Quark mass effects in the flowed action density

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Aim

Compute the NNLO mass effects of

$S(t) = \langle\bar{\chi}(t)\chi(t)\rangle,\quad R(t) = \langle\bar{\chi}(t)\overleftrightarrow{D}_\mu\gamma^\mu\chi(t)\rangle,\qquad E(t) = \langle G^{\mu\nu}(t)G_{\mu\nu}(t)\rangle$

in the gradient flow

Outline

- The gradient flow in perturbation theory
- Motivation for mass effects
- Computation
- Results

The Gradient Flow

Definition:
$E(t) = \langle G^{\mu\nu}(t) G_{\mu\nu}(t) \rangle$

The Gradient Flow

[Narayanan, Neuberger 2006]
[Lüscher 2010]
[Lüscher, Weisz 2011]

Introduce new heat equations dependent on the ‘flowtime’ $[t] = -2$,

$$\partial_t B_\mu = D_\nu G_{\nu\mu} + \kappa D_\mu \partial_\mu B_\nu \qquad \text{such that} \qquad B_\mu(t=0,x) = A_\mu(x)$$

Applications:

- Scale-setting: [Lüscher 2010]

$$t^2 E(t)|_{t=t_0} = 0.3$$

Or common alternative in [Borsànyi et al. 2012]

- Lattice-matching: $\alpha_s \sim E(t)|_{t=\rho(\mu)}$

$$a^2 \ll 8t \ll \Lambda_{\mathrm{QCD}}^{-2}$$

Renormalization

Definition:

$E(t) = \langle G^{\mu\nu}(t) G_{\mu\nu}(t) \rangle$

$S(t) = \langle \bar{\chi}(t) \chi(t) \rangle$

$R(t) = \langle \bar{\chi}(t) \overleftrightarrow{D}_\mu \gamma^\mu \chi(t) \rangle$

We can define our renormalization constants by VEVs

$$\alpha_{GF} = \mu^{-2\epsilon} Z_\alpha^{-1} \alpha_0 = \frac{8\pi t^2}{3N_A} E(t) \Big|_{t=\frac{\rho}{\mu^2}, m=0} \qquad Z_A = 1$$

[Lüscher 2010]

$$\mathring{\chi} = \mathring{Z}_\chi^{-\frac{1}{2}} \chi_0 \quad \text{s.t.} \quad \mathring{R}(t) \Big|_{m=0} = -\frac{2N_c n_F}{(4\pi t)^2}$$

[Makino, Suzuki 2014]

$$m_f^{GF} = Z_m^{-1} m_0 = -\frac{8\pi}{N_c t} m_f \frac{d}{dm_f} \mathring{S}(t) \Big|_{t=\frac{\rho}{\mu^2}, m=0}$$

[Artz et al. 2019]

Perturbation Theory	
$m = 0$	$\overline{MS}$

Lattice Field Theory	
$m \neq 0$	Not $\overline{MS}$

Definition:

$S(t) = \langle \bar{\chi}(t)\chi(t)\rangle$

$R(t) = \langle \bar{\chi}(t)\overleftrightarrow{D}_\mu \gamma^\mu \chi(t)\rangle$

Measurable Ratios

[Takaura, Harlander, Lange 2025]

Mass effects of $S(t)$ and $R(t)$ can be used precision determination of quark masses.

See Fabians talk

No divergent gluon renormalization required:

$$\mathring{S}(t) = \mathring{Z}_\chi S_0(t) \qquad \mathring{R}(t) = \mathring{Z}_\chi R_0(t)$$

Define 'measurable' ratios, e.g.

$$r_a(t) = \frac{S(t)}{R(t)} = \frac{S_0(t)}{R_0(t)} \longrightarrow z_m(r_a, t, m_a)$$

Set by mass of e.g. light scalar/ pseudo-scalar

Matching between PT and LFT

Mass effects calculated at NLO in [Takaura, Harlander, Lange 2025].

Definition:
$E(t) = \langle G^{\mu\nu}(t) G_{\mu\nu}(t) \rangle$

Mass Dependent Schemes

[Kluth 2026] defines a non-minimal but gauge-invariant subtraction scheme that absorbs all power law divergences

$$\beta = \left(\frac{d-4}{2}\right)\alpha + \frac{\alpha^2}{4\pi}\left[-\frac{11}{3}C_A + \underbrace{\sum_{r=1}^{n_f}\frac{2}{3}e^{-m^2/\mu^2}}_{\beta_r}\right]$$

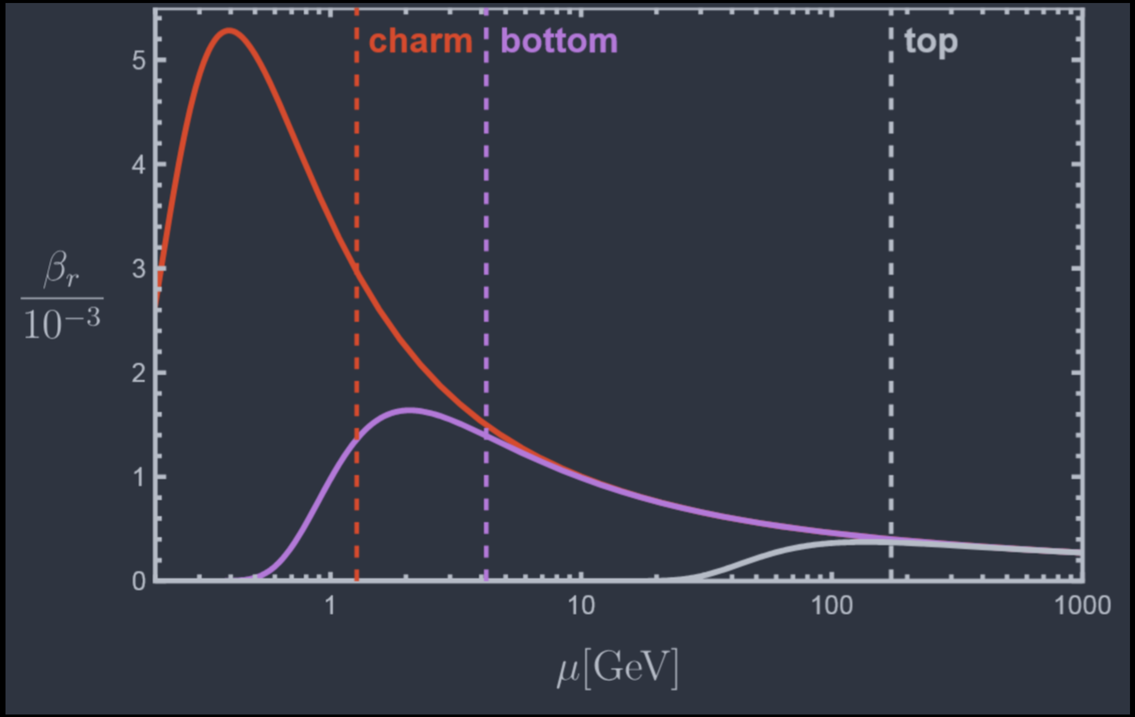
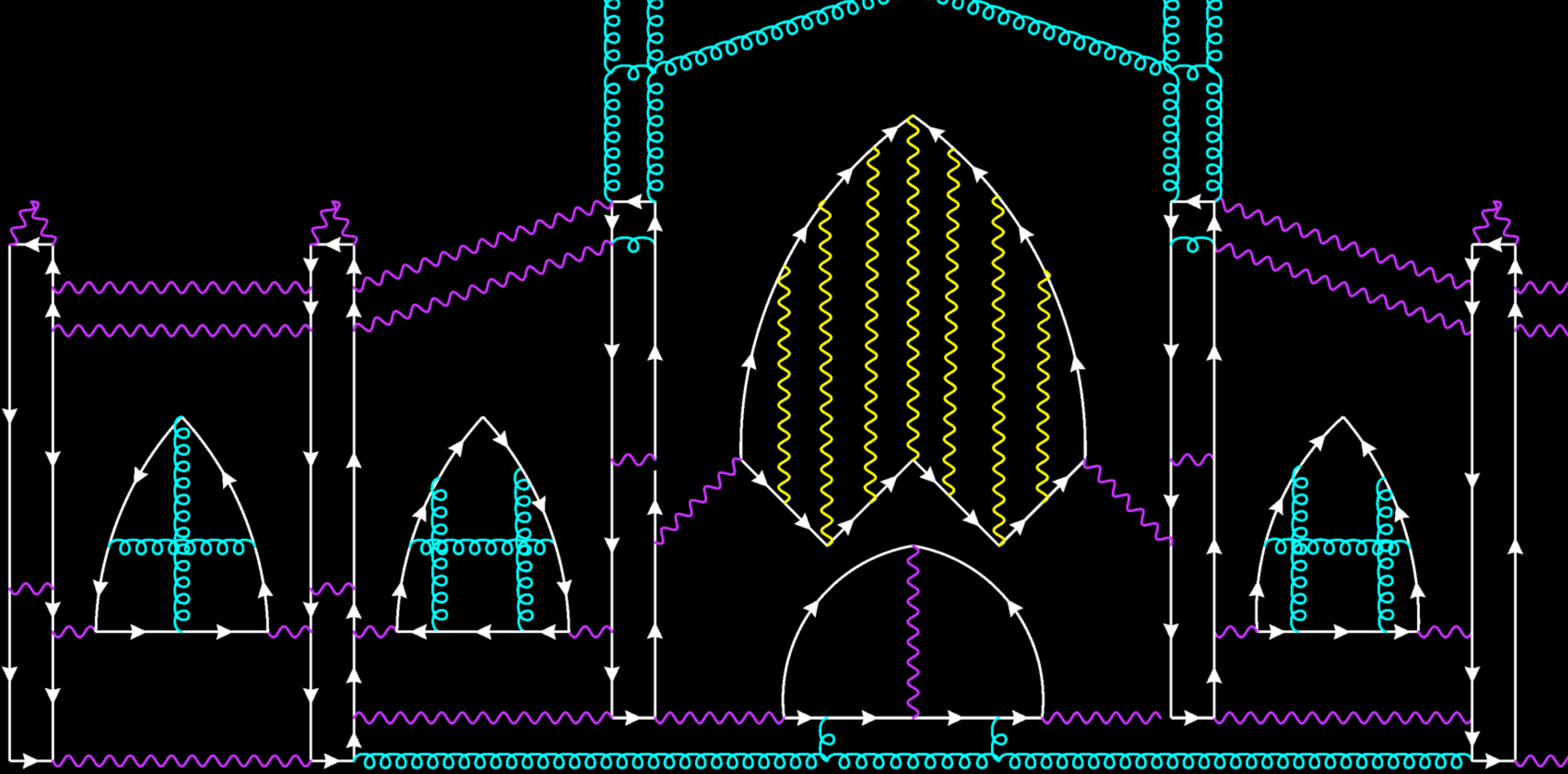


Figure and β equation from [Kluth 2026]
[2604.23526]

Gradient flow coupling?

$$\alpha_{GF} = \frac{8\pi t^2}{3N_A} E(t)\bigg|_{t=\frac{\rho}{\mu^2}}$$ [Lüscher 2010]

Similar to application of gradient flow for Ricci flow – see talk by Henry Werthenbach



Computation

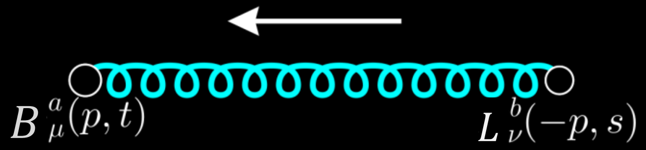
Feynman Rules [Lüscher, Weisz 2011]

Flowed propagators modified by exponential suppression

$$= \frac{\delta_{ab}}{p^2}\,\delta_{\mu\nu}e^{-(t+s)p^2}$$

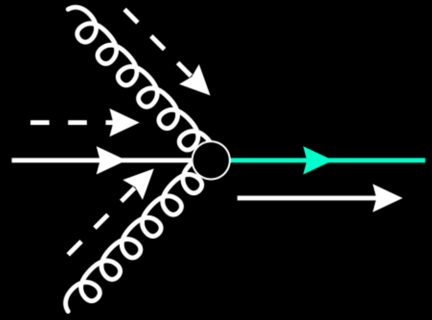
In Feynman gauge

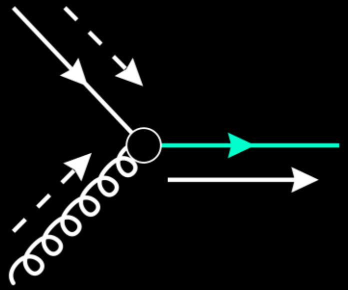
New ‘flow lines’ involving Heaviside step function



$$= \delta_{ab}\delta_{\mu\nu}\theta(s-t)e^{-(s-t)p^2}$$

Additional vertices





All diagrams made with Feyngame [Bündgen et al. 2025]

The Gluon Condensate

One quantity we are interested in is $E(t) = \langle G^{\mu\nu}(t) G_{\mu\nu}(t) \rangle$

Schematically:

$E(t) =$ 

$$= \frac{3 N_A \alpha_s}{8 \pi t^2} \int_p e^{-2tp^2} + O(\alpha^2)$$

$$= \frac{3 N_A}{8 \pi t^2} \alpha_{GF}$$

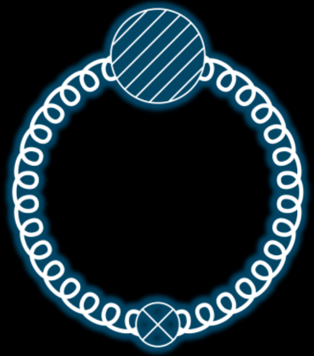
[Lüscher, Weisz 2011]

See also talk by Jonas Kohnen

Gradient Flow Setup

[To be released 2026]

Operator Feynman Rules → Setup → Integral level VEV



$$-g^2\delta^{ab}\left(\delta_{\mu\nu}p.q - p_\mu q_\nu\right)$$

[Lüscher, Weisz 2011]

Setup

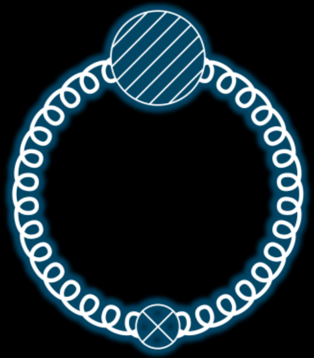
QGRAF [Nogueira 1991]

[Harlander, et. al., 1998] EXP / TAPIR [Gerlach, et. al., 2022]

FORM [Vermaseren 1989]

Gradient Flow Setup [To be released 2026]

Operator Feynman Rules



$$-g^2\delta^{ab}\left(\delta_{\mu\nu}p.q - p_\mu q_\nu\right)$$

[Lüscher, Weisz 2011]

Setup

QGRAF [Nogueira 1991]

```
[Et,Et,+,external]
```

```
[Et,g,g]
```

[Harlander, et. al., 1998] EXP / TAPIR [Gerlach, et. al., 2022]

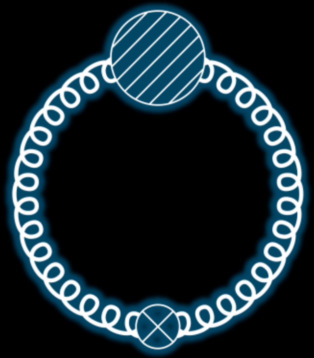
FORM [Vermaseren 1989]

Integral level VEV

Gradient Flow Setup

[To be released 2026]

Operator Feynman Rules



$$-g^2\delta^{ab}\left(\delta_{\mu\nu}p.q - p_\mu q_\nu\right)$$

[Lüscher, Weisz 2011]

Setup

QGRAF [Nogueira 1991]

[Harlander, et. al., 1998] EXP / TAPIR [Gerlach, et. al., 2022]

```
{Et,g,g:*VEtgg(<indices>)
*Flow(<flow indices>)
|*CEtgg(<colour indices>)}
```

```
id CEtgg( a1?(s1?),a2?(s2?))
= +prop(a1(s1),a2(s2));
```

FORM [Vermaseren 1989]

Integral level VEV

Gradient Flow Setup

[To be released 2026]

Operator Feynman Rules

$$-g^2\delta^{ab}\left(\delta_{\mu\nu}p.q - p_\mu q_\nu\right)$$

[Lüscher, Weisz 2011]

Setup

QGRAF [Nogueira 1991]

[Harlander, et. al., 1998] EXP / TAPIR [Gerlach, et. al., 2022]

FORM [Vermaseren 1989]

```
id VEtgg(mu1?,L,x1?,?a,x1?,L,mu2?,L,x2?,?b,x2?,L)
= -4*d_(mu1,mu2)*scalar(?a,product,?b)
+4*Vec(mu2,?a)*Vec(mu1,?b)*M1;
```

```
#include flowexternalexp

#include identifyintegrals
```

Integral level VEV

```
(3/2*na*g0^2-na*ep*g0^2)*
IF(ftpar,ftexp(2),ftmom(0))
```

and up to the three-loop level

Definition:

$R(t) = \langle \bar{\chi}(t) \overleftrightarrow{D}_\mu \gamma^\mu \chi(t) \rangle$

The Quark Kinetic Operator

Schematically:

$R(t) =$  $+O(\alpha_s)$

$$= -\frac{2N_c}{(4\pi t)^2} + \frac{2m^2 t N_c}{(4\pi t)^2}\left(1 - 2m^2 t e^{-2m^2 t}\Gamma(0,2m^2 t)\right) + O(\alpha_s)$$

$$= t^{-\frac{[R]}{2}}(\mathrm{const} + f(m^2 t))$$

Computation of Integrals

General gradient flow integral:

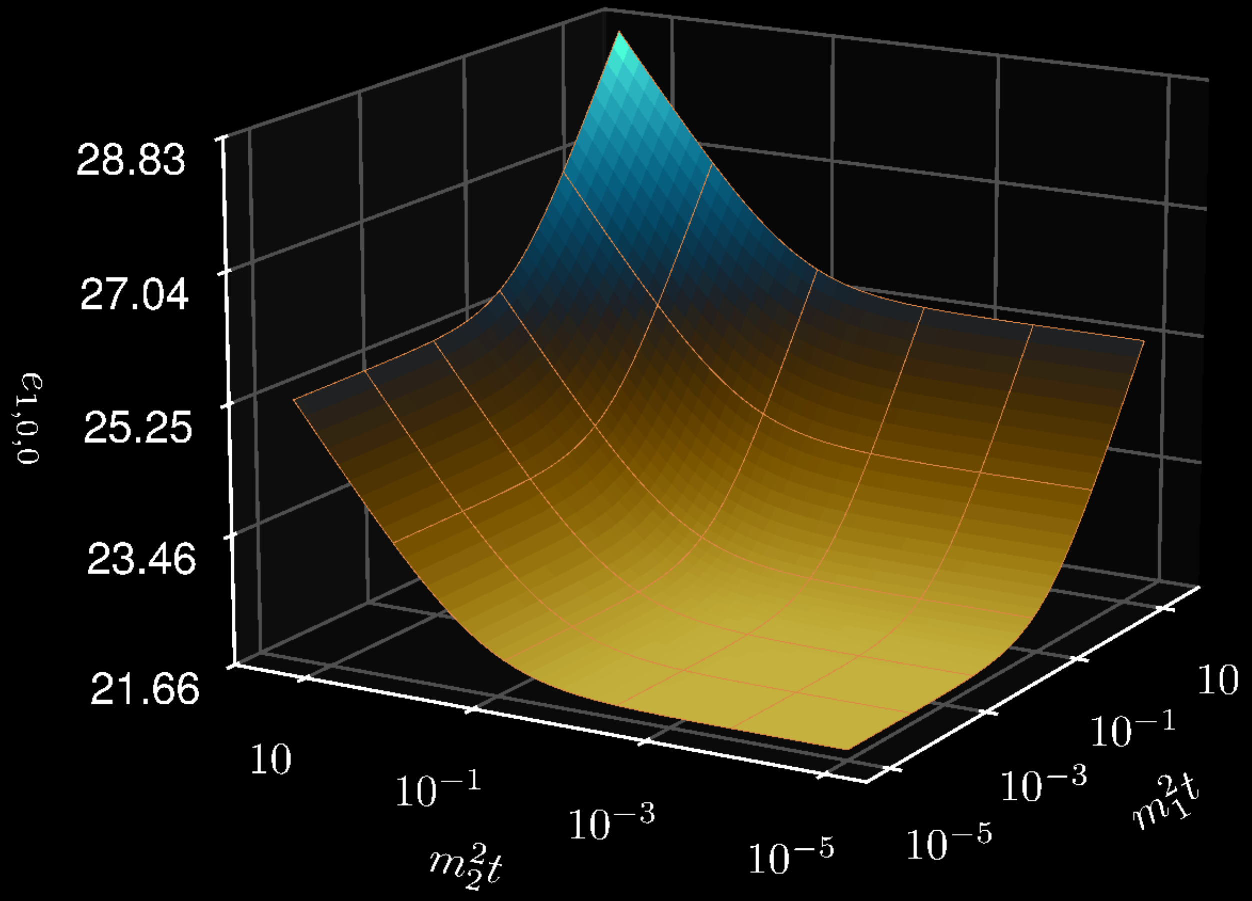
$$f(c,b,n) \sim \int_0^1 d\boldsymbol{u}\boldsymbol{u}^{\boldsymbol{c}} \int_{k_i} \frac{e^{-tb_j p_j^2}}{\left(p_j^2 - m_j\right)^{2n_j}}$$

Massless case: at two loop reduction to six known masters

Massive case: numerical integral with ftint [Harlander et al. 2025]

Based on pySecDec [Borowka et al. 2018]

For more information see talk by Lars Georg



28.83
27.04
25.25
23.46
21.66
$e_{1,0,0}$
10
10^{-1}
10^{-3}
10^{-5}
$m_2^2 t$
10
10^{-1}
10^{-3}
10^{-5}
$m_1^2 t$

Results

Summary

We calculated

- $S(t) = \langle \bar{\chi}(t)\chi(t) \rangle$
- $R(t) = \langle \bar{\chi}(t)\overleftrightarrow{D}_{\mu}\gamma^{\mu}\chi(t) \rangle$
- $E(t) = \langle G^{\mu\nu}(t)G_{\mu\nu}(t) \rangle$

Integral Order	$\mathbf{O}(\alpha_s^0)$	$\mathbf{O}(\alpha_s^1)$	$\mathbf{O}(\alpha_s^2)$
No. Diagrams	$\mathbf{O}(1)$	$\mathbf{O}(10)$	$\mathbf{O}(100)$
No. Integrals	$\mathbf{O}(1)$	$\mathbf{O}(10)$	$\mathbf{O}(1000)$
Time for sector decomposition (s)	$\mathbf{O}(1)$	$\mathbf{O}(10)$	$\mathbf{O}(600)$

to NNLO for systems with 1 massive quark and N_l massless quarks and one massive quark at ~200 values of the flow time t in range:

$$0.001 \leq m^2 t \leq 63$$

for OS and $\overline{MS}$ schemes. [Harlander, *RM* 2025]

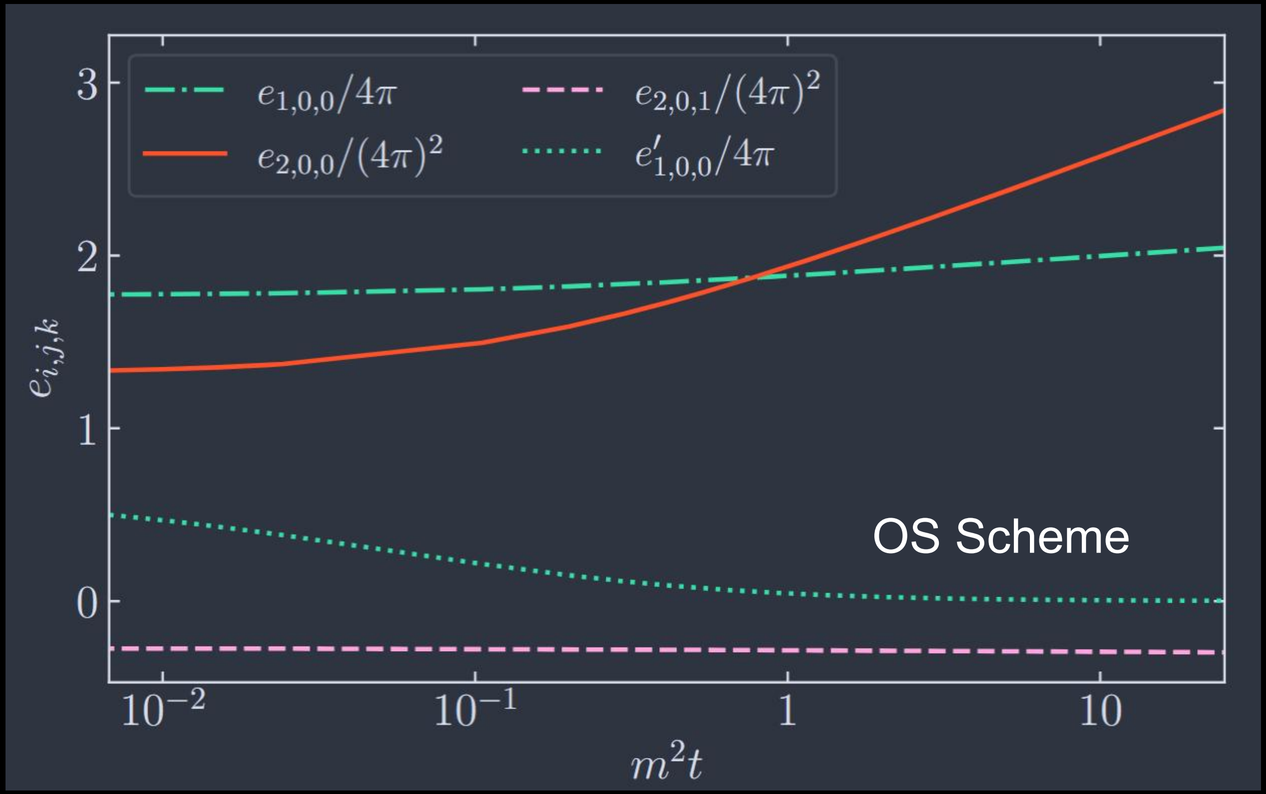
Definition:
$E(t) = \langle G^{\mu\nu}(t) G_{\mu\nu}(t) \rangle$

Flowed Gluon Condensate

$$E(t) = \frac{3\alpha}{4\pi t^2} \frac{n_A}{8} \sum_{n=0}^{N} \left(\frac{\alpha}{4\pi}\right)^n L_{\mu t}^k N_L^j e_{n,k,j}(m^2 t)$$

$$L_{\mu t} = \ln(2\mu^2 t) + \gamma_E$$

$N_L = N_F - 1 =$ no. light quark flavours



Definition:
$E(t) = \langle G^{\mu\nu}(t) G_{\mu\nu}(t) \rangle$

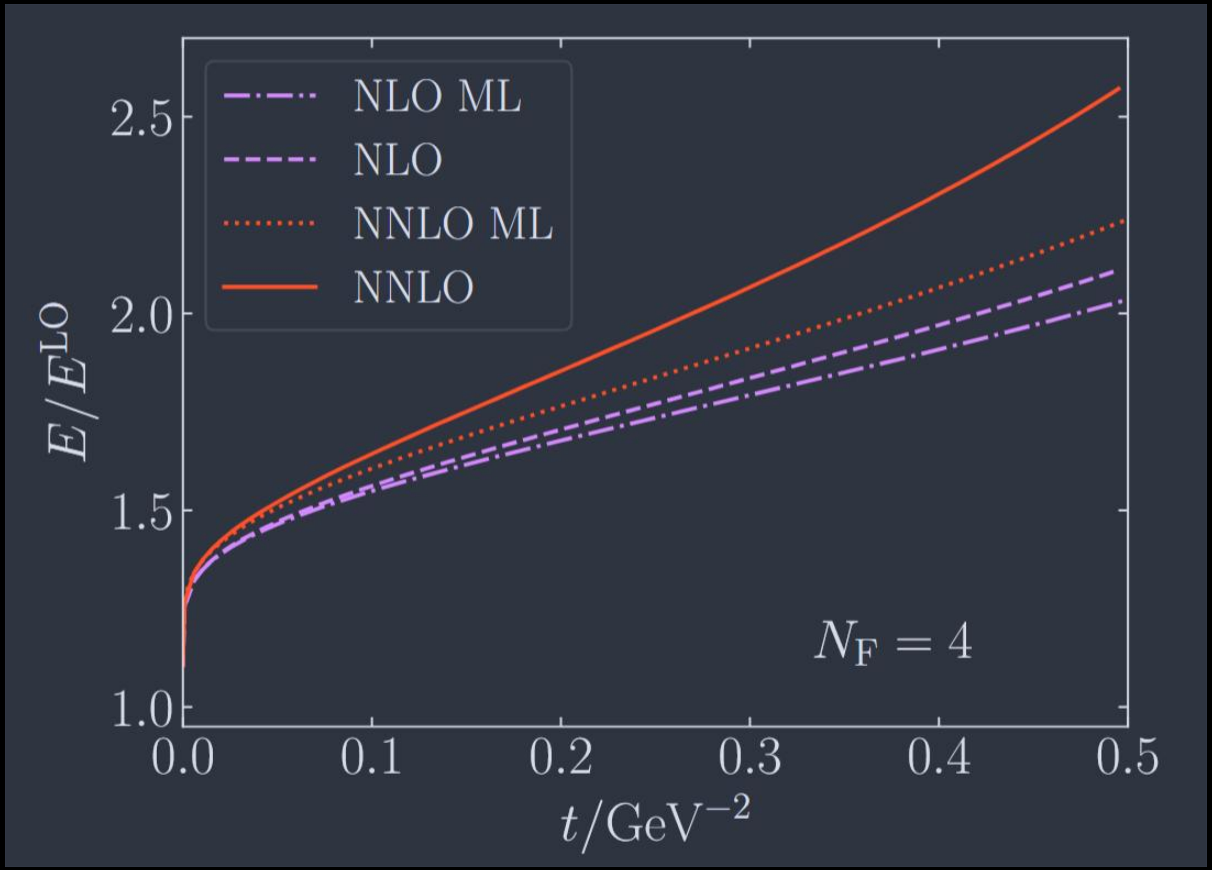
Flowed Gluon Condensate

$\overline{MS}$ running with rundec
$m_c(m_c) = 1.273\text{GeV}$

$$E(t) = \frac{3\alpha}{4\pi t^2} \frac{n_A}{8} \sum_{n=0}^{N} \left(\frac{\alpha}{4\pi}\right)^n L_{\mu t}^k N_L^j e_{n,k,j}(m^2 t)$$

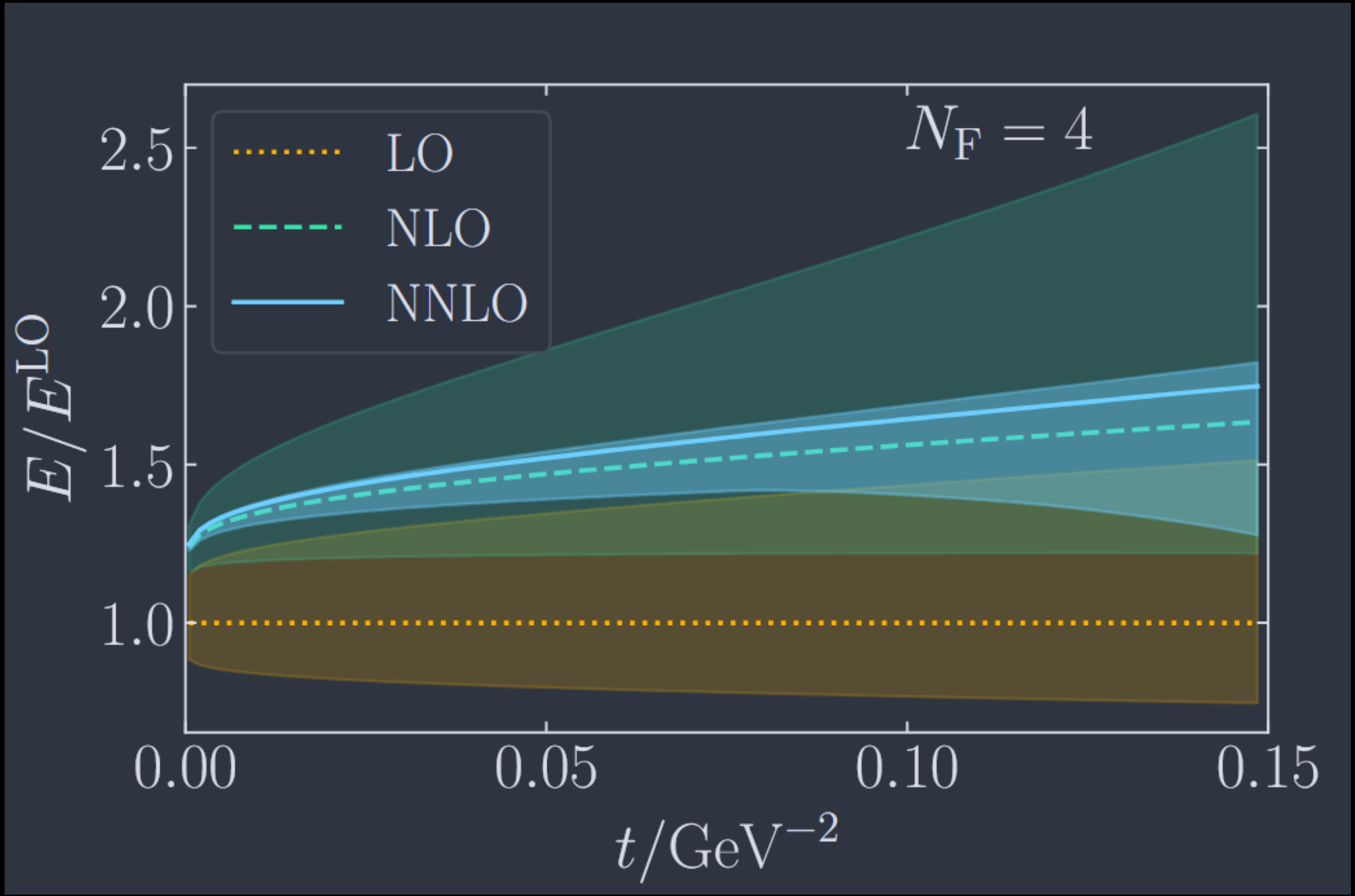
Linear interpolation

$$\mu = \mu_t = \frac{1}{2 e^{\gamma_E} t}$$



Definition:
$E(t) = \langle G^{\mu\nu}(t) G_{\mu\nu}(t) \rangle$

Flowed Gluon Condensate



Scale uncertainty

$$\mu = \kappa \mu_t = \frac{\kappa}{2 e^{\gamma_E} t}$$

for $\frac{1}{2} \leq \kappa \leq 2$.

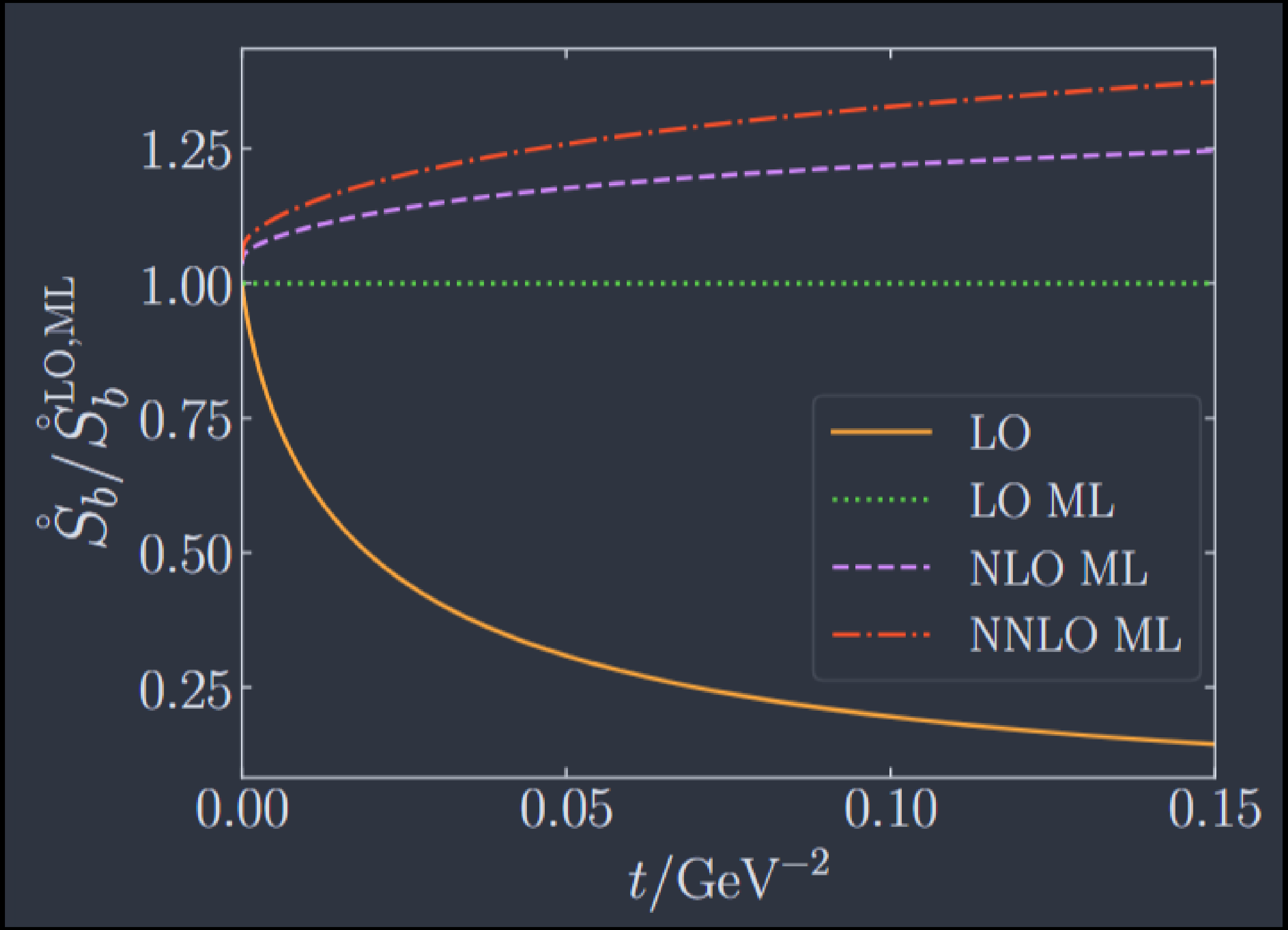
Definition:
$S(t) = \langle \bar{\chi}(t)\chi(t)\rangle$

Flowed Quark Condensate

$$S_f(t) = -\frac{n_c m_f}{8\pi t}\sum_{n=0}^{N}\left(\frac{\alpha}{4\pi}\right)^n L_{\mu t}^k N_L^j s_{n,k,j}(m^2 t)$$

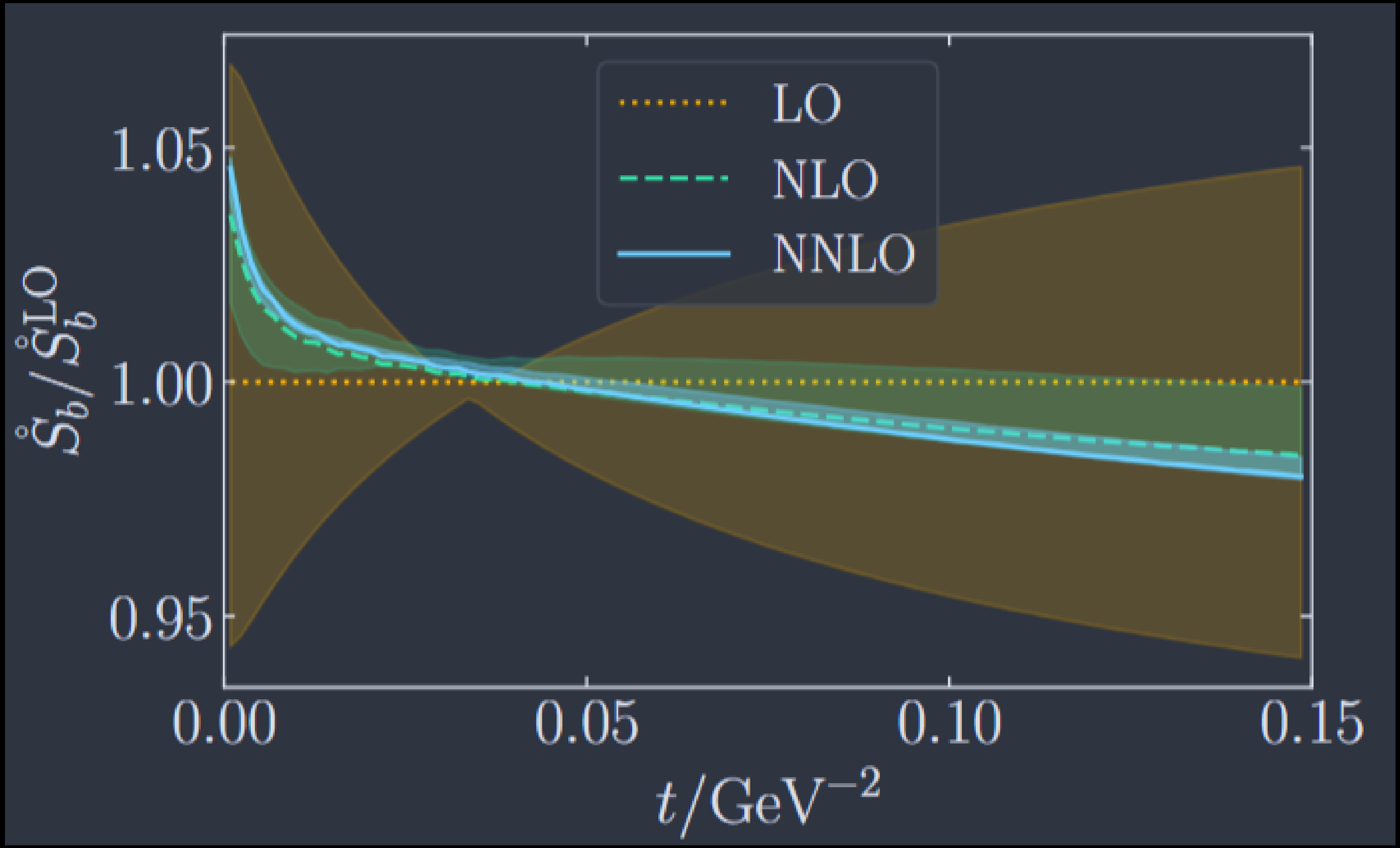
$$\mu_{\text{int}} = \sqrt{\mu_t^2 + m_f^2(m_f)}$$

$$S_{ML}(t) = m_f \frac{d}{dm_f} S(t)\Big|_{,m=0}$$



Definition:
$S(t) = \langle \bar{\chi}(t)\chi(t) \rangle$

Flowed Quark Condensate



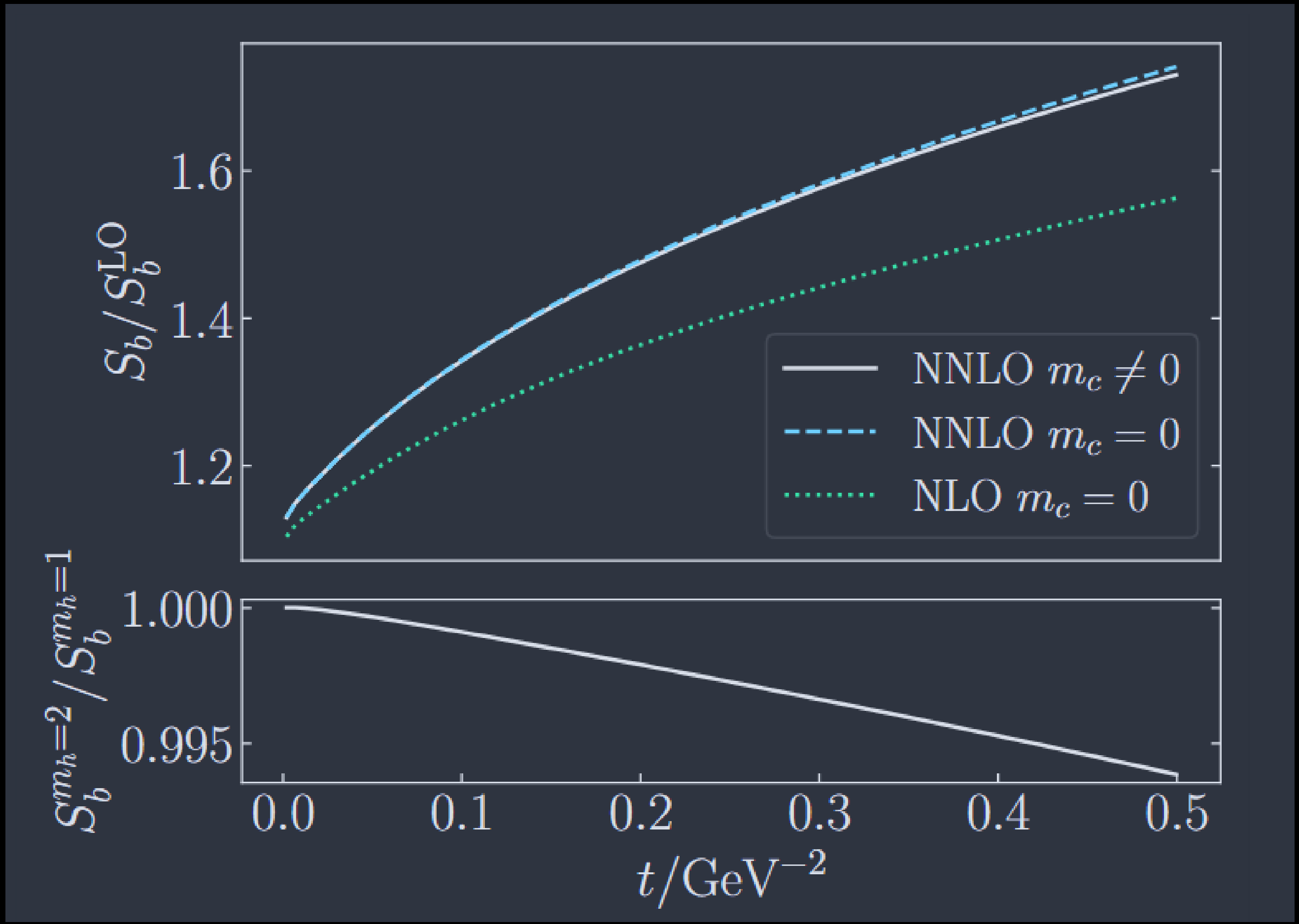
Scale uncertainty

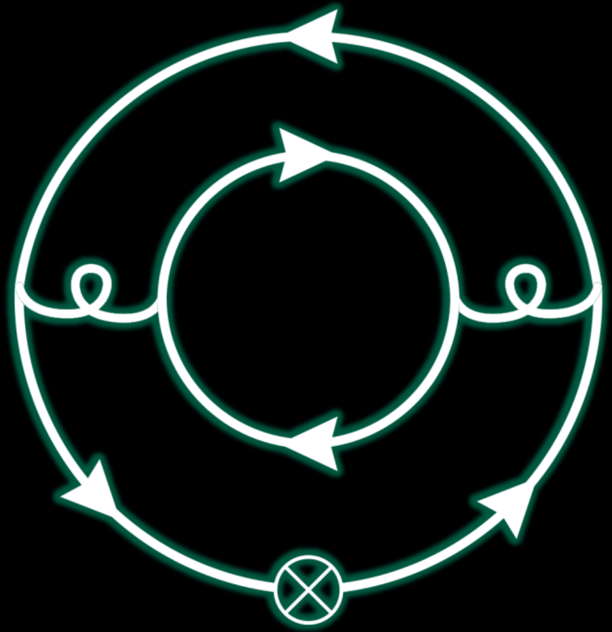
$\mu = \kappa\mu_{\mathrm{int}}$

for $\frac{1}{2} \leq \kappa \leq 2$.

Definition:
$S(t) = \langle\bar{\chi}(t)\chi(t)\rangle$

Flowed Quark Condensate





Thank you for your attention!

Computed the quark mass effects of the following gradient flow quantities to the three-loop level

- $S(t) = \langle \bar{\chi}(t)\chi(t) \rangle$
- $R(t) = \langle \bar{\chi}(t)\overleftrightarrow{D}_{\mu}\gamma^{\mu}\chi(t) \rangle$
- $E(t) = \langle G^{\mu\nu}(t)G_{\mu\nu}(t) \rangle$