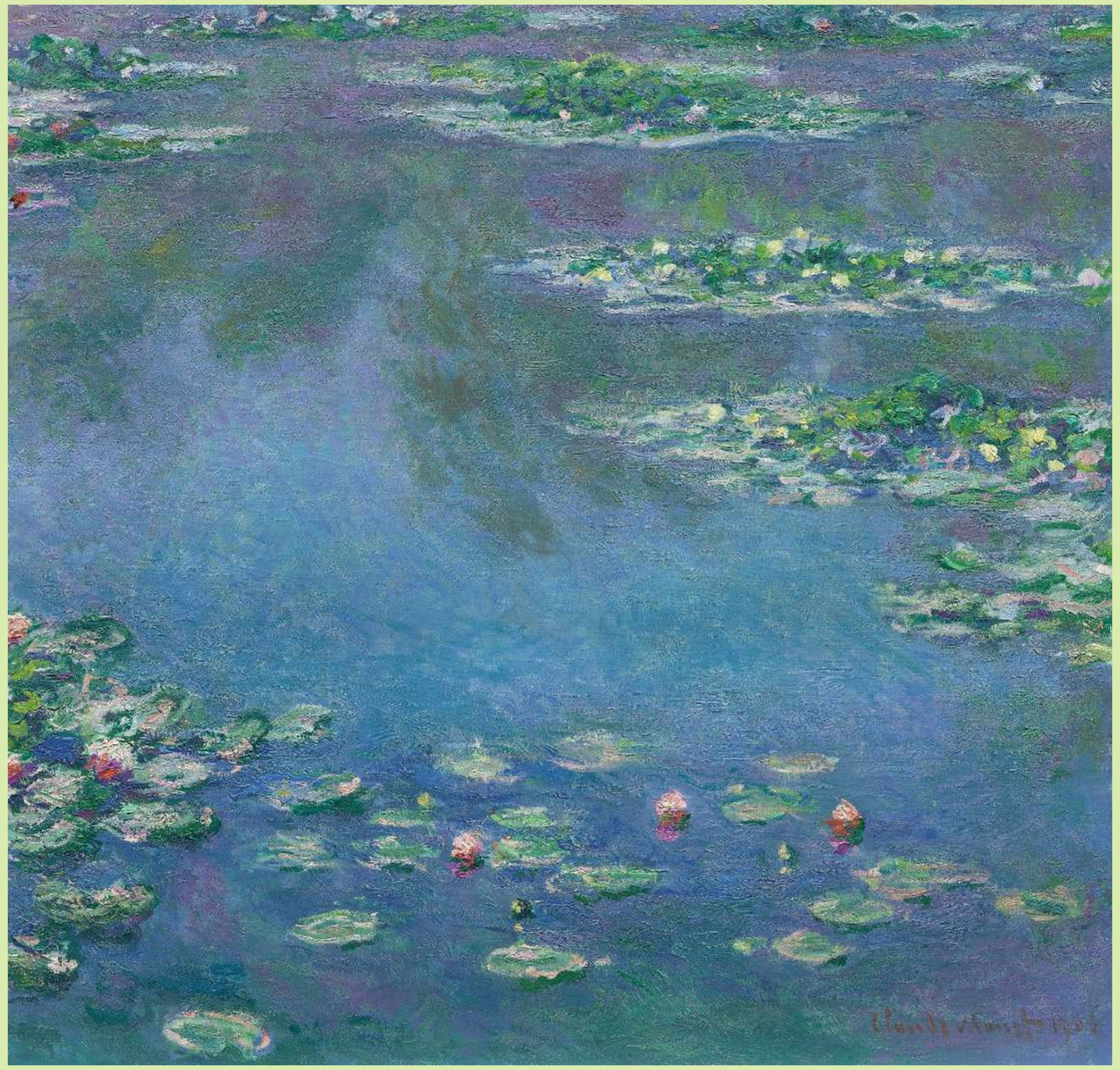
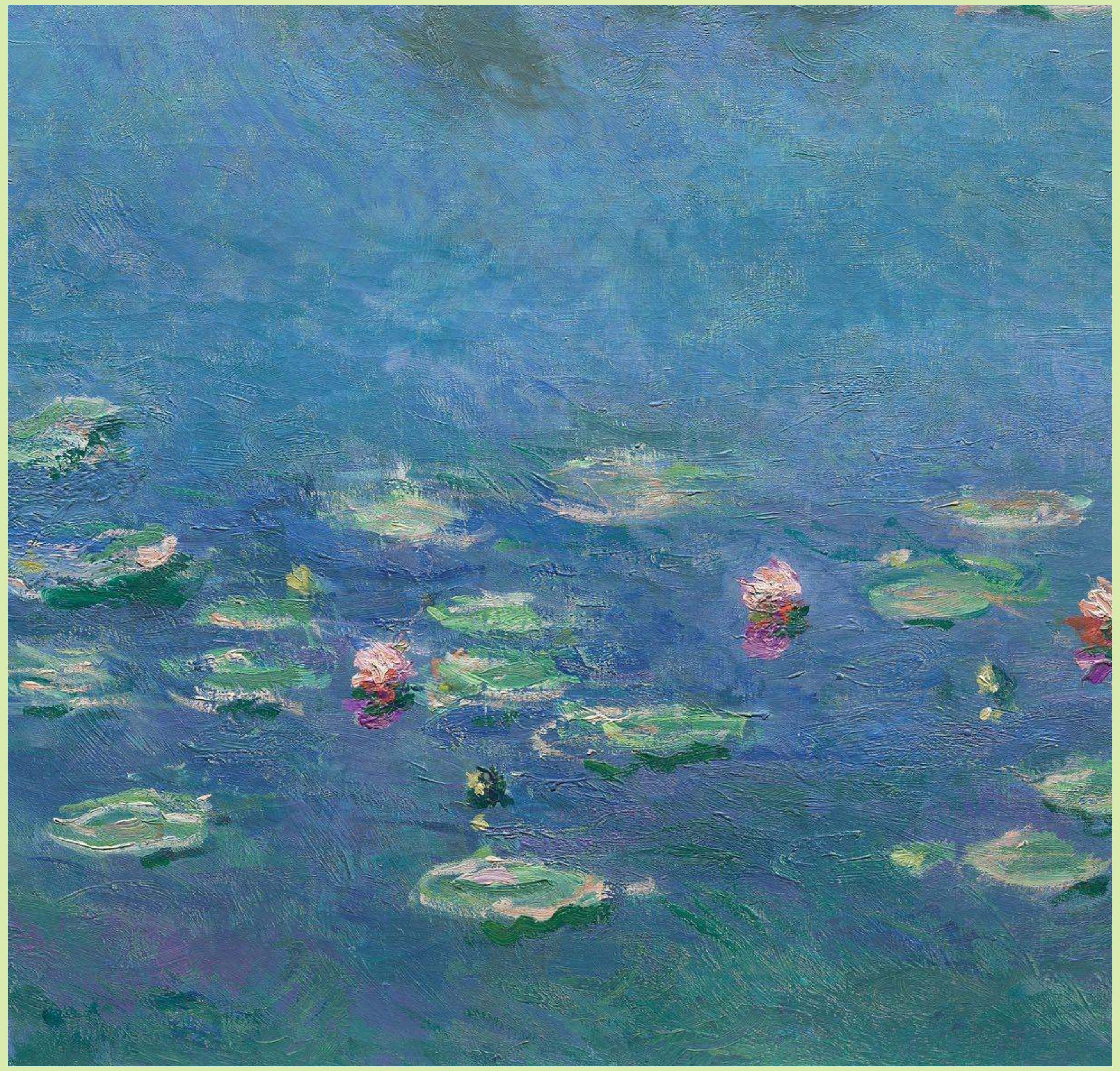
Smoothing properties of the gradient flow at large N





Sofie Martins, University of Graz

15th of May 2026

Standard Model parameters and observables from gradient flow

University of Edinburgh

The project

Work by Pietro Butti, Michele Della Morte, Ben Jäger, me,

J. Tobias Tsang

- Discretisation effects of gradient flows in QCD-like theories on the lattice, arXiv:2603.05155, PoS, 2026
- Comparison of smoothening flows for the topological charge in QCD-like theories, arXiv:2504.10197, Phys. Rev. D, 2025
- Smoothing properties of the Wilson flow and the topological charge, 2501.16043, PoS, 2025

NAWI Graz Natural Sciences

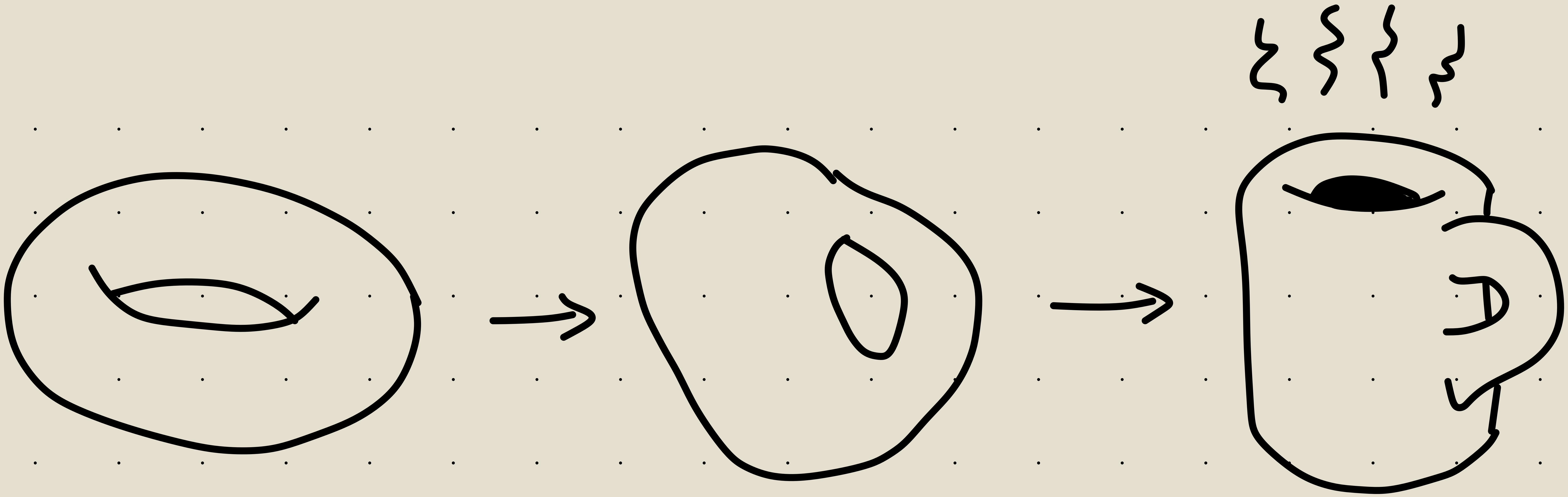
UNI GRAZ

FWF Österreichischer Wissenschaftsfonds

SDU Syddansk Universitet

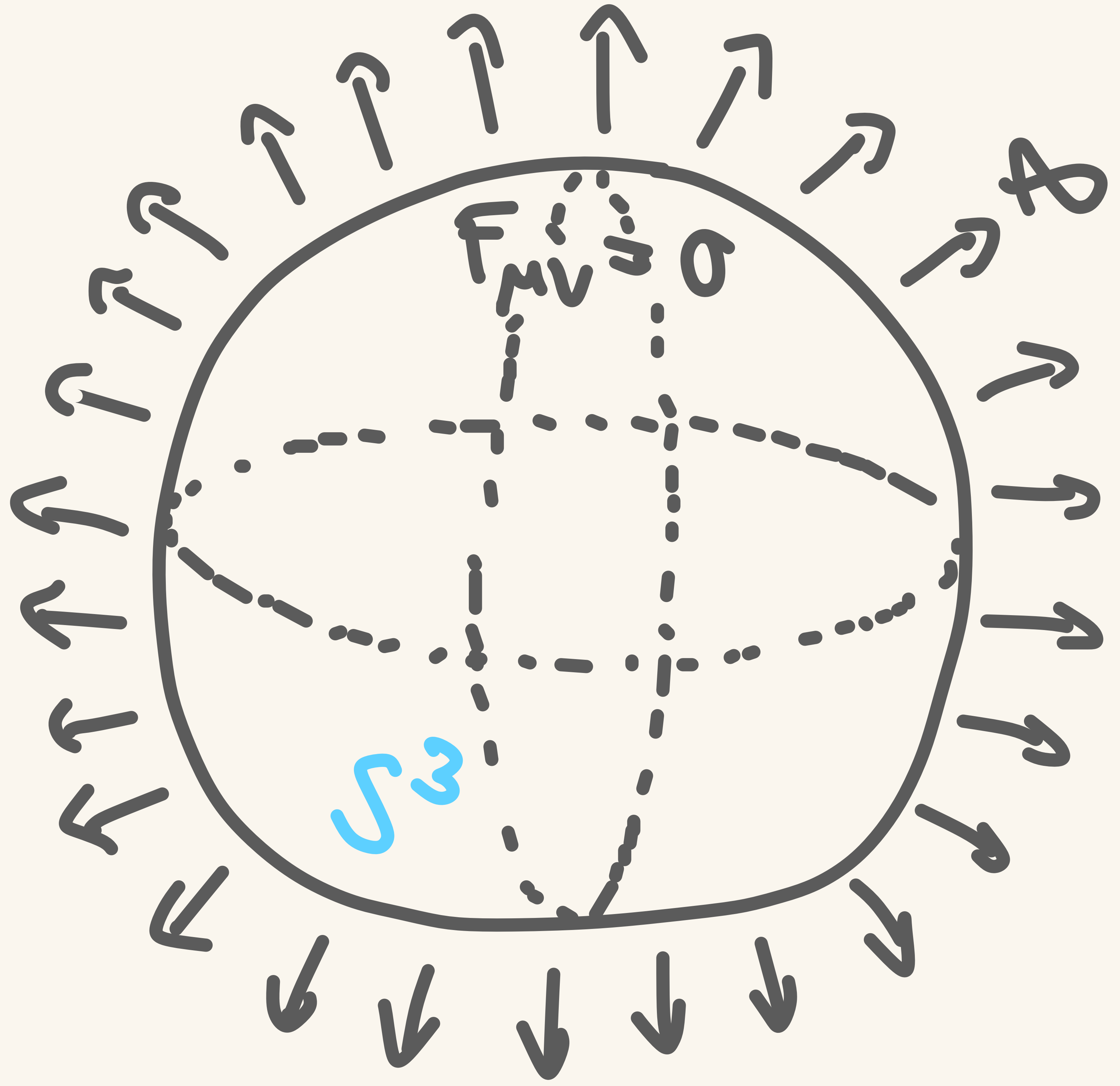
Introduction

The topological charge



What is gauge topology?

$F_{\mu\nu} \to 0$ at spatial and temporal infinity



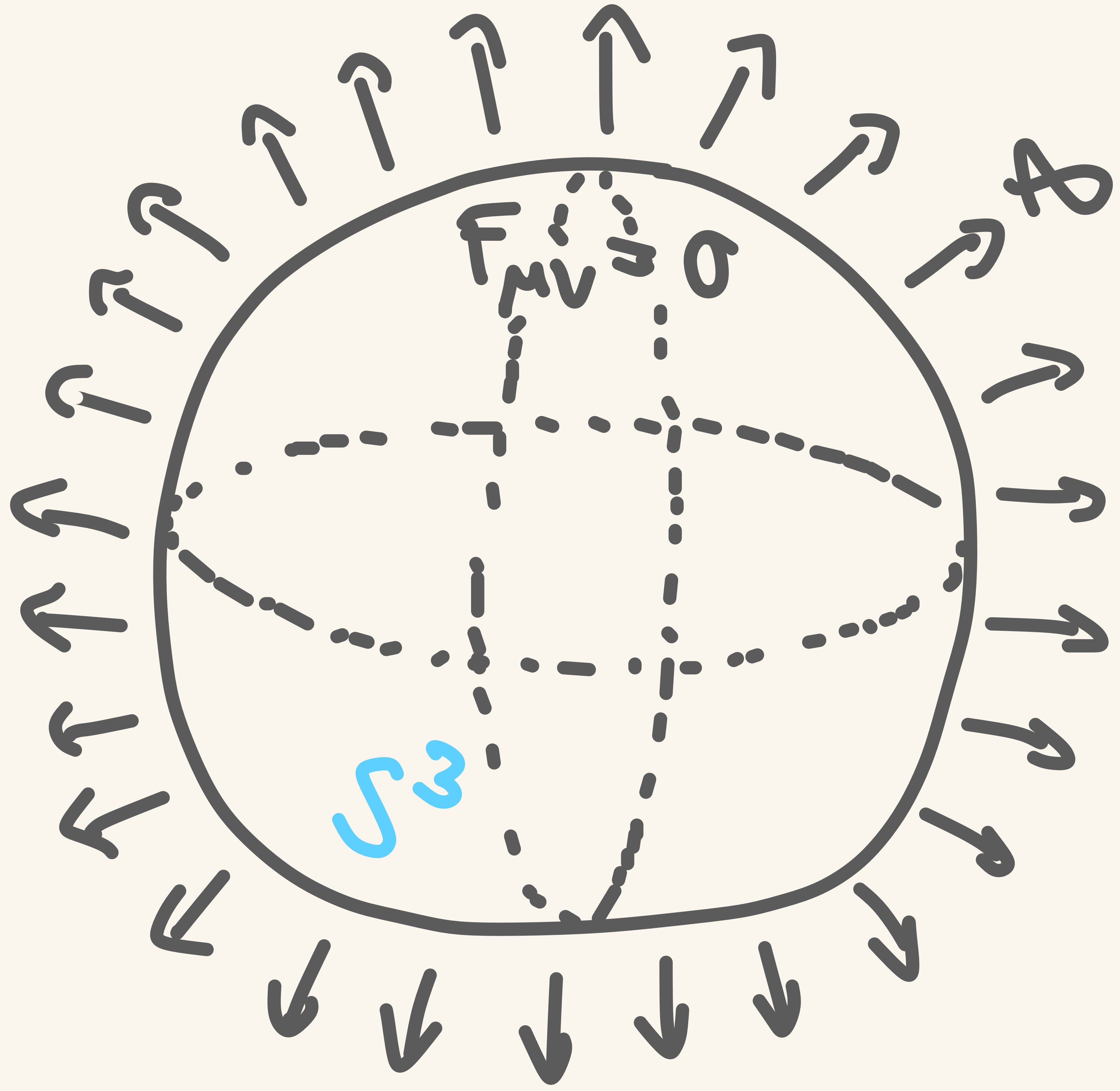
See for example: A. Smilga, Lectures on Quantum Chromodynamics

What is gauge topology?

$F_{\mu\nu} \to 0$ at spatial and temporal infinity

described by pure gauge potential

$$A_\mu(\{\phi_i\}) = -\mathrm{i}\Omega^\dagger(\{\phi_i\})(\partial_\mu \Omega(\{\phi_i\}))$$



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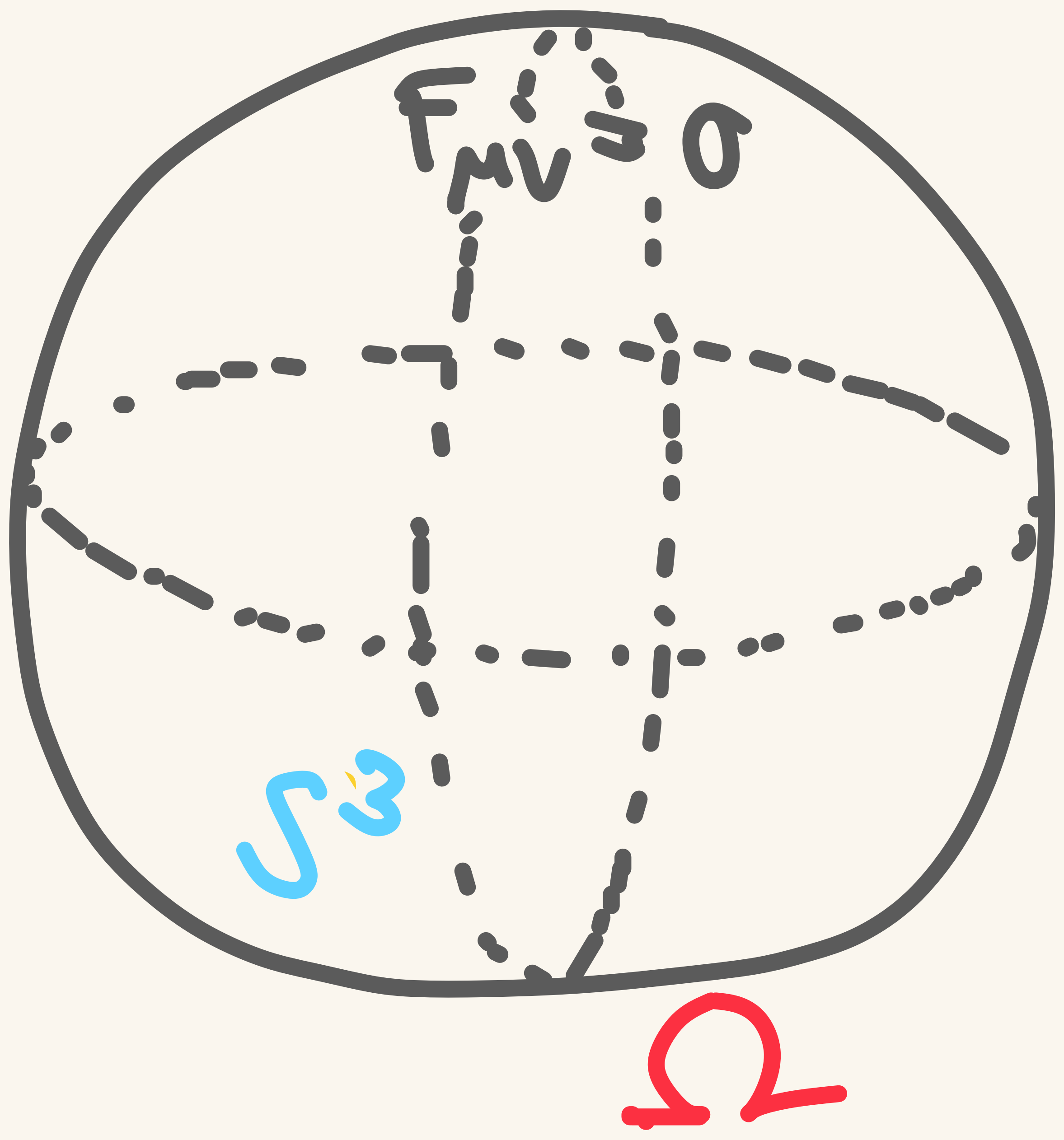
What is gauge topology?

The map

$$\Omega : S^3 \to SU(N)$$



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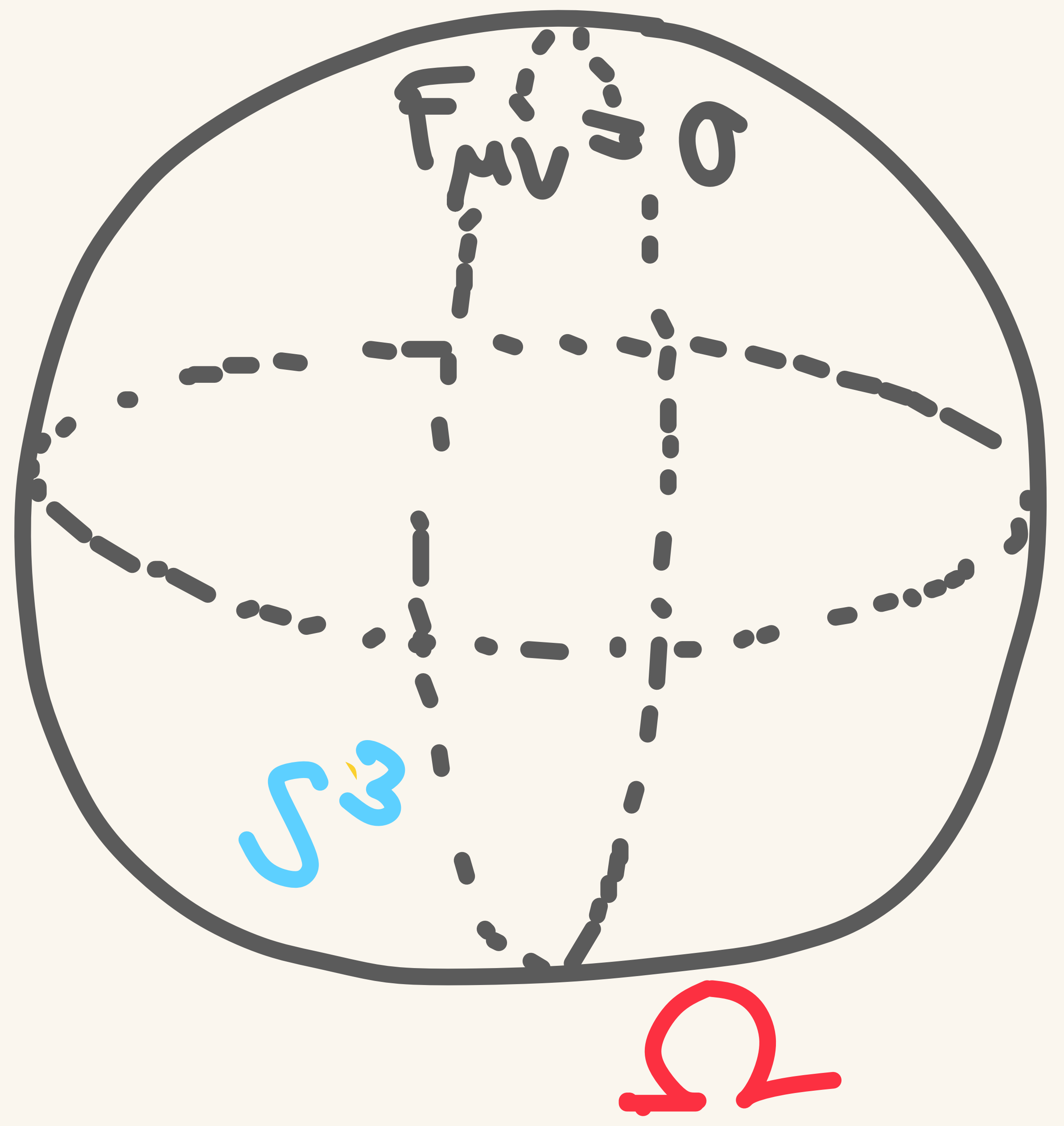

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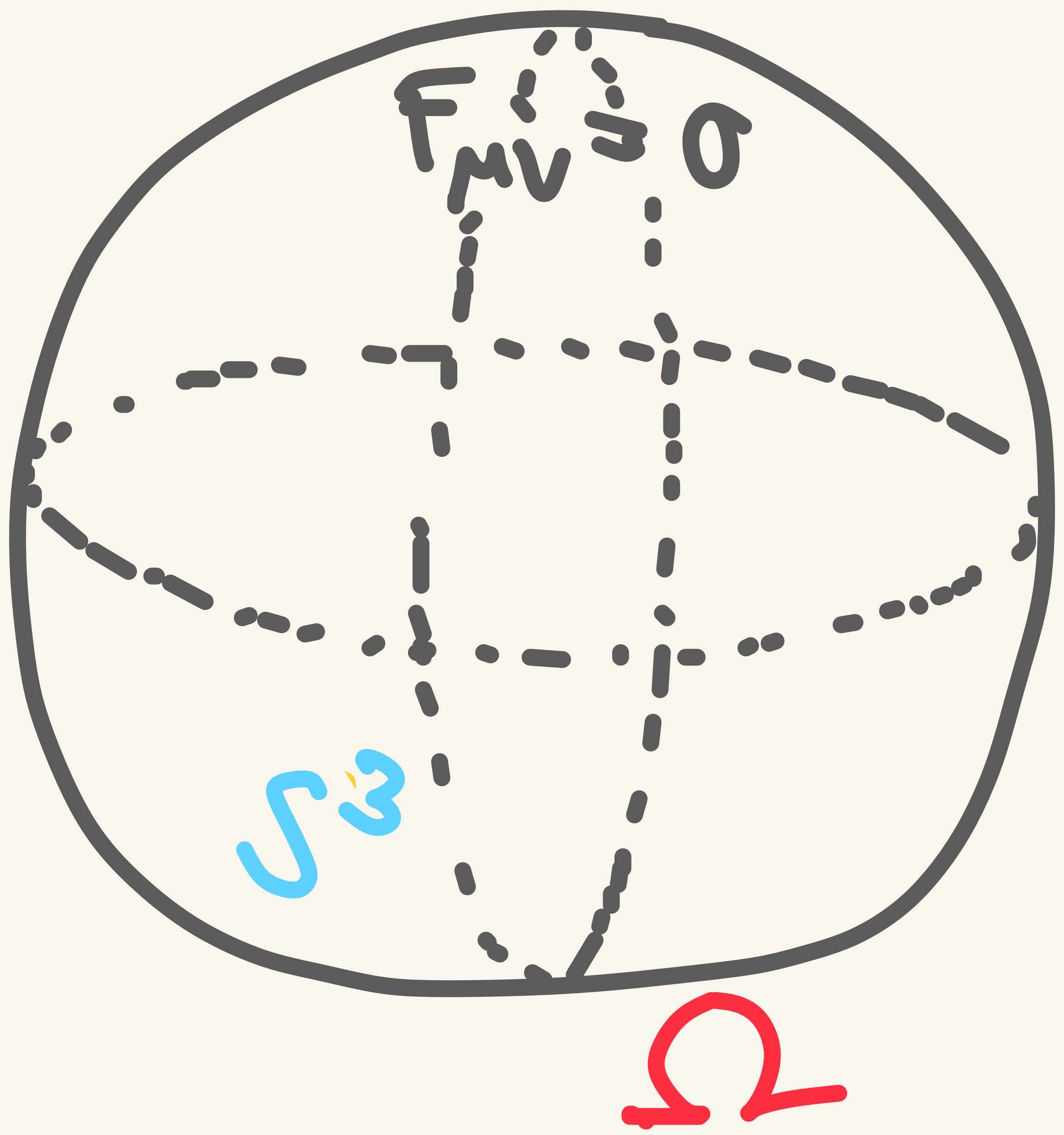

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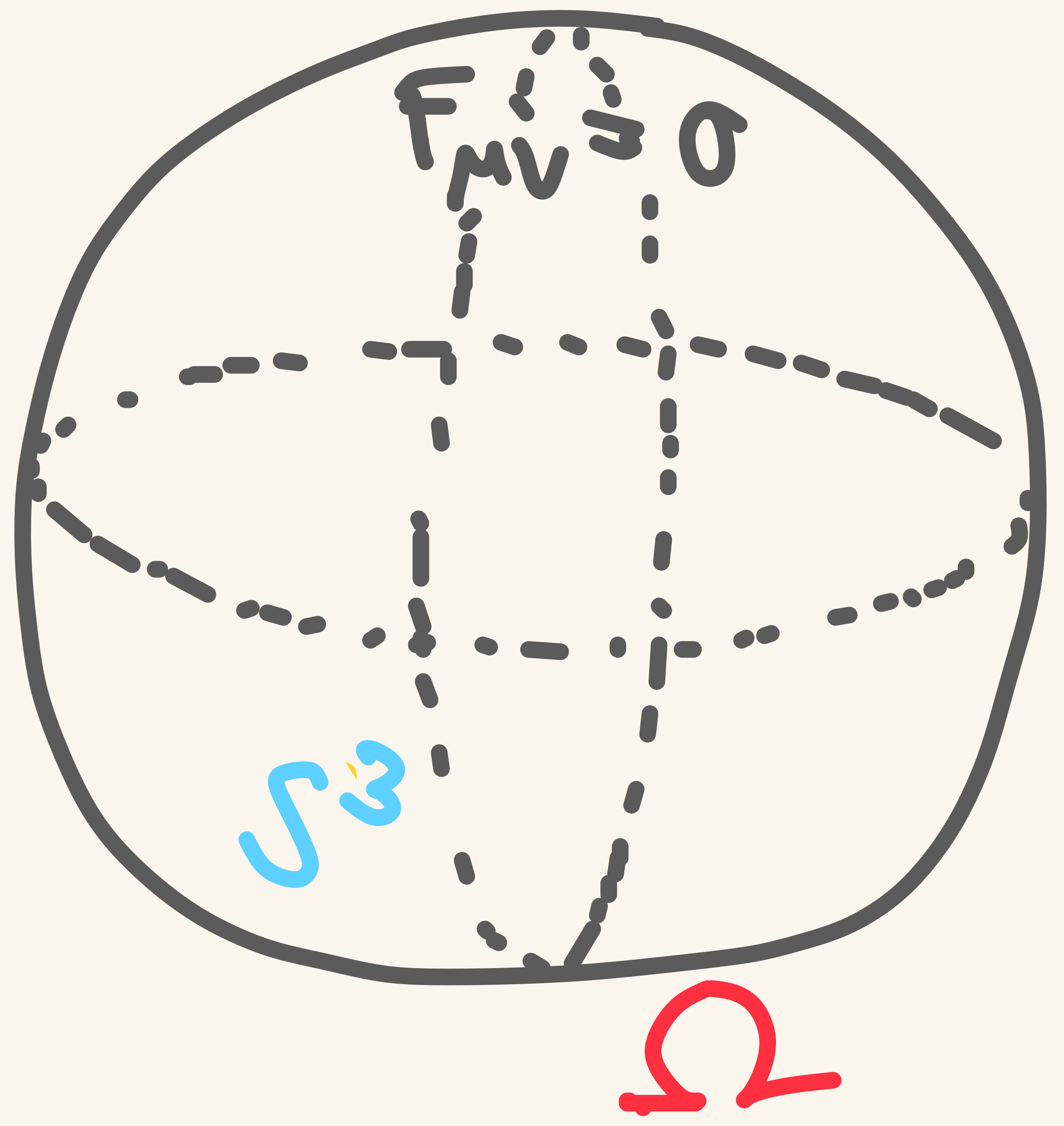

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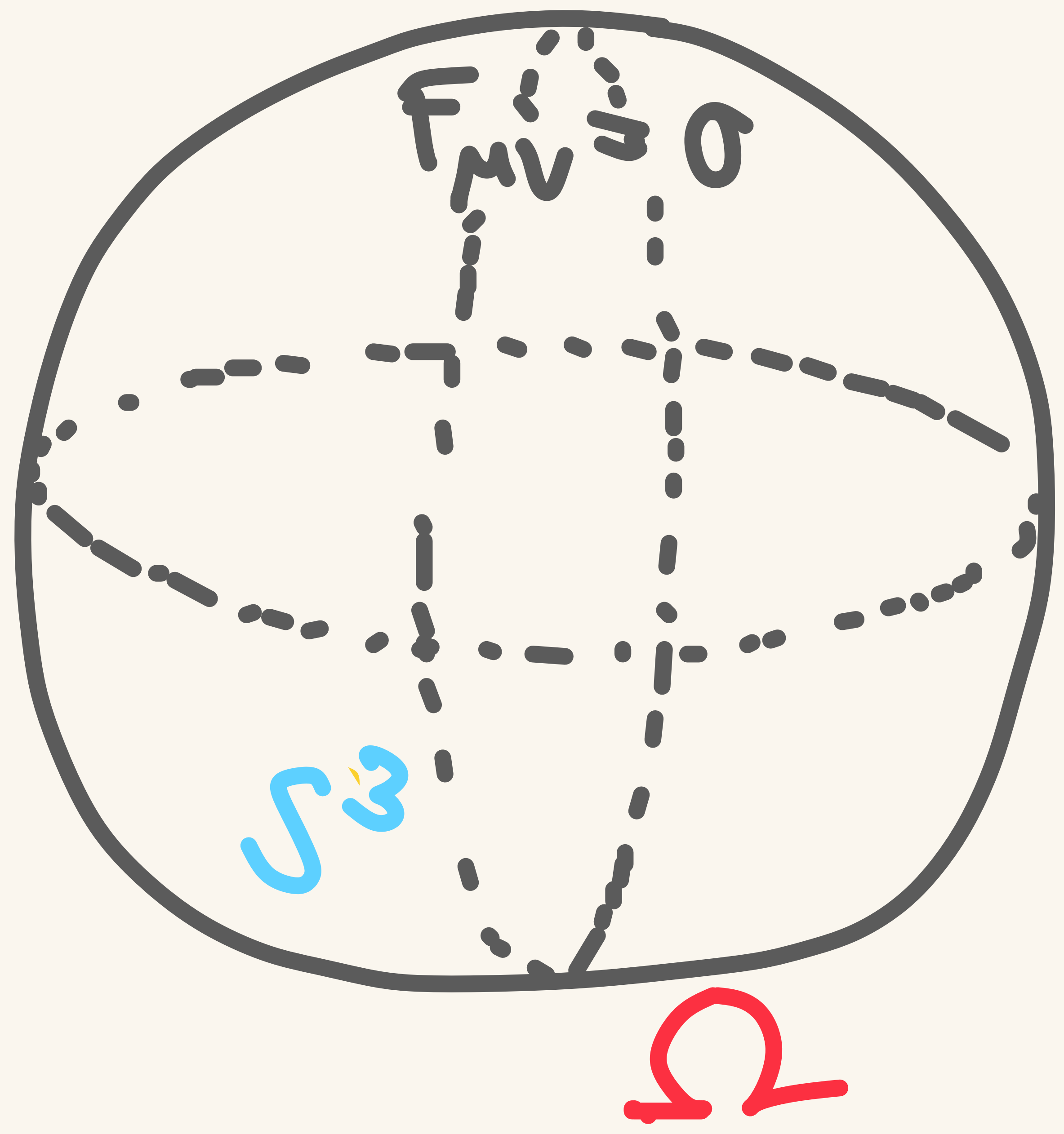


has the topological property **degree of mapping** which

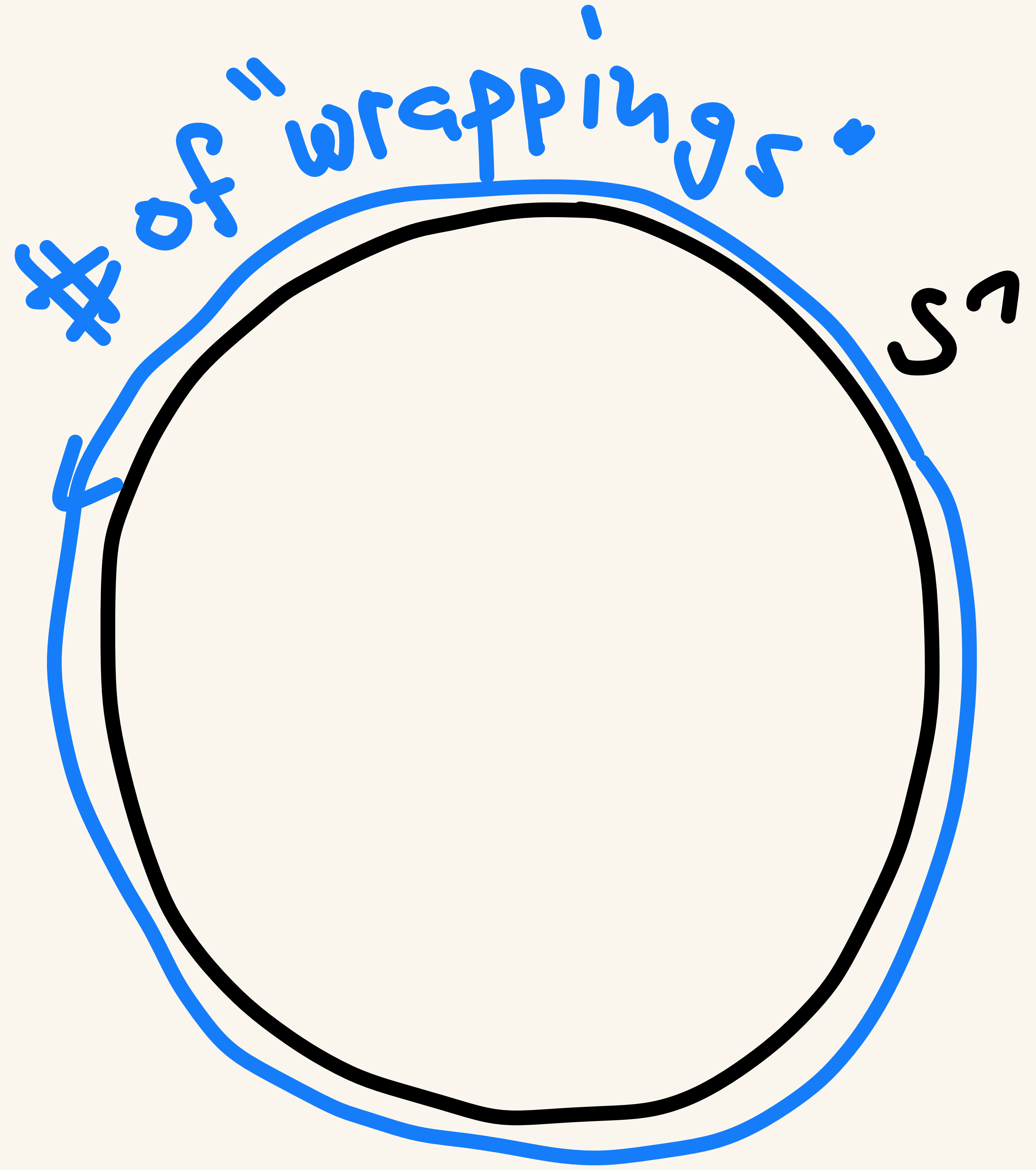
- defines homotopy classes
- integer number classification
- Classifies fields topologically according to their behavior on the edge of the universe

See for example: A. Smilga, Lectures on Quantum Chromodynamics

What is gauge topology? Simple example

1+1D U(1) gauge theory

$$\Omega(\phi) = e^{ia(\phi)}$$



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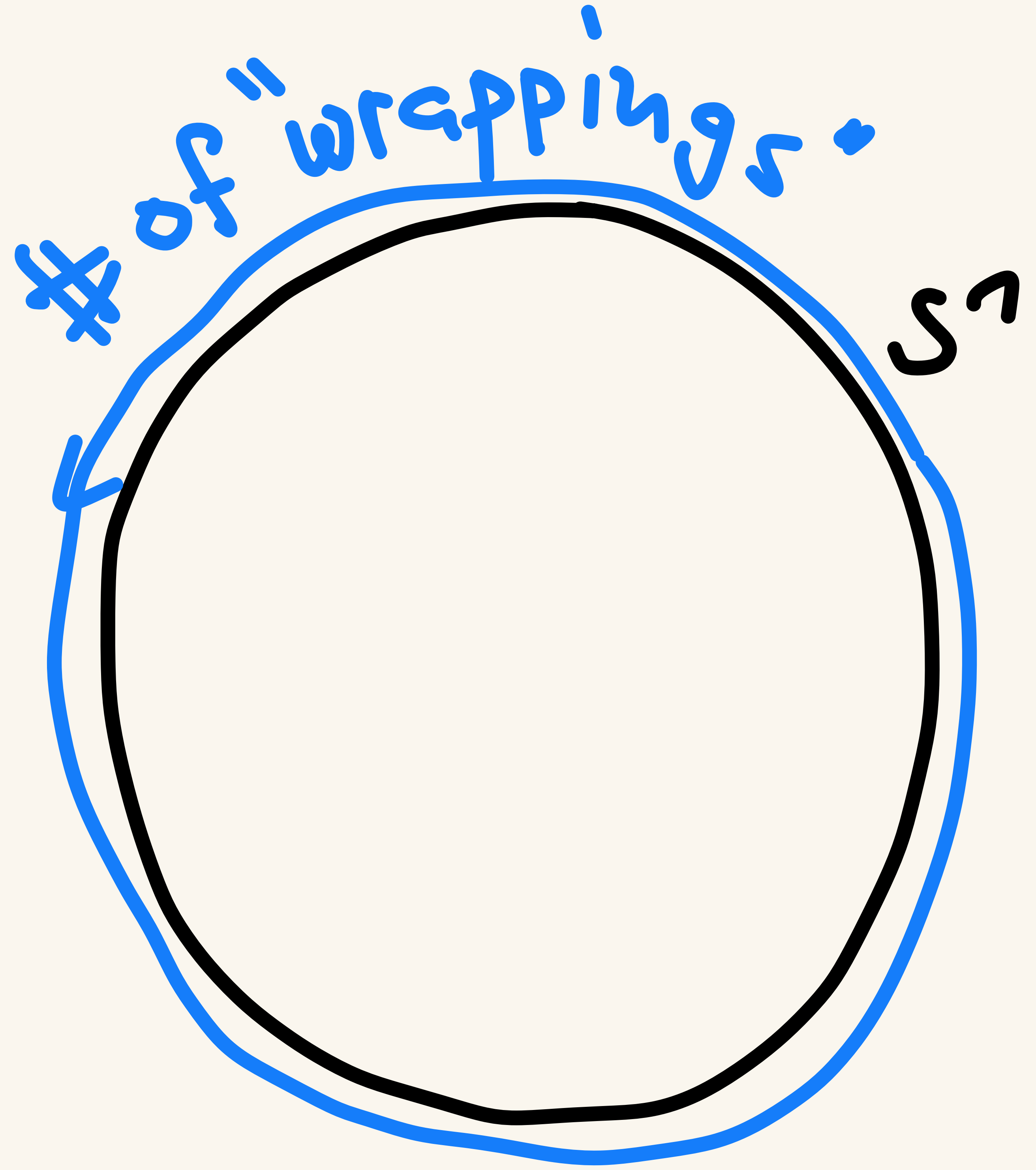
$$a(\phi + 2\pi) = \phi + 2\pi Q$$



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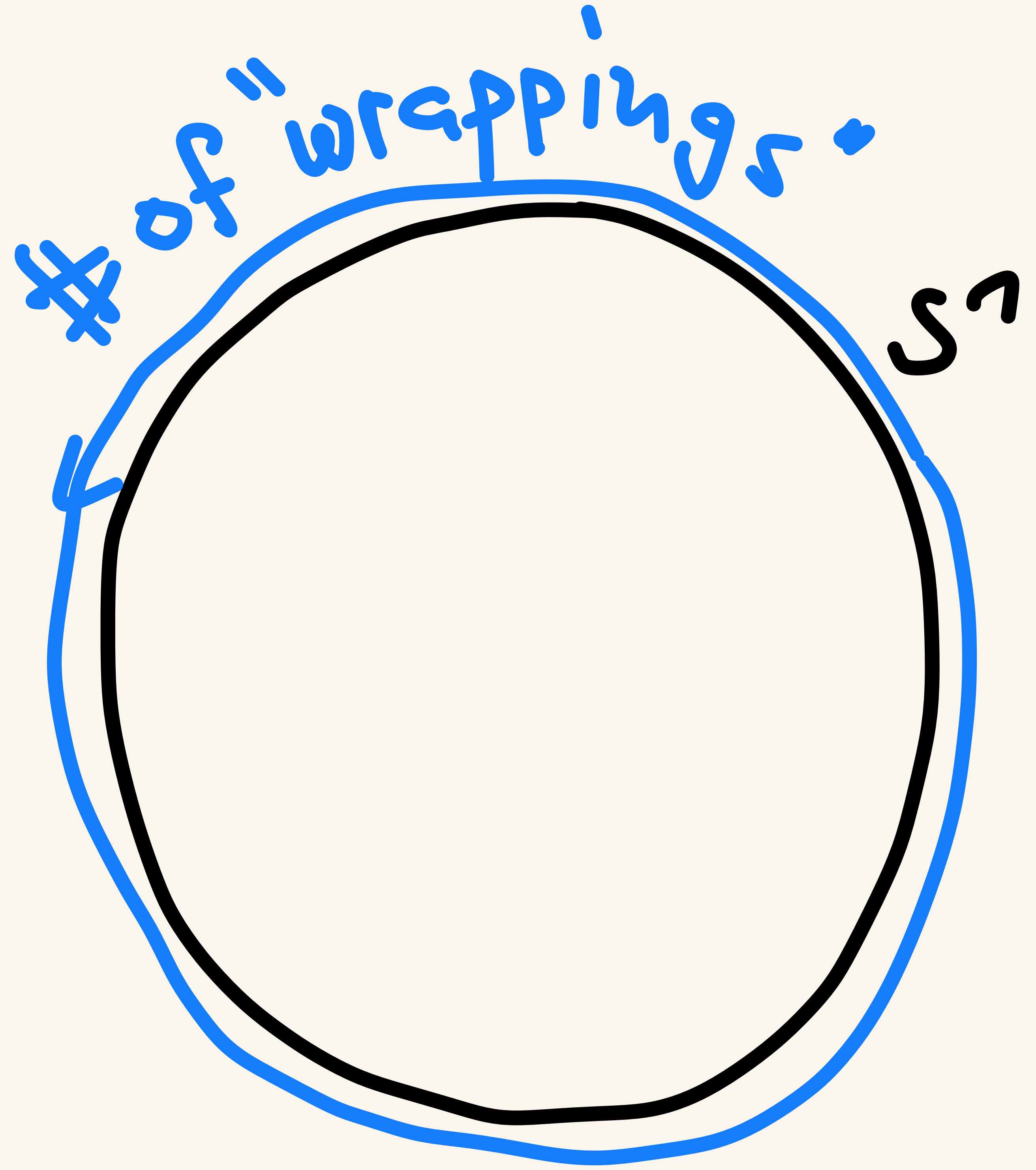
$$a(\phi + 2\pi) = \phi + 2\pi Q$$

which defines a degree of mapping Q

$$Q = \frac{1}{2\pi}\int_{S^1} \mathrm{d}x_\mu A_\mu(\phi) = \frac{1}{2\pi}\int_{S^1} \mathrm{d}\phi \partial_\phi a(\phi)$$



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What is gauge topology? 4D SU(N)

Take the dual field tensor

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}$$

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What is gauge topology? 4D SU(N)

Take the dual field tensor

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}$$

Winding number: Chern number/Pontryagin index

$$Q = \frac{1}{32\pi^2}\int \mathrm{tr}[\tilde{F}F]\mathrm{d}^4x \quad \in \mathbb{Z}$$

See for example: A. Smilga, Lectures on Quantum Chromodynamics

What is gauge topology? Classical solutions

$$S = \frac{1}{2g^2} \int \mathrm{d}^4x \, \mathrm{tr}[F^2] = \frac{1}{4g^2} \int \mathrm{d}^4x \, \mathrm{tr}[(F - \tilde{F})(F - \tilde{F})] + \frac{8\pi^2 Q}{g^2}$$

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Classically solved by self-dual solutions

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Classically solved by self-dual solutions

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Turn out to be extremely localized in space and time: instantons

See for example: A. Smilga, Lectures on Quantum Chromodynamics

Finite-volume definition

No pure gauge potential at infinity

Simply use Chern number naively

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Finite-volume definition

No pure gauge potential at infinity

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But: Atiyah Singer index theorem

relates the mode count of the Dirac operator to the same topological charge

$$\mathrm{Index}(D) = n_+ - n_- = 2T_R Q$$

[Atiyah, Singer, Annals Math., 1968]

Atiyah Singer index theorem vs. boundary conditions

$$Q = \frac{n_+ - n_-}{2T_R}$$

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The torus

With the definition

$$a(\phi + 2\pi) = \phi + 2\pi Q$$

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$$Q = \frac{1}{2\pi}(a(0) - a(2\pi))$$

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Twisted boundary conditions

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$$a(\phi + 2\pi) = \phi + 2\pi Q + \chi$$

Atiyah Singer index theorem vs. boundary conditions

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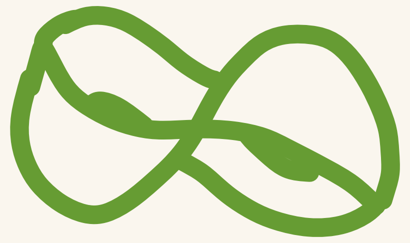

Twisted boundary conditions

With the definition

$$a(\phi + 2\pi) = \phi + 2\pi Q + \chi$$

which allows fractional charges

$$Q = \frac{1}{2\pi}(a(0) - a(2\pi)) - \frac{\chi}{2\pi}$$

So can the topological charges be fractional or not?

The role of the gradient flow

Separation of topological sectors in finite volume

- Not actually separated, except if you limit how smooth configurations have to be

[M. Lüscher, JHEP, 2010]

Separation of topological sectors in finite volume

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- Define *smoothness*

$$h = \max_{V_4,\mu,\nu} \mathrm{Re}\,\mathrm{tr}[1 - P_{\mu\nu}]$$

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$$h = \max_{V_4,\mu,\nu} \mathrm{Re}\,\mathrm{tr}[1 - P_{\mu\nu}]$$

- A configuration between sectors has a high value (not smooth)
- The probability to find such a config decreases with a^6 (topological freezing)

[M. Lüscher, JHEP, 2010]

Gradient flow

$$\dot{B}_\nu = D_\mu G_{\mu\nu} \qquad\qquad B_\mu\,|_{t=0} = A_\mu$$

steepest descend towards stationary configuration

= instantons

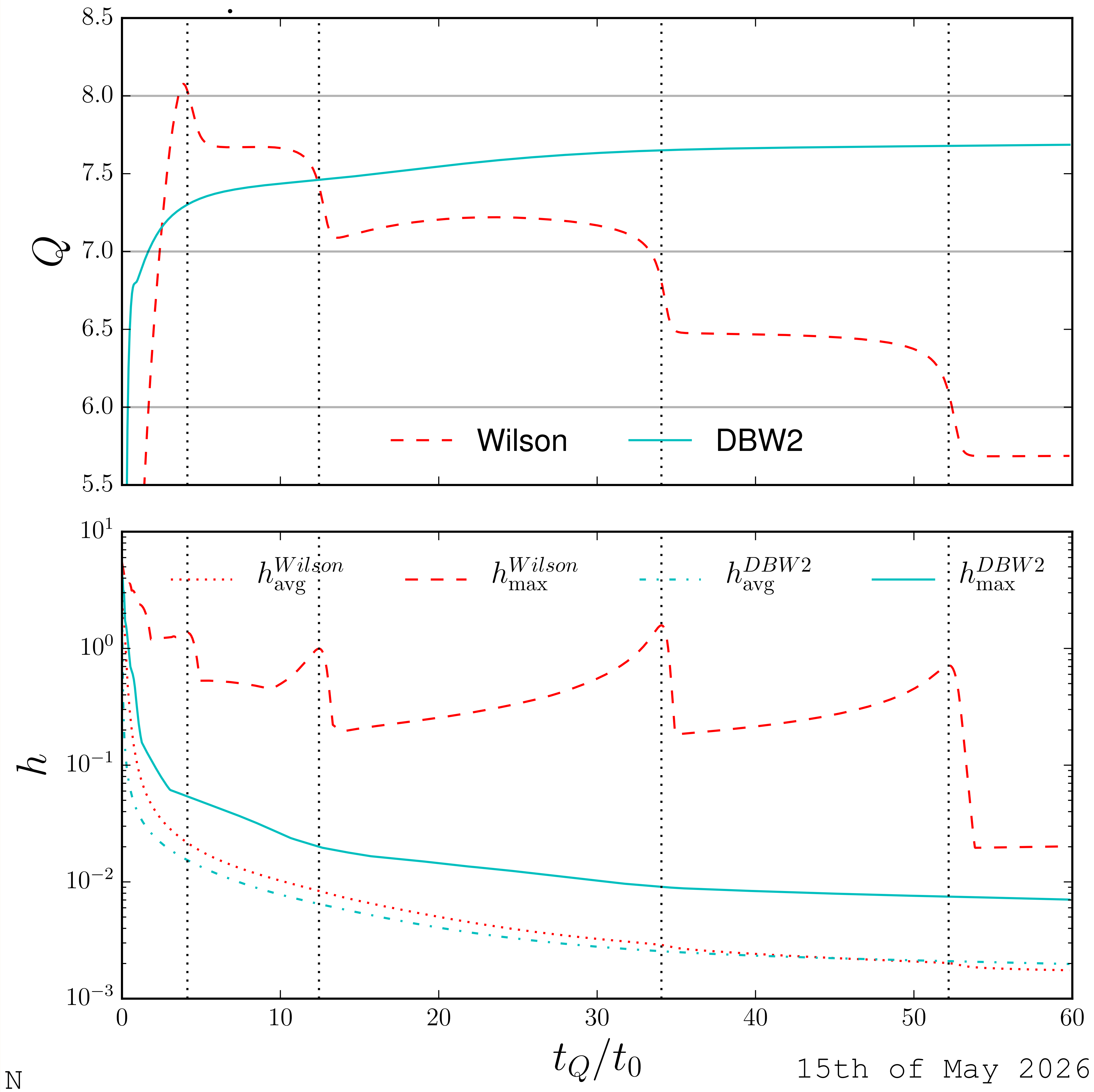
= smoothest configurations

= clearest integer signal of the topological charge

Smoothness and topology

- Topology changes

[Butti, SM, et. al, Phys. Rev. D, 2025]



Smoothness and topology

- Topology changes
- Everytime it changes, smoothness spikes

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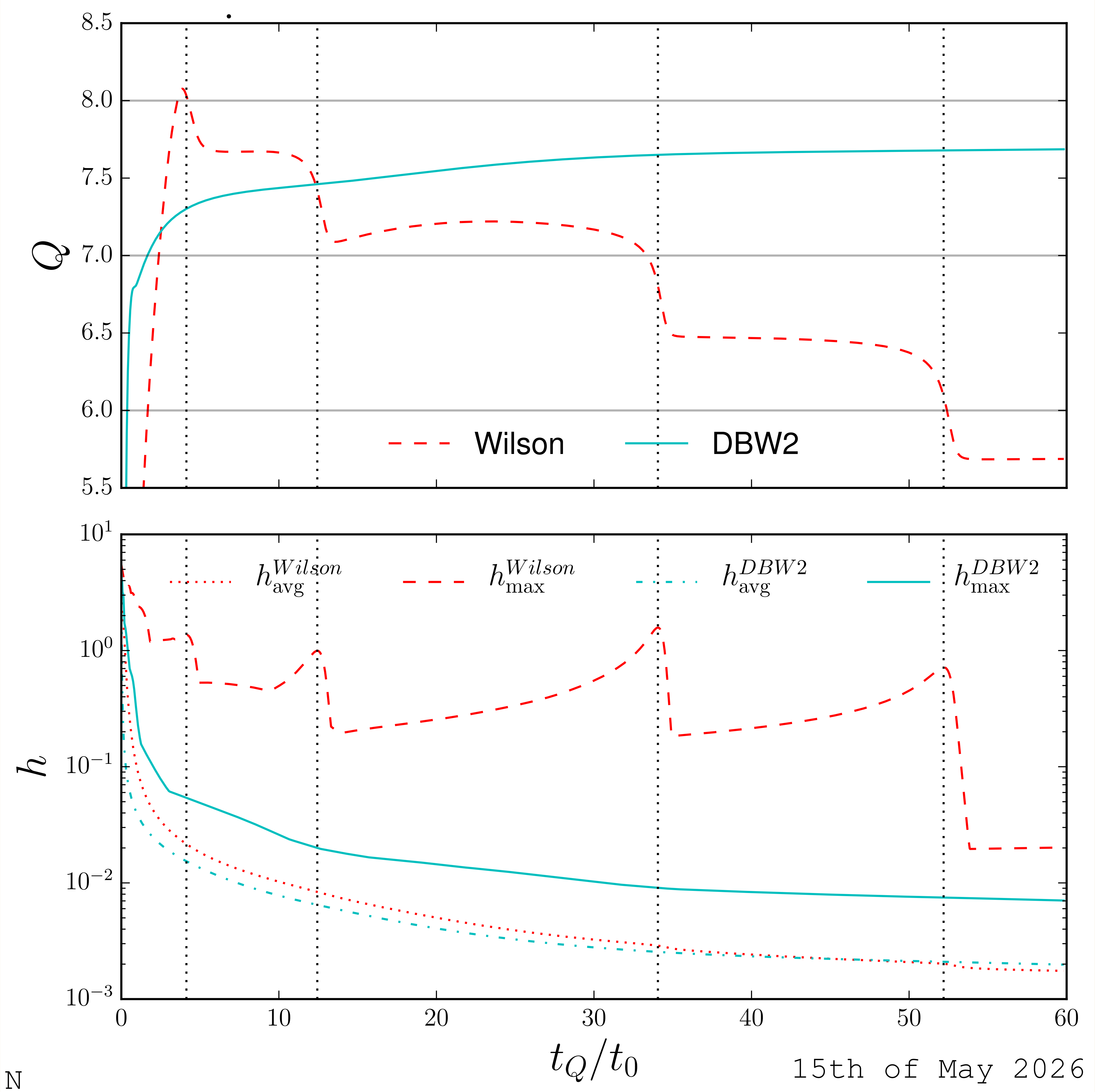
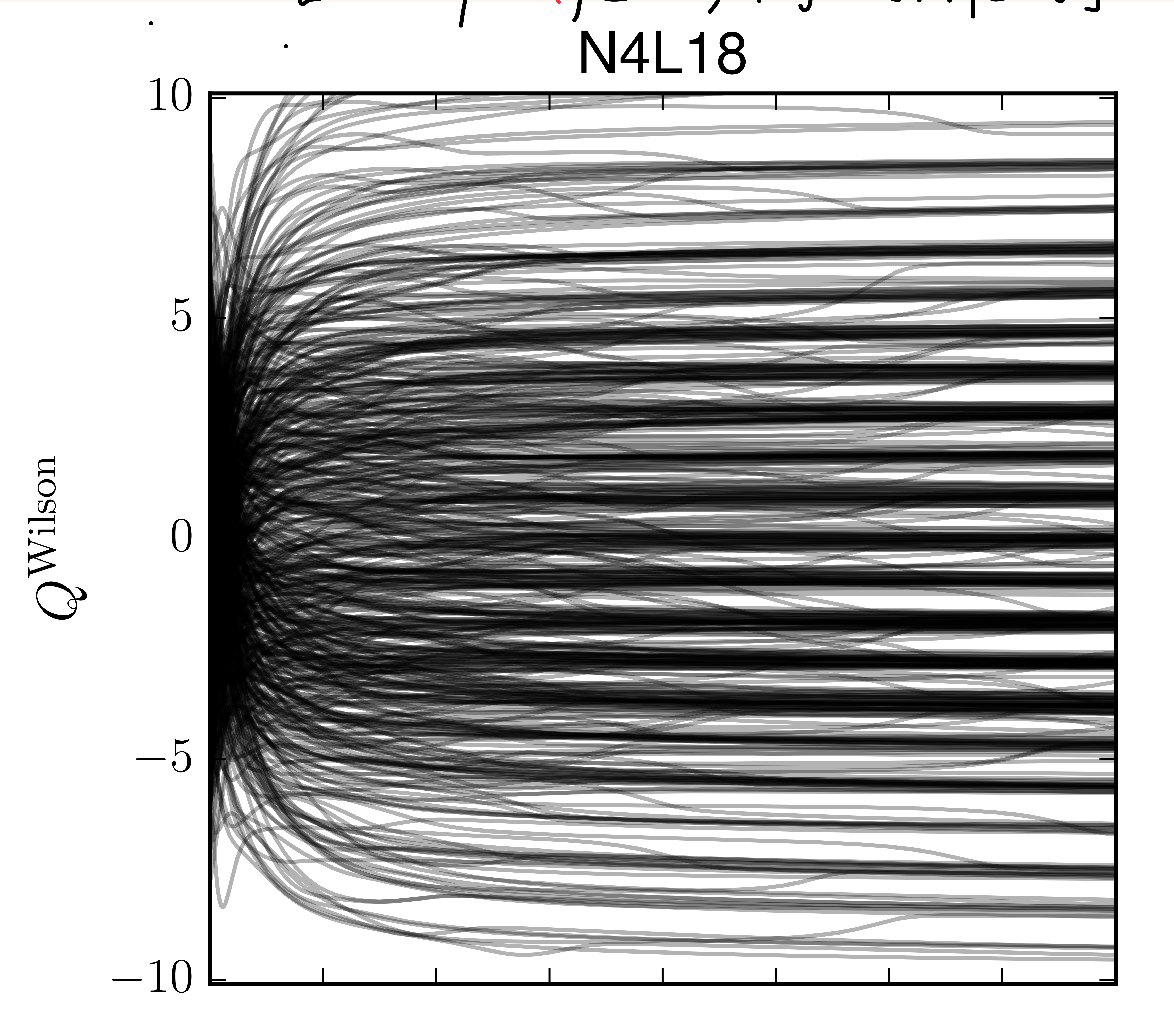


Smoothness and topology

- Topology changes
- Everytime it changes, smoothness spikes
- however: Jumping between integers (up to discretizaton effects) not fractions

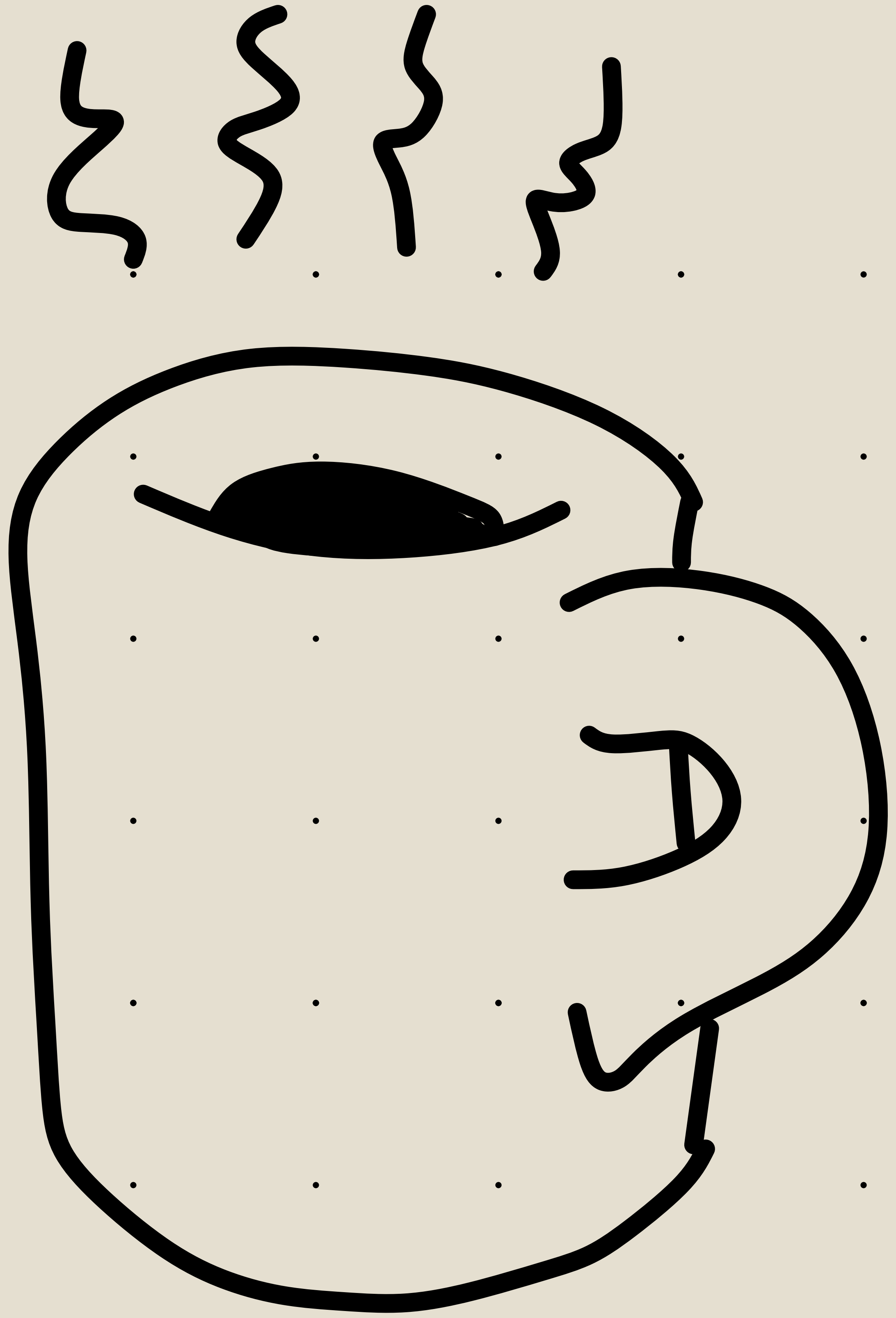
[Butti, SM, et. al, Phys. Rev. D, 2025]



When do we stop measuring for the topological charge?

Smoothing with the flow

The core issue are more foundational discretization effects



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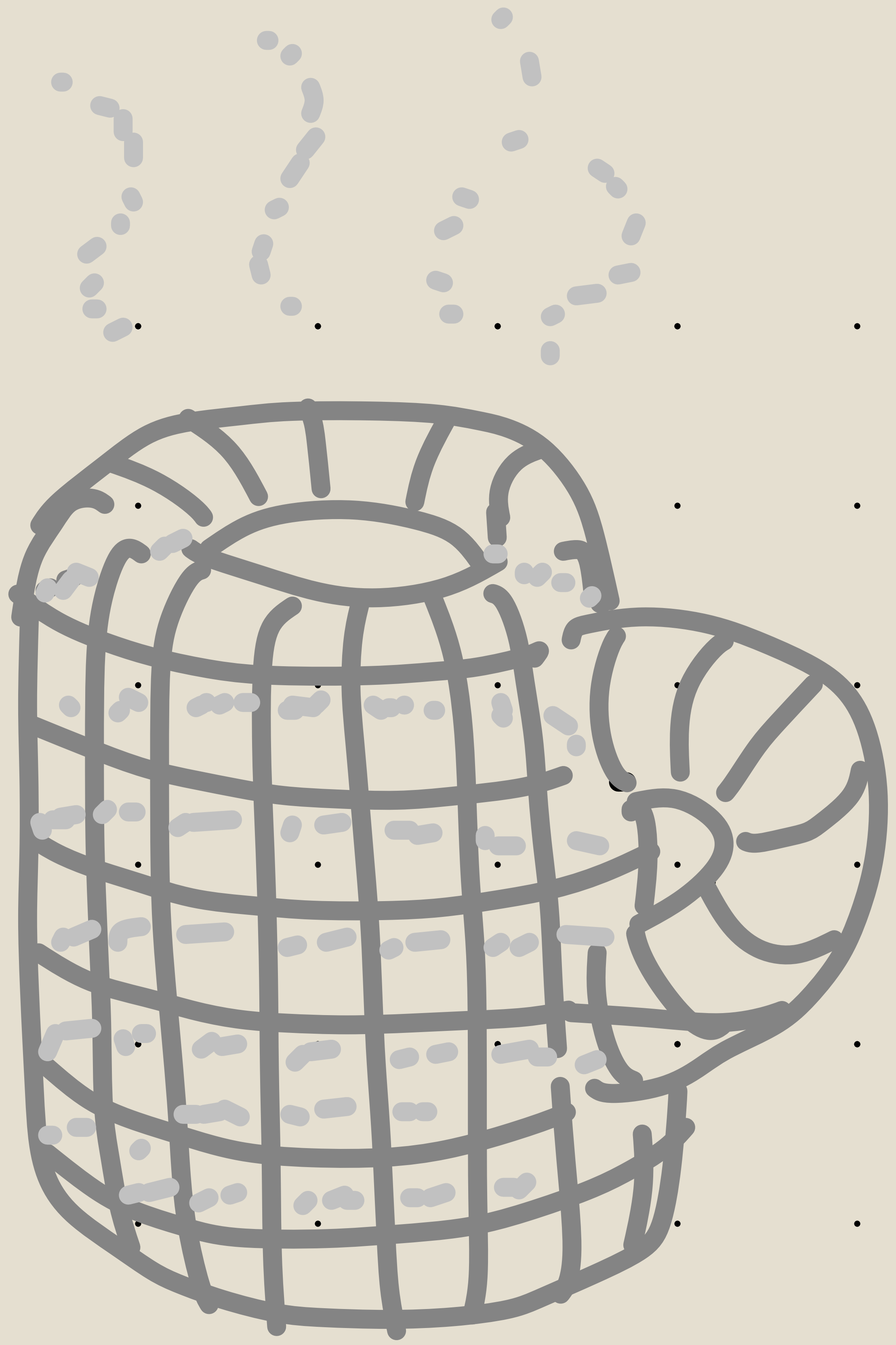
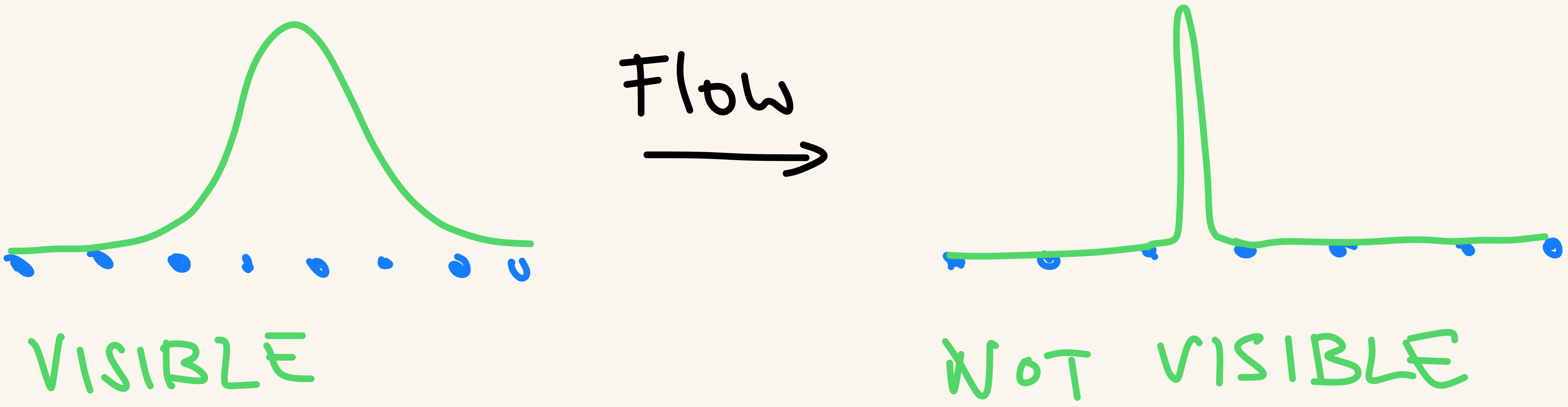
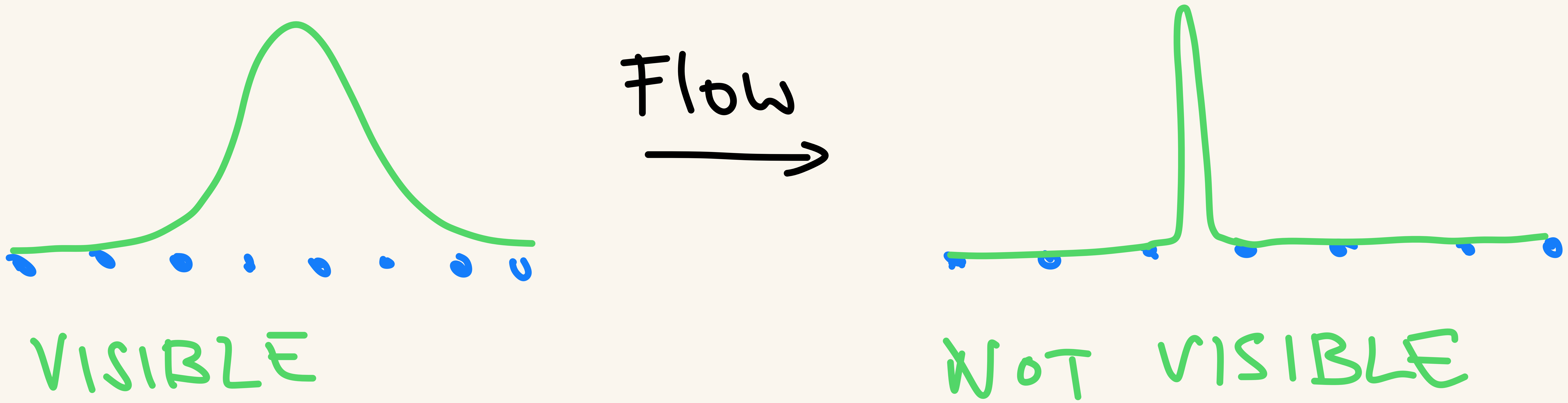


"Instabilities" = instanton falls through the lattice



Discussion with Antonio Gonzalez Arroyo

"Instabilities" = instanton falls through the lattice



Put differently: Small-sized semiclassical objects contribute negatively to the lattice action, inducing instabilities in Q (under-improvement)

[Garcia Perez, González-Arroyo, Snippe, van Baal, Nucl. Phys. B, 1994]

Large-N scale setting

$$2a \ll \sqrt{8t_0} \ll \frac{L}{2}$$

[M. Lüscher, JHEP, 2010]

Large-N scale setting

$$2a \ll \sqrt{8t_0} \ll \frac{L}{2}$$

Need to flow for increasingly long times, requires:

- Larger lattices
- Something that prevents the loss of instantons during smearing

[M. Lüscher, JHEP, 2010]

[Lüscher, Weisz, Commun. Math. Phys., 1985]

Improved schemes: Gradient flow

$$a^2\left(\frac{\mathrm{d}}{\mathrm{d}x}U_{x,\mu}(t)\right)U^{\dagger}_{\mu,x}(t) = -\,g_0^2\partial_{\mu,x}S_{\text{flow}}(U)$$

$$S = c_0 \sum_n \mathrm{tr}\left(1 - \Box\right) + \underbrace{c_1 \sum_n \mathrm{tr}\left(1 - \Box\!\Box\right)}$$

Improvement with additional rectangle term

[Lüscher, Weisz, Commun. Math. Phys., 1985]

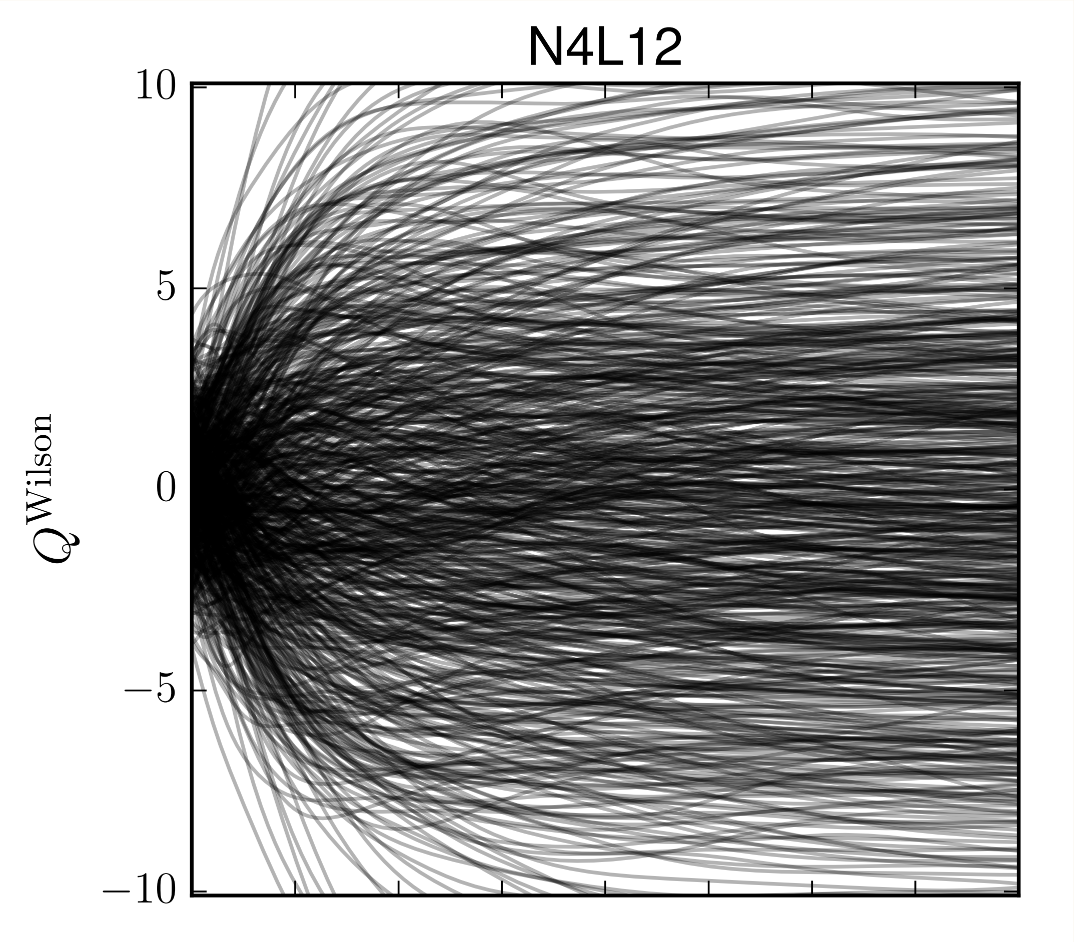
Improved schemes: Gradient flow

$$S = c_0 \sum_n \mathrm{tr}\left(1 - \Box\right) + c_1 \sum_n \mathrm{tr}\left(1 - \Box\!\Box\right)$$

	c_0	c_1
Wilson	1	0
Lüscher-Weisz	5/3	-1/12
Iwasaki	3.648	-0.331
DBW2	12.2704	-1.4088

Large flow time behavior of topological charge

coarsest spacing



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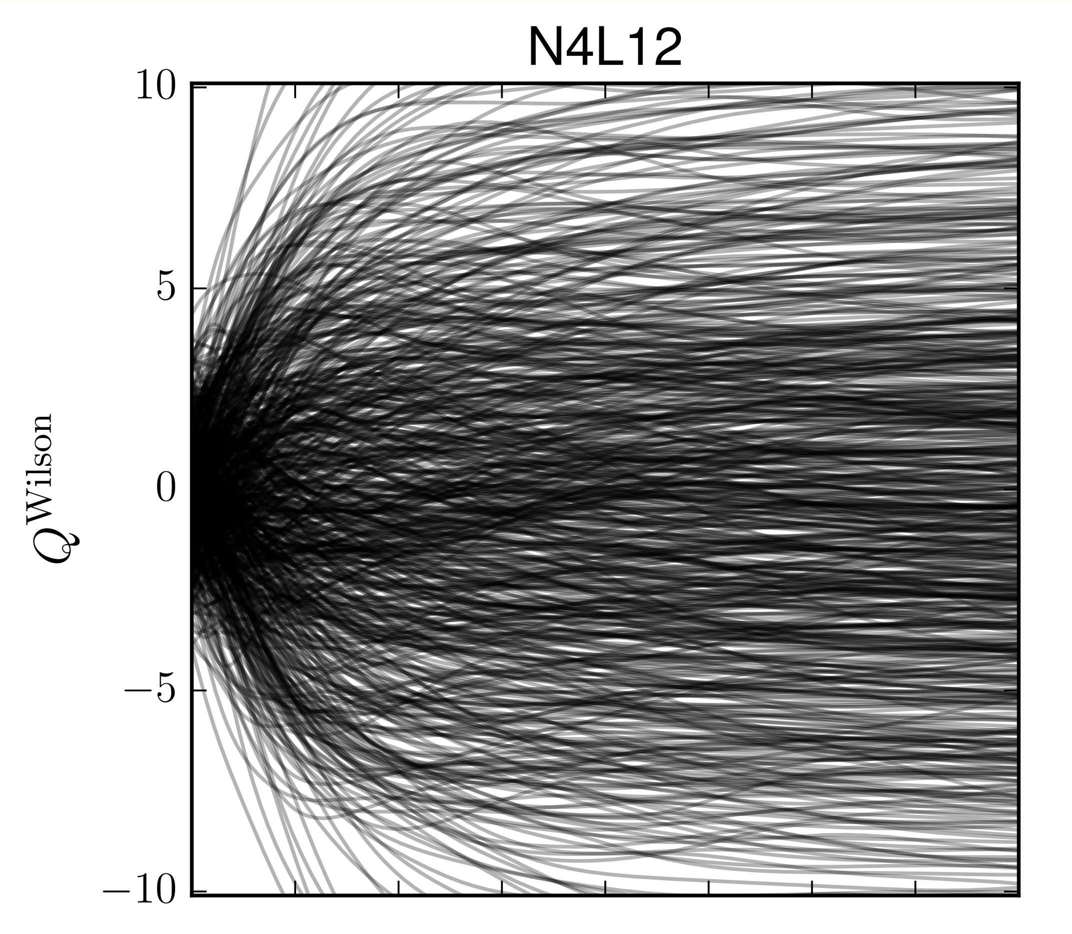
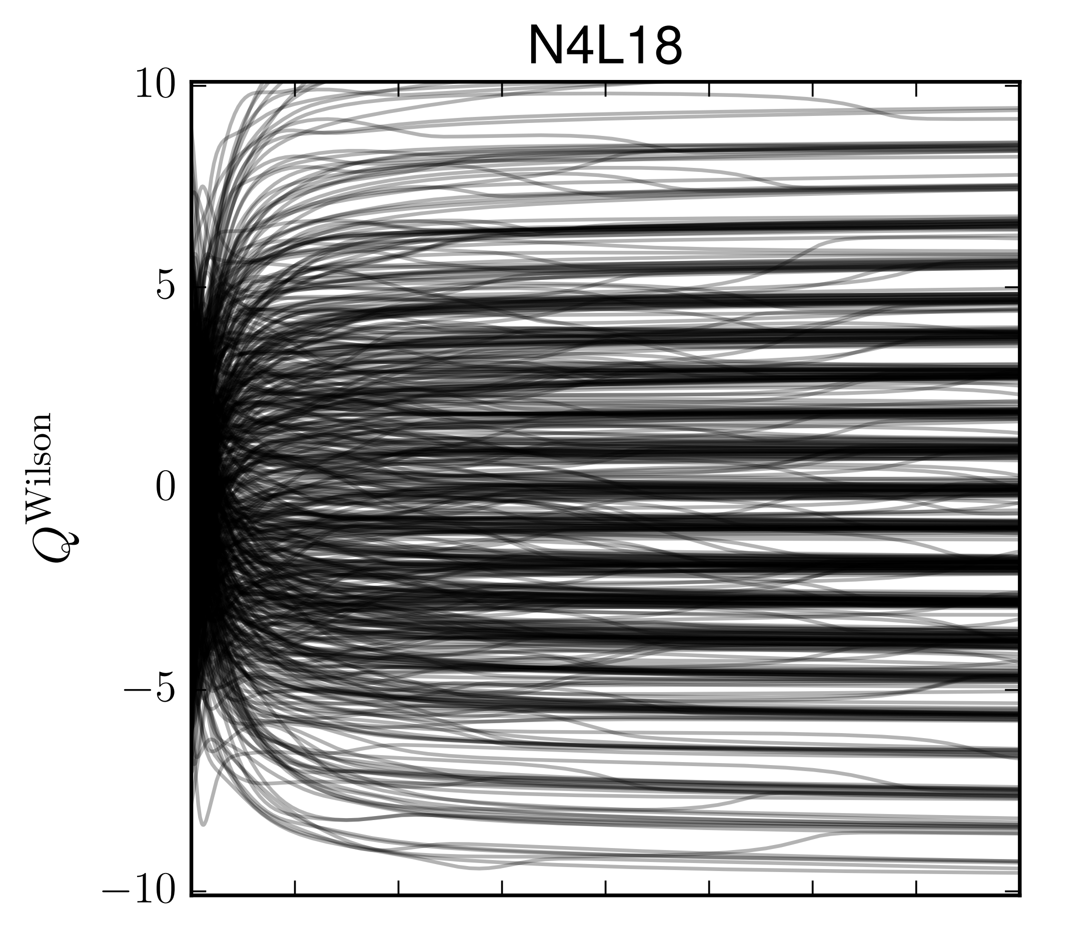


finer spacing



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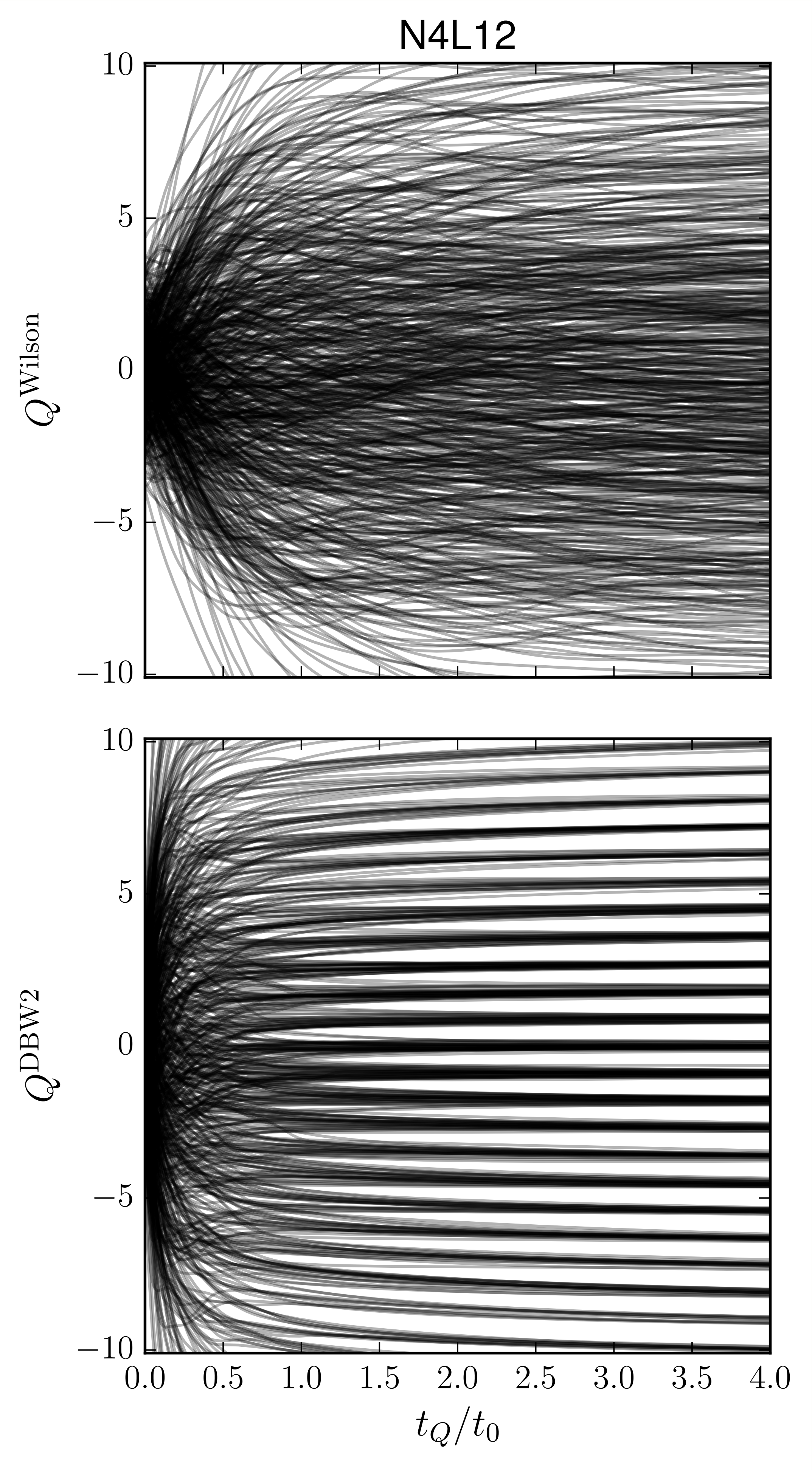

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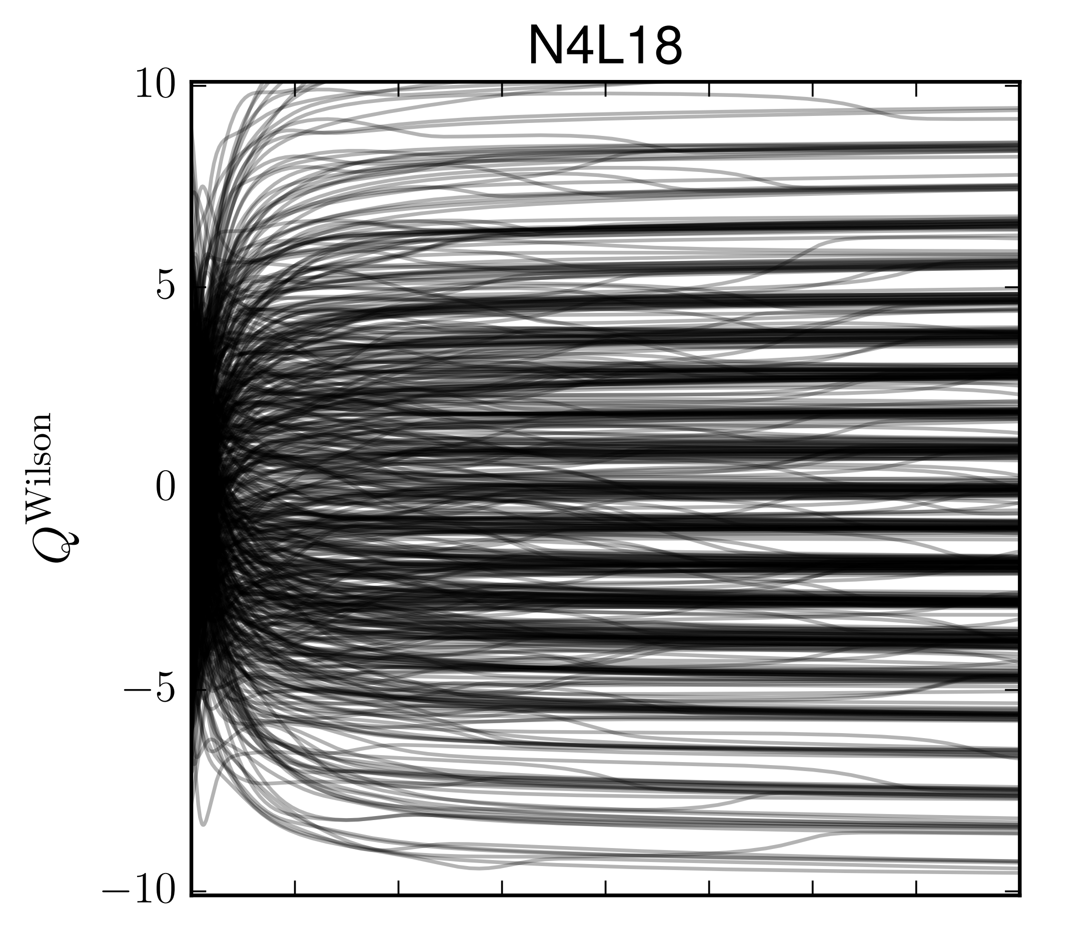

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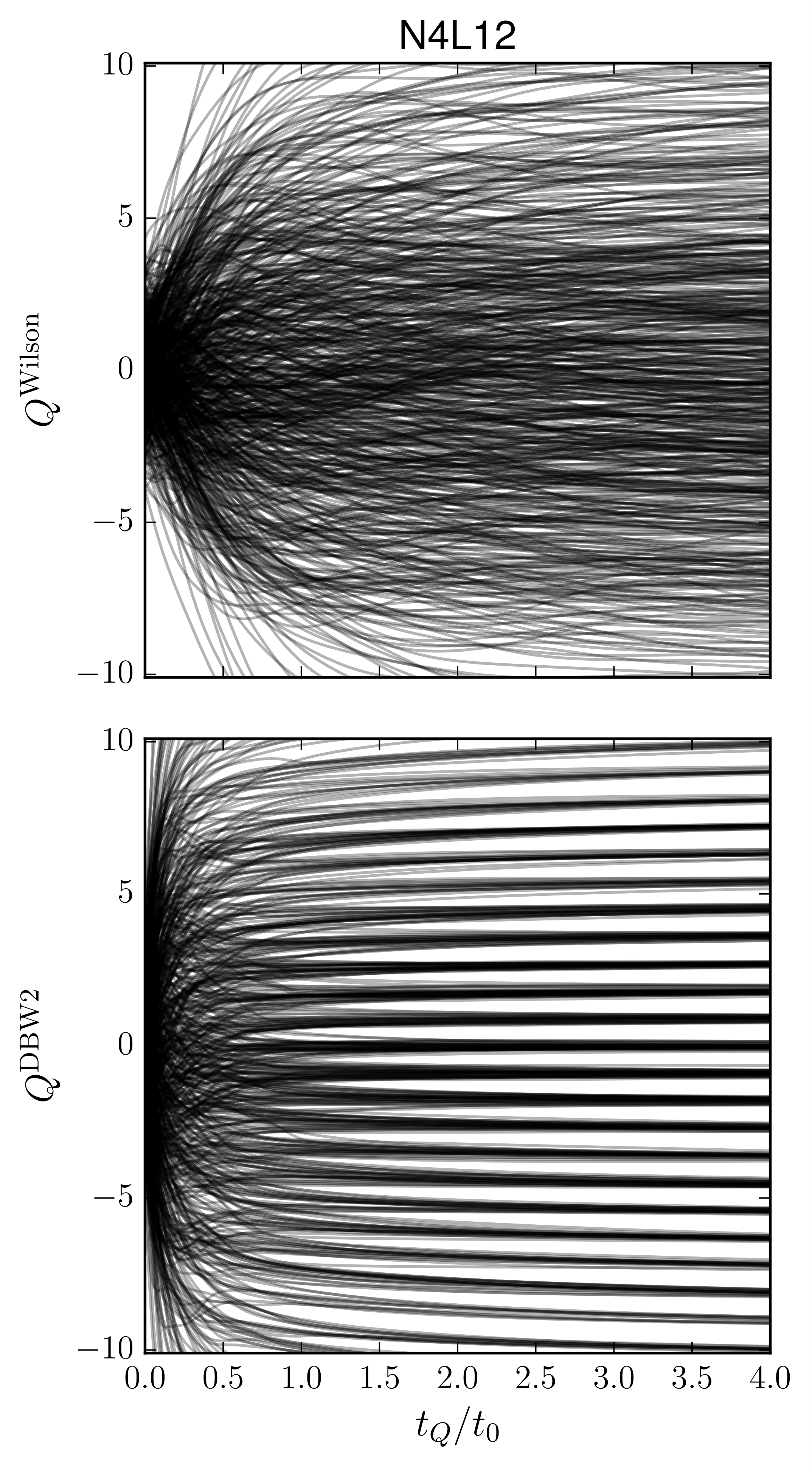

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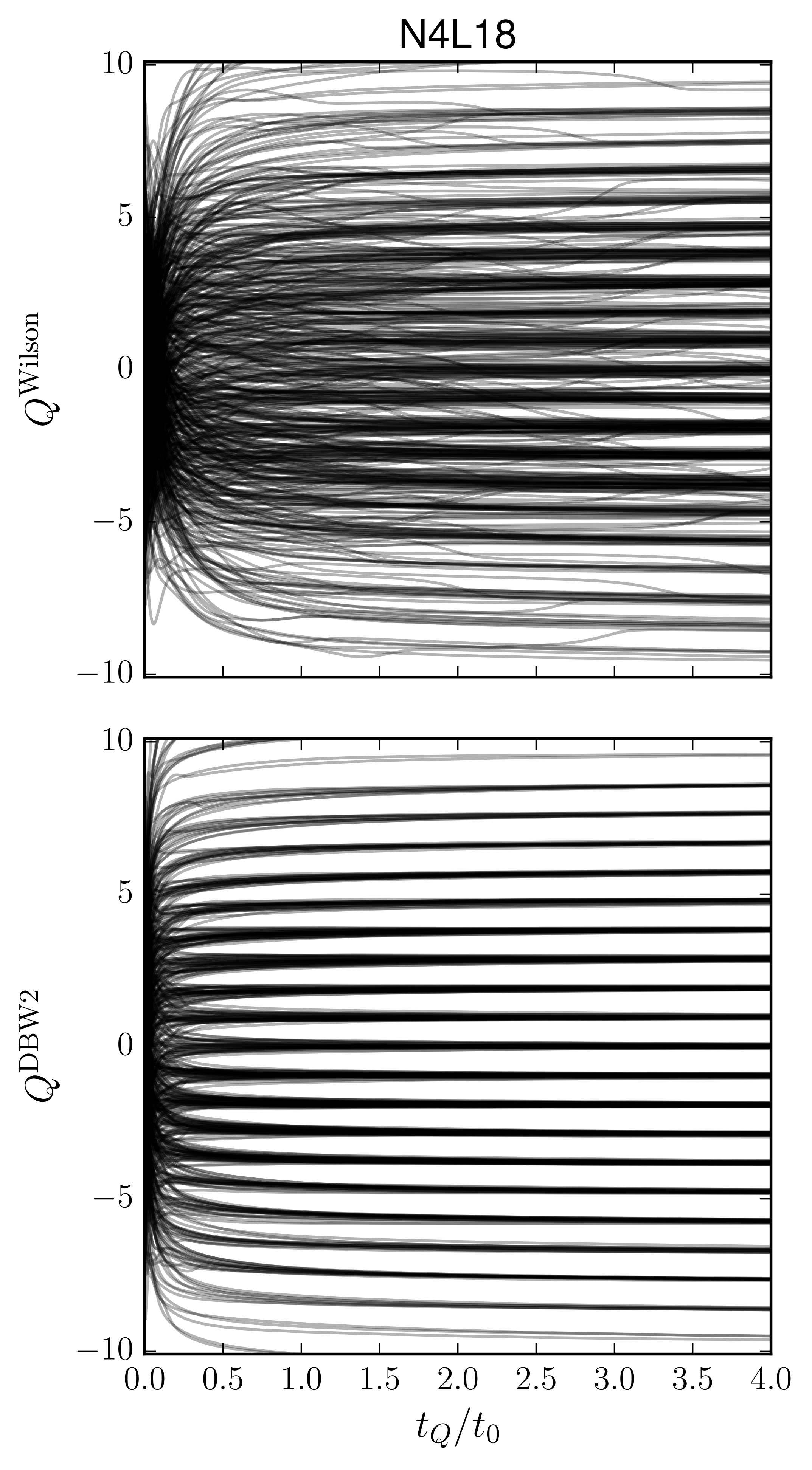

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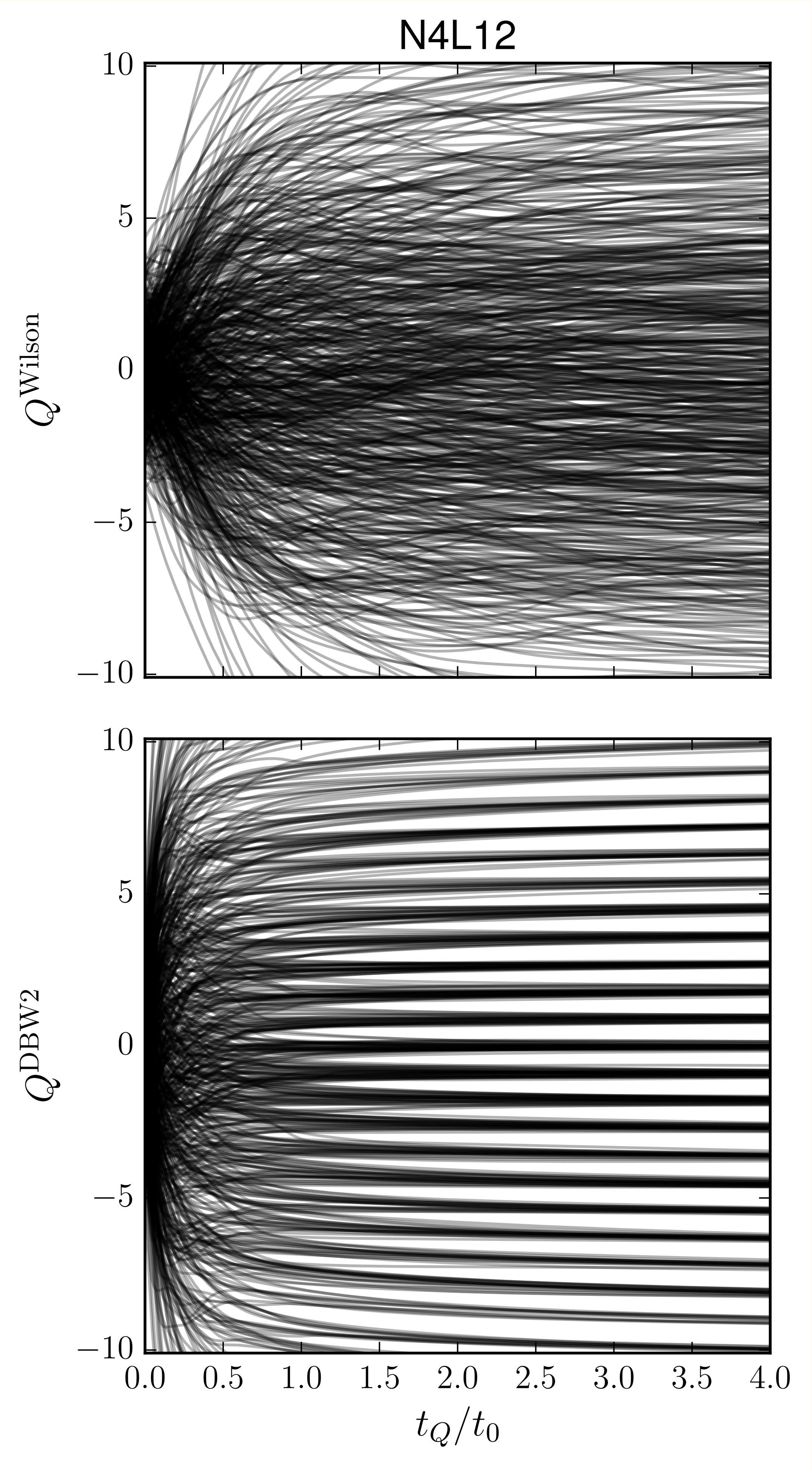

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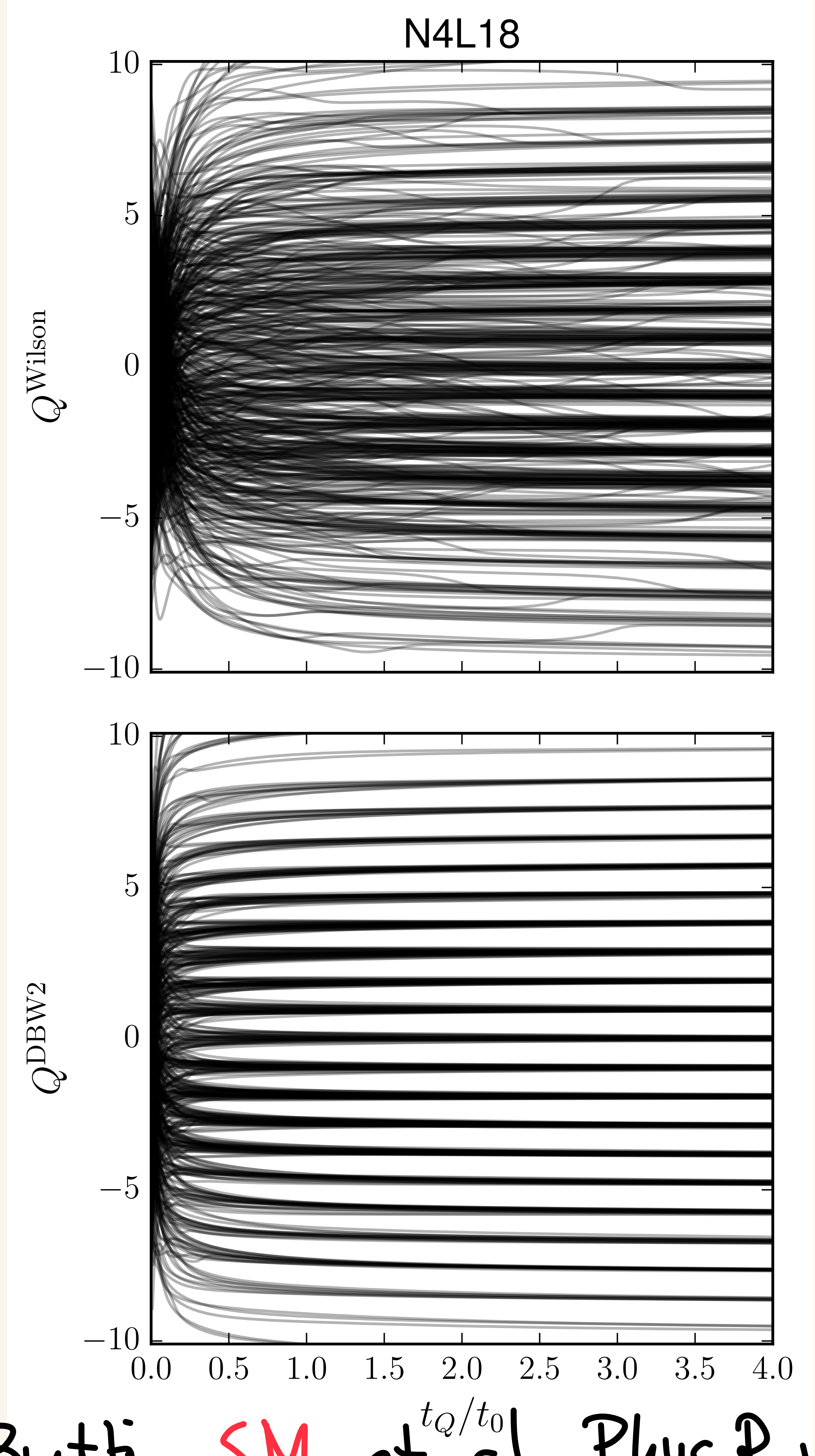
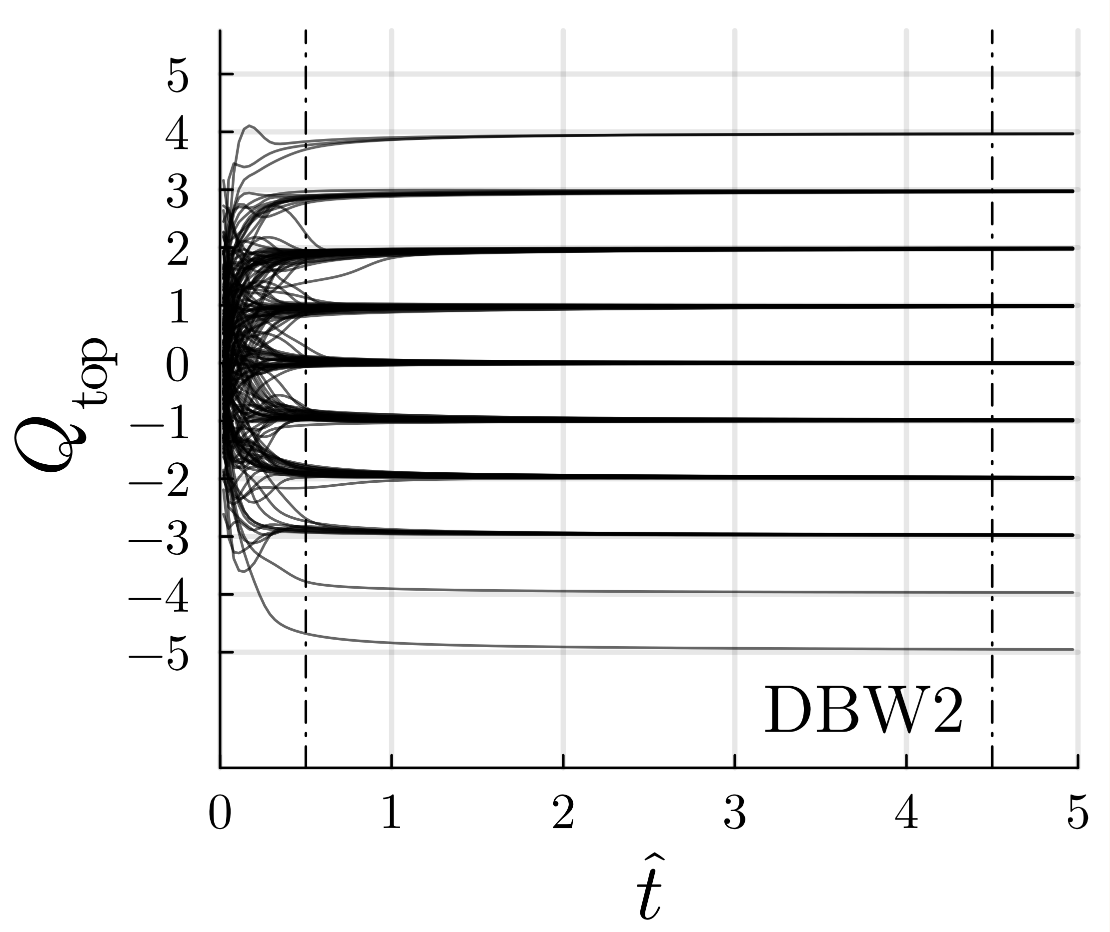


different flows



[Butti, SM, et. al, Phys. Rev. D, 2025] [Tanizaki, Tomiya, Watanabe JHEP, 2025]

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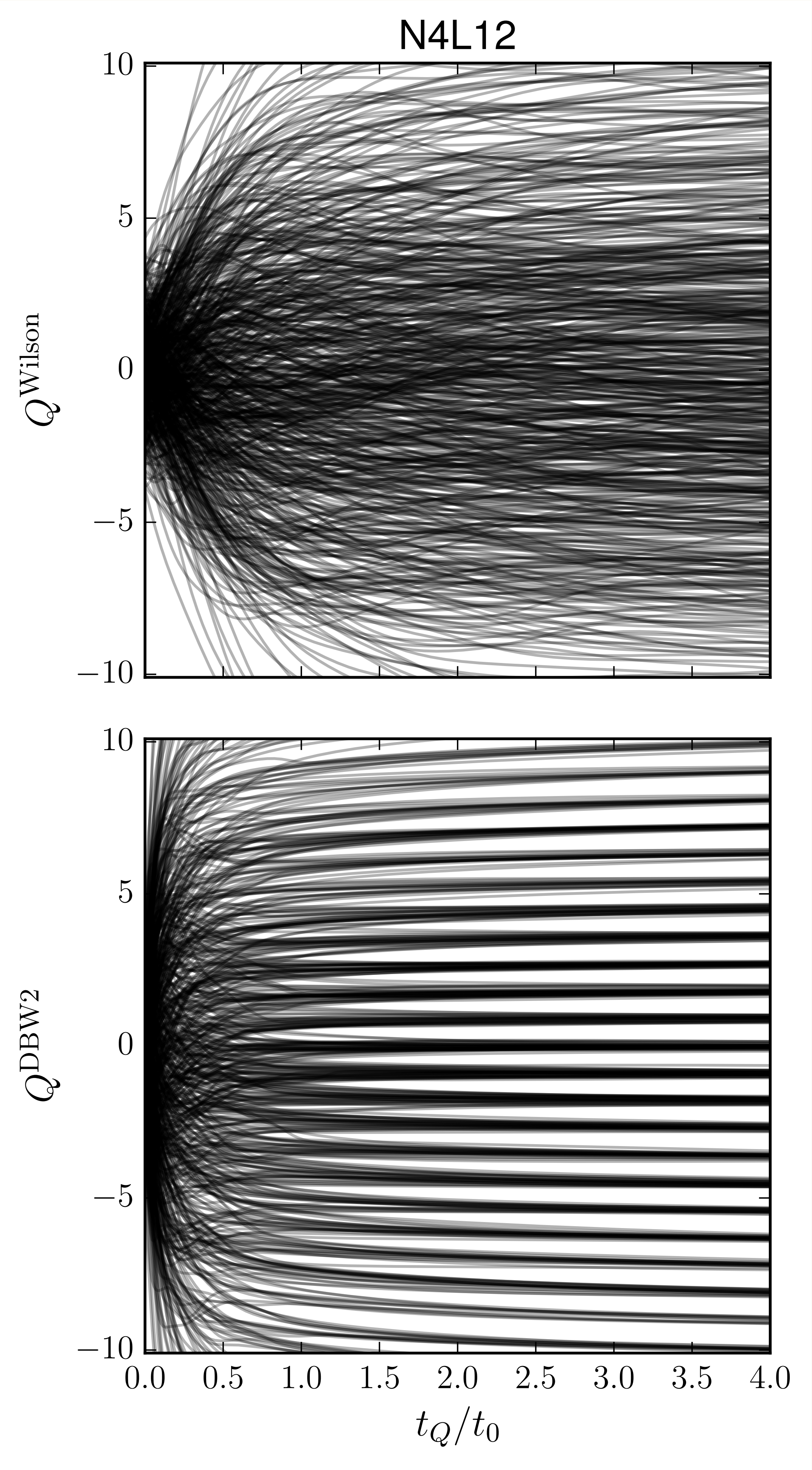

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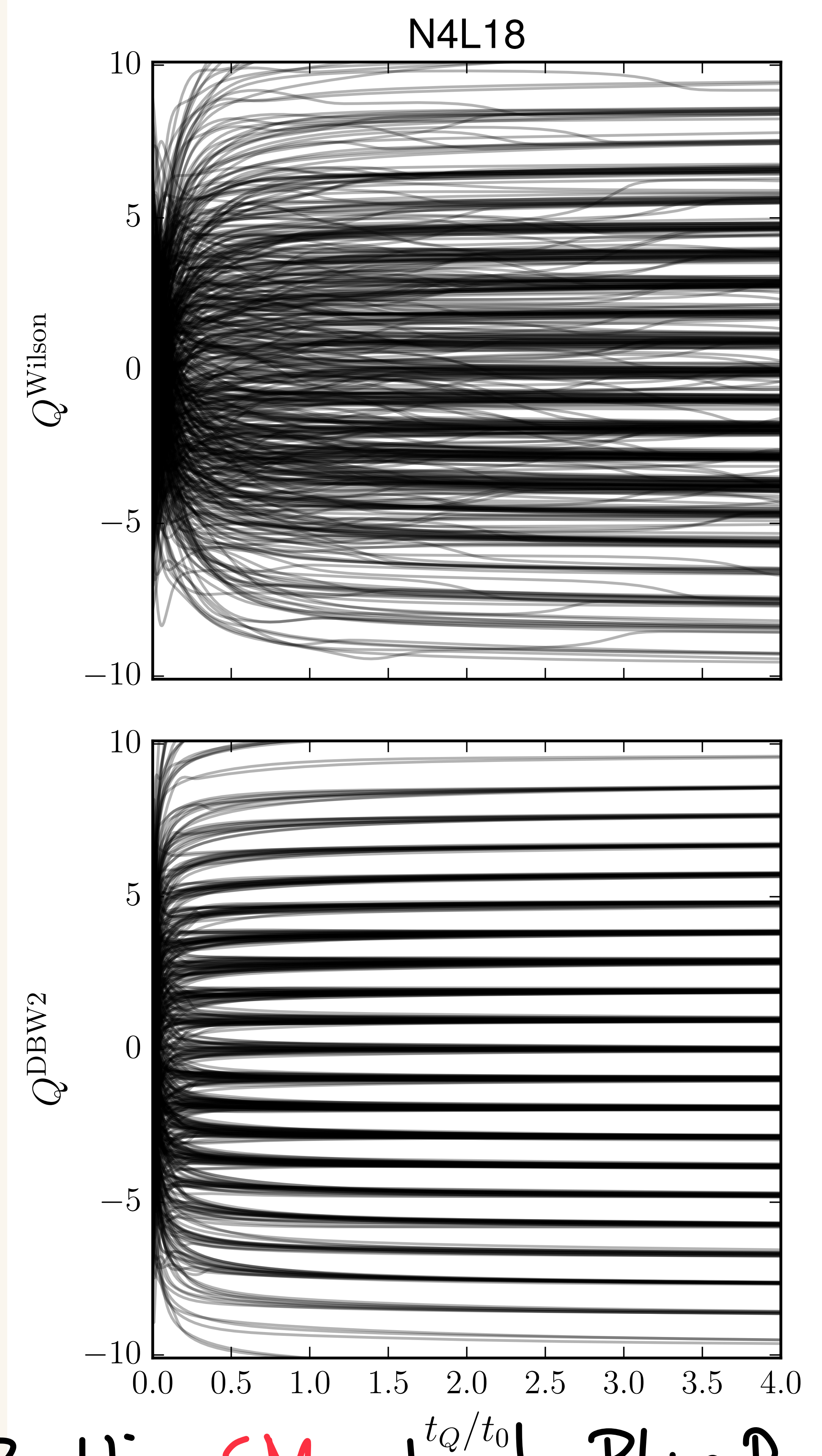

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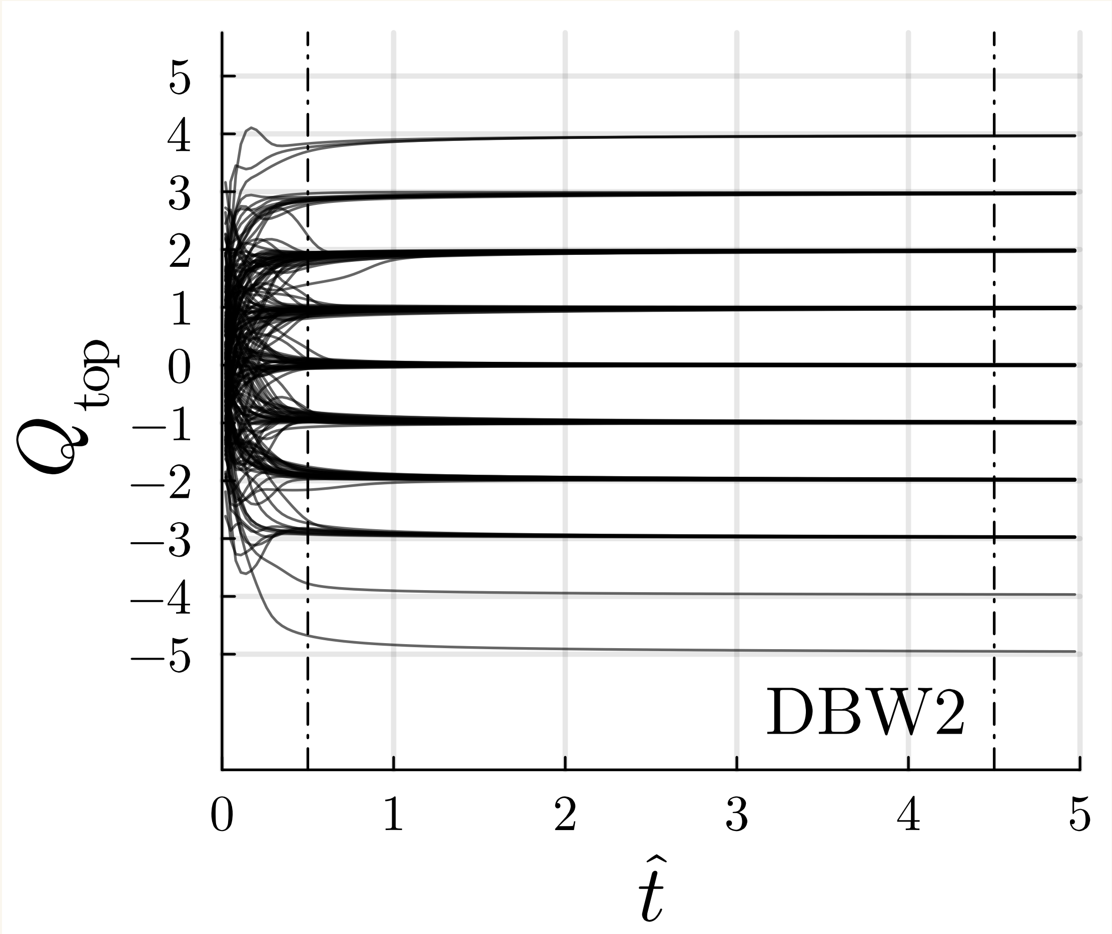
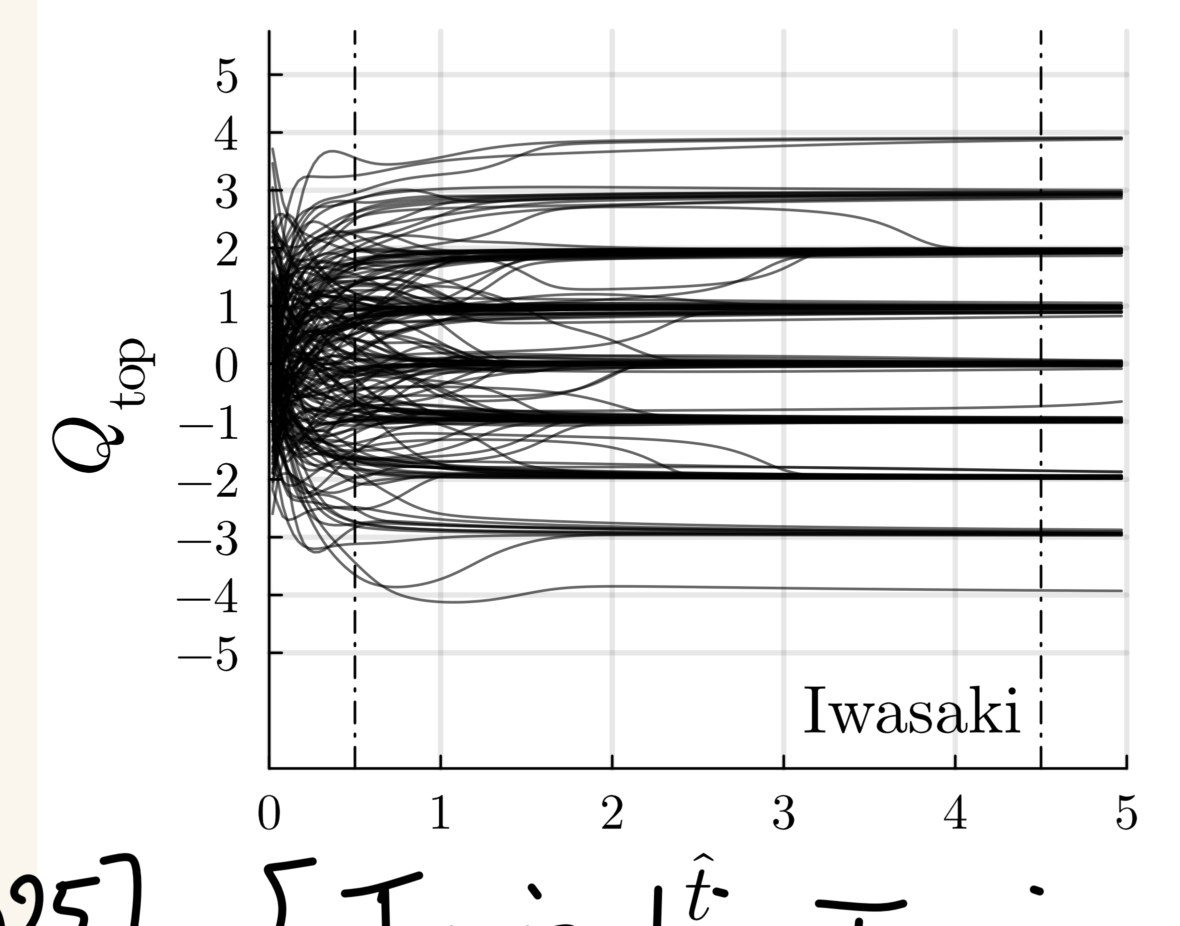
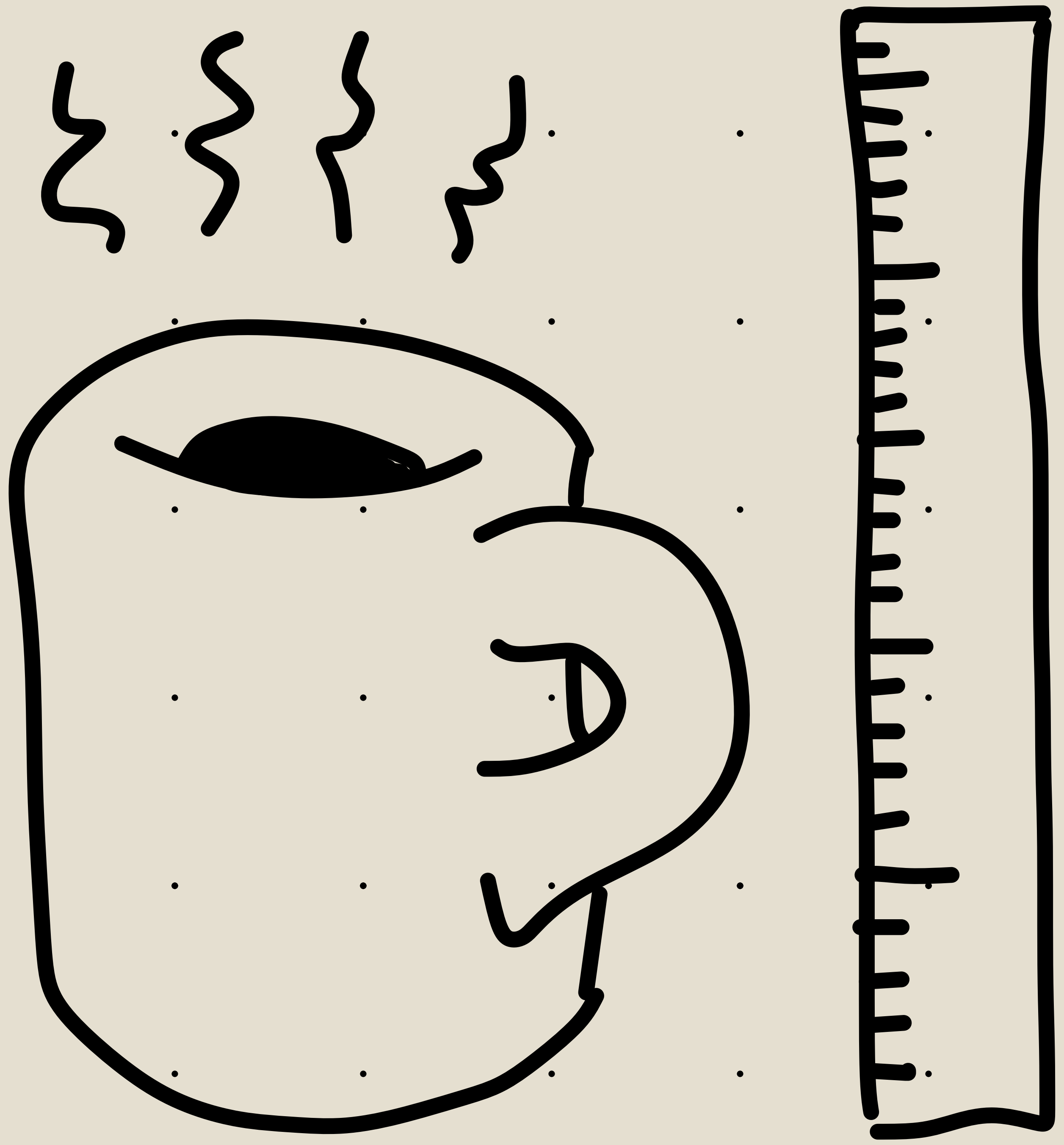




[Butti, SM, et. al, Phys. Rev. D, 2025] [Tanizaki, Tomiya, Watanabe JHEP, 2025]

Avoiding discretization effects

Setting the lattice spacing



Large-N lattice spacing

$$t^2\langle E\rangle = \frac{3}{128\pi}\frac{N_c^2-1}{N_c} \times \texttt{ some chosen value}$$

$$t_0^2\langle E\rangle = \begin{cases} 0.3 & \texttt{for } N_c = 3 \\ 0.421 & \texttt{for } N_c = 4 \\ 0.54 & \texttt{for } N_c = 5 \\ \vdots & \vdots \end{cases}$$

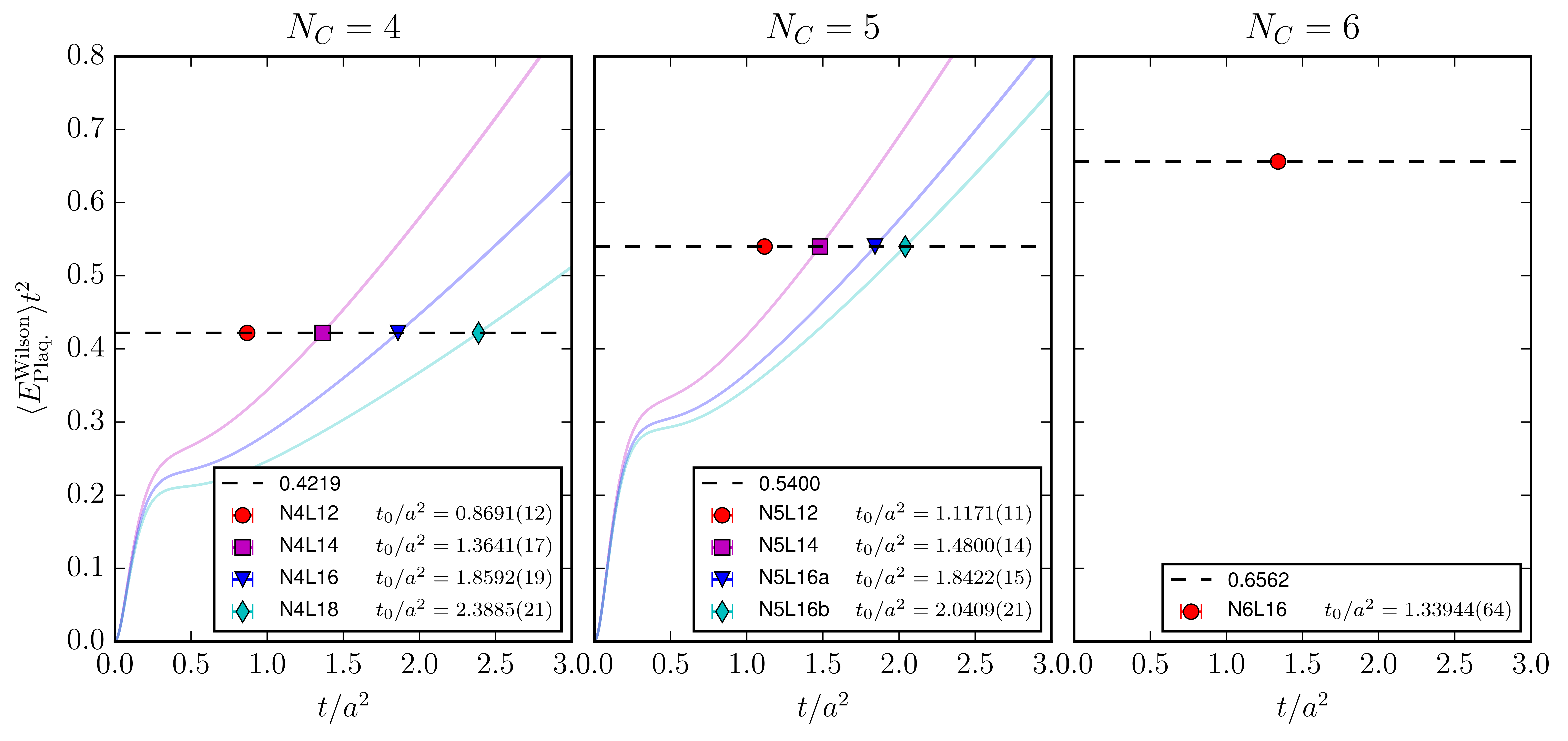
But also:

$$r_{\text{smear}} = \sqrt{8t_0} \ll \frac{L}{2}$$

[M. Lüscher, JHEP, 2010]

Large-N lattice spacing

[Butti, SM, et. al, Phys. Rev. D, 2025]



One flavor in the two-index antisymmetric representation

Resolution

How to control discretization effects for topological quantities

Probing effect on topology over susceptibility

- Atiyah Singer Index theorem slightly misleading

Probing effect on topology over susceptibility

- Atiyah Singer Index theorem slightly misleading

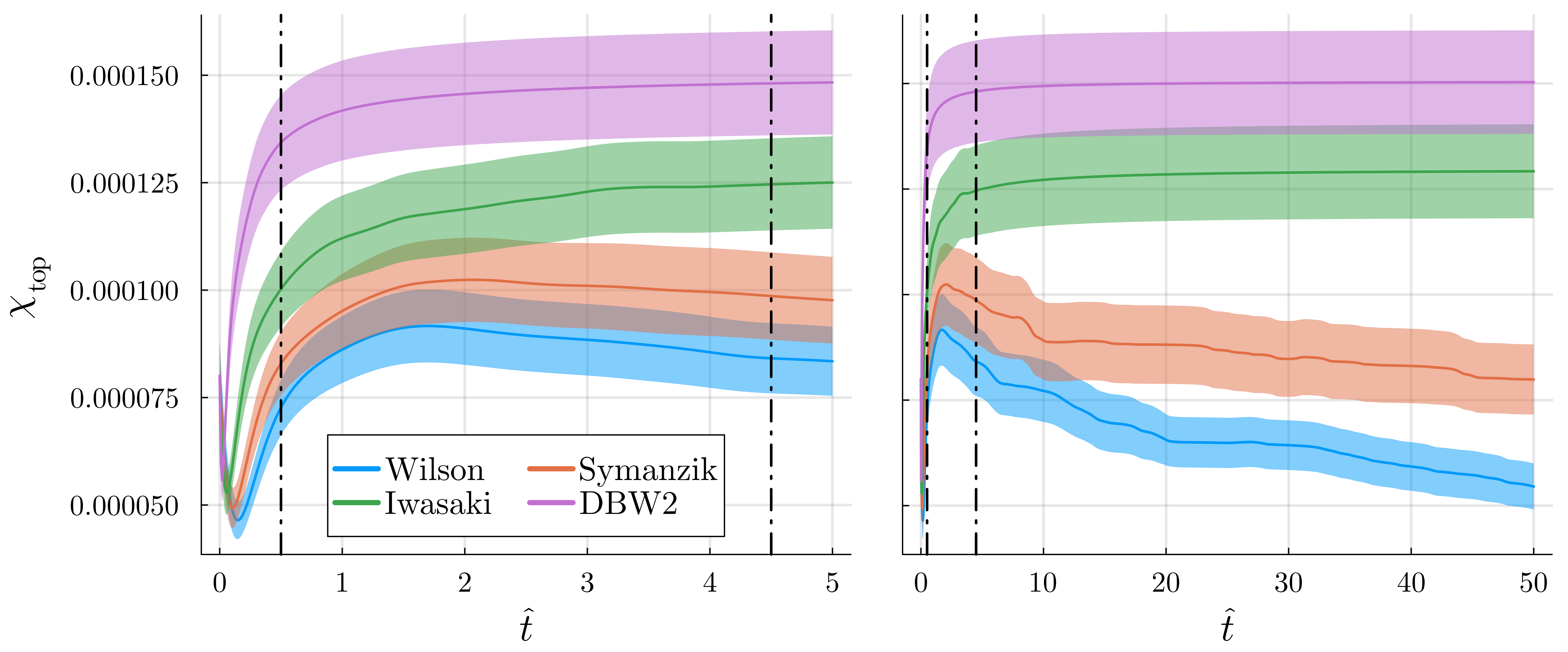
- non-integer topological charges are a sign of discretization effects

Probing effect on topology over susceptibility

- Atiyah Singer Index theorem slightly misleading
- non-integer topological charges are a sign of discretization effects
- **Are all improvement schemes made equal?**

[Tanizaki, Tomiya, Watanabe, JHEP, 2025]

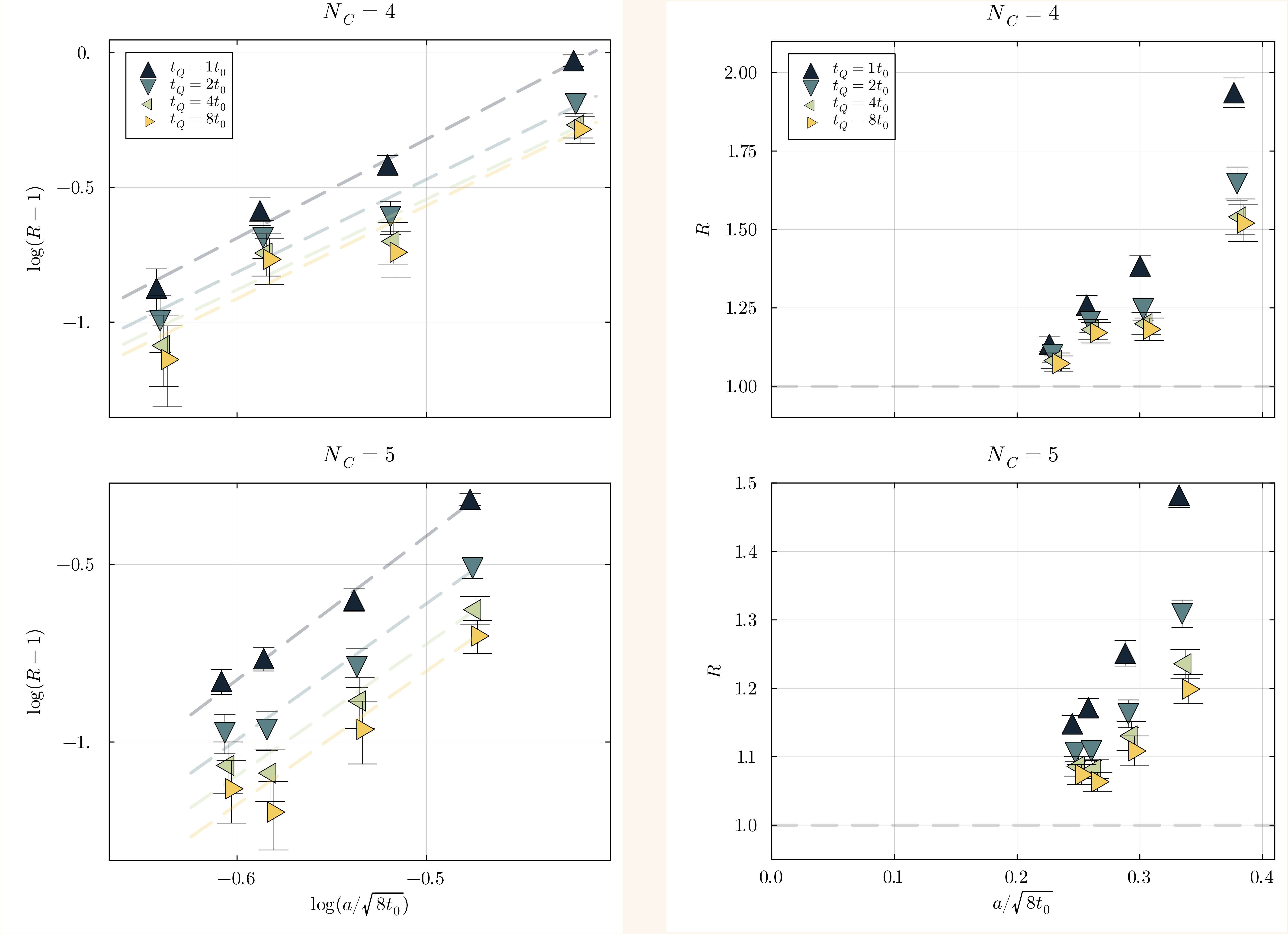
See: Topological susceptibility



The topological susceptibility behaves very differently for larger flow times

See: Topological susceptibility [Butti, SM, et. al, Phys. Rev. D, 2025]

The susceptibility from the Wilson flow strives with power 3-4 from the one of DBW2



Conclusions

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- DBW2 gradient flow controls best the smearing out of instantons and hence the discretization effects associated with topological observables

Thank you for your attention!