Gradient-flowed Operator Product Expansion of the Adler function without Infrared Renormalons

M. Beneke (TU München)

Standard Model parameters and observables from gradient flow
Edinburgh, UK, May 12 – 15, 2026

MB, H. Takaura, arXiv:2510.12193 and 2309.10853

Introduction

External current correlation functions

$$\Pi(Q) = i \int d^4x\, e^{iqx}\, \langle J(x)J(0)\rangle$$

- Electromagnetic or electroweak currents $\bar{\psi}\Gamma\psi$
- $\rangle = |\Omega\rangle$, or $|\bar{B}\rangle$ or $|\text{nucleon}\rangle$ (DIS)
- $Q \gg \Lambda_{\text{QCD}}$ large momentum or heavy quark mass $\rightarrow$ perturbative expansion

Introduction

External current correlation functions

$$\Pi(Q) = i \int d^4x\, e^{iqx}\, \langle J(x) J(0) \rangle$$

- Electromagnetic or electroweak currents $\bar{\psi}\Gamma\psi$
- $\rangle = |\Omega\rangle$, or $|\bar{B}\rangle$ or $|\text{nucleon}\rangle$ (DIS)
- $Q \gg \Lambda_{\text{QCD}}$ large momentum or heavy quark mass → perturbative expansion

Adler function (flavour non-singlet vector current, vacuum, $n_f = 3$ massless flavours)

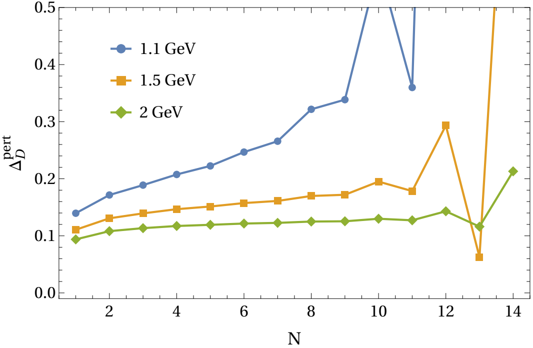
$$D(Q^2) = Q^2 \frac{d\Pi}{dQ^2} = \frac{N_c}{12\pi^2}\left[1 + \sum_{n=1}^{\infty} d_n\, a_Q^n\right] \qquad (a_Q = \alpha_s(Q)/\pi)$$

$$= \frac{N_c}{12\pi^2}\left[1 + a_Q + 1.64 a_Q^2 + 6.37 a_Q^3 + 49.08 a_Q^4 + \ldots\right]$$

[5-loop d_4: Baikov, Chetyrkin, Kühn, 2008]

Adler function, perturbative approximation

$$\Delta_D^{\text{pert}}(Q^2) \equiv \sum_{n=1}^{N} d_n\, a_Q^n$$



Asymptotic behaviour in dimensional regularization

Factorial growth, zero radius of convergence.
Several components of factorial divergence of form

$$c_n \overset{n\gg 1}{=} \alpha_s^{n+1} K(a\beta_0)^n n!\, n^b \left(1 + \frac{s_1}{n} + O(1/n^2)\right)$$

Borel transform

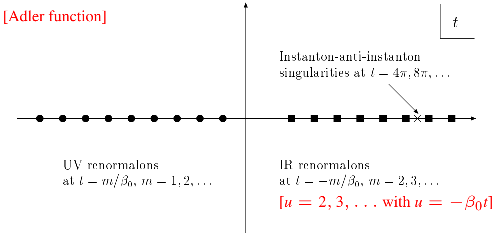
$$F = \sum_{n=0}^{\infty} c_n \alpha_s^{n+1} \implies B[F](t) = \sum_{n=0}^{\infty} c_n \frac{t^n}{n!} \implies F(\alpha) = \int_0^{\infty} dt\, \mathrm{e}^{-t/\alpha} B[F](t)$$

$$c_n = K a^n \Gamma(n+1+b) \iff B[F](t) = \frac{K\Gamma(1+b)}{(1-at)^{1+b}}.$$

Minimal term at $n \approx 1/(|a\beta_0|\alpha_s(Q))$ of size

$$\Delta \approx e^{-1/(|a\beta_0|\alpha_s(Q))} \approx \left(\frac{\Lambda^2}{Q^2}\right)^{1/a} \approx \text{size of ambiguity}$$

Renormalon (and instanton) singularities



- UV renormalons – from large loop momentum
  Sign-alternating, singularity structure related to higher-dim operators in the cut-off QCD Lagrangian [Parisi, 1977; MB, Kivel, Braun 1997]
- IR renormalons – from small loop momentum
  Fixed-sign, singularity structure related to higher-dim operators in the OPE [Gross, Neveu, 1974; Lautrup, 1977; ‘t Hooft, 1977; David, 1984; Mueller, 1985; Zakharov, 1992; MB, 1993]
- Instanton-anti-instanton – number of diagrams
  [Bogomolny, Fadeev, 1977; Balitsky, 1991]

IR renormalons and the operator product expansion (OPE)

Large momentum/small coupling expansion of the Adler function:

$$D(Q^2) = -Q^2 \frac{d\Pi(Q^2)}{dQ^2} = \sum_{k=0} \underbrace{C_k(\alpha_s(Q))}_{\text{perturbative series}} \times \frac{1}{(Q^2)^k} \times \underbrace{\langle \mathcal{O}_k \rangle}_{\substack{\text{condensates } (k>0) \\ \text{“power corrections”}}}$$

$$= \sum_{k=0} \underbrace{\left[\frac{\langle \tilde{\mathcal{O}}_k \rangle}{\Lambda_{\rm QCD}^{2k}}\right]}_{\text{pure number}} \times \left[e^{-\frac{1}{(-\beta_0)\alpha_s(Q)}}\right]^k (-\beta_0\alpha_s(Q))^{k\beta_1/\beta_0^2 - \gamma_{0,k}/\beta_0} \times \sum_{n=0} c_k^{(n)} \alpha_s(Q)^n$$

IR renormalons and the operator product expansion (OPE)

Large momentum/small coupling expansion of the Adler function:

$$D(Q^2) = -Q^2 \frac{d\Pi(Q^2)}{dQ^2} = \sum_{k=0} \underbrace{C_k(\alpha_s(Q))}_{\text{perturbative series}} \times \frac{1}{(Q^2)^k} \times \underbrace{\langle \mathcal{O}_k \rangle}_{\substack{\text{condensates } (k>0) \\ \text{“power corrections”}}}$$

$$= \sum_{k=0} \underbrace{\left[\frac{\langle \tilde{\mathcal{O}}_k \rangle}{\Lambda_{\rm QCD}^{2k}}\right]}_{\text{pure number}} \times \left[e^{-\frac{1}{(-\beta_0)\alpha_s(Q)}}\right]^k (-\beta_0 \alpha_s(Q))^{k\beta_1/\beta_0^2 - \gamma_{0,k}/\beta_0} \times \sum_{n=0} c_k^{(n)} \alpha_s(Q)^n$$

Transseries structure: Series $c_k^{(n)}$ divergent and not Borel-summable (IR renormalons) but the entire double expansion is unambiguous.

- Condensates must be ambiguous in dim reg and their values related to the summation prescription (or vice versa!) [Can be checked in the 2d O(N) non-linear sigma model, David, 1984, 1986; MB, 1998, MB, Braun, Kivel, 1998; Marino et al., 2024, 2025]
- But in QCD cannot do nonperturbative computations in dim reg
- Ambiguities related to UV power divergences of local operators

Asymptotic behaviour and cut-off / Wilsonian OPE

Main obstruction from IR renormalons

Cut-off factorization [advocated by Novikov, Shifman, Vainshtein, Zakharov (1984, 1985)]

$$D(Q^2) = \sum_{k=0} \left[\frac{\langle \mathcal{O}_k \rangle (\Lambda_f)}{Q^{2k}} \right] \times \sum_{n=0} c_k^{(n)}(\Lambda_f, Q) \alpha_s(Q)^n$$

- Restricts loop momenta to $k > \Lambda_f$, where $\Lambda_f \gg \Lambda_{\text{QCD}}$ is a hard factorization scale.
- Perturbative series ($c_k^{(n)}(\Lambda_f, Q)$) don't have IR renormalon factorial divergence.
- Condensates unambiguous but contain terms proportional to Λ_f^{2k} from power UV divergences.

Asymptotic behaviour and cut-off / Wilsonian OPE

Main obstruction from IR renormalons

Cut-off factorization [advocated by Novikov, Shifman, Vainshtein, Zakharov (1984, 1985)]

$$D(Q^2) = \sum_{k=0} \left[\frac{\langle \mathcal{O}_k \rangle (\Lambda_f)}{Q^{2k}} \right] \times \sum_{n=0} c_k^{(n)}(\Lambda_f, Q) \alpha_s(Q)^n$$

- Restricts loop momenta to $k > \Lambda_f$, where $\Lambda_f \gg \Lambda_{\text{QCD}}$ is a hard factorization scale.
- Perturbative series ($c_k^{(n)}(\Lambda_f, Q)$) don't have IR renormalon factorial divergence.
- Condensates unambiguous but contain terms proportional to Λ_f^{2k} from power UV divergences.

Has never been used in practice: Calculation with hard cut-offs too difficult beyond one loop. No gauge invariant practical implementation of a hard cut-off, but a gauge-invariant regulator is crucial.

Gradient flow [Lüscher, 1006.4518; Lüscher, Weisz 1101.0963]

Here: continuum version (“Wilson flow” on lattice)
Define “flowed” gluon field $B_\mu(t,x)$ by

$$\partial_t B_\mu = \tilde{D}_\nu \tilde{G}_{\nu\mu} + \xi_0 \tilde{D}_\mu \partial_\nu B_\nu \,, \qquad B_\mu|_{t=0} = A_\mu$$

$t =$ flow “time”, $\tilde{G}_{\mu\nu}$, $\tilde{D}_\mu$ usual definitions but with B_μ.

Interpretation: Smeared gluon field over distance $\sqrt{2t}$. LO solution

$$B_\mu(t,x) = \int d^d y \, K(t, x-y) \, A_\mu(x) \qquad K(t,z) = \frac{e^{-z^2/(4t)}}{(4\pi t)^{d/2}}$$

Gradient flow [Lüscher, 1006.4518; Lüscher, Weisz 1101.0963]

Here: continuum version ("Wilson flow" on lattice)
Define "flowed" gluon field $B_\mu(t,x)$ by

$$\partial_t B_\mu = \tilde{D}_\nu \tilde{G}_{\nu\mu} + \xi_0 \tilde{D}_\mu \partial_\nu B_\nu, \qquad B_\mu|_{t=0} = A_\mu$$

$t =$ flow "time", $\tilde{G}_{\mu\nu}$, $\tilde{D}_\mu$ usual definitions but with B_μ.

Interpretation: Smeared gluon field over distance $\sqrt{2t}$. LO solution

$$B_\mu(t,x) = \int d^d y\, K(t, x-y)\, A_\mu(x) \qquad K(t,z) = \frac{e^{-z^2/(4t)}}{(4\pi t)^{d/2}}$$

Key idea: $1/\sqrt{8t}$ provides a gauge-invariant, Wilsonian cut-off. Formulate OPE in terms of gradient-flow regularized local operators at finite flow time $\Lambda_{\rm QCD} \ll 1/\sqrt{2t} \ll Q$

- Well-defined condensates and short-distance coefficients: Divergent series from IR renormalon disappear
- Can be done non-perturbatively in the continuum and on the lattice ($a \ll \sqrt{2t} \ll L$). Decouples continuum limit from power divergences and factorization

$$D(Q) = C_0(\alpha_s, Q/\mu) + C_{GG}(\alpha_s, Q/\mu)\,\frac{1}{Q^4}\,\langle\frac{\alpha_s}{\pi}G^2\rangle(\mu) + O(1/Q^6)$$

[C_0 known to $\mathcal{O}(\alpha_s^4)$, see above, C_{GG} to $\mathcal{O}(\alpha_s^2)$ (Harlander 1998, Brüser et al., 2408.03989)]

Any definition / subtraction of the divergent perturbative series implies a renormalization scheme for the quartic power-divergences of the operator $\langle\frac{\alpha_s}{\pi}G^2\rangle(\mu)$.

Action density

$$E(t) = \frac{g^2}{4}G^A_{\mu\nu}(t)G^{A\mu\nu}(t)$$

Its expectation value, $\langle E(t)\rangle$, can be regarded as a gauge-invariant non-perturbative definition of the gluon condensate with cut-off $\Lambda_{\rm UV} \propto 1/\sqrt{t}$, which can replace the ill-defined $\langle\frac{\alpha_s}{\pi}G^2\rangle(\mu)$ in the $\overline{\rm MS}$-OPE.

[The idea to replace local operators in an OPE by gradient-flowed operators was explored in a different context by [Monahan, Originos, 1410.3393, 1501.05348] – (“locally smeared OPE”)]

OPE of the action density / subtracted Adler function

Small flow-time expansion $t \ll 1/\Lambda^2_{\rm QCD}$ [Lüscher, 1006.4518; Harlander, Neumann 1606.03756 (NNLO)]

$$\frac{1}{\pi^2}\,\langle E(t)\rangle = \frac{\widetilde{C}_0(t)}{t^2} + \widetilde{C}_{GG}(t)\langle\frac{\alpha_s}{\pi}GG\rangle + \mathcal{O}(t\times \text{dim-6})$$

$\widetilde{C}_0$ known to NLO [Lüscher, 1006.4518] and NNLO [Harlander, Neumann 1606.03756], $\widetilde{C}_{GG}(t)$ to NNLO [Harlander, Kluth, Lange, 2007.01057]

OPE of the action density / subtracted Adler function

Small flow-time expansion $t \ll 1/\Lambda_{\rm QCD}^2$ [Lüscher, 1006.4518; Harlander, Neumann 1606.03756 (NNLO)]

$$\frac{1}{\pi^2}\,\langle E(t)\rangle = \frac{\widetilde{C}_0(t)}{t^2} + \widetilde{C}_{GG}(t)\langle \frac{\alpha_s}{\pi} GG\rangle + \mathcal{O}(t \times \text{dim-6})$$

$\widetilde{C}_0$ known to NLO [Lüscher, 1006.4518] and NNLO [Harlander, Neumann 1606.03756], $\widetilde{C}_{GG}(t)$ to NNLO [Harlander, Kluth, Lange, 2007.01057]

Eliminate $\langle \frac{\alpha_s}{\pi} G^2\rangle(\mu)$ in the standard OPE for the Adler function

$$D(Q) = \underbrace{\left[C_0(Q) - \frac{1}{t^2 Q^4}\frac{C_{GG}(Q)}{\widetilde{C}_{GG}(t)} \times \widetilde{C}_0(t)\right]}_{u\,=\,2\text{ renormalon cancels}} + \frac{1}{Q^4}\,\underbrace{\frac{C_{GG}(Q)}{\widetilde{C}_{GG}(t)}\,\frac{1}{\pi^2}\,\langle E(t)\rangle}_{\text{non-perturbatively defined}} + O(1/Q^6)$$

Only fixed-order calculations needed to obtain a better-behaved expansion.

Borel function models [MB, Jamin, 2008]

Incorporate the knowledge of asymptotic behaviour into an Ansatz for the Adler function that reproduces known $c_{n,1}$ to $n = 4$ and $c_{5,1} = 283$.

$$B[D](u) \ = \ B[D_1^{\mathrm{UV}}](u) + B[D_2^{\mathrm{IR}}](u) + B[D_3^{\mathrm{IR}}](u) + d_0^{\mathrm{PO}} + d_1^{\mathrm{PO}} u$$

- Fit Stokes constants K_p for $u = -1, 2, 3$ to $c_{3,1}$, $c_{4,1}$ and $c_{5,1}$, and adjust $d_{0,1}^{\mathrm{PO}}$ to reproduce $c_{1,1}$ and $c_{2,1}$.
- Pole ansatz works well already at $n = 2$ (d_1^{PO} small). Apparently the series is very regular.

Borel function models [MB, Jamin, 2008]

Incorporate the knowledge of asymptotic behaviour into an Ansatz for the Adler function that reproduces known $c_{n,1}$ to $n = 4$ and $c_{5,1} = 283$.

$$B[D](u) \;=\; B[D_1^{\mathrm{UV}}](u) + B[D_2^{\mathrm{IR}}](u) + B[D_3^{\mathrm{IR}}](u) + d_0^{\mathrm{PO}} + d_1^{\mathrm{PO}} u$$

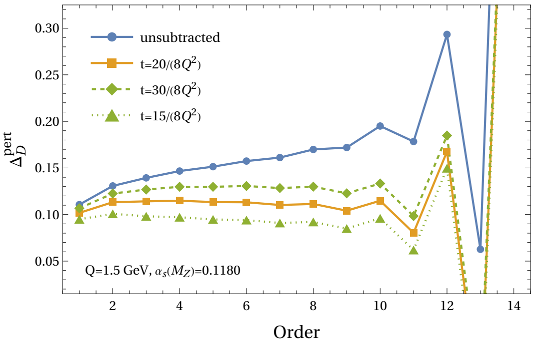
- Fit Stokes constants K_p for $u = -1, 2, 3$ to $c_{3,1}$, $c_{4,1}$ and $c_{5,1}$, and adjust $d_{0,1}^{\mathrm{PO}}$ to reproduce $c_{1,1}$ and $c_{2,1}$.
- Pole ansatz works well already at $n = 2$ (d_1^{PO} small). Apparently the series is very regular.

The gradient flow action density has no UV renormalons. Ansatz (in practice for the entire subtraction term including $C_{GG}(Q)/\widetilde{C}_{GG}(t)$):

$$B[E](u) \;=\; B[E_2^{\mathrm{IR}}](u) + B[E_3^{\mathrm{IR}}](u) + e_0^{\mathrm{PO}} + e_1^{\mathrm{PO}} u$$

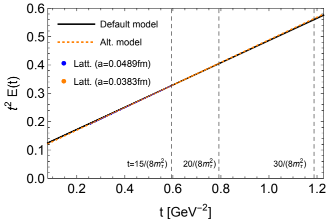
Three unknowns (one fixed by $u = 2$ cancellation). Matches the available three exactly known coefficients of $\tilde{C}_0$

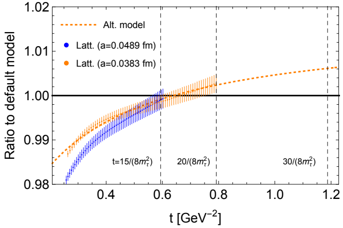
Adler function series with gradient-flow subtraction



Perturbative growth stops around second to third order, until UV renormalon divergence sets in.

Lattice data from 1411.3982, 1712.04884 (CLS), reweighting according to Kuberski, 2306.02385, kindly provided by Saez Gonzalvo





Almost linear dependence of $t^2E(t)$ on t in the t-range of interest. Parameterize as

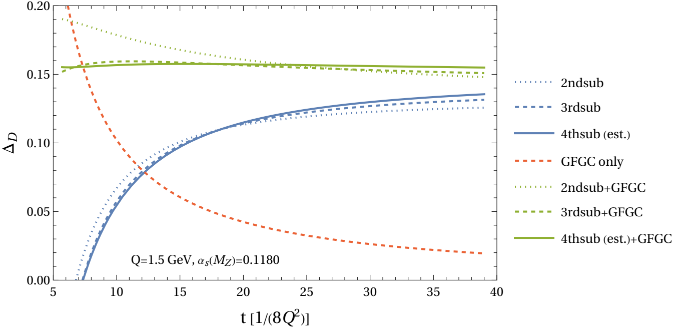
$$t^2E(t)|_{\rm lin} = 0.3 + \frac{0.3}{w_0^2}(t - t_0)$$
$$t^2E(t)|_{\rm fine\ latt.} = 0.29929 + 0.39855(t - t_0)$$

[t_0, w_0 defined from

$$t^2E(t)|_{t=t_0} = 0.3, \quad t\frac{d}{dt}\left(t^2E(t)\right)\Big|_{t=w_0^2} = 0.3,$$

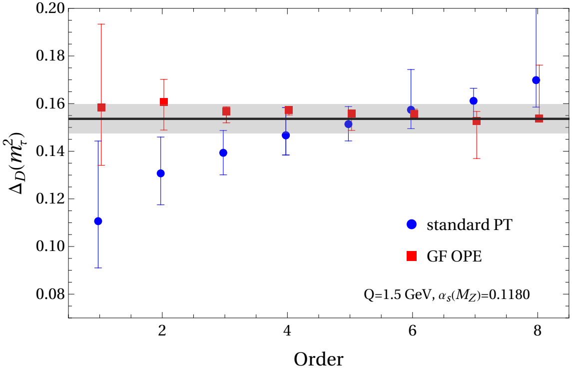
and given by the FLAG averages $\sqrt{t_0} = 0.14292$ fm, $w_0 = 0.17256$ fm]

Adler function series with gradient-flow subtraction + gluon condensate



Independence of t in the $8tm_\tau^2$ window 15 – 30, little dependence on perturbative truncation order.

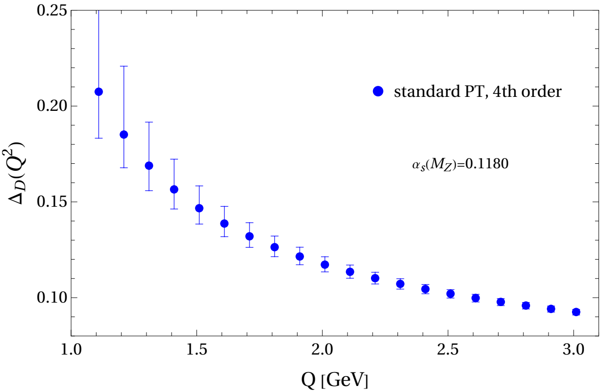
Adler function, perturbative vs. non-perturbative (I)



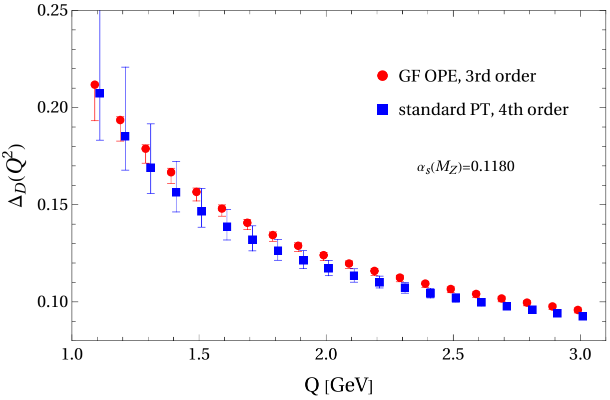
Best approximation at 3rd or 4th order. Increasing uncertainty at higher orders due to UV renormalon (removable).

In grey: Borel sum and ambiguity of the unsubtracted perturbative Adler function

Adler function, perturbative vs. non-perturbative (II) - Q dependence



Adler function, perturbative vs. non-perturbative (II) - Q dependence



Extends the range where perturbation theory is reliable to smaller Q, larger coupling. Applications to QCD sum rules, α_s and strange quark mass determinations

Improvements / Wish-list for lattice

Gluon condensate / action density

- We use lattice data at $t = 0.25 \ldots 0.8\,\mathrm{GeV}^{-2}$. Rely on extrapolation to larger t when analyzing the most interesting region $Q = 1.0 \ldots 1.8\,\mathrm{GeV}$.

  Are there continuum-extrapolated lattice calculations for t up to $t \approx 3\,\mathrm{GeV}^{-2}$?
  Can you do them? (Small t would also be helpful ....)

Improvements / Wish-list for lattice

Gluon condensate / action density

- We use lattice data at $t = 0.25\ldots 0.8\,\mathrm{GeV}^{-2}$. Rely on extrapolation to larger t when analyzing the most interesting region $Q = 1.0\ldots 1.8\,\mathrm{GeV}$.

Are there continuum-extrapolated lattice calculations for t up to $t \approx 3\,\mathrm{GeV}^{-2}$?
Can you do them? (Small t would also be helpful ....)

Beyond the gluon condensate – dimension-six operators

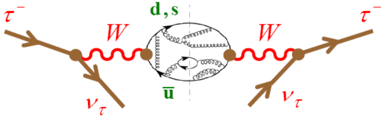
- Important check that there exists a t window where the GF-OPE converges.
- Need lattice calculations of flowed four-light-quark operators in various colour, flavour and Dirac matrix combinations (and some gluonic, less important).

$$\frac{E(t)}{\pi^2} = \frac{Y_1(t)}{t^2} + Y_{GG}(t)\,\langle \frac{\alpha_s}{\pi} GG\rangle + t\,Y_{4q}(t)\,\langle(\bar{q}q)^2\rangle + \mathcal{O}(t^2\Lambda^8_{\mathrm{QCD}})$$

$$\langle(\bar{\chi}\chi)^2\rangle(t) = \frac{X_1(t)}{t^3} + \frac{X_{GG}(t)}{t}\langle \frac{\alpha_s}{\pi} GG\rangle + X_{4q}(t)\,\langle(\bar{q}q)^2\rangle + \mathcal{O}(t\Lambda^8_{\mathrm{QCD}})$$

$$R_\tau \equiv \frac{\Gamma[\tau^- \to \text{hadrons}\,\nu_\tau(\gamma)]}{\Gamma[\tau^- \to e^-\overline{\nu}_e\nu_\tau(\gamma)]} = \frac{1-\mathcal{B}_e-\mathcal{B}_\mu}{\mathcal{B}_e} = R_{\tau,V}+R_{\tau,A}+R_{\tau,S} = 3.6381 \pm 0.0075$$

[HFLAV, 2206.07501]

$$\Gamma_{\tau\to\nu_\tau+\text{had}} \sim \text{Im}\left\{ \right.$$ 

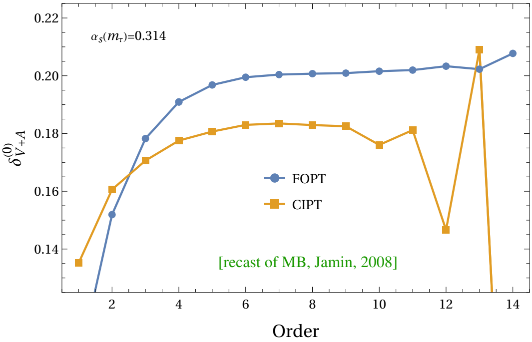
$$\left. \right\}$$

Focus on non-strange final states

$$R_\tau = 12\pi \int_0^{M_\tau^2} \frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2 \left[\left(1+2\frac{s}{M_\tau^2}\right)\text{Im}\,\Pi^{(1)}(s)+\text{Im}\,\Pi^{(0)}(s)\right]$$

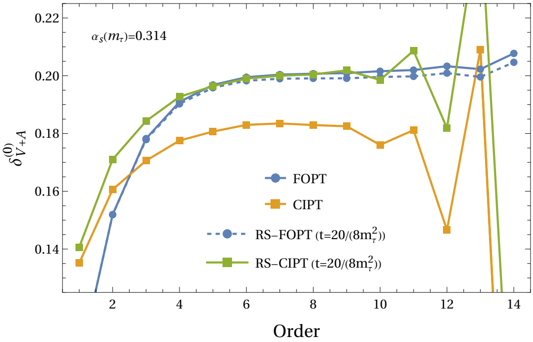
$$\Pi_{\mu\nu}^{V/A}(p) \equiv i\int dx\, e^{ipx}\,\langle\Omega|\,T\{J_\mu^{V/A}(x)\,J_\nu^{V/A}(0)^\dagger\}|\Omega\rangle = (p_\mu p_\nu - g_{\mu\nu}p^2)\,\Pi^{V/A,(1)} + p_\mu p_\nu\,\Pi^{V/A,(0)}$$

Hadronic τ decay with gradient-flow subtraction [MB, Takaura, 2309.10853]



- FO/CI difference *increases* by adding more orders. Systematic problem.
- Problem arises from renormalon pole related to the gluon condensate [MB, Boito, Jamin, 2012]
  CIPT is inconsistent with OPE [Regner, Hoang, 2021; Gracia, Hoang, Mateu, 2022]

Hadronic τ decay with gradient-flow subtraction [MB, Takaura, 2309.10853]



- CI and FO approach now similar values.
  (How well, depends on choice of t.)
- CI converges more quickly than FO at low orders, now to the correct value.

Conclusion

- The gradient-flowed OPE attains non-perturbative accuracy while maintaining the benefits of dim reg calculations.
- $1/\sqrt{t}$ acts as a “hard cut-off”. Eliminates IR renormalons and makes condensates well-defined in the continuum limit.
- Works in low orders without explicit knowledge of asymptotic behaviour at a low subtraction scale.
- “Simple” scheme for lattice calculations of the condensates (with continuum extrapolation at finite t)

Conclusion

- The gradient-flowed OPE attains non-perturbative accuracy while maintaining the benefits of dim reg calculations.
- $1/\sqrt{t}$ acts as a "hard cut-off". Eliminates IR renormalons and makes condensates well-defined in the continuum limit.
- Works in low orders without explicit knowledge of asymptotic behaviour at a low subtraction scale.
- "Simple" scheme for lattice calculations of the condensates (with continuum extrapolation at finite t)

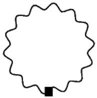
- Adler function: smaller Q (large coupling) is possible, smaller theoretical errors. Applications to QCD sum rules, α_s and strange quark mass determinations
- FOPT/CIPT resolved by gluon-condensate renormalon subtraction. Affects α_s value from τ decay.

Backup slides

Condensate ambiguities at order $\mathcal{O}(1/N)$

Define condensates with a momentum cut-off $\mu_f \gg m$ [Novikov, Shifman, Vainshtein, Zakharov (1984)]

$$\langle \alpha^2 \rangle(\mu_f, m) \;=$$



$$= \int\limits_{p^2<\mu^2} \frac{d^2p}{(2\pi)^2}\, D_\alpha(p)$$

$$= \; m^4 \left[F(\ln A) + F(-\ln A) - 2\gamma_E \right]$$

$$A = \left(\sqrt{1 + \frac{\mu_f^2}{4m^2}} + \sqrt{\frac{\mu_f^2}{4m^2}} \right)^4, \qquad \ln A \stackrel{\mu_f \gg m}{\rightarrow} \frac{2}{g(\mu_f)} \gg 1$$

$F(x) = \mathrm{Ei}(-x) - \ln x$ has an essential singularity at $x = 0$ but no discontinuity. Asymptotic expansion for large x has a Stokes discsontinuity at negative arguments

$$F(-x) = e^x \left[\sum_{n=0}^{\infty} \frac{n!}{x^{n+1}} - e^{-x} \left(\ln x \mp i\pi \right) \right]$$

$$\langle \alpha^2 \rangle = \mu^4 \sum_{n=0}^{\infty} \left(\frac{g}{2} \right)^{n+1} n! \, + \, 2g\, \mu^2 m^2 + \, m^4 \left[-2 \ln \frac{2}{g} \pm i\pi \, - 2\gamma_E - 4g + \frac{g^2}{2} \right] + \mathcal{O}\left(\frac{m^2}{\mu^2} \right)$$

Condensate ambiguities at order $\mathcal{O}(1/N)$

Define condensates in dimensional regularization [David (1982, 1984, 1986)]
Computation of the ambiguity [MB (1998)]

$$\langle\alpha^2\rangle(\mu,m) = \mu^\epsilon \int \frac{d^dp}{(2\pi)^d}\, D_\alpha(p,d) = \frac{m^2/(4\pi)}{\Gamma(1-\epsilon/2)} \left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon} \int_0^\infty d\left(\frac{p^2}{m^2}\right) \left(\frac{p^2}{m^2}\right)^{-\epsilon} D_\alpha(p,d)$$

Can set $d=2$ in $D_\alpha(p,d)$. Define $u = p^2/m^2$ and use

$$D_\alpha(p) = 4\pi m^2 u \sum_{k,l} \frac{c_{kl}}{u^k}\, \frac{1}{\ln^l u}, \qquad \frac{1}{\ln^l u} = \int_0^\infty dv\, v^{l-1} u^v .$$

Then (keeping only singular terms in v-integral)

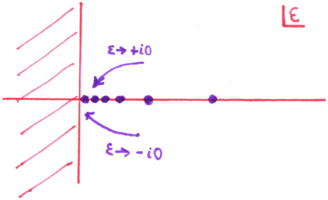
$$\langle\alpha^2\rangle(\mu,m) \sim \frac{m^4}{\Gamma(1-\epsilon/2)} \left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon} \sum_{k,l} c_{kl} \int_0^\infty dv\, \frac{v^{l-1}}{-2+\epsilon+k+v}$$

$$\left[\lim_{\epsilon\to+i0} - \lim_{\epsilon\to-i0}\right] \langle\alpha^2\rangle(\mu,m) = 2\pi i\, m^4 \sum_{k=0}^{1} \sum_{l=1} c_{kl}\,(2-k)^{l-1} = 2\pi i\, m^4$$

Condensate ambiguities at order $\mathcal{O}(1/N)$

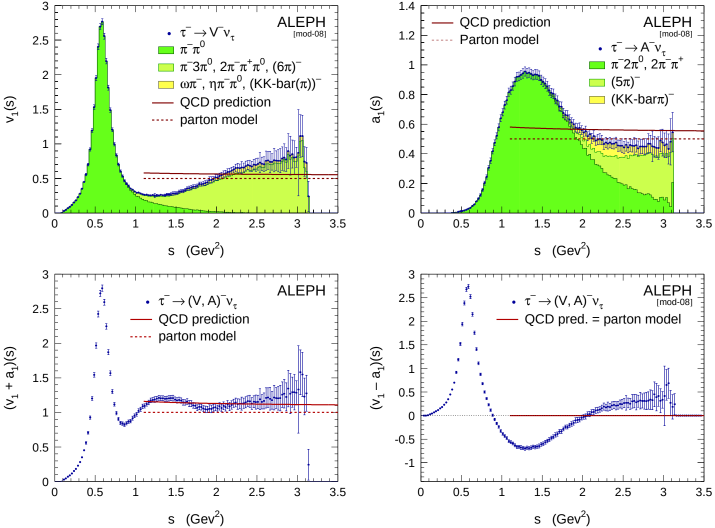
The ambiguity arises from power divergences. They do not lead to $1/\epsilon$ poles but to accumulating poles near $\epsilon = 0$.

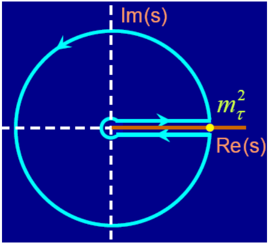
For a condensate of dimension D, the poles of the v-integral are located at $\epsilon = 2k/l$ ($k < d$, l positive integer).



No ambiguity at finite ϵ. No renormalon divergence either. Limits $\epsilon \to 0$ and $n \to \infty$ don't commute.

ALEPH spectral functions





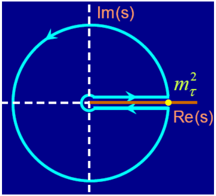
- Analyticity
- Condensate expansion
- Slightly Euclidean [$(1-x)^3$ suppression]
- $D^{(1+0)}(s) \equiv -s\,\frac{d}{ds}\left[\Pi^{(1+0)}(s)\right]$ (Adler fn)

$$
\begin{aligned}
R_\tau &= 6\pi i \oint_{|s|=M_\tau^2} \frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2\left[\left(1+2\frac{s}{M_\tau^2}\right)\Pi^{(1)}(s)+\Pi^{(0)}(s)\right] \\
&= -i\pi \oint_{|x|=1} \frac{dx}{x}\,(1-x)^3\left[3\,(1+x)\,D^{(1+0)}(M_\tau^2 x)+4\,D^{(0)}(M_\tau^2 x)\right] \\
&= N_c\,S_{\mathrm{EW}}\,|V_{ud}|^2\left[1+\delta^{(0)}+\delta'_{\mathrm{EW}}+\sum_{D\geq 2}\frac{C_D(s,\mu)\langle O_D(\mu)\rangle}{(-s)^{D/2}}\right]
\end{aligned}
$$

[Braaten, Narison, Pich, 1992]

FO and CI perturbation theory

$$R_\tau = -i\pi \oint_{|x|=1} \frac{dx}{x} (1-x)^3 \left[3\,(1+x)\, D(M_\tau^2 x) + 4\, D^{(0)}(M_\tau^2 x) \right]$$



FOPT $\qquad \delta_{\rm FO}^{(0)} = \sum_{n=1}^{\infty} a(M_\tau^2)^n \sum_{k=1}^{n} k\, c_{n,k}\, J_{k-1} \qquad J_l \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x) \ln^l(-x)$

CIPT $\qquad \delta_{\rm CI}^{(0)} = \sum_{n=1}^{\infty} c_{n,1}\, J_n^a(M_\tau^2) \qquad J_n^a(M_\tau^2) \equiv \frac{1}{2\pi i} \oint_{|x|=1} \frac{dx}{x} (1-x)^3 (1+x)\, a^n(-M_\tau^2 x)$

[Le Diberder, Pich, 1993] - Sums π^2 terms

Condensate expansion

- D=2 m_q^2
  $\longrightarrow (3.1 \pm 8.6) \cdot 10^{-5}$
- D=4 $m_q^4,\, m_q\langle\bar{q}q\rangle,\, \langle\frac{\alpha_s}{\pi}GG\rangle$
  $\longrightarrow (6.3 \pm 3.3) \cdot 10^{-4}$

  Suppression of the D=4 contribution due to the kinematic weight function
  $(1-x)^3(1+2x) = 1 - 2x + 2x^3 - x^4$
- D=6 $\langle\bar{q}q\bar{q}q\rangle,\, \langle\alpha_s G^3\rangle$
  $\longrightarrow (-4.8 \pm 2.9) \cdot 10^{-3}$ – dominant

  Factor 10 cancellation between V and A. This explains $R_{\tau,V-A} \approx 0.08$ ($\rightarrow$ Fig.)
- S+P $D^{(0)}(s)$ contribution dominated by the calculable pion pole contribution
  $\longrightarrow (-2.64 \pm 0.05) \cdot 10^{-3}$

Non-perturbative terms very small [3.5% of perturbative contribution!] due to V+A cancellation and kinematic suppression

$$\delta_{\rm PC} = (-\,6.8 \pm 3.5) \cdot 10^{-3}$$

Nevertheless, the gluon condensate will play an important role in the following.