A non-perturbative definition of Magnetostatic QCD

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High temperature 4D from 3D EFT

High Temperature QCD
Static QCD physics

3D
Effective Field Theory

- Tune 3D couplings, $g_i(T)$
  MATCHING
- Simulate 3D (cheap!)

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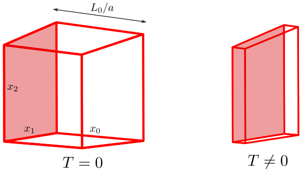
Non-Perturbative matching required!

Outline

- Introduction
  - Dimensional Reduction
  - Perturbative matching
- Matching on the lattice
  - Magnetostatic QCD on the lattice
  - Gradient flow finite volume matching
- Preliminary results
  - NP Matching vs PT
  - Predicting 4D physics

Dimensional reduction



- System does no 'have time' to fluctuate
- Thermal fluctuations dominate

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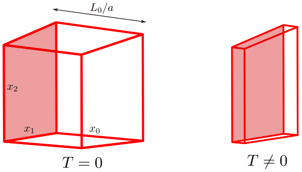

$$A_\mu(x) = T \sum_n \int_{\vec{p}} e^{i(\omega_n x_0 + \vec{p}\cdot\vec{x})} \tilde{A}_\mu(\omega_n, \vec{p}), \;\; \omega_n = 2\pi T n$$

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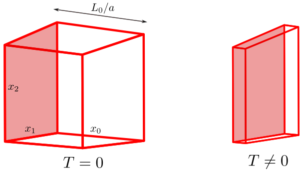

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- Large T
  - Hard modes decouple $n\neq 0$
  - Static! mode survives $n=0$

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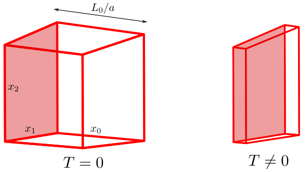

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- Static quantities captured by bosonic 3D EFT

$$\mathcal{L}_{3\mathrm{D}} = \sum_i c_i(g(T)) f_i\left(A_i, A_0\right)\Big|_{n=0} \longrightarrow \langle \mathcal{O} \rangle_{3\mathrm{D}} = \langle \mathcal{O} \rangle_{4\mathrm{D}} + \mathcal{O}(1/T)$$

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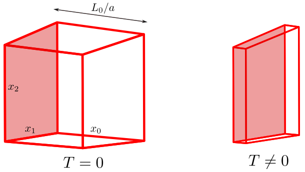

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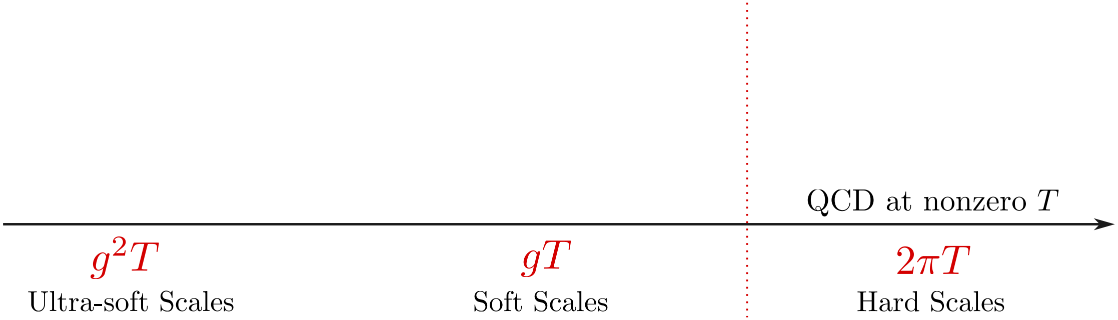
- Broad applications: QCD thermodynamics, EWPT & Baryogenesis, BSM
  - Pressure, screening masses, jet quenching, ...

    [Kajantie, M. Laine, Rummukainen, et al. 2001; Caron-Huot 2009; Moore et al. 2021 ...]
  - Properties of phase transitions (order, critical temperature), ...

    [Kajantie, Rummukainen, and Shaposhnikov 1993 ...]

Dimensional reduction & Separation of Scales



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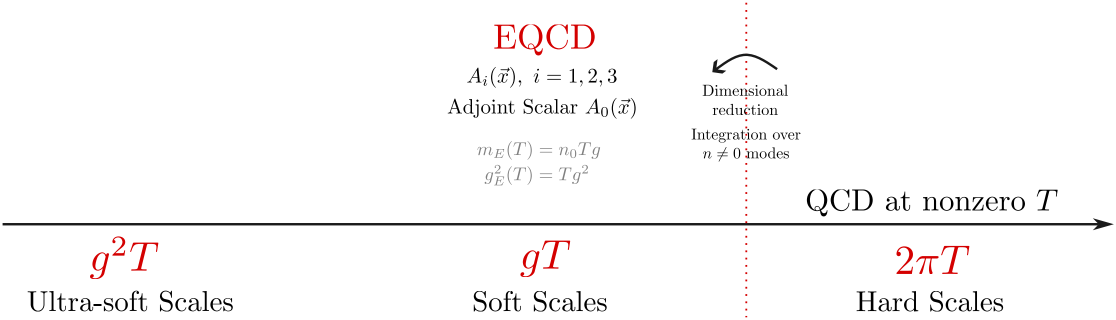

$$\mathcal{L}_{\text{EQCD}} = -\frac{1}{2 g_E^2} \operatorname{Tr}\{F_{ij} F_{ij}\} + \operatorname{Tr}\Big\{(D_i A_0)^\dagger D_i A_0\Big\} - m_E^2 \operatorname{Tr}\{A_0^2\} + \lambda_E \operatorname{Tr}\{A_0^2\}^2 + \mathcal{O}(1/T)$$

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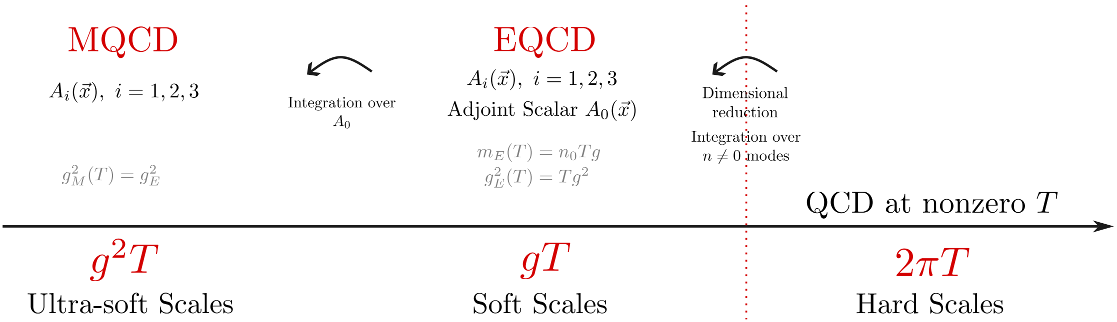

$$\mathcal{L}_{\mathrm{MQCD}} = -\frac{1}{2 g_M^2} \mathrm{Tr}\{F_{ij} F_{ij}\} + \mathcal{O}(1/T)$$

Status of the Dimensional Reduced EFT

1. Ideas of DR EFT

[Ginsparg 1980; Appelquist and Pisarski 1981; Arnold and Yaffe 1995; Braaten and Nieto 1995 ...]

- Perturbative estimates [Braaten and Nieto 1996 ...]

$$g^2T \ll gT \ll 2\pi T \quad \longrightarrow \quad \alpha_{\overline{MS}}(2\pi T) \ll 1 \quad \longrightarrow T/T_c \gtrsim 10 \ldots 20$$

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All studies rely on the perturbative matching!

$g_E^{\mathrm{PT}}(T), m_E^{\mathrm{PT}}(T)$ or $g_M^{\mathrm{PT}}(T)$ to 2-loop in $g_{\overline{\mathrm{MS}}}$

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Answer two questions:

- Validity of the perturbative matching (even at at high T)
- Where do we find separation of scales? (range of validity of the EFT)

Matching on the lattice

Magnetostatic QCD on the Lattice

$$\mathcal{L}_{\mathrm{MQCD}} = -\frac{1}{2g_M^2}\,\mathrm{Tr}\{F_{ij}F_{ij}\} + \ldots$$

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Magnetic action density:

$$\langle E_m\rangle = -\frac{1}{4}\langle G^a_{ij}G^a_{ij}\rangle \xrightarrow{\text{large } T} \frac{g_0^2(N^2-1)}{2}T\int_{\vec{p}} e^{-2t\vec{p}^2}\left(\vec{p}^2\delta_{ij} - p_ip_j\right)D_{ji}(\vec{p},0)$$

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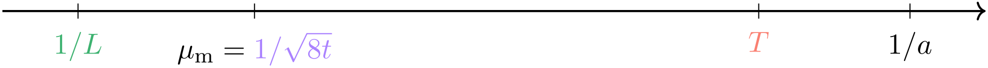

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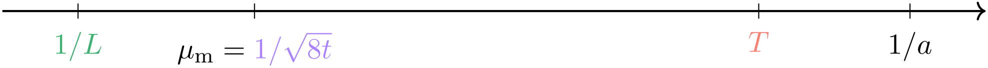

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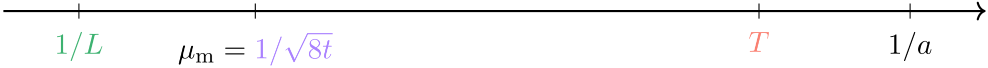

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  - Switch to a finite volume matching scheme

Finite volume Gradient Flow matching

- Define finite volume flow scale $\sqrt{8t_c} = cL$, $c = 0.3$ $(t_c \leftrightarrow L)$
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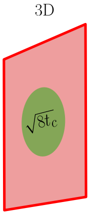
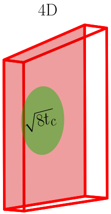
Continuum matching:





$$g_M^2(T) \text{ such that: } t_c^2 \langle E(t_c) \rangle \Big|_{3D}^{g_M^2} = t_c^2 \langle E_m(t_c) \rangle \Big|_{4D}^{T \neq 0}$$

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3D Lattice matching

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  - Matching:

$$\frac{g_M^2}{T} = \frac{1}{R} \lim_{\beta_3 \to \infty} \left(\frac{L}{a} \right) (ag_3^2) \qquad 1 \ll \frac{L_0}{a} \ll \frac{\sqrt{8t_c}}{a} = cR \frac{L_0}{a}$$

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- Setup
  - $L_0/a = L'/2a = 4, \ldots, 12$
  - $T/T_c \sim 1.5, \ldots, 300$
  - $R = L/L_0 = 10$

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  - $T/T_c \sim 1.5, \ldots, 300$
  - $R = L/L_0 = 10$
- GF Finite Volume LCP

  [Dalla Brida and Ramos 2019]

  $\{\beta, L'/a\} \longrightarrow g^2_{\mathrm{GF,m}}(\beta, L') = \#$
  - TL improved
  - NP β-function GF scheme
  - Precise tuning

$3\mathrm{D} - L^3$

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  Z_g for $\mathcal{O}(a)$ improvement
- $L/a = 64, \ldots, 160$

Lattice setup

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$$t^2 \langle E \rangle_{\mathrm{imp}} = \frac{\mathcal{N}_{\mathrm{TBC}}(c)}{\mathcal{N}^{\mathrm{latt.}}_{\mathrm{TBC}}(c, a)} t^2 \langle E \rangle$$

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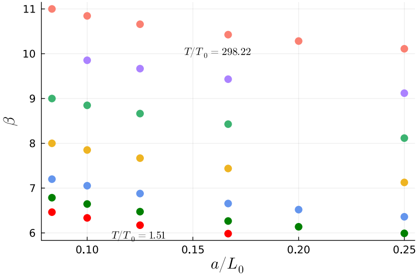
- Wilson/Zeuthen flow;

  E-discretizations: Pl, Cl, Imp.
- https://gitlab.com/guitelo96/latticegpu.jl

  [A. Ramos, LatticeGPU]

Results

4D continuum limit (PRELIMINARY)



- Some on-going simulations

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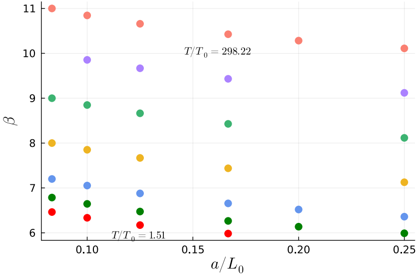

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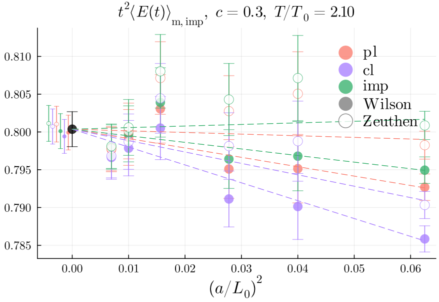

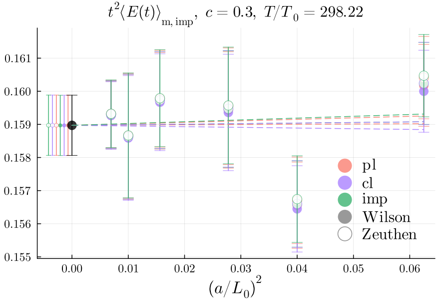

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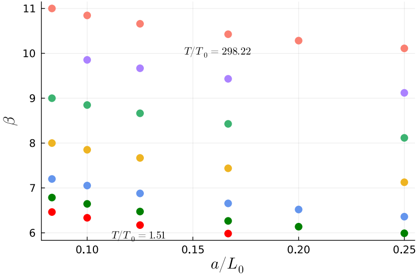

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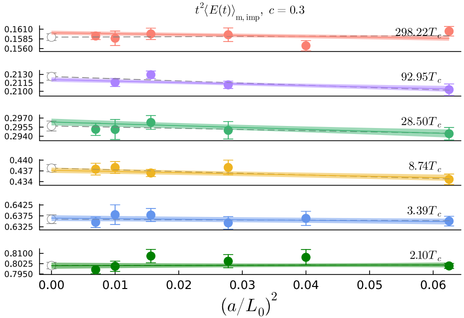
- Mild discretization effects (IR scales)
- Controlled extrapolations



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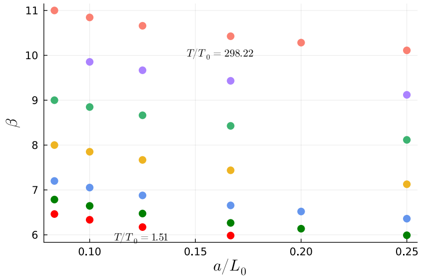

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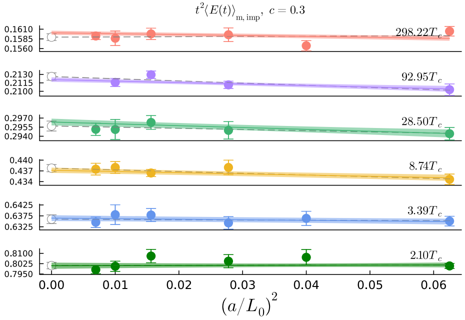

- Start exploring global fits (preliminary^2)

$$\frac{\mathcal{N}^{\rm latt.}_{\rm TBC}(c,a)}{t^2\langle E(t)\rangle} - \frac{1}{g^2} = \sum_{n=0} A_n g^{2n} + (a/L_0)^2\left[\sum_{k=1} B_k g^{2k}\right]$$

- TL imp. removes B_0

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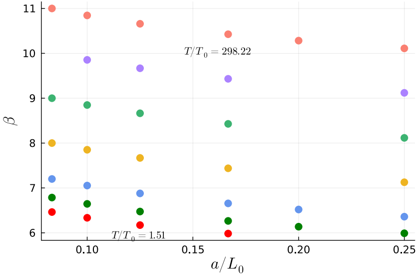

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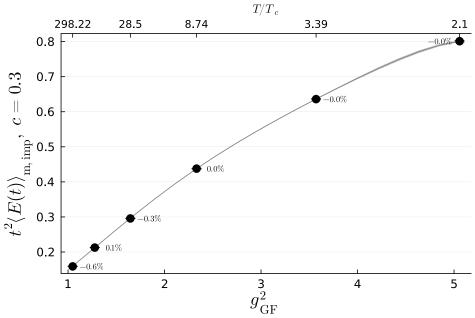

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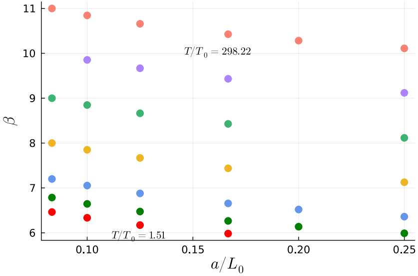

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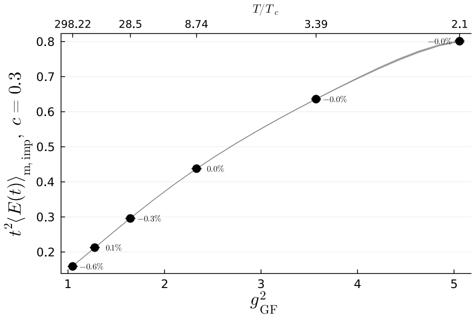
- For now take individual continuum limits at each T



Tuning the 3D theory (PRELIMINARY)

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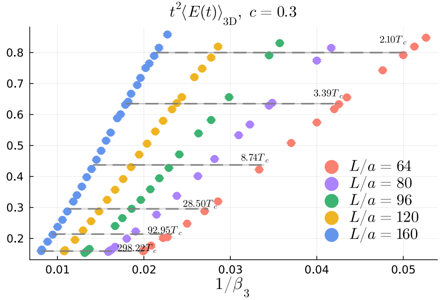
- Super-renormalizability $t^2 \langle E(t) \rangle \big|_{\mathrm{3D}} = \# a g_3^2$
- Good control over the 3D tuning



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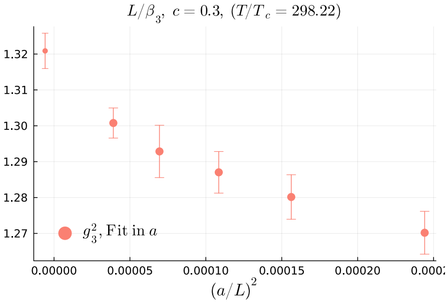

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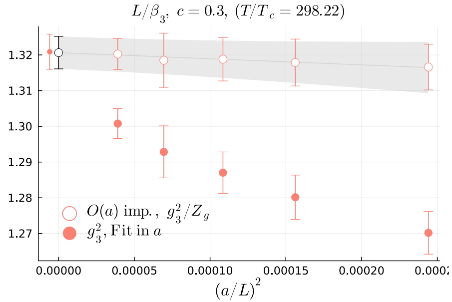
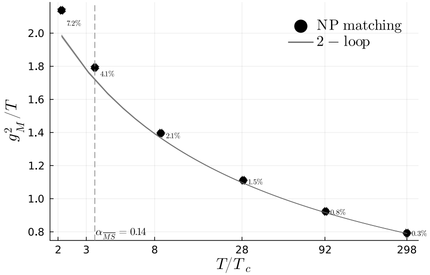
- $g_3^2 \longrightarrow g_3^2 / Z_g(g_3)$
  - $\mathcal{O}(a)$ improvement works
  - Clean extrapolations in a^2



Matching results (PRELIMINARY)



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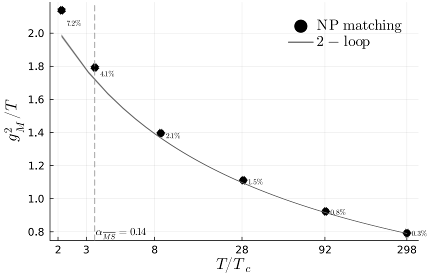

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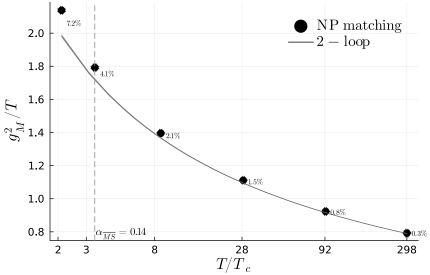

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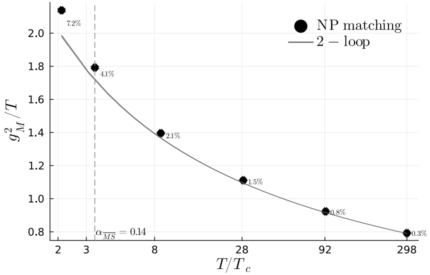

Fit w/ perturbative parametrization

$$g_M^2 = g_E^2\left(1 + c_1 K + c_2 K^2 + \mathcal{O}\left(K^3\right)\right)$$

$$K = 3g_E^2/\pi m_E$$

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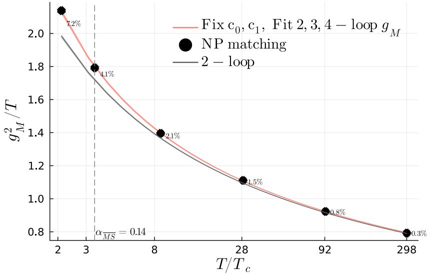

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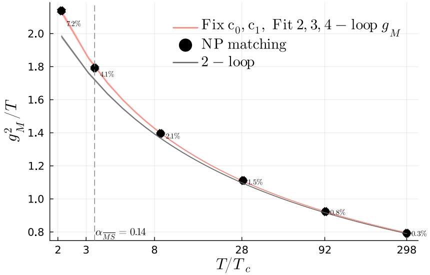

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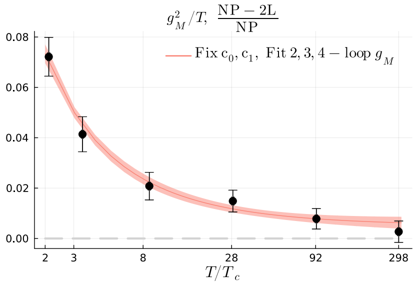

4D predictions

4D predictions – Spatial string tension (PRELIMINARY)

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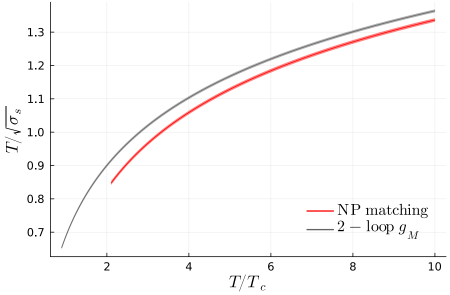

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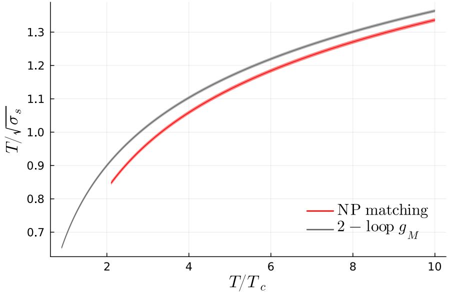

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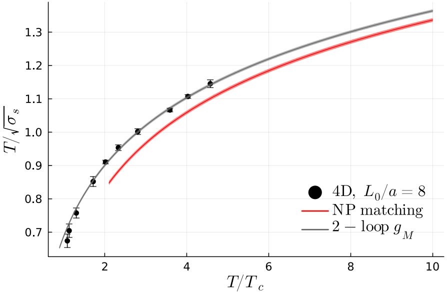

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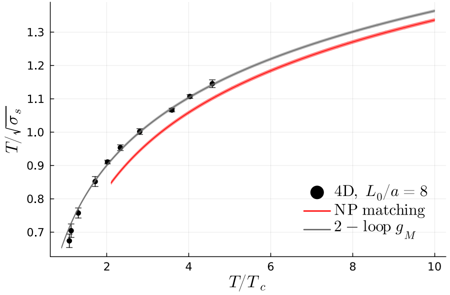

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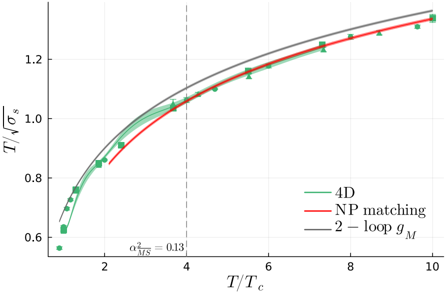

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- $\mathcal{O}(a)$ improvement m_G/g_M^2, $\sqrt{\sigma_s}/g_M^2$
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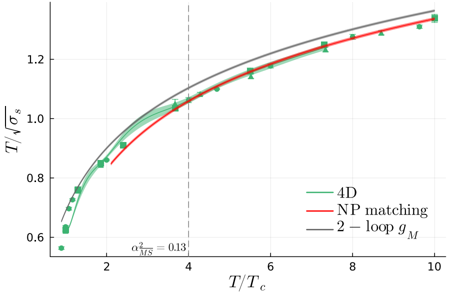

Status

- Non-perturbative study of dimensional-reduction needed
  - Difficult to improve PT
  - Problem overlap: applicability of EFT vs PT
- First Non-Perturbative matching of MQCD
  - $T/T_c \sim 1.5, \ldots, 300$
  - Gradient Flow + Twisted FV – good control of lattice systematics
- (3D) $\times$ (NP matching) reproduce 4D continuum results $T/T_c \gtrsim 4$
  - Requires the comparison for a larger range in T
  - Above $4T_c$ – statistically significant NP effects below $\sim 4\%$
  - Are higher dim. operators required below $\sim 4T_c$?
- Understand the reliability of the EFT & the applicability of the PT matching
  - Lack of 4D continuum results

Future plans

- MQCD
  - Statistics
  - Study systematics: c, discretizations, L/L_0
  - Continuum matching – control over mistuning errors
  - Precise computation of 4D quantities (screening masses, string tension, r_c)
  - Matching QCD
- EQCD
  - Working code
  - Tuning m_E, λ_E?
  - $\mathcal{O}(a)$ improvement?
  - Phase space
- Higher order operators (?)

Matching 3D – 4D

4D at $T \neq 0$

- QCD static physics
  - spatial string tension – σ_s
  - screening masses – m_G
  - ...

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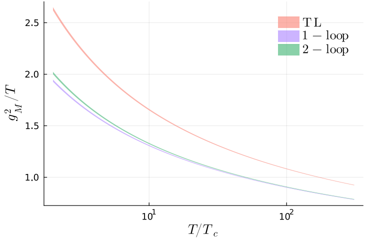
Matching only known perturbatively

[Giovannangeli 2006; Ghisoiu, Moller, and Schroder 2015 ...]

$$g_E^2(T) = T\left(g^2 + \cdots + n_2 g^8 + \dots\right)$$
$$m_E^2(T) = T^2\left(n_3 g^2 + \cdots + n_5 g^6 + \dots\right)$$
$$g_M^2(T) = g_E^2\left(1 + n_6 \frac{g_E^2}{m_E} + n_7 \frac{g_E^4}{m_E^2} + \dots\right)$$



Matching MQCD

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- m_G, $\sqrt{\sigma_s}$ – not so easy to compute
- 'wasting' a prediction of the theory

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  - Define a flow scale $\sqrt{8t^*}T = c \longrightarrow t^*(T)$

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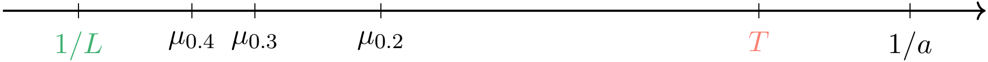
- Window problem:

$$\overbrace{1 \ll}^{\text{cutoff effects}} \underbrace{L_0/a \ll \sqrt{8t^*}/a}_{\text{matching condition}} \overbrace{\ll L/a}^{\text{FV effects}}$$

Scales – Finite volume matching



$$L = RL_0 \qquad\qquad \mu_c = \frac{1}{\sqrt{8t_c}} = \frac{1}{cL} > \frac{1}{L}$$

$$\begin{aligned} t^2 \langle E(t) \rangle_{4\mathrm{D}} &= t^2 \langle E(t) \rangle_{3\mathrm{D}} + \mathcal{O}\left(\frac{1}{\sqrt{8t}T}\right) + \mathcal{O}\left(\frac{1}{LT}\right) \\ &= t^2 \langle E(t) \rangle_{3\mathrm{D}} + \mathcal{O}\left(\frac{1}{cR}\right) + \mathcal{O}\left(\frac{1}{R}\right) \end{aligned}$$

Scales – Finite volume matching

- Cutoff effects:

$$T \ll \frac{1}{a} \longrightarrow 1 \ll \frac{L_0}{a}$$

- Matching condition:

$$\mu_c \ll gT \longrightarrow \frac{1}{g} \ll cR$$

  Here g is the 4D coupling at the scale used for the matching, μ_c. If c defines an IR scale, then g will be $\mathcal{O}(1)$ and the condition is easier to satisfy.

- Spatial volume:

$$1/L \ll 1/L_0 \longrightarrow R \gg 1$$

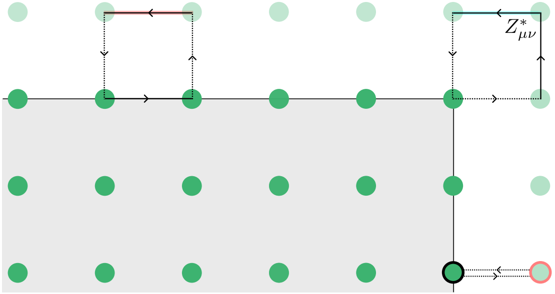
  Although the condition $cR \gg 1/g$ is stronger than $R \gg 1$ in the sense of the matching, the corrections have different coefficients.

Twisted Boundary Conditions on the lattice

$$U_\mu(x + L\hat{\nu}) = \Omega_\nu(x) U_\mu(x) \Omega^\dagger_\nu(x + \hat{\mu})$$

Considering the space-independent twist matrices

$$\Omega_\mu(x) = \Gamma_\mu \longrightarrow \Gamma_\mu \Gamma_\nu = Z_{\mu\nu} \Gamma_\nu \Gamma_\mu$$



The lattice Wilson action is then modified to

$$S_{\text{Wilson}} \propto \sum_x \sum_{\mu \neq \nu} \left(N - Z^*_{\mu\nu} \operatorname{Tr} P_{\mu\nu}(x) \right). \tag{1}$$

Tree-level norms – TBC

Continuum Infinite volume norm

$$N_\infty^{\mathrm{4D,m}} = \frac{3\times 8}{(16\pi t)^2}, \qquad N_\infty^{4D} = \frac{1}{2}\frac{N^2-1}{(8\pi t)^{D/2}}(D-1) = \frac{3}{16(\pi t)^2}, \qquad N_\infty^{3D} = \frac{8}{(8\pi)^{3/2}}$$

Continuum/Lattice finite volume $\sqrt{8t} = cL$ w/ finite temperature $R = L/L_0$

$$N_T^{4D}(c,R) = \frac{t^2\langle E(t)\rangle}{g_0^2} \qquad\qquad N_T^{3D}(c) \equiv \frac{t^{3/2}\langle E(t)\rangle}{g_0^2}$$

$$N_T^{4D}(c,R,a/L) \qquad\qquad N_T^{3D}(c,a/L)$$

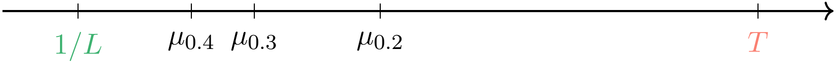
Static QCD quantities

- Spatial string tension:
  - extracted from the asymptotic behaviour of large spatial Wilson loops, which show an area law behaviour even above T_c

$$\langle W(R,S)\rangle \sim e^{-\sigma_s RS}$$

  - correlator of spatial Polyakov loop

Possible Systematics (PRELIMINARY)



Matching condition: $\mu_c = \frac{1}{\sqrt{8t_c}} \ll gT \longrightarrow \frac{1}{g} \ll cR$

$$m_G(c_i) = \# g_M^2(c_i) + \mathcal{O}(1/T) \qquad \longrightarrow \qquad \frac{m_G(c_i)}{m_G(c_j)} = \frac{g_M^2(c_i)}{g_M^2(c_j)} = 1 + \mathcal{O}(1/T)$$

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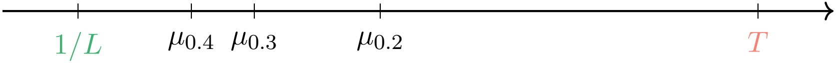

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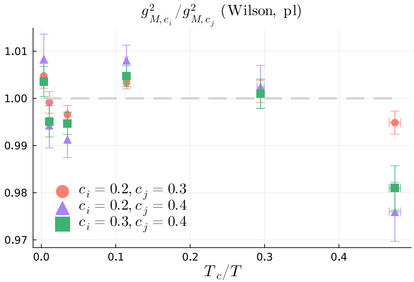

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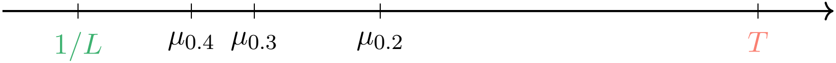

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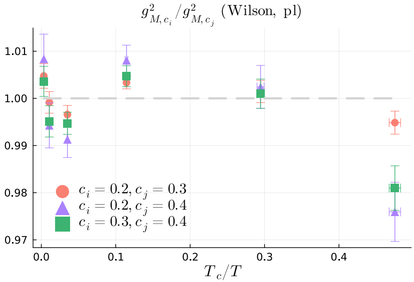

- Mild dependence on c
  - cR controls matching
  - Explore dependence in R

NP vs PT (PRELIMINARY^2)

$$g_M^2 = g_E^2\Big(1 + c_1 K + c_2 K^2 + \mathcal{O}\big(K^3\big)\Big), \quad K = \frac{3g_E^2}{\pi m_E}, \quad \frac{g_M^2}{T} = g^2\Big(1 + \#g + \mathcal{O}\big(g^2\big)\Big)$$

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- Fix n PT coeffs. – Fit $> n$ loops
  - ~~Fix c_{0-2}, Fit 3L: $\chi^2/\chi^2_{\text{exp.}} = 15.3$~~
  - Fix c_{0-2}, Fit (3+4)L: $\chi^2/\chi^2_{\text{exp.}} = 2.09$
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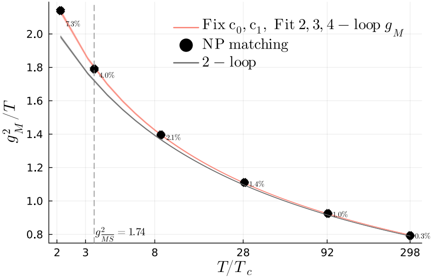

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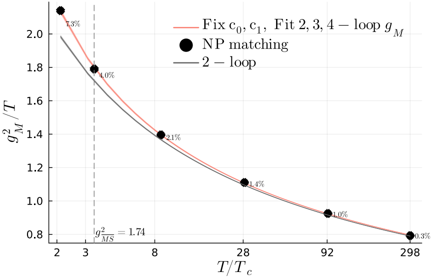

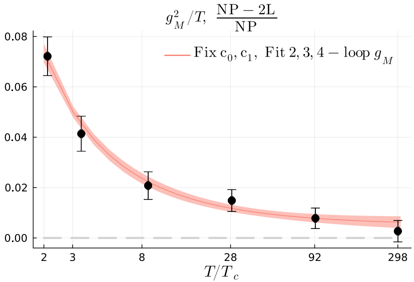

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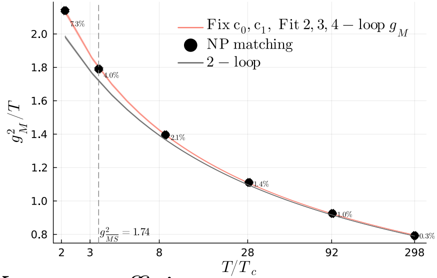
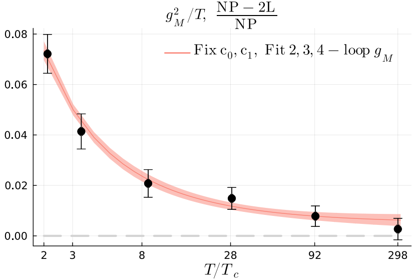




- Known coefficients:
  - $c_0 = 1, \; c_1 = 0.00208, \; c_2 = 0.0037$
- Estimated coefficients:
  - $c_3 = -0.0338(46), \; c_4 = 0.0451(37)$
  - $c_2 = 0.072(32), \; c_3 = -0.164(57)$
    $c_4 = 0.100(24)$

4D LCP

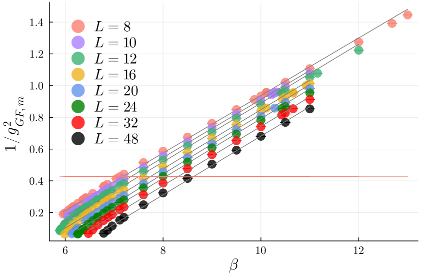
$$\bar{g}^2_{\mathrm{m,e}}(\mu) = t^2 \hat{\mathcal{N}}^{-1}_{\mathrm{e,m}}(c, a/L) \frac{\langle E_{\mathrm{m,e}}(t,x)\hat{\delta}_Q \rangle}{\langle \hat{\delta}_Q \rangle}\Bigg|_{\mu=1/\sqrt{8t},\, \sqrt{8t}=cL,\, x_0=T/2} \qquad (c = 0.3)\,. \tag{3.8}$$

To define the topological charge on the lattice we use the clover discretization of the flow strength tensor [11]:

$$Q = -\frac{1}{16\pi^2} \sum_x \epsilon_{\mu\nu\rho\sigma}\, \mathrm{tr}\{G^{\mathrm{cl}}_{\mu\nu}(t,x) G^{\mathrm{cl}}_{\rho\sigma}(t,x)\}\,, \tag{3.9}$$

measured at flow time $\sqrt{8t} = cL$. For the discretization of the flow equations used for Q we will always use the one employed for the discretization of $E_{\mathrm{e/m}}$ entering the coupling definition. In addition, since on the lattice Q is not integer-valued, we replace the Dirac δ-function with:

$$\hat{\delta}_Q = \begin{cases} 1\,, & \text{if } |Q| < 0.5, \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{3.10}$$



Exact Matching condition

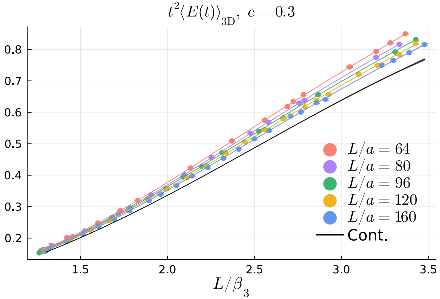
4D:

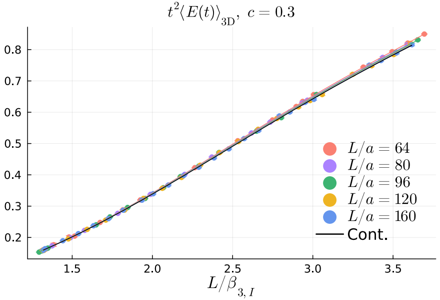
$$\left.\frac{N_T^{\mathrm{m}}(c,R)}{N_T^{\mathrm{m}}(c,R,a/L)}t^2\;\frac{\left\langle E_{\mathrm{m}}(t)\hat{\delta}_Q\right\rangle}{\left\langle\hat{\delta}_Q\right\rangle}\right|_{\sqrt{8t}=cL}$$

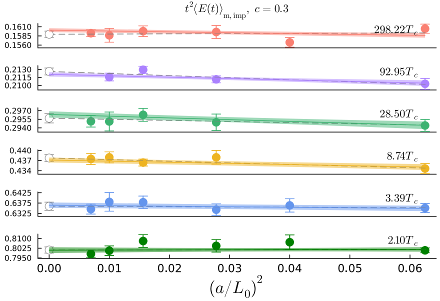
3D:

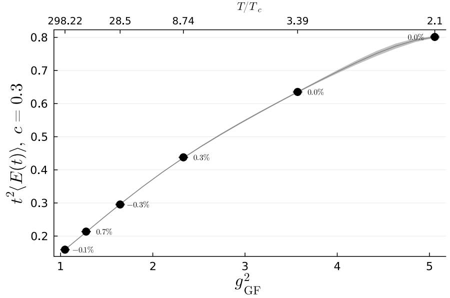
$$\left.\frac{N_T^{\mathrm{3D}}(c)}{N_T^{\mathrm{3D}}(c,a/L)}t^2\;\langle E_{\mathrm{m}}(t)\rangle\right|_{\sqrt{8t}=cL}$$

Matching in the continuum







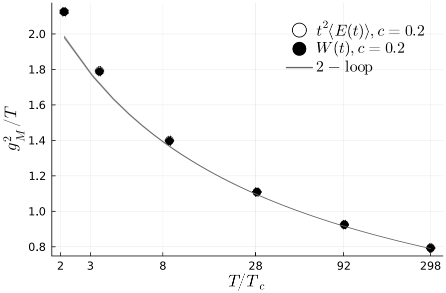


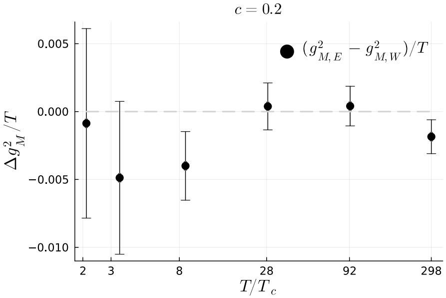
Matching with other observables

$$t^2 \langle E(t) \rangle \longrightarrow g^2_{M,E}$$

Consider also:

$$W(t) = t \frac{\mathrm{d}}{\mathrm{d}t} t^2 \langle E(t) \rangle \longrightarrow g^2_{M,W}$$





3D observables

$$\left.\frac{\sqrt{\sigma_s}}{T}\right|_{\text{4D}} = \underbrace{\frac{\sqrt{\sigma_s}}{g_M^2}}_{\text{3D obs.}} \times \overbrace{\frac{g_M^2}{T}}^{\text{matching}} + \mathcal{O}(1/T)$$

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3D lattice results

[Athenodorou and Teper 2017]

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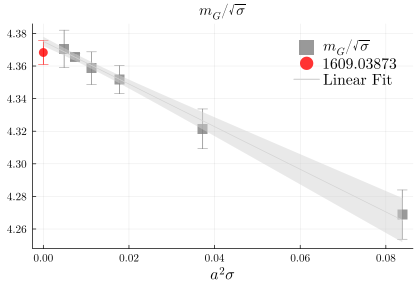


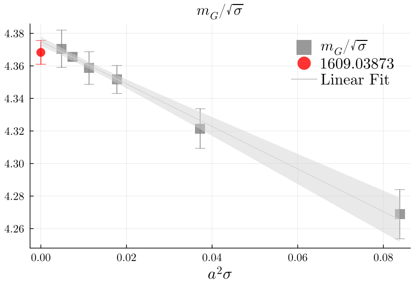
3D lattice results

[Athenodorou and Teper 2017]

- $m_G/\sqrt{\sigma_s}$ free of $\mathcal{O}(a)$

3D observables

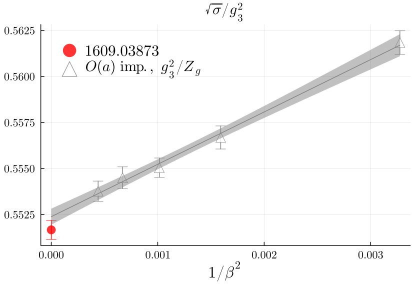
$$\left.\frac{\sqrt{\sigma_s}}{T}\right|_{4\mathrm{D}} = \underbrace{\frac{\sqrt{\sigma_s}}{g_M^2}}_{\text{3D obs.}} \times \overbrace{\frac{g_M^2}{T}}^{\text{matching}} + \mathcal{O}(1/T)$$



3D lattice results

[Athenodorou and Teper 2017]

- $m_G/\sqrt{\sigma_s}$ free of $\mathcal{O}(a)$
- $\mathcal{O}(a)$ improvement $m_G/g_M^2,\ \sqrt{\sigma_s}/g_M^2$
- $\sqrt{\sigma}/g_M^2 = 0.55239(43)$
  - inherently non-perturbative
  - magnetic scale contribution only



3D observables

- Re-analyze 4D continuum limit:

$$\sqrt{\sigma_s}/T = \sum_{n=1}^{3} A_n g^{2n} + (a/L_0)\left[\sum_{k=0}^{2} B_k g^{2k}\right]$$

  - Take g as, e.g., g_{GF}
- Can we see the $1/T$ corrections

$$\langle \mathcal{O} \rangle_{\mathrm{3D}} - \langle \mathcal{O} \rangle_{\mathrm{4D}} \propto \frac{1}{T}$$

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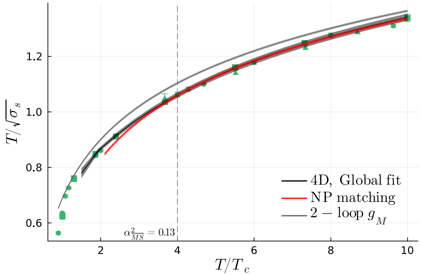

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