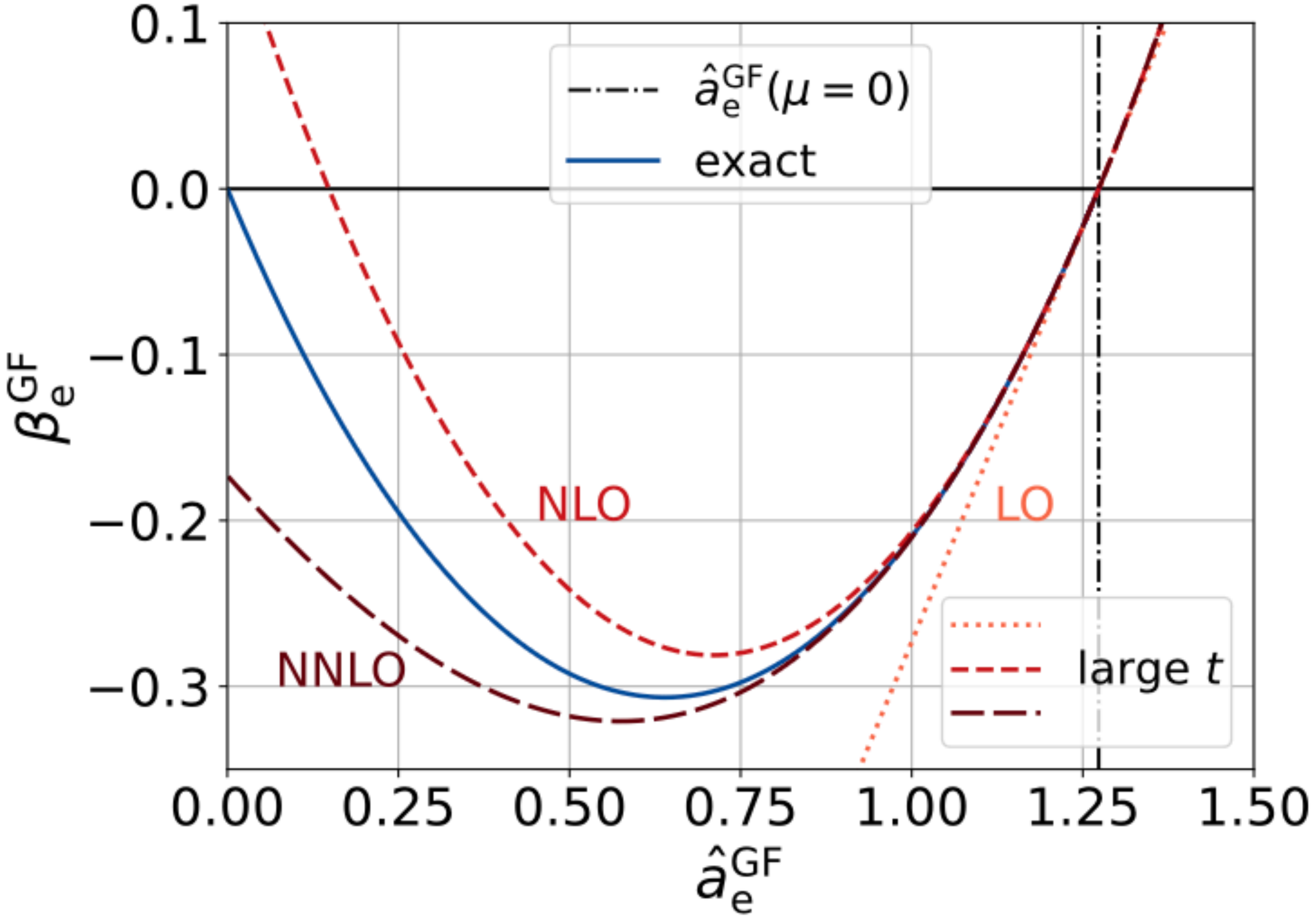
strongly interacting in IR

IR fixed point at large n_f

dynamical fermion mass generation

Application to QED$_3$

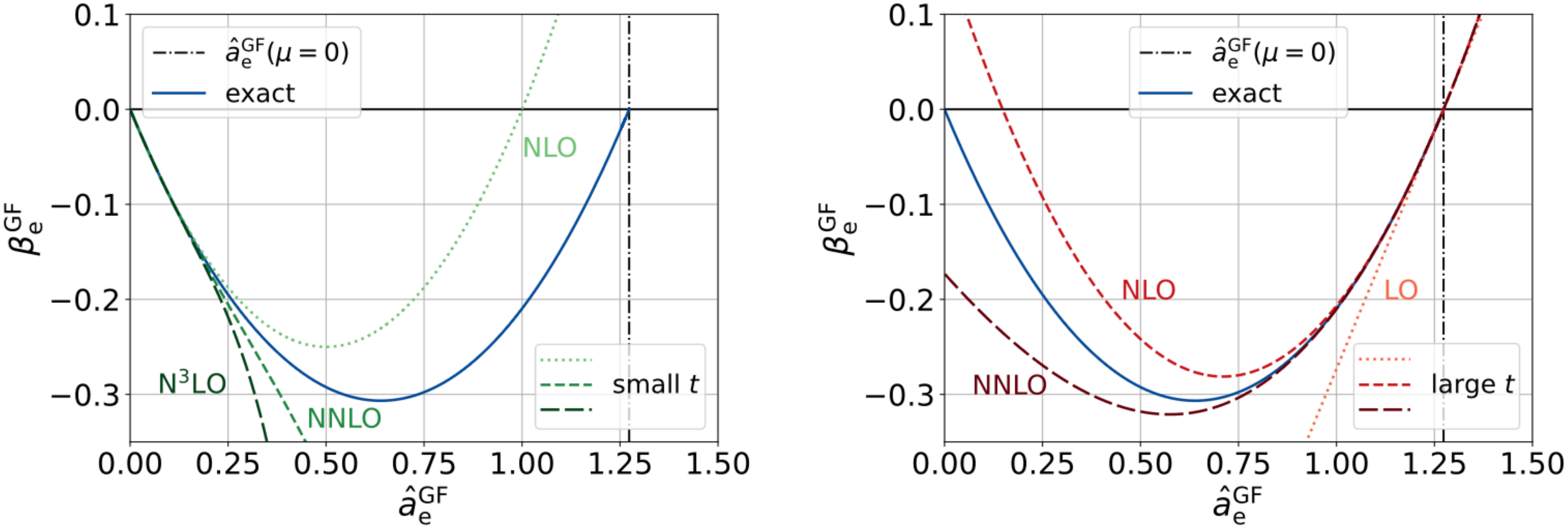




Georg, RH, Mason 2025

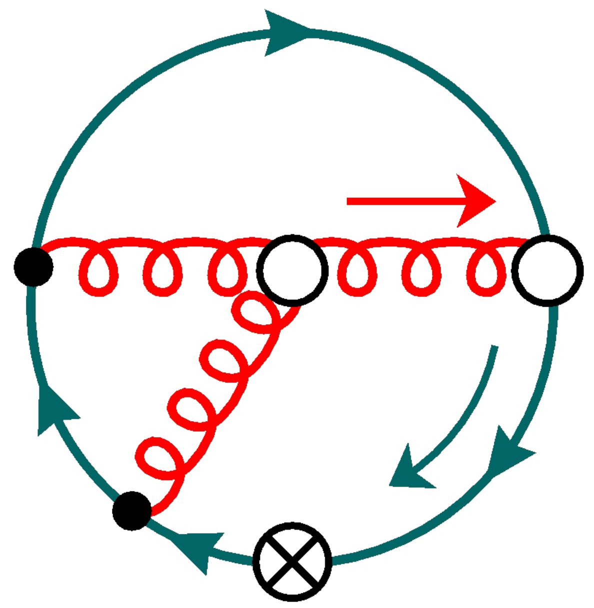
Application to QED$_3$

Lars Georg (Thu 16:30)



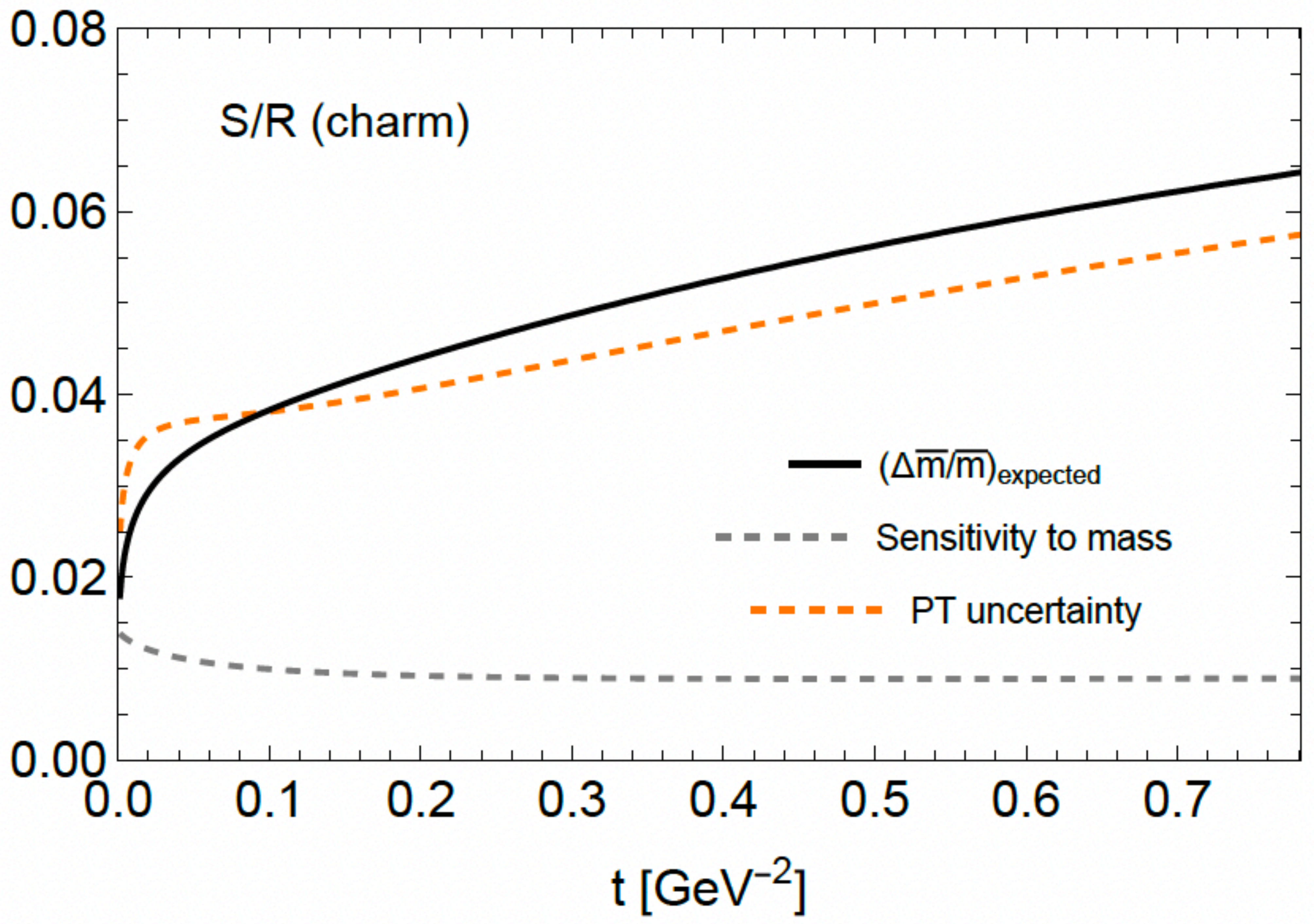
Georg, RH, Mason 2025

Quark masses



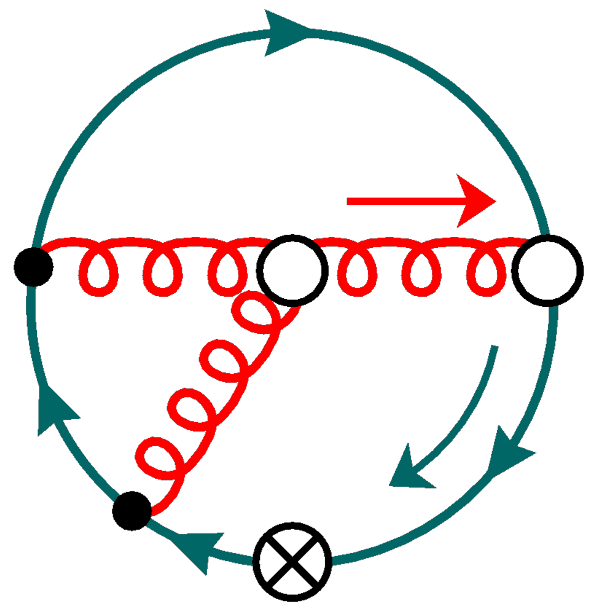
$$\equiv \quad \langle \bar{\chi}(t)\chi(t)\rangle = -\frac{3}{8\pi^2 t} m(\mu) \quad + \text{higher orders}$$

Artz, RH, Lange, Neumann, Prausa ‘19
Georg, RH, Mason ‘26



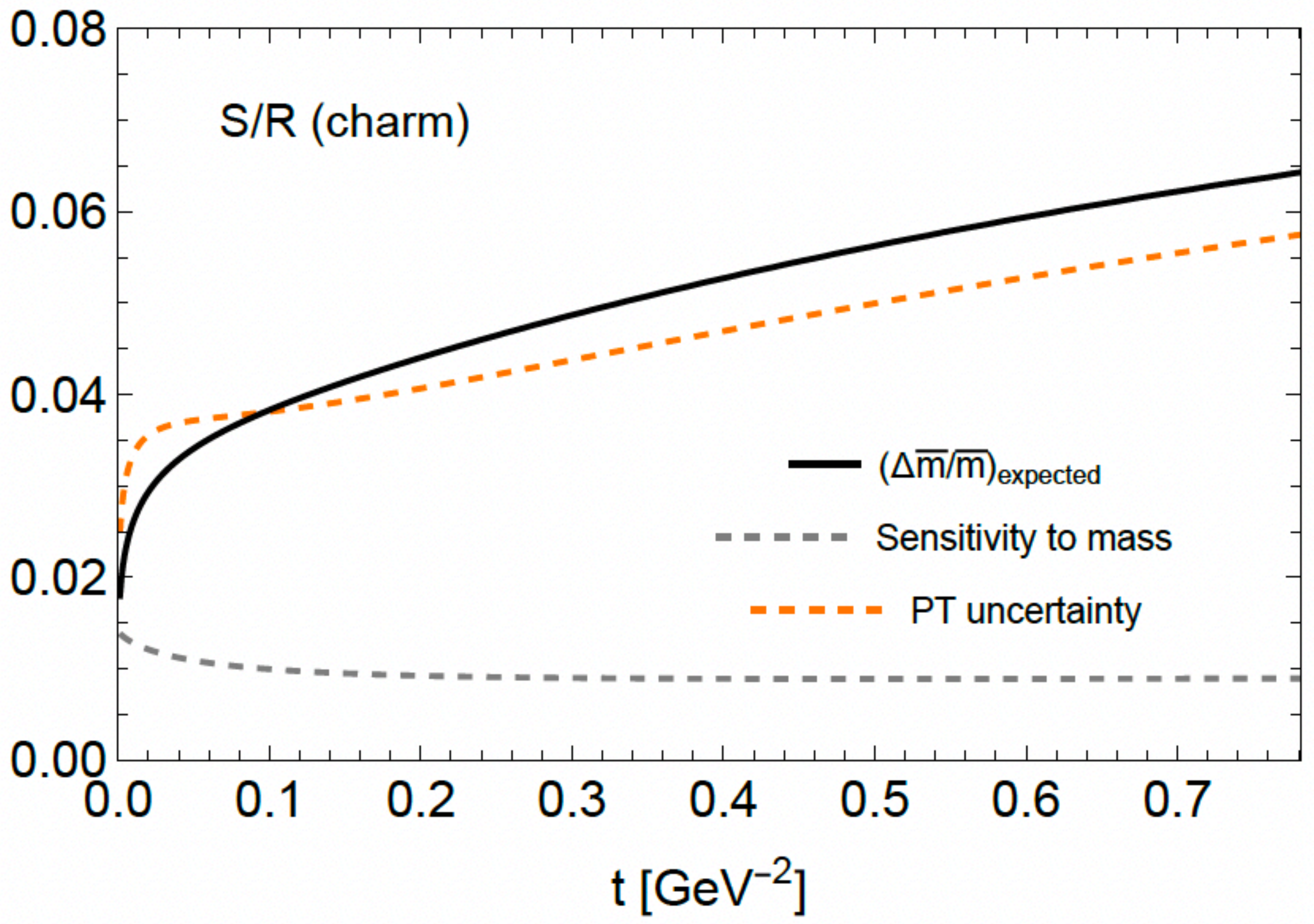
Takaura, RH, Lange ‘25

Quark masses



$$\equiv \quad \langle\bar\chi(t)\chi(t)\rangle = -\frac{3}{8\pi^2 t}m(\mu) \quad + \text{higher orders}$$

Artz, RH, Lange, Neumann, Prausa '19
Georg, RH, Mason '26



Fabian Lange (Tue 11:00)
Robert Mason (Wed 11:00)

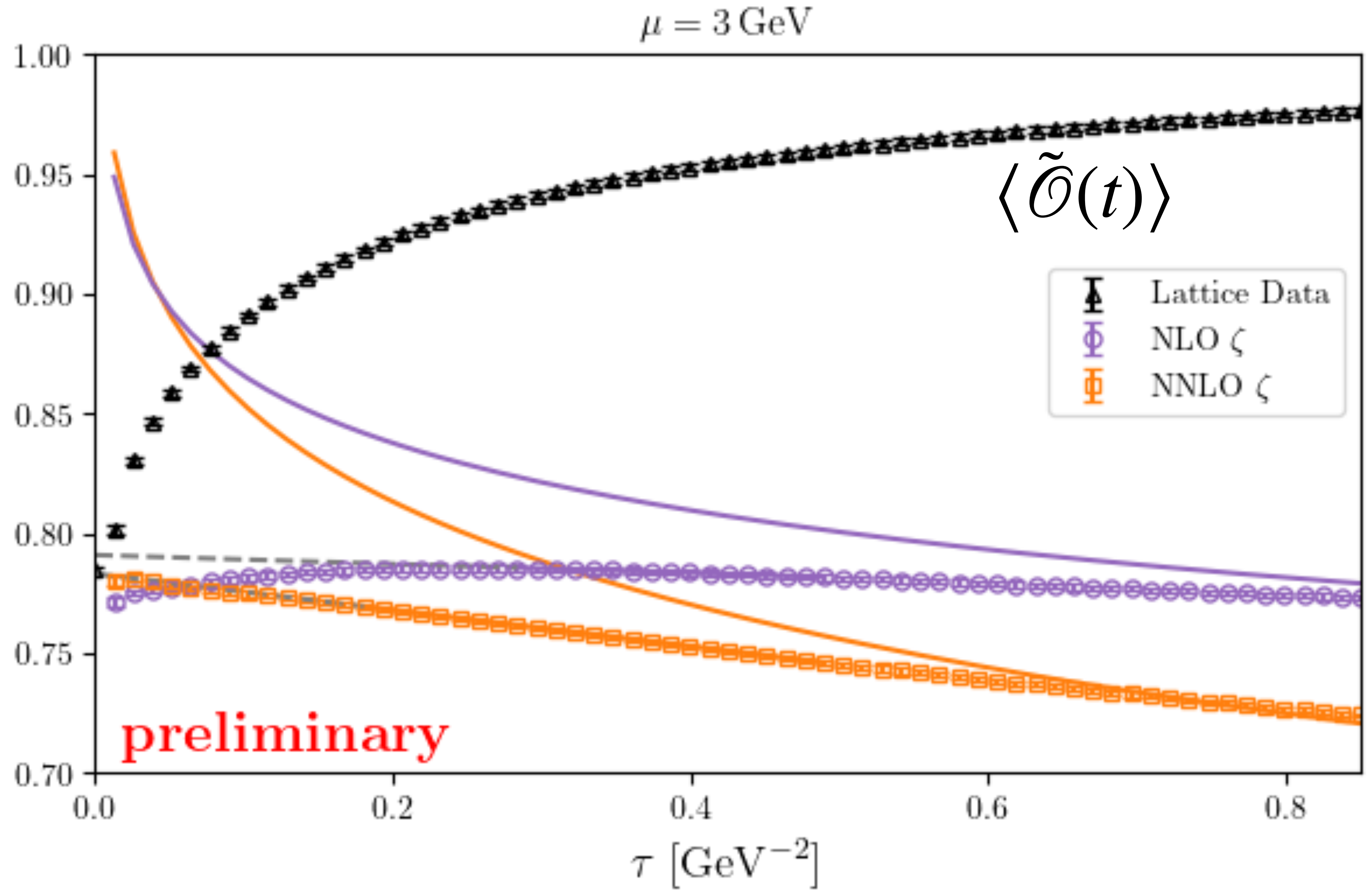
Takaura, RH, Lange '25

The gradient flow scheme

From Zürich 2025 Workshop

$$\Delta\Gamma \sim \sum_n C_n^{\rm R}(\mu)\,\langle \mathcal{O}_n^{\rm R}\rangle(\mu)$$

$$= \sum_n C_n^{\rm R}(\mu)\left[\zeta^{-1}(t,\mu)\langle\tilde{\mathcal{O}}_n\rangle(t)\right]$$

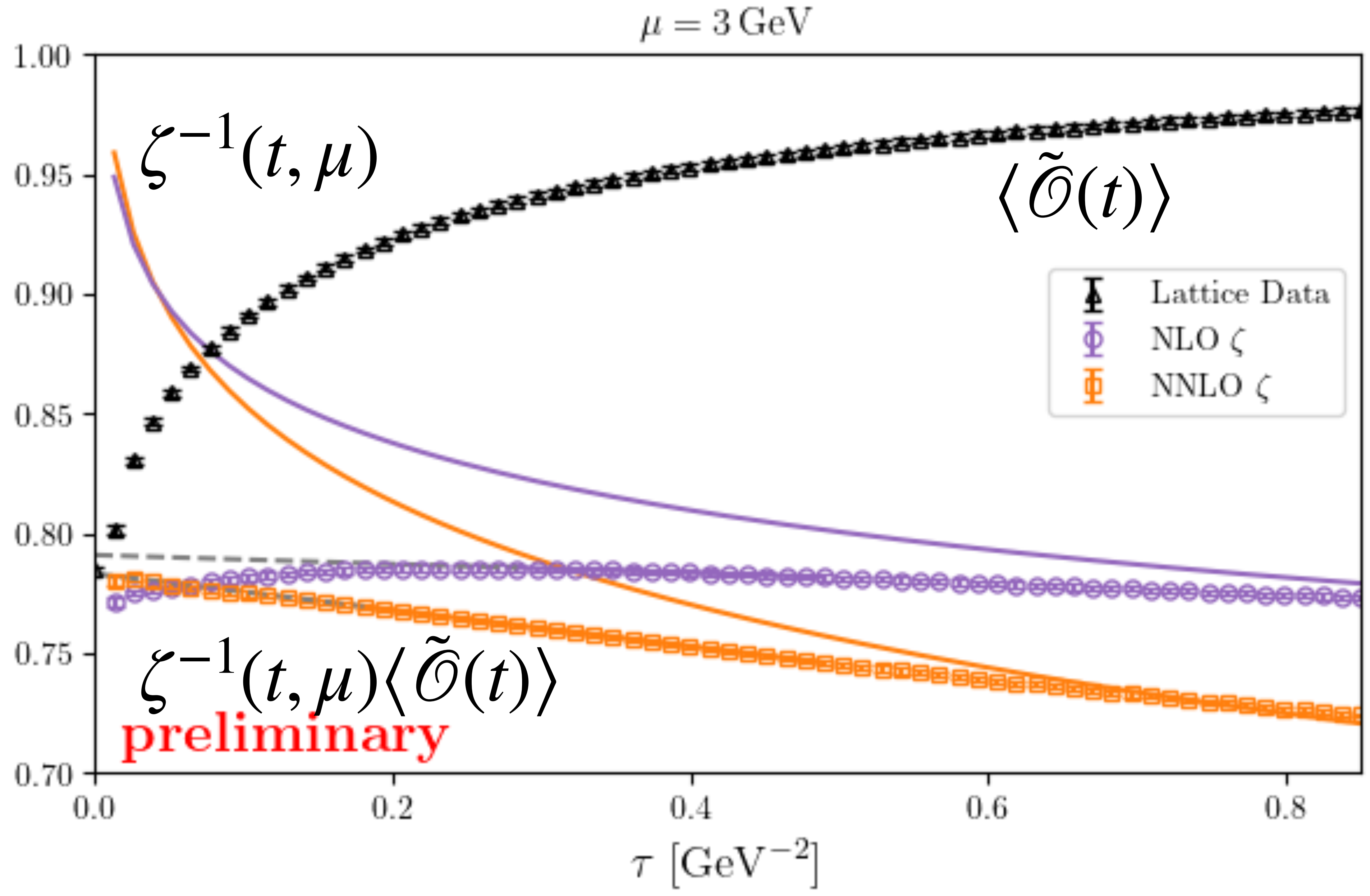


The gradient flow scheme

From Zürich 2025 Workshop

$$\Delta\Gamma \sim \sum_n C_n^{\mathrm{R}}(\mu)\,\langle \mathcal{O}_n^{\mathrm{R}}\rangle(\mu)$$

$$= \sum_n C_n^{\mathrm{R}}(\mu)\left[\zeta^{-1}(t,\mu)\langle\tilde{\mathcal{O}}_n\rangle(t)\right]$$



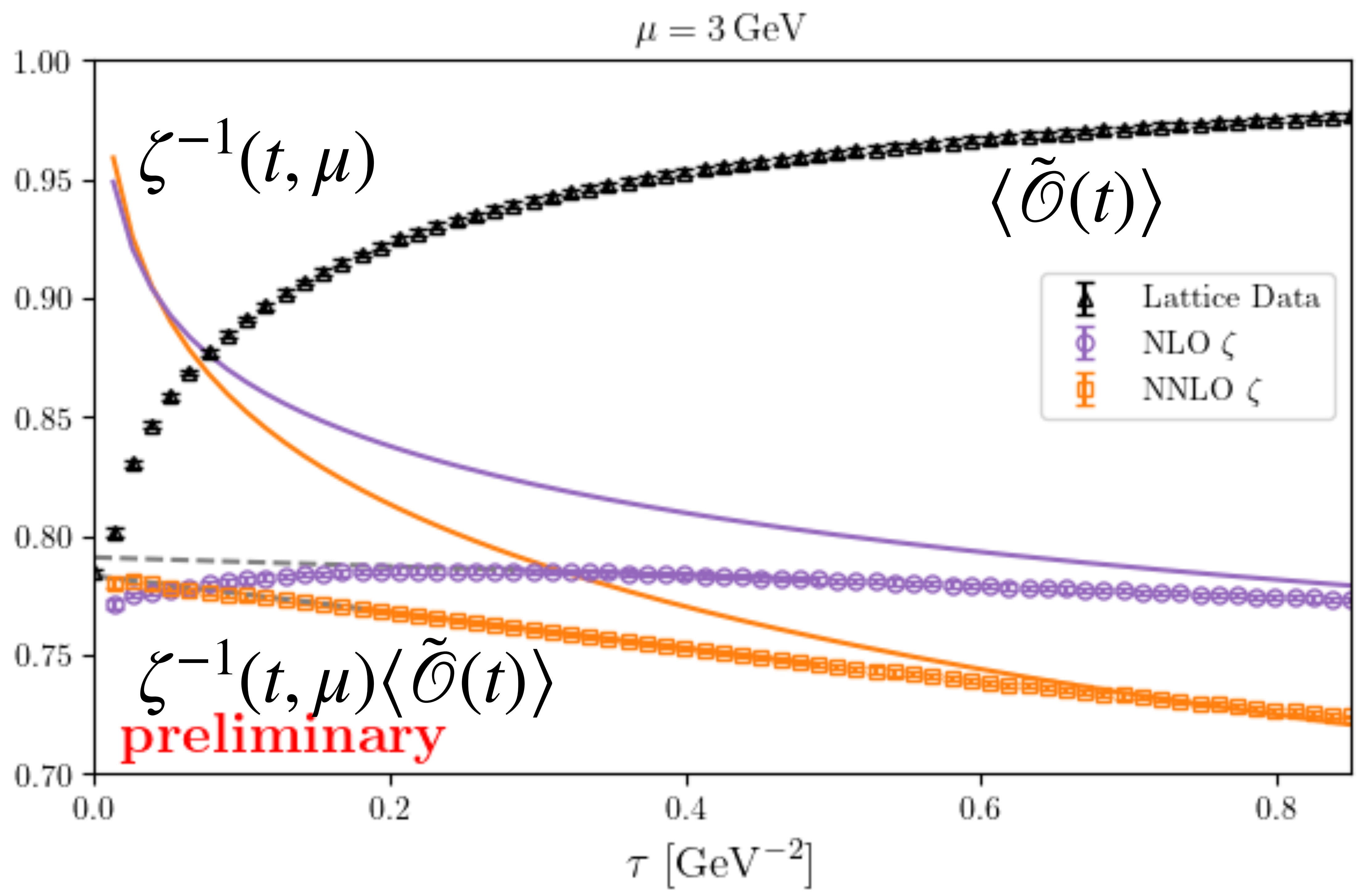
The gradient flow scheme

From Zürich 2025 Workshop

$$\Delta\Gamma \sim \sum_n C_n^{\mathrm{R}}(\mu)\,\langle \mathcal{O}_n^{\mathrm{R}}\rangle(\mu)$$

$$= \sum_n C_n^{\mathrm{R}}(\mu)\left[\zeta^{-1}(t,\mu)\langle\tilde{\mathcal{O}}_n\rangle(t)\right]$$

$$= \sum_n \left[C_n^{\mathrm{R}}(\mu)\,\zeta^{-1}(t,\mu)\right]\langle\tilde{\mathcal{O}}_n\rangle(t)$$



The gradient flow scheme

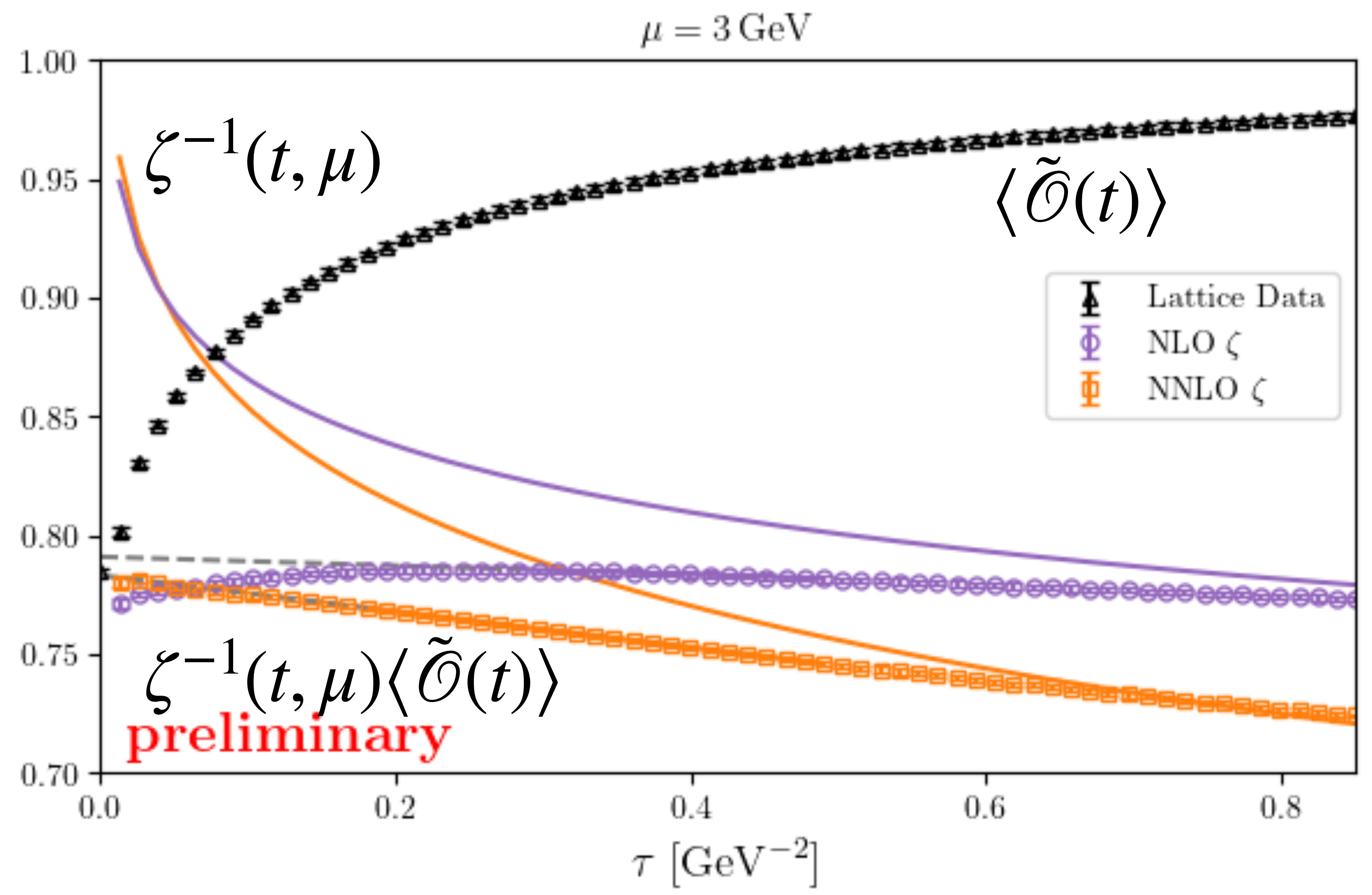
From Zürich 2025 Workshop

$$\Delta\Gamma \sim \sum_n C_n^{\mathrm{R}}(\mu)\,\langle \mathcal{O}_n^{\mathrm{R}}\rangle(\mu)$$

$$= \sum_n C_n^{\mathrm{R}}(\mu)\left[\zeta^{-1}(t,\mu)\langle \tilde{\mathcal{O}}_n\rangle(t)\right]$$

$$= \sum_n \left[C_n^{\mathrm{R}}(\mu)\,\zeta^{-1}(t,\mu)\right]\langle \tilde{\mathcal{O}}_n\rangle(t)$$

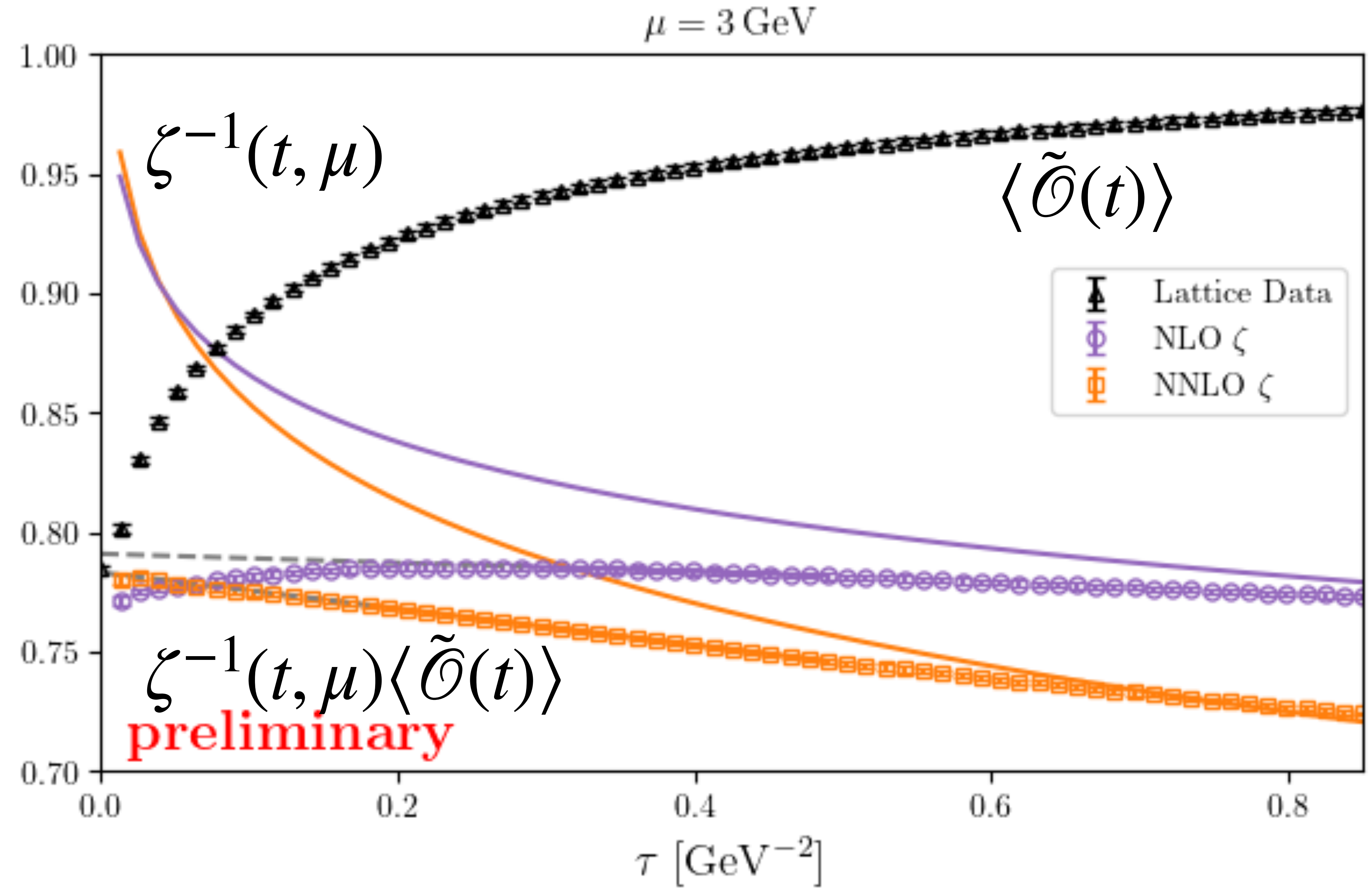
$$= \sum_n \tilde{C}_n(t)\langle \tilde{\mathcal{O}}_n\rangle(t)$$



The gradient flow scheme

From Zürich 2025 Workshop

$$\Delta\Gamma \sim \sum_n C_n^{\mathrm{R}}(\mu)\,\langle \mathcal{O}_n^{\mathrm{R}}\rangle(\mu)$$
$$= \sum_n C_n^{\mathrm{R}}(\mu)\left[\zeta^{-1}(t,\mu)\langle\tilde{\mathcal{O}}_n\rangle(t)\right]$$
$$= \sum_n \left[C_n^{\mathrm{R}}(\mu)\,\zeta^{-1}(t,\mu)\right]\langle\tilde{\mathcal{O}}_n\rangle(t)$$
$$= \sum_n \tilde{C}_n(t)\langle\tilde{\mathcal{O}}_n\rangle(t)$$



[…] Hence, no conversion to the MS scheme is needed any more, the advantage being that the renormalization scheme employed is well-defined beyond perturbation theory.

Ammer, Dürr '24

The GF scheme

$$\mathscr{L}_{\text{eff}} = \sum_n C_n \langle \mathscr{O}_n \rangle$$

GF	$\overline{\text{MS}}$

The GF scheme

$$\mathscr{L}_{\text{eff}} = \sum_n C_n \langle \mathcal{O}_n \rangle$$

GF	$\overline{\text{MS}}$
$= \sum_n (C\,\zeta^{-1}(t))_n \,\langle \zeta(t)\,\mathcal{O}\rangle_n$	$= \sum_n (C\,Z^{-1})_n \langle Z\,\mathcal{O}\rangle_n$

The GF scheme

$$\mathscr{L}_{\text{eff}} = \sum_n C_n \langle \mathcal{O}_n \rangle$$

GF	$\overline{\text{MS}}$
$= \sum_n (C\,\zeta^{-1}(t))_n \, \langle \zeta(t)\, \mathcal{O} \rangle_n$	$= \sum_n (C\,Z^{-1})_n \langle Z\, \mathcal{O} \rangle_n$
$= \sum_n \tilde{C}_n(t)\, \langle \tilde{\mathcal{O}}_n(t) \rangle$	$= \sum_n C_n^{\text{R}}(\mu)\, \langle \mathcal{O}_n^{\text{R}} \rangle(\mu)$

The GF scheme

$$\mathscr{L}_{\text{eff}} = \sum_n C_n \langle \mathcal{O}_n \rangle$$

GF	$\overline{\text{MS}}$
$= \sum_n (C\,\zeta^{-1}(t))_n \, \langle \zeta(t)\, \mathcal{O} \rangle_n$	$= \sum_n (C\,Z^{-1})_n \langle Z\, \mathcal{O} \rangle_n$
$= \sum_n \tilde{C}_n(t)\, \langle \tilde{\mathcal{O}}_n(t) \rangle$	$= \sum_n C_n^{\text{R}}(\mu)\, \langle \mathcal{O}_n^{\text{R}} \rangle(\mu)$
$t\frac{d}{dt}\tilde{C}(t) = \tilde{C}(t)\,\tilde{\gamma}$	$\mu\frac{d}{d\mu}C^{\text{R}}(\mu) = C^{\text{R}}(\mu)\,\gamma$

The GF scheme

$$\mathscr{L}_{\rm eff} = \sum_n C_n \langle \mathcal{O}_n \rangle$$

GF	$\overline{\rm MS}$
$= \sum_n (C\,\zeta^{-1}(t))_n\,\langle \zeta(t)\,\mathcal{O}\rangle_n$	$= \sum_n (C\,Z^{-1})_n \langle Z\,\mathcal{O}\rangle_n$
$= \sum_n \tilde{C}_n(t)\,\langle \tilde{\mathcal{O}}_n(t)\rangle$	$= \sum_n C_n^{\rm R}(\mu)\,\langle \mathcal{O}_n^{\rm R}\rangle(\mu)$
$t\frac{d}{dt}\tilde{C}(t) = \tilde{C}(t)\,\tilde{\gamma}$	$\mu\frac{d}{d\mu}C^{\rm R}(\mu) = C^{\rm R}(\mu)\,\gamma$
$\tilde{\gamma} = -\,t\frac{d}{dt}\ln\zeta(t)$	$\gamma_{nm} = -\,\mu\frac{d}{d\mu}\ln Z$

The GF scheme

$$\mathscr{L}_{\text{eff}} = \sum_n C_n \langle \mathcal{O}_n \rangle$$

GF	$\overline{\text{MS}}$
$= \sum_n (C\,\zeta^{-1}(t))_n \langle \zeta(t)\,\mathcal{O}\rangle_n$	$= \sum_n (C\,Z^{-1})_n \langle Z\,\mathcal{O}\rangle_n$
$= \sum_n \tilde{C}_n(t)\,\langle \tilde{\mathcal{O}}_n(t)\rangle$	$= \sum_n C_n^{\text{R}}(\mu)\,\langle \mathcal{O}_n^{\text{R}}\rangle(\mu)$
$t\frac{d}{dt}\tilde{C}(t) = \tilde{C}(t)\,\tilde{\gamma}$	$\mu\frac{d}{d\mu}C^{\text{R}}(\mu) = C^{\text{R}}(\mu)\,\gamma$
$\tilde{\gamma} = -\,t\frac{d}{dt}\ln\zeta(t)$	$\gamma_{nm} = -\,\mu\frac{d}{d\mu}\ln Z$

RH, Lange, Neumann ‘20

Borgulat, Felten, RH, Kohnen ‘25

Left out

SFTX applied to

LEFT (QCD x QED) Crosas, Stoffer 2026

scalar QCD Borgulat, Felten, RH, Kohnen 2025

Standard Model Borgulat (Diss), Borgulat et al. (in prep)

. . .

Left out

SFTX applied to

LEFT (QCD x QED)	Crosas, Stoffer 2026	Oscar Crosas (Wed 11:30)
scalar QCD	Borgulat, Felten, RH, Kohnen 2025	
Standard Model	Borgulat (Diss), Borgulat et al. (in prep)	

. . .

Conclusions

Gradient flow: rich field theoretical concept

Established tool in **lattice calculations**

Institute for Theoretical Particle Physics and Cosmology
RWTH AACHEN UNIVERSITY

Conclusions

Gradient flow: rich field theoretical concept

Established tool in **lattice calculations**

Lattice/PT interplay picks up momentum:

Conclusions

Gradient flow: rich field theoretical concept

Established tool in **lattice calculations**

Lattice/PT interplay picks up momentum:
first lattice **PDF moments** beyond $\langle x^3 \rangle$

Conclusions

Gradient flow: rich field theoretical concept

Established tool in **lattice calculations**

Lattice/PT interplay picks up momentum:
first lattice **PDF moments** beyond $\langle x^3 \rangle$
first meson lifetime **bag parameters** with full error budget

Conclusions

Gradient flow: rich field theoretical concept

Established tool in **lattice calculations**

Lattice/PT interplay picks up momentum:

- first lattice **PDF moments** beyond $\langle x^3 \rangle$
- first meson lifetime **bag parameters** with full error budget
- **renormalon**-free OPEs

Conclusions

Gradient flow: rich field theoretical concept

Established tool in **lattice calculations**

Lattice/PT interplay picks up momentum:

first lattice **PDF moments** beyond $\langle x^3 \rangle$

first meson lifetime **bag parameters** with full error budget

renormalon-free OPEs

quark masses, strong coupling, …

Conclusions

Gradient flow: rich field theoretical concept

Established tool in **lattice calculations**

Lattice/PT interplay picks up momentum:

first lattice **PDF moments** beyond $\langle x^3 \rangle$

first meson lifetime **bag parameters** with full error budget

renormalon-free OPEs

quark masses, strong coupling, …

Full potential not yet fully explored (I think…)

Conclusions

Gradient flow: rich field theoretical concept

Established tool in **lattice calculations**

Lattice/PT interplay picks up momentum:

first lattice **PDF moments** beyond $\langle x^3 \rangle$

first meson lifetime **bag parameters** with full error budget

renormalon-free OPEs

quark masses, strong coupling, …

Full potential not yet fully explored (I think…)

Many things not even discussed:
Static QCD potential, EDMs, …