Factorization in QCD and Beyond, Edinburgh, 6 May 2026

Beyond Facet Regions in Parameter Space

w/ Gardi, Herzog, Ma
[2211.14845] [2407.13738] [26xx.xxxxx]

w/ Olsson, Stone [2506.24073]
+ Bennett, Chargeishvili, Magerya
[26xx.xxxxx]

Stephen Jones
IPPP Durham / Royal Society URF

Durham University

THE ROYAL SOCIETY

Artwork: Milad Fakurian on Unsplash

Outline

Motivation

Expansion of Feynman Integrals

Goals

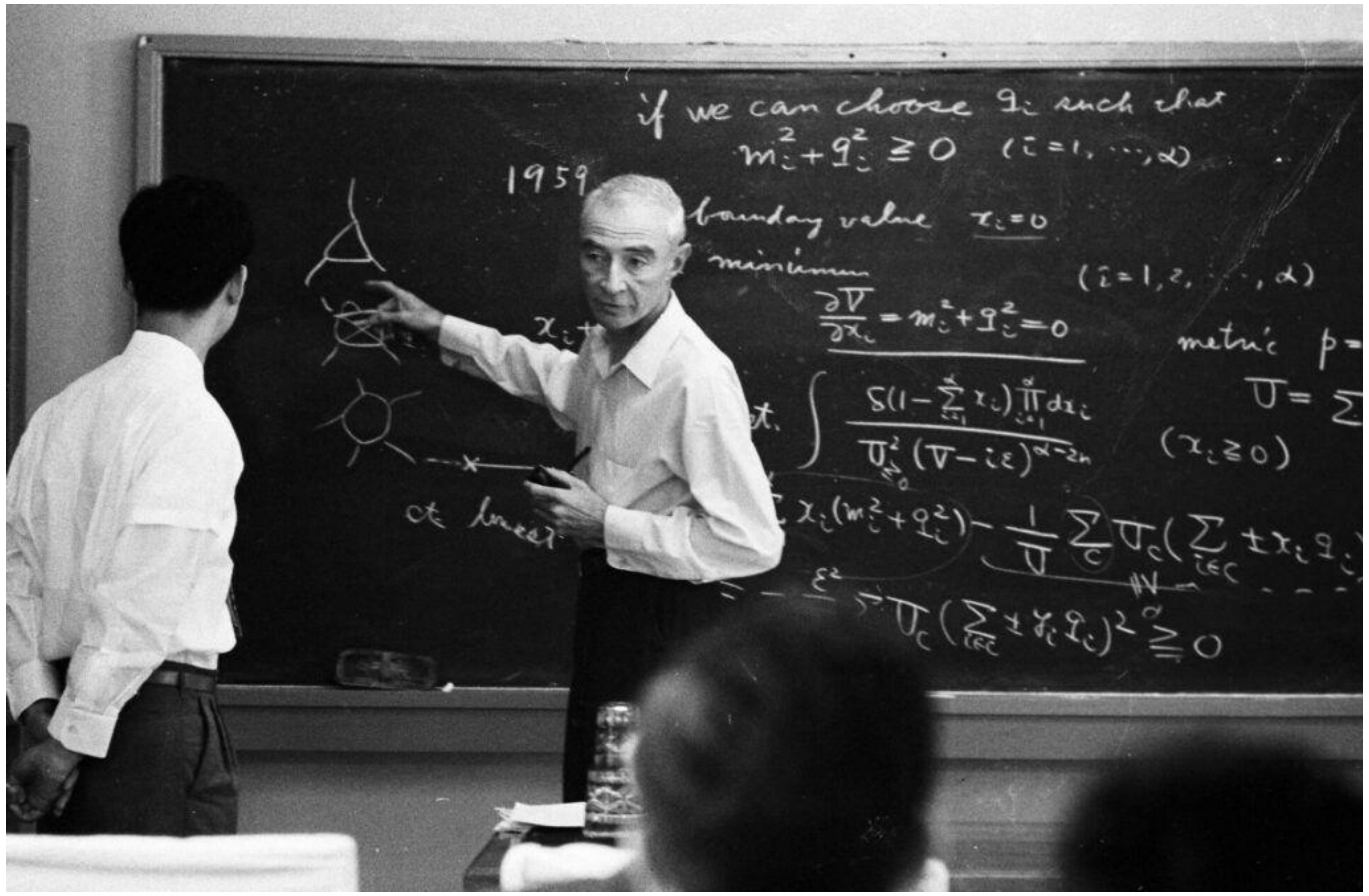
Clarify what we mean by ``facet'' and ``hidden'' regions

Finding ``facet'' regions

Finding ``hidden'' regions (discussion)

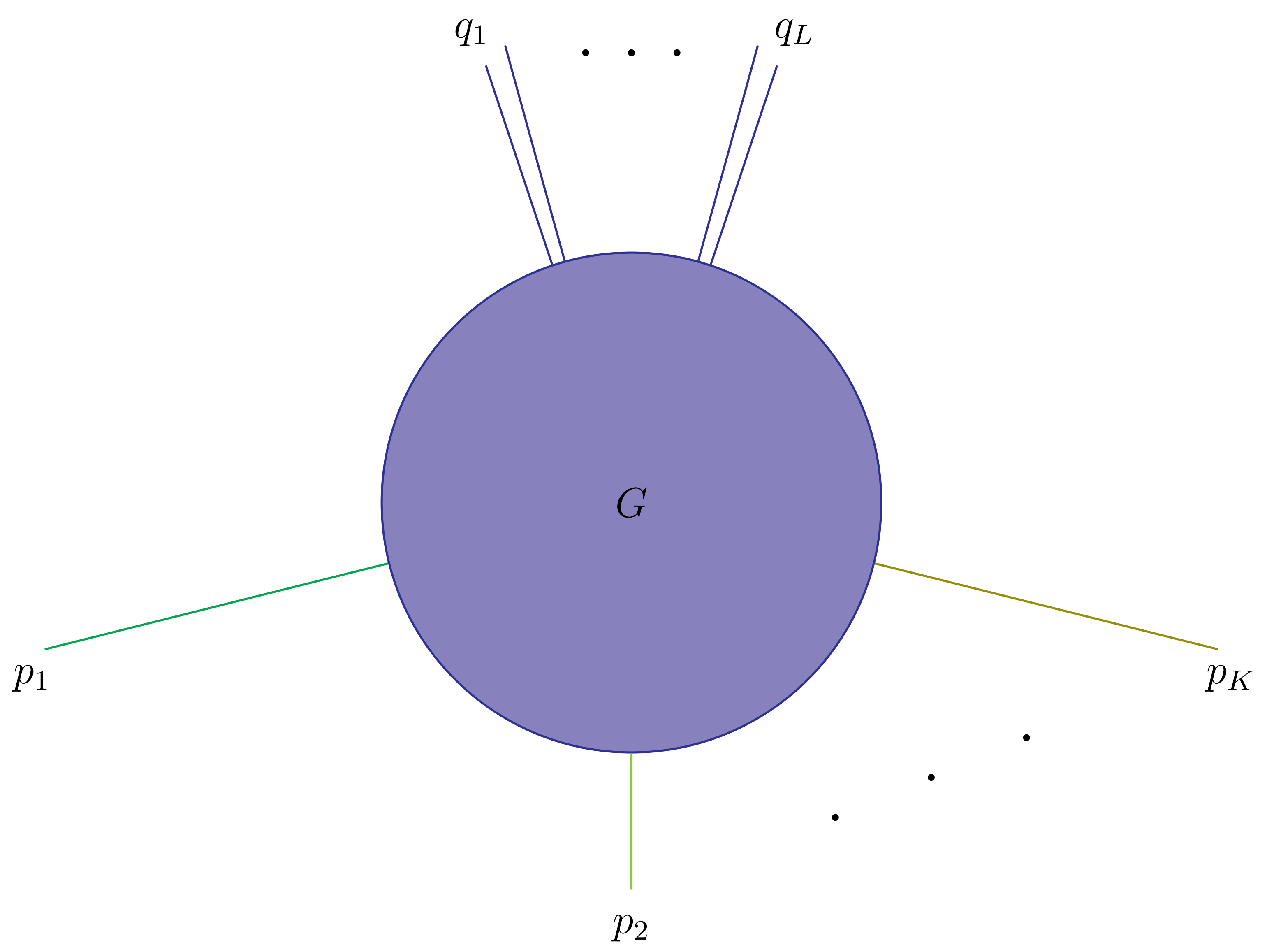
Overview

1. Facet Regions
2. Thresholds
3. Beyond Facet Regions (``hidden'' regions)



Oppenheimer (Kyoto University, Japan), 1960

Introduction



Feynman Integrals

How do we compute loop integrals anyway?

Difficult to compute analytically

Various methods to approximate/evaluate them numerically

Numerical differential equations	**ODE/PDE**
Series solutions of differential equations (AMFlow, DiffExp, Seasyde)	**Series Solutions**
Taylor expansion in Feynman parameters (TayInt)	
Asymptotic expansions (AsyInt, TAPIR, EXP)	
Numerical Mellin-Barnes (MB, Ambre)	**~Monte Carlo Integration**
Tropical sampling (Feyntrop)	
Numerical Loop-Tree Duality (cLTD, Lotty)	
Sector decomposition (Sector_decomposition, FIESTA, pySecDec)	

Expansions key to many evaluation methods

Feynman Integrals in Parameter Space

Loop momenta

$$[\mathrm{d}\mathbf{x}] = \prod_{e\in G} \frac{\mathrm{d}x_e}{x_e}$$

$$I(\mathbf{s}) \sim \int \mathrm{d}^d k_1 \ldots \mathrm{d}^d k_l \frac{1}{\prod_{i=1}^{N} (q_i^2 - m_i^2 + i\delta)^{\nu_i}} \leftrightarrow \int_{\mathbb{R}^N_{>0}} [\mathrm{d}\mathbf{x}] \; \frac{[\mathcal{U}(\mathbf{x})]^{N-(L+1)D/2}}{[\mathcal{F}(\mathbf{x};\mathbf{s}) - i\delta]^{N-LD/2}} \delta(1-\alpha(\mathbf{x}))$$

Propagator momentum (depend on $k_1, \ldots, k_l$)

$\mathcal{U}, \mathcal{F}$ *polynomials* in Feynman parameters

$$\mathbf{x} = (x_1, x_2, \ldots, x_N)$$

Feynman Integrals

$$\mathcal{U}(\mathbf{x}) = \sum_{T^1} \prod_{e \notin T^1} x_e,$$

Linear in x_e

$$\mathcal{F}_0(\mathbf{x};\mathbf{s}) = \sum_{T^2} (-s_{T^2}) \prod_{e \notin T^2} x_e, \qquad \mathcal{F}(\mathbf{x};\mathbf{s}) = \mathcal{F}_0(\mathbf{x};\mathbf{s}) + \mathcal{U}(\mathbf{x}) \sum_e m_e^2 x_e,$$

Linear in x_e

Quadratic in x_e

The signs of the monomials of $\mathcal{F}$ depend on **kinematic invariants** and **masses**

Singularities

Propagators

Loops

Hyperplane bounding integral for at least one $x_i \geq 0$

$$I(\mathbf{s}) \sim \int_{\mathbb{R}^N_{\geq 0}} [\mathrm{d}\mathbf{x}] \, \frac{[\mathscr{U}(\mathbf{x})]^{N-(L+1)D/2}}{[\mathscr{F}(\mathbf{x};\mathbf{s}) - i\delta]^{N-LD/2}} \, \delta(1 - \alpha(\mathbf{x}))$$

Dimensional regulator $D = 4 - 2\epsilon$

Singularities

1. UV/IR singularities when some $x \to 0$ (or $x \to \infty$) simultaneously
2. Thresholds when $\mathscr{F}$ vanishes inside integration region, taking $\lim_{\delta \to 0^+}$ gives causal Feynman prescription

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Dimensional regulator $D = 4 - 2\epsilon$

$$\mathscr{F}(\boldsymbol{x};\mathbf{s}) = 0$$

$$x_j \frac{\partial \mathscr{F}(\boldsymbol{x};\mathbf{s})}{\partial x_j} = 0 \quad \forall j$$

Solve Landau Equations on the Boundary

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\# Propagators

\# Loops

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$\mathscr{F}(\boldsymbol{x};\mathbf{s}) = 0$

Landau Equations

Third representation of Analytic S-Matrix

Eden, Landshoff, Olive, Polkinghorne 66

1 $$\mathcal{F}(\mathbf{x};\mathbf{s}) = 0$$

2 $$x_j \frac{\partial \mathcal{F}(\mathbf{x};\mathbf{s})}{\partial x_j} = 0 \quad \forall j$$

$x_j = 0$ or $\frac{\partial \mathcal{F}(\mathbf{x};\mathbf{s})}{\partial x_j}$

For fixed $\mathbf{s} = (s_1, \ldots, s_M, m_1^2, \ldots, m_N^2)$ each equation (variety of $\mathcal{F}$ or $\partial \mathcal{F}/\partial x_j$) defines a codim-1 hypersurface

Landau singularities dictate the properties of asymptotic expansions

Homogeneity / Euler's Theorem

$$\mathcal{F}(\mathbf{x};\mathbf{s}) \propto \sum_i x_i \frac{\partial \mathcal{F}(\mathbf{x};\mathbf{s})}{\partial x_j}$$

Dealing with Singularities

Let us first consider singularities on the boundary (i.e. for $x_j \to 0$)

$t_j \in \mathbb{C}$

$$I(\mathbf{s}) = \int_{\mathbb{R}^N_{\geq 0}} [\mathrm{d}\boldsymbol{x}] \left(\prod_i^N x_i^{p_i} \right) \left(\prod_j^M f_j(\mathbf{x};\mathbf{s})^{t_j} \right)$$

$\mathbf{v}_{ji} \in \mathbb{Z}^N$

$$f_j(\mathbf{x}) = \sum_i^m c_{ji}(\mathbf{s})\, \mathbf{x}^{\mathbf{v}_{ji}}$$

Assume: $c_{ji}(\mathbf{s}) \in \mathbb{R}_{\geq 0}$ **``Euclidean region"**

Change variables to $z_i = \log(x_i)$

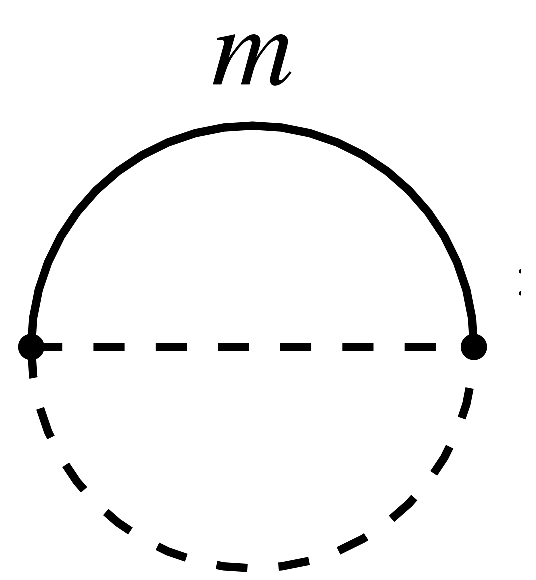
Singularities encoded by Newton Polytope

$\mathcal{N}(f) = \mathrm{conv}(\mathbf{v}_{j1}, \ldots, \mathbf{v}_{jm})$

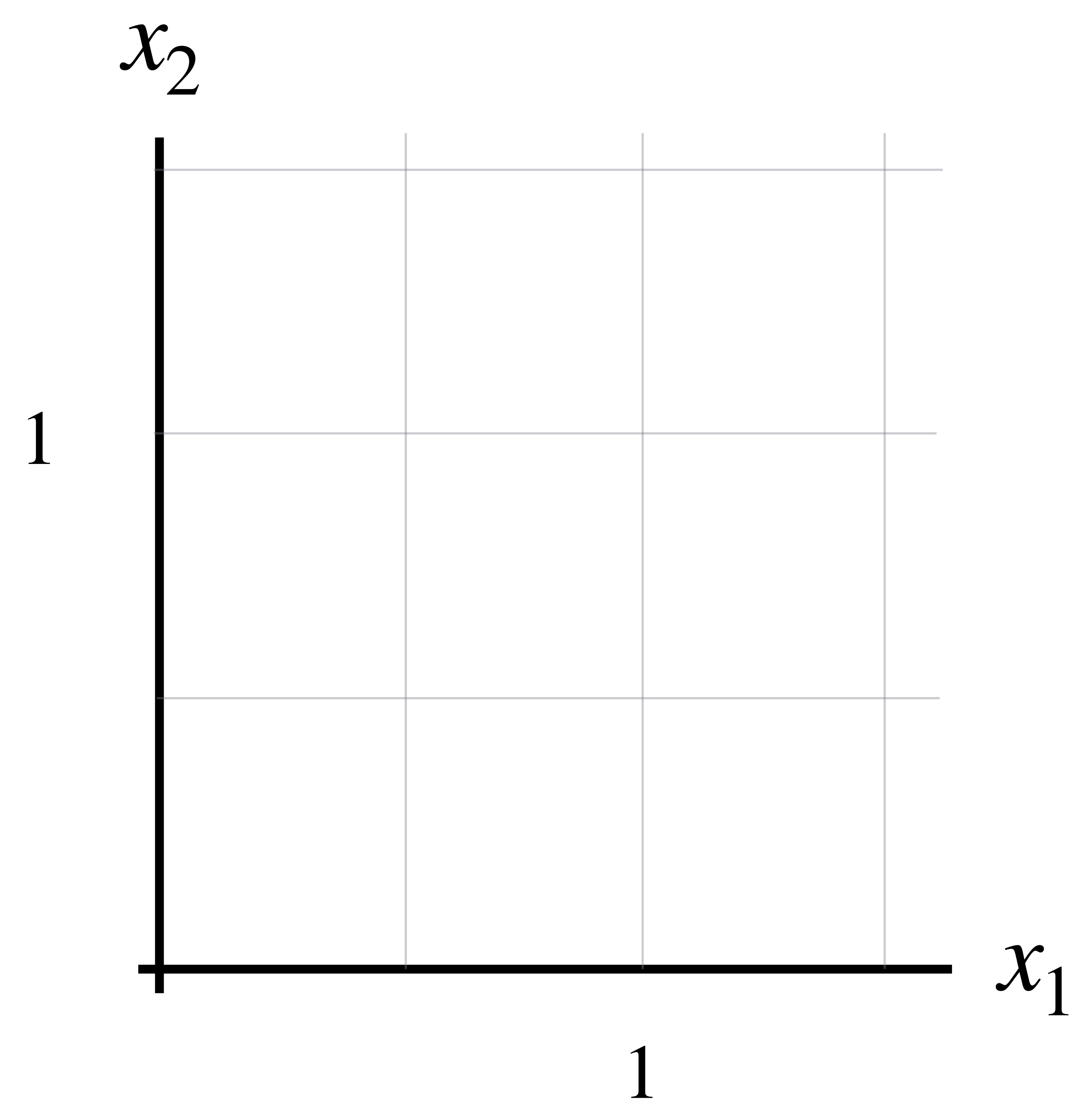
$$I(\mathbf{s}) = \int_{\mathbb{R}^N} \mathrm{d}\boldsymbol{z}\; e^{\langle \mathbf{p},\mathbf{z}\rangle} \left(\prod_j^M f_j(\mathbf{z};\mathbf{s})^{t_j} \right)$$

$$f_j(\mathbf{z}) = \sum_i^m c_{ji}(\mathbf{s})\, e^{\langle \mathbf{v}_{ji},\mathbf{z}\rangle}$$

Dealing with Singularities



$$\mathcal{U}(\mathbf{x}) = x_1^1 x_2^0 + x_1^1 x_2^1 + x_1^0 x_2^1$$



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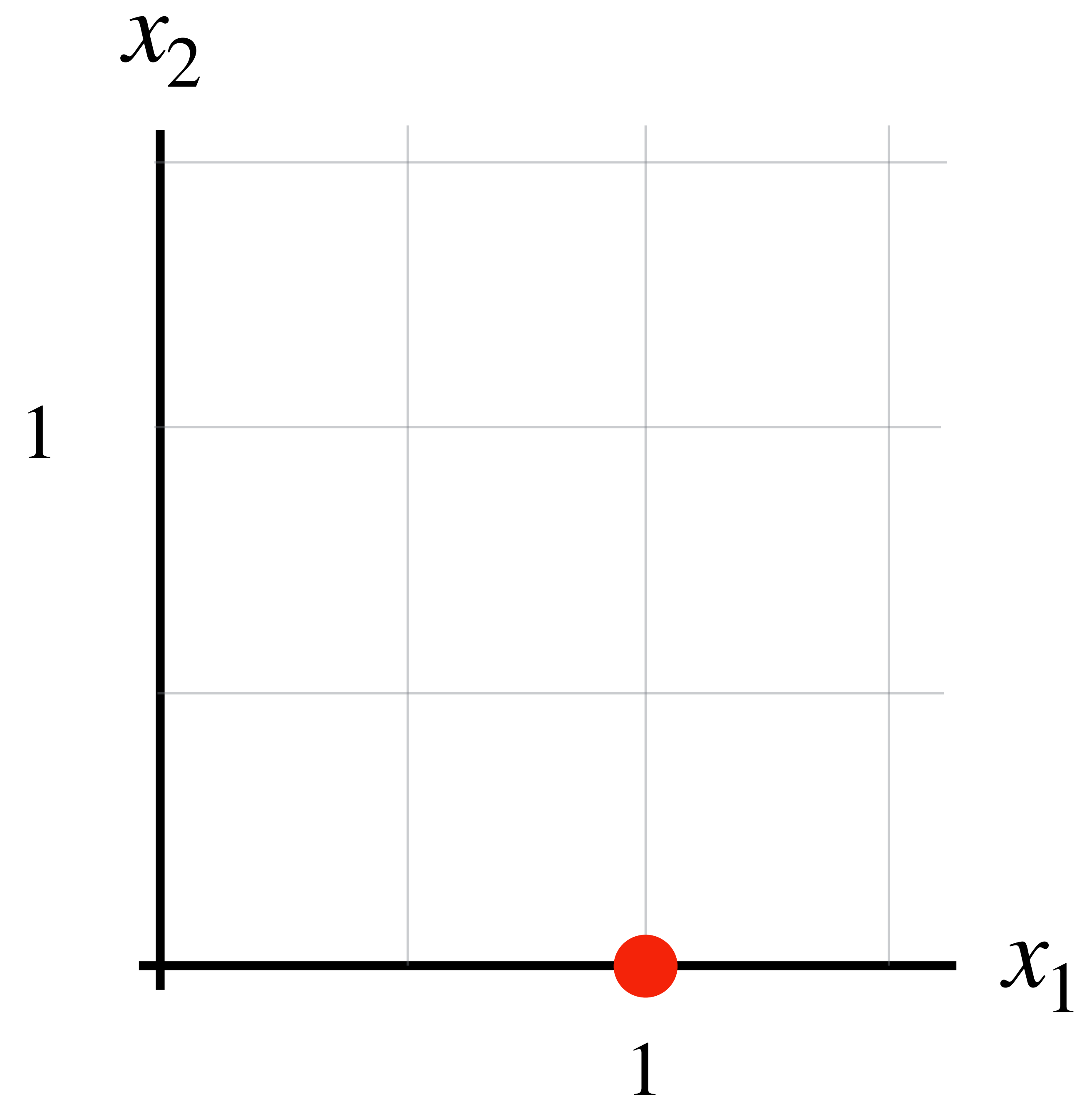

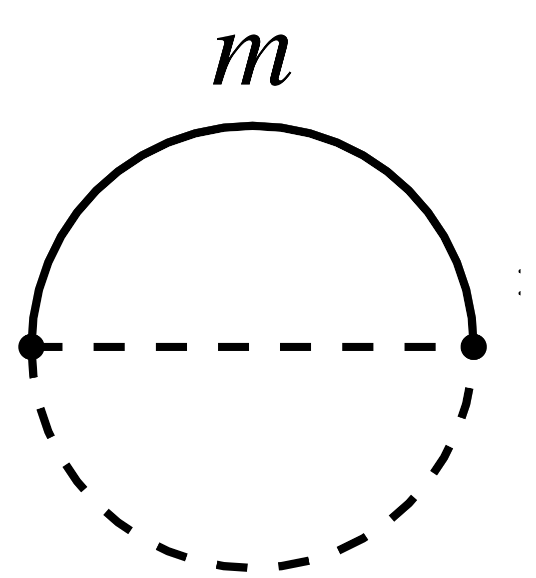

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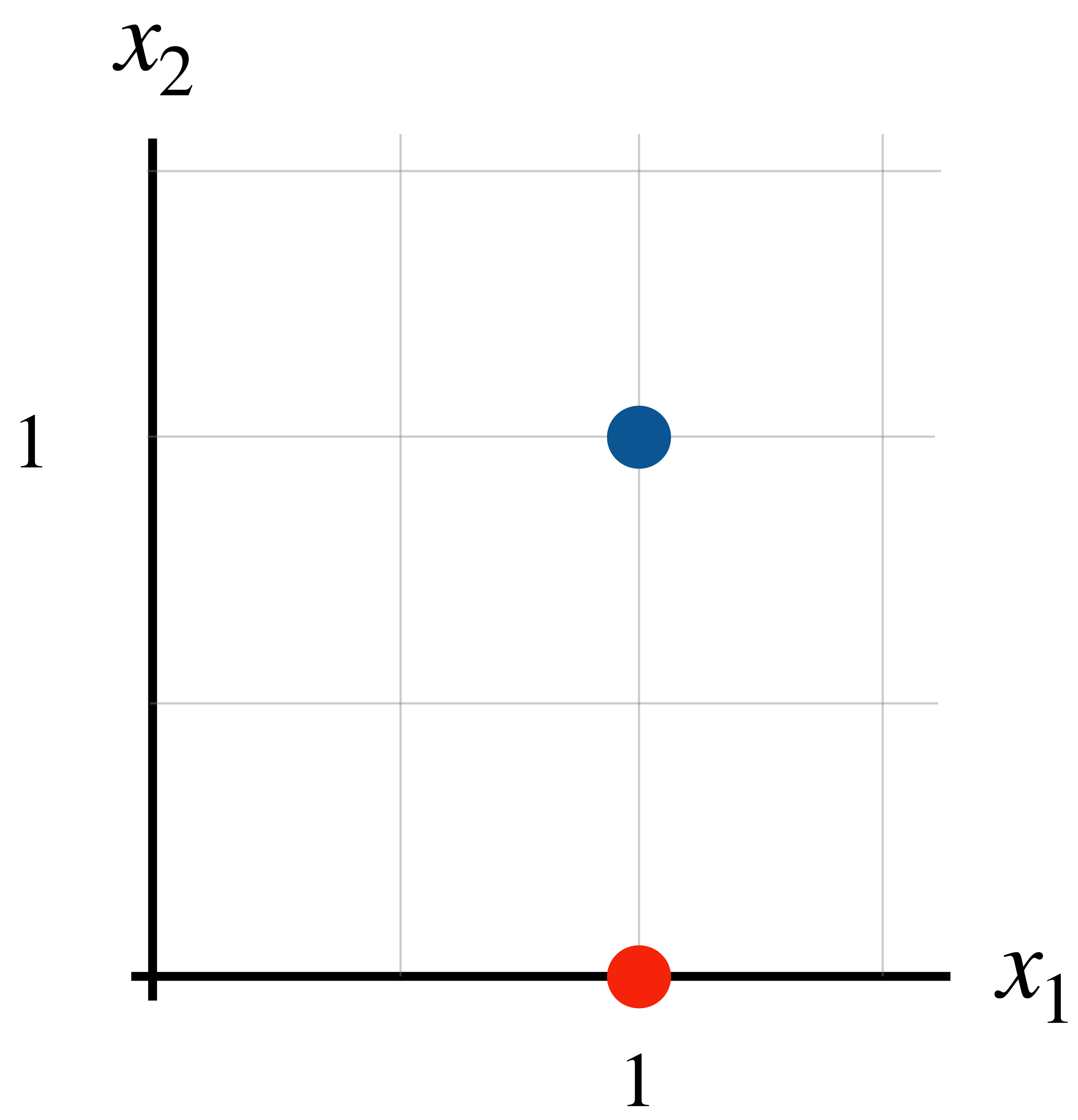

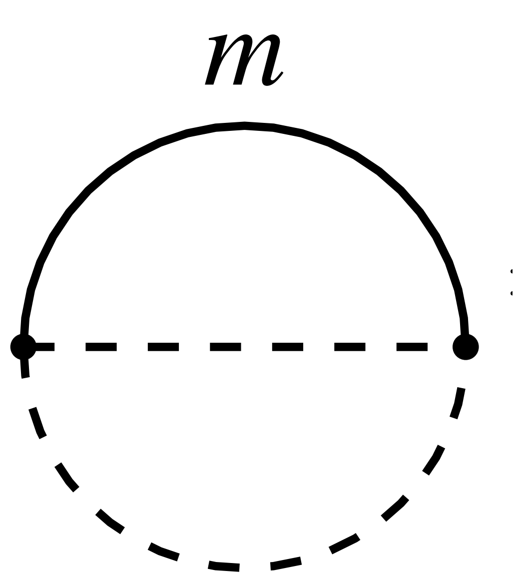

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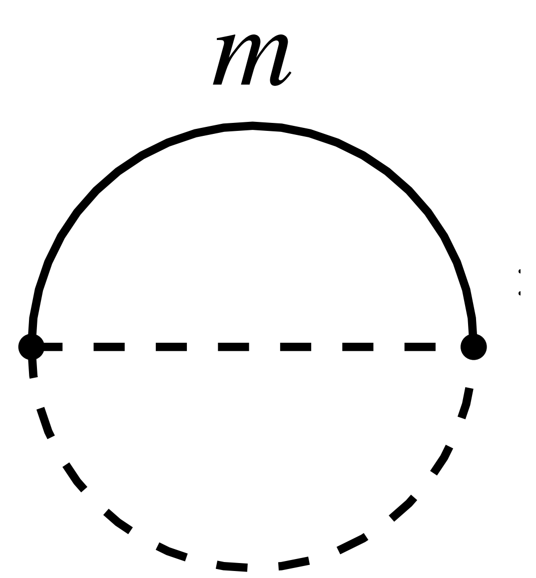

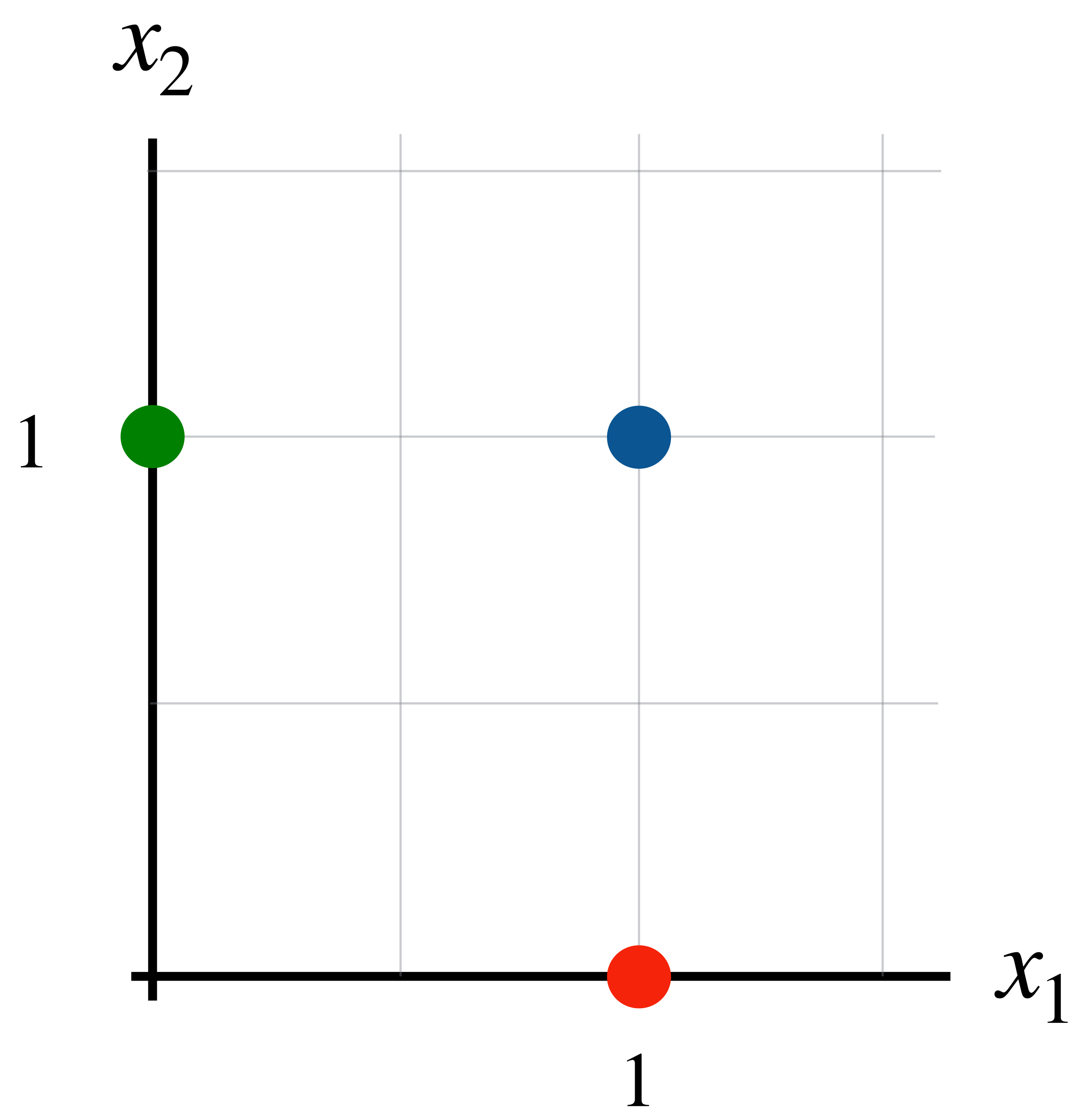

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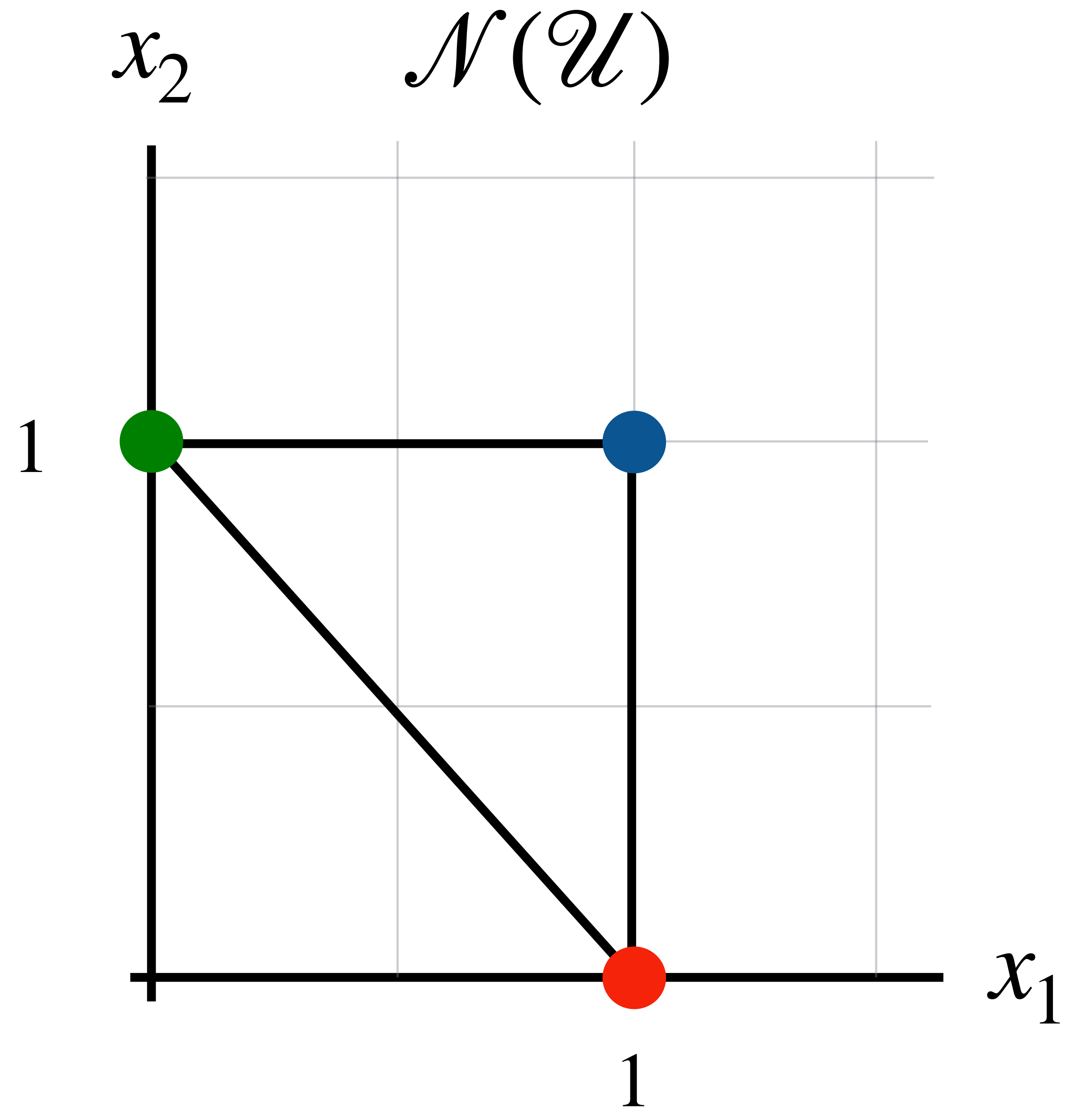

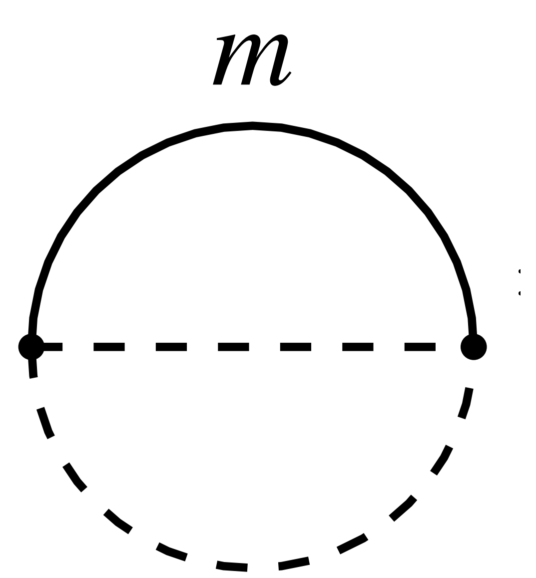

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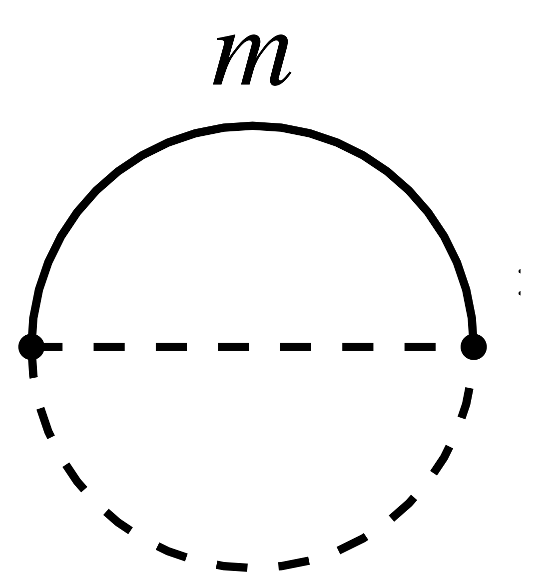

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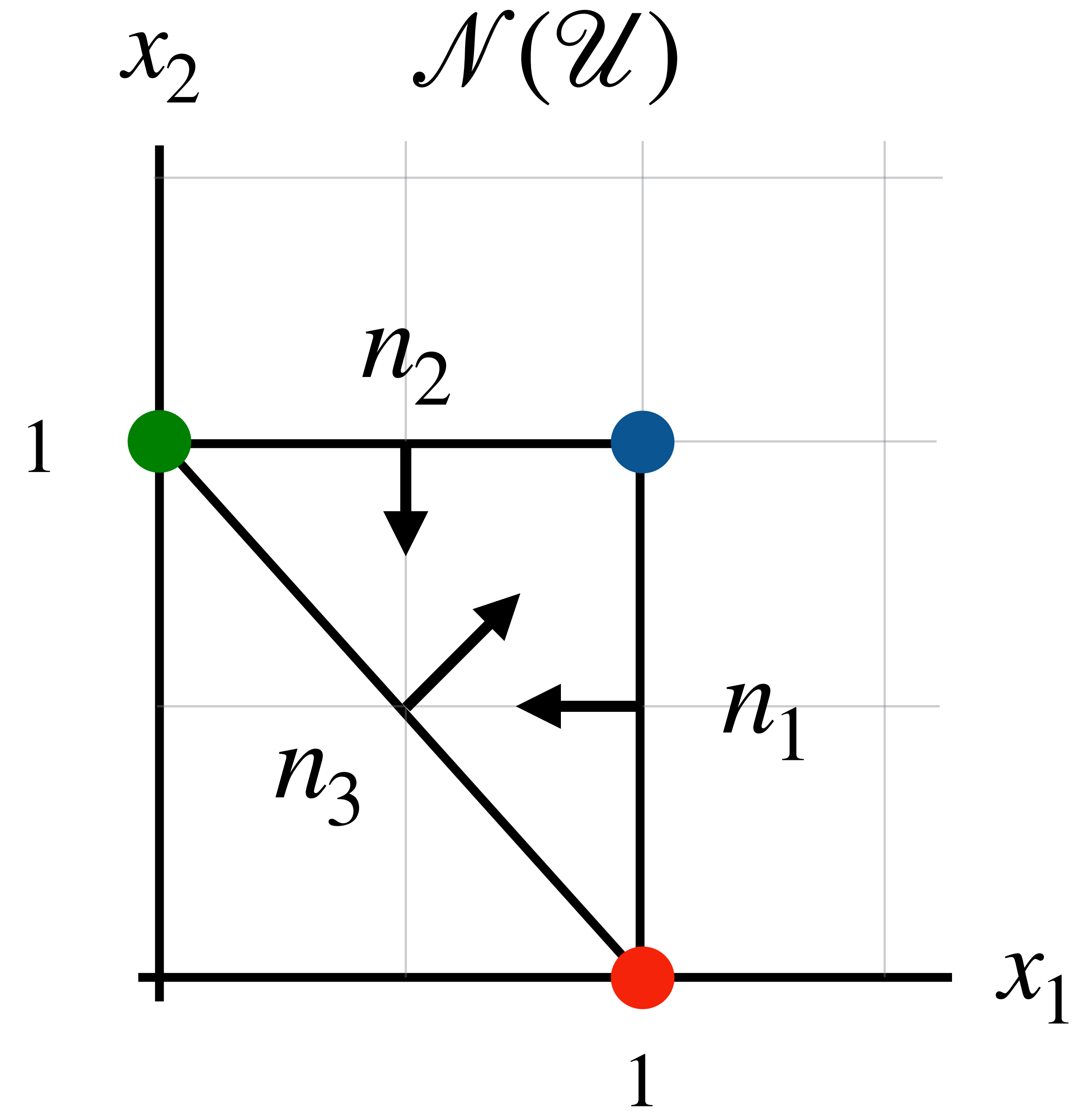

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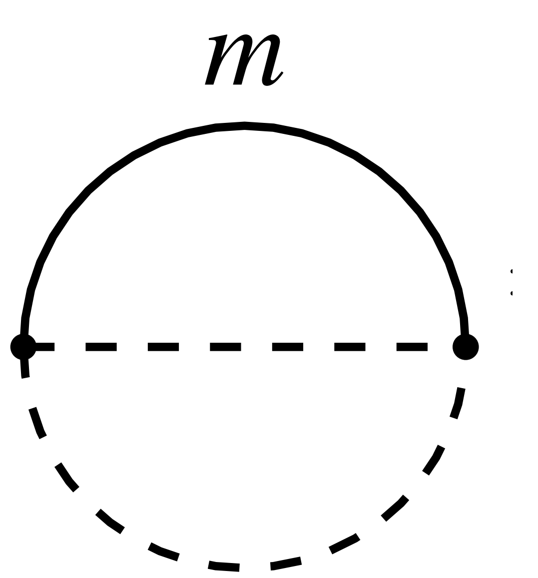

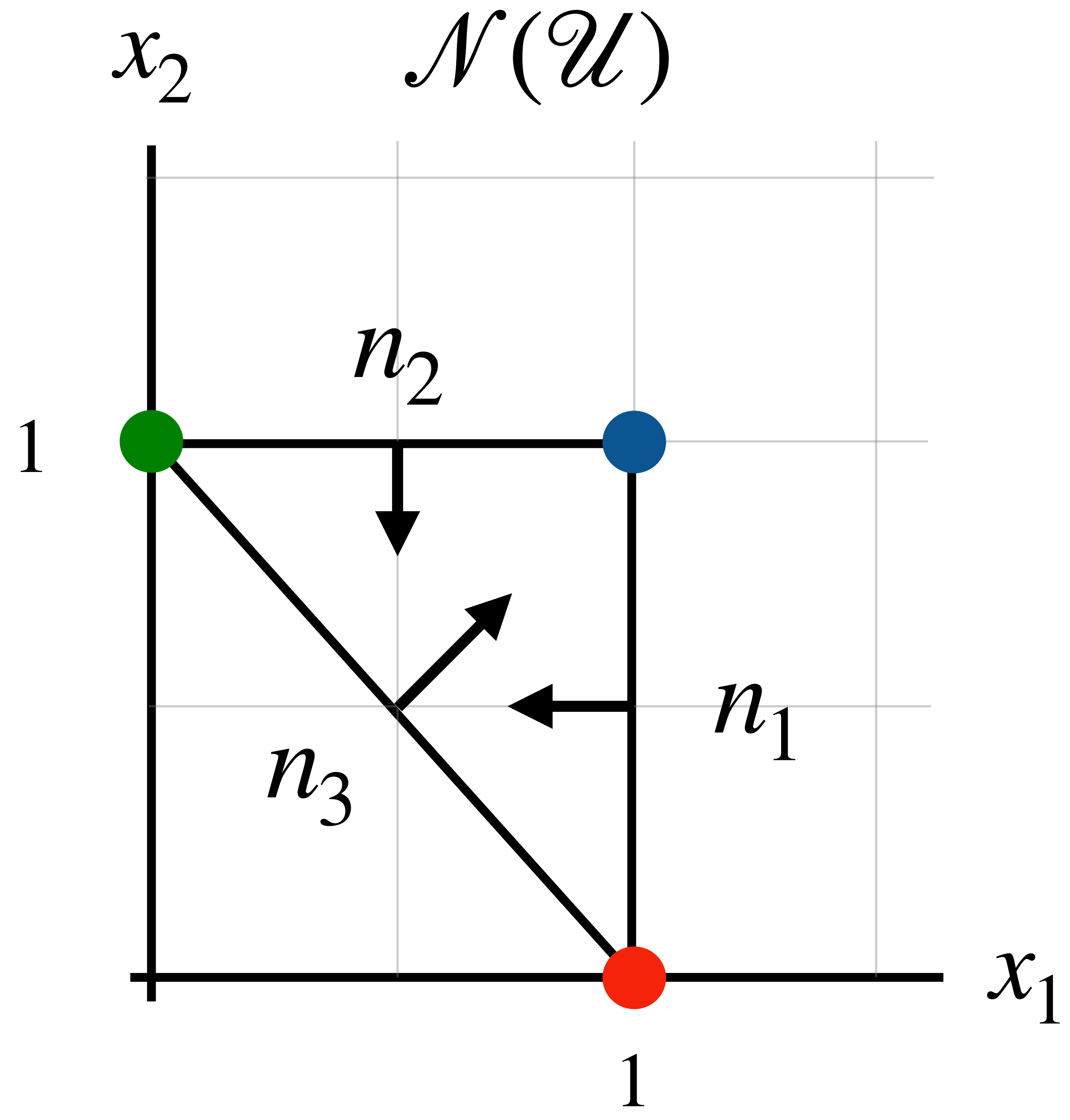

Sector decomposition

Facets define a local change of coordinates for each singularity that factorises it (blow-up)

Hepp 66; Roth, Denner 96; Binoth, Heinrich 00; Heinrich 08; Kaneko, Ueda 10; Schlenk 16

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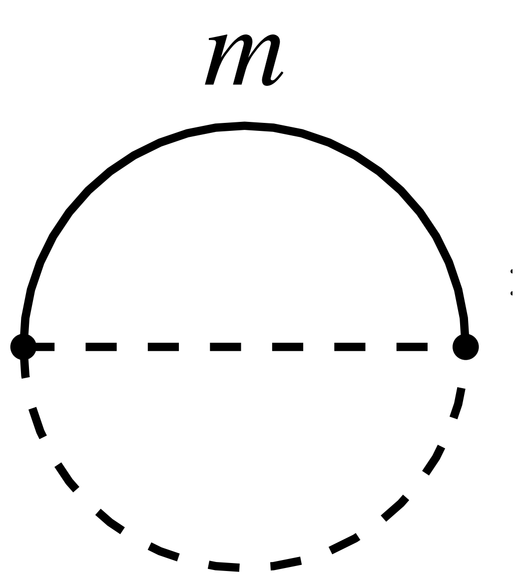

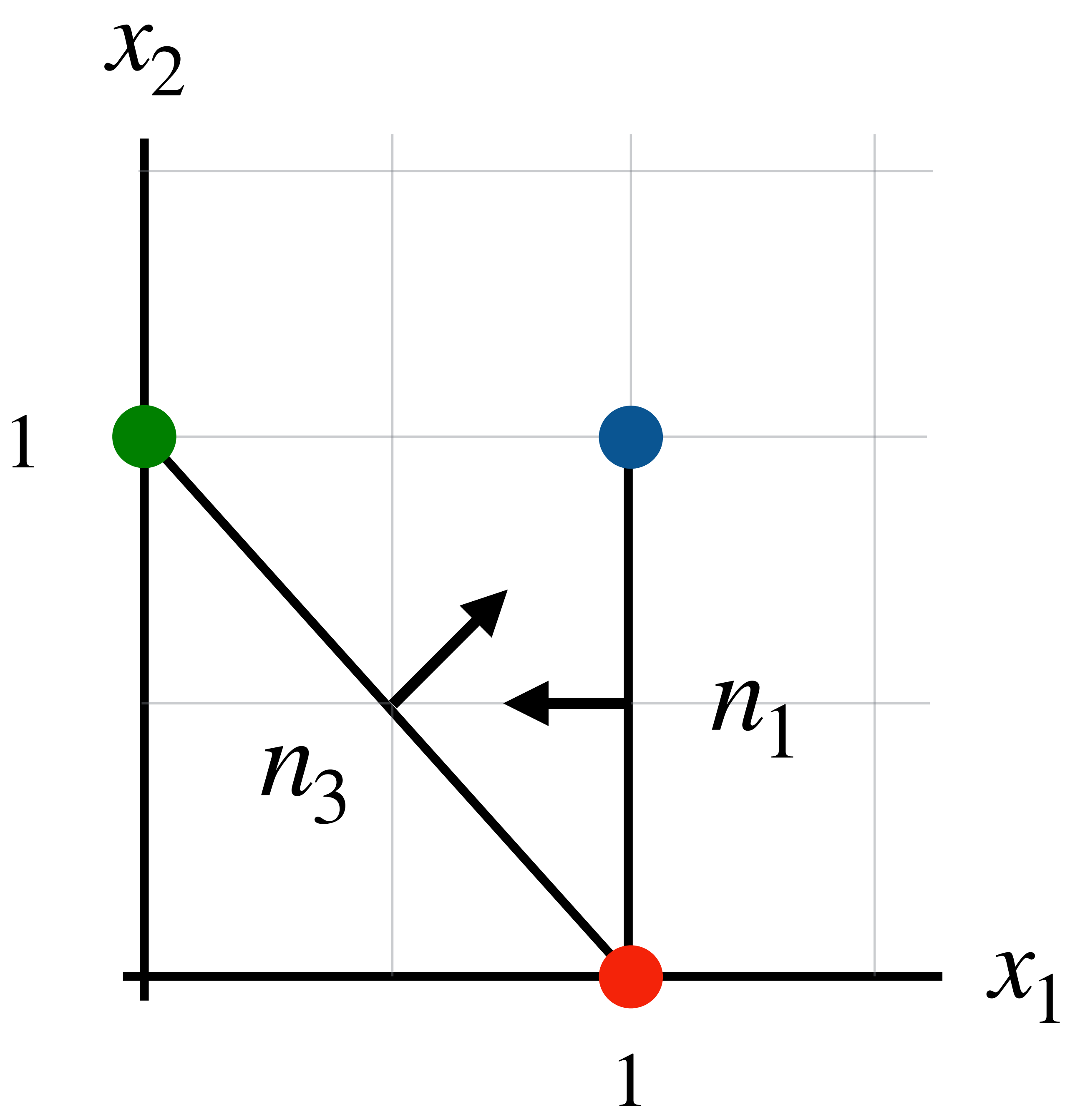

● $x_1 = y_1^{-1} y_3 \quad x_2 = y_3$ $\qquad \mathcal{U}_1(\mathbf{x}) = y_3 y_1^{-1}(1 + y_1 + y_3)$

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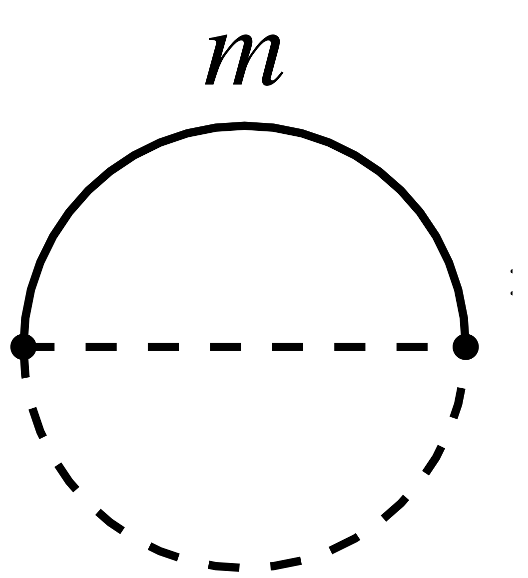

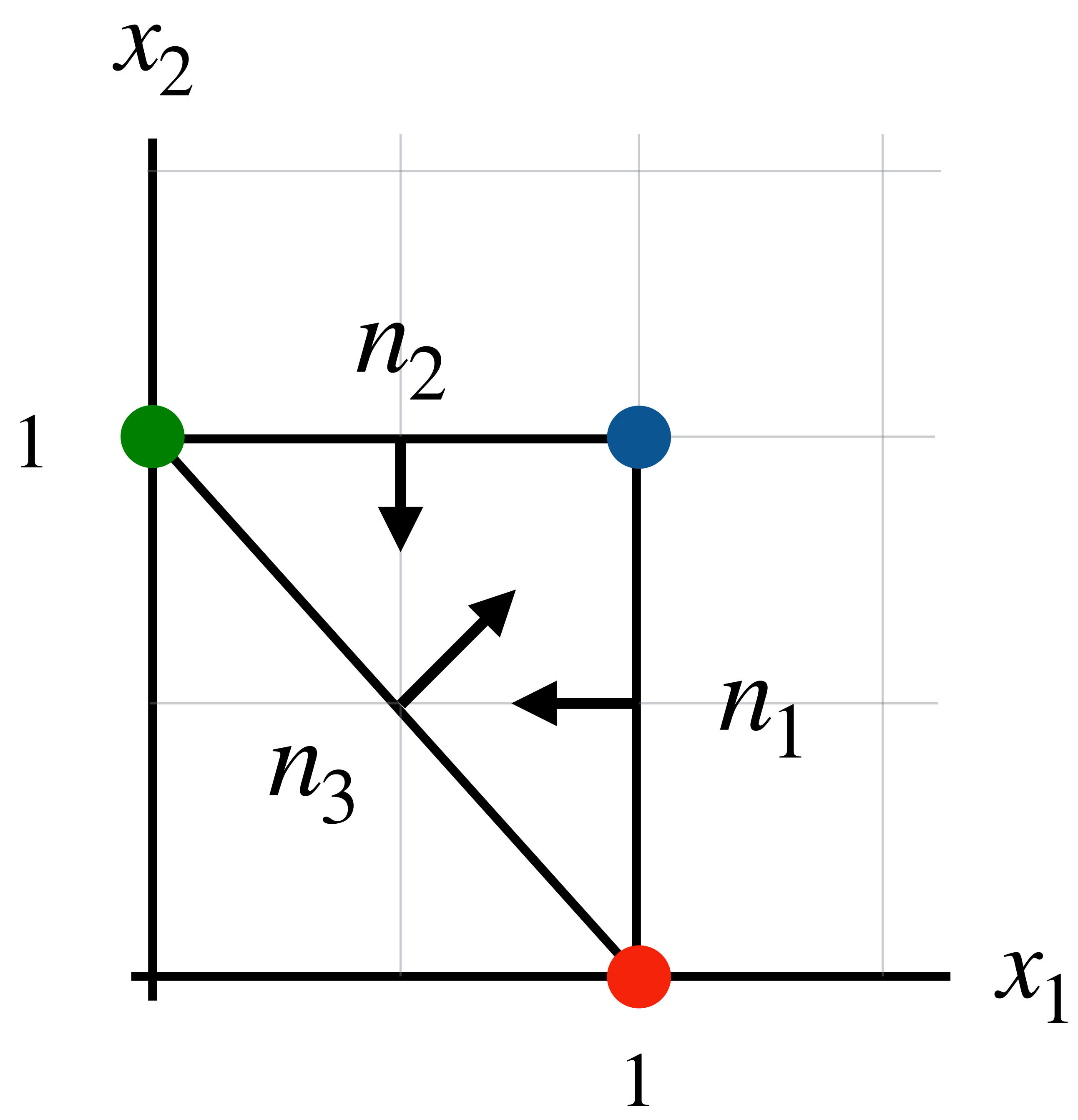

● $x_1 = y_1^{-1} y_3 \quad x_2 = y_3$ $\qquad \mathcal{U}_1(\mathbf{x}) = y_3 y_1^{-1}(1 + y_1 + y_3)$

● $x_1 = y_1^{-1} \quad x_2 = y_2^{-1}$ $\qquad \mathcal{U}_2(\mathbf{x}) = y_1^{-1} y_2^{-1}(1 + y_1 + y_2)$

● $x_1 = y_3 \quad x_2 = y_2^{-1} y_3$ $\qquad \mathcal{U}_3(\mathbf{x}) = y_3 y_2^{-1}(1 + y_2 + y_3)$

Sector decomposition

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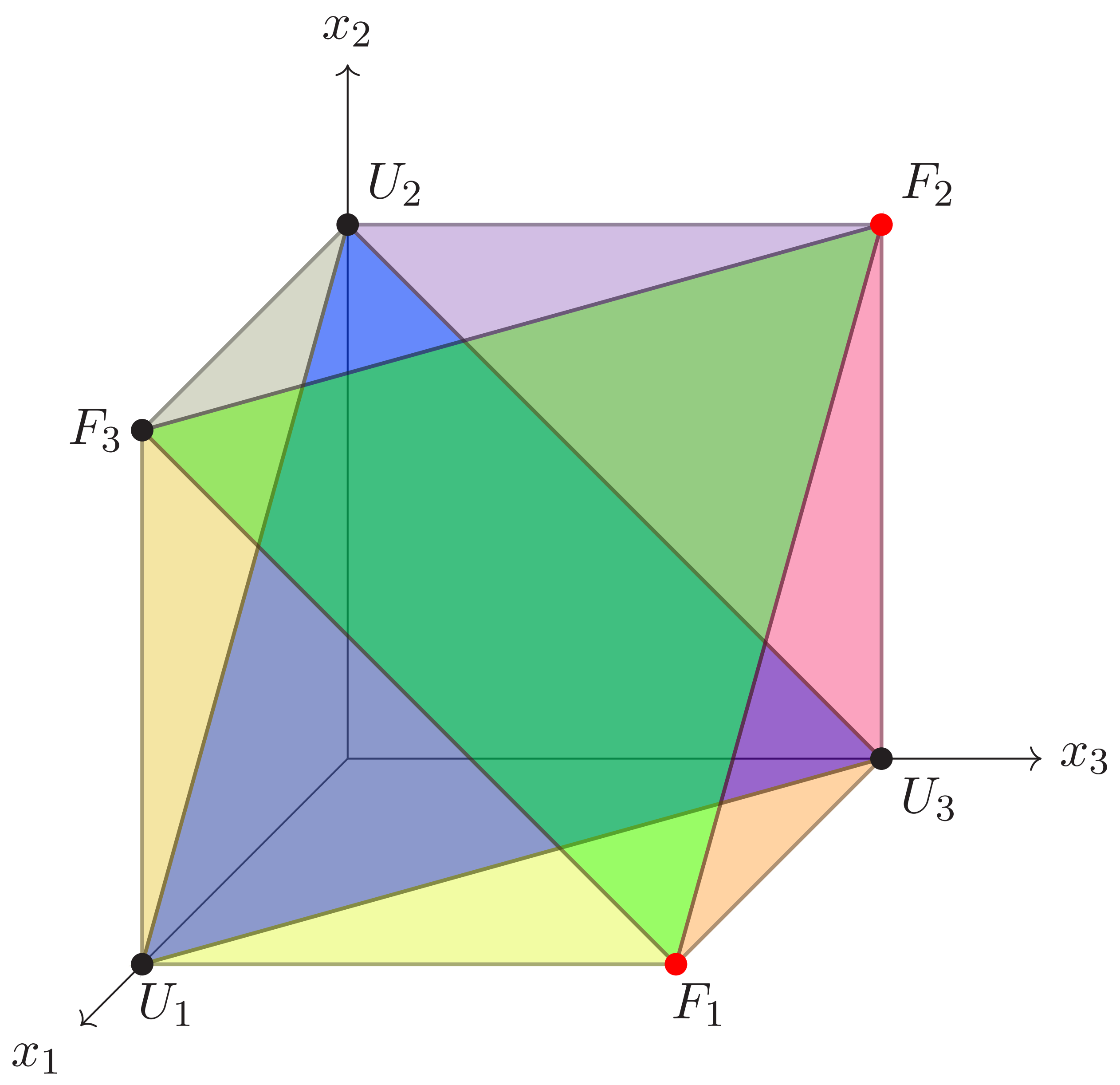
Method of Regions: Facet Regions



Method of Regions

Useful to expand about some limit (λ) before integration

Integration and series expansion do not commute → Method of Regions

Smirnov 91; Beneke, Smirnov 97; Smirnov, Rakhmetov 99; Pak, Smirnov 11; Jantzen 2011; …

$$I(\mathbf{s}) = \sum_R I^{(R)}(\mathbf{s}) = \sum_R T_{\mathbf{t}}^{(R)} I(\mathbf{s})$$

Define expansion by specifying scaling of each invariant in $\mathbf{s}$, for example $p_1^2 \to \lambda p_1^2$ or $m_1^2 \to \lambda m_1^2$ or $s_{12} \to \lambda s_{12}$

Polytopes allow us to find regions → consider the exponent space of $(\mathbf{x}, \lambda)$ rather than just $\mathbf{x}$

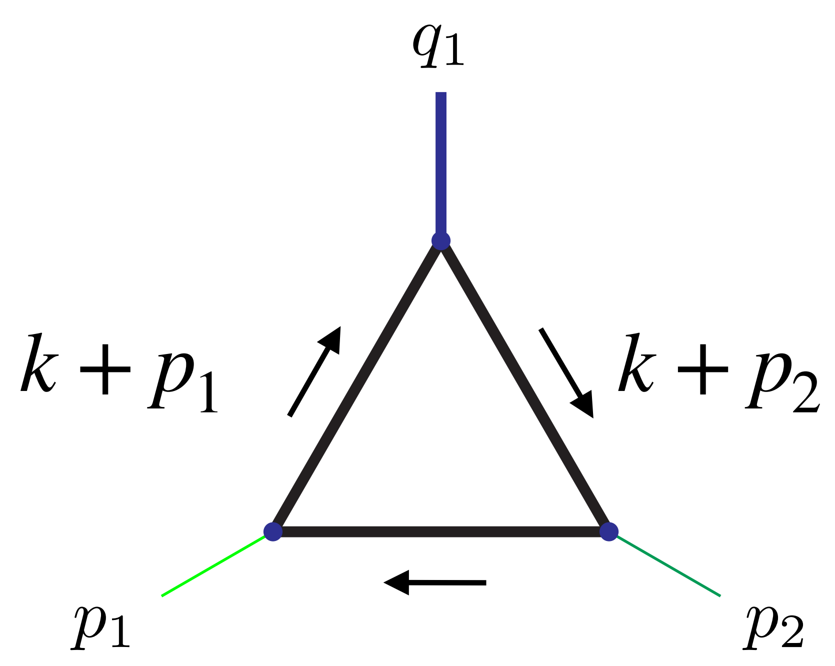
Pak, Smirnov 11; Semenova, A. Smirnov, V. Smirnov 18

Facet Regions

Each region will be defined by a **region vector** $\mathbf{v} = (v_1, \ldots, v_N; 1)$, in each region we will perform a change of variables $x_i \to \lambda^{v_i} x_i$ and series expand about $\lambda = 0$

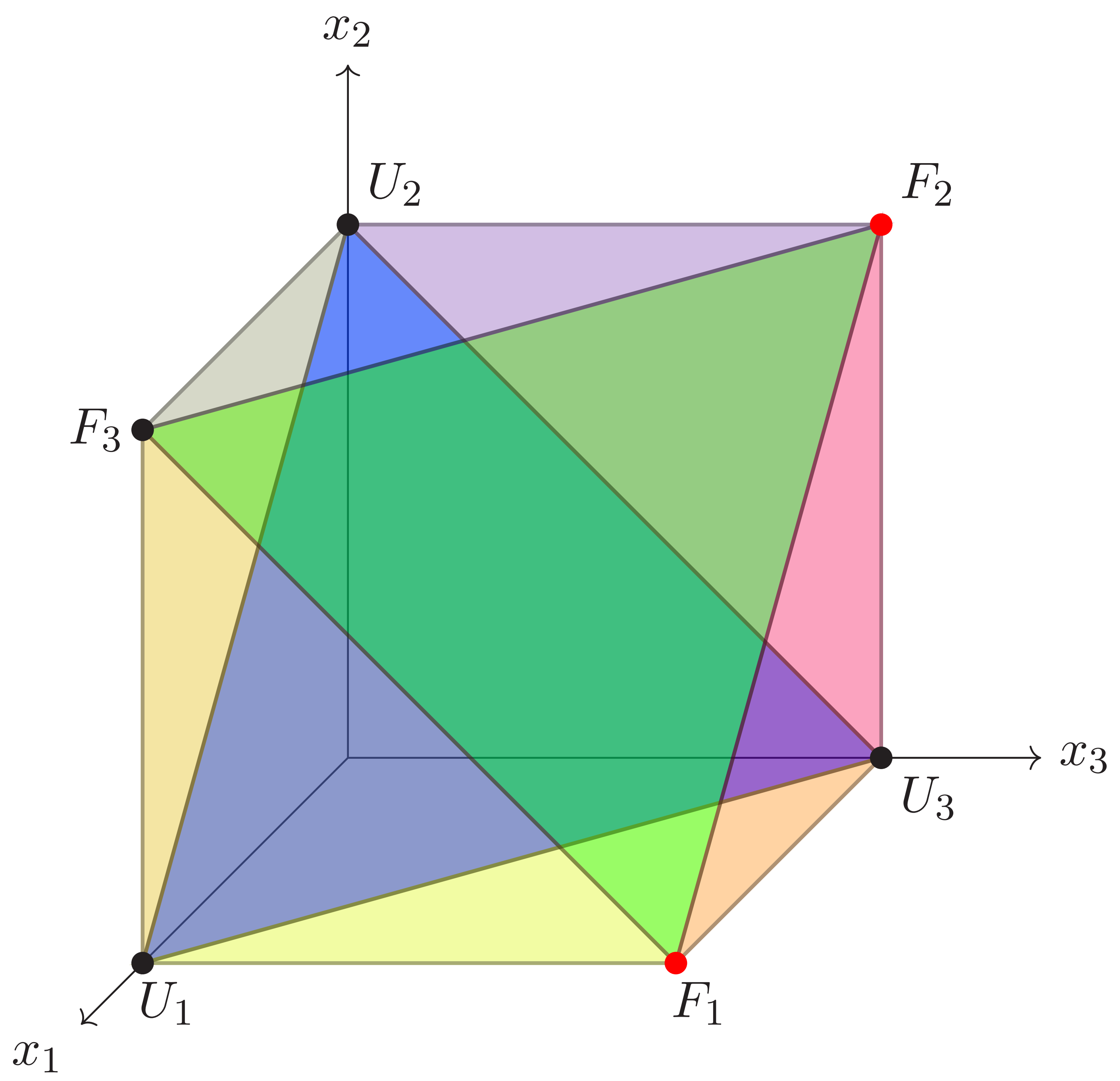
Consider the on-shell limit

$p_1^2 \to \lambda p_1^2,\ p_2^2 \to \lambda p_2^2$ for $\lambda \to 0$



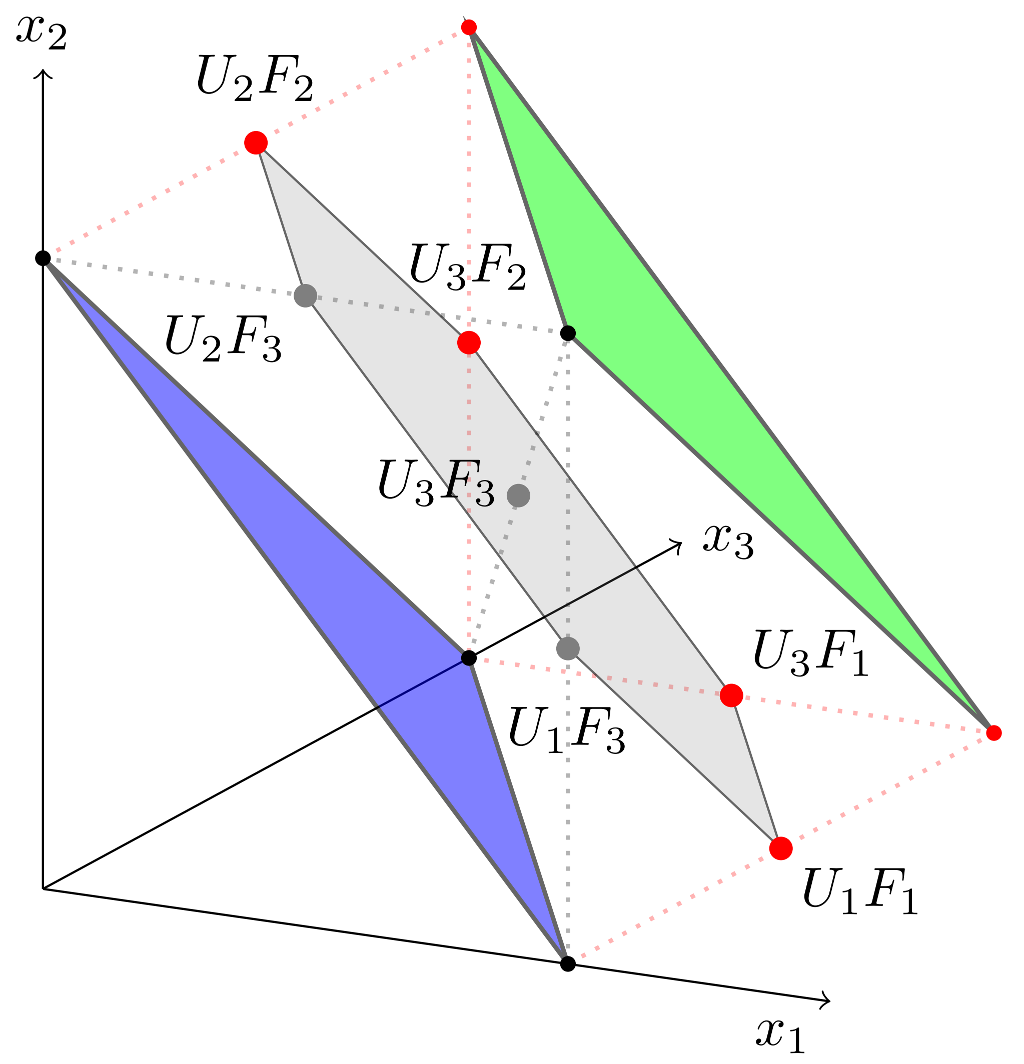
$\mathcal{U}(\mathbf{x}) = x_1 + x_2 + x_3$

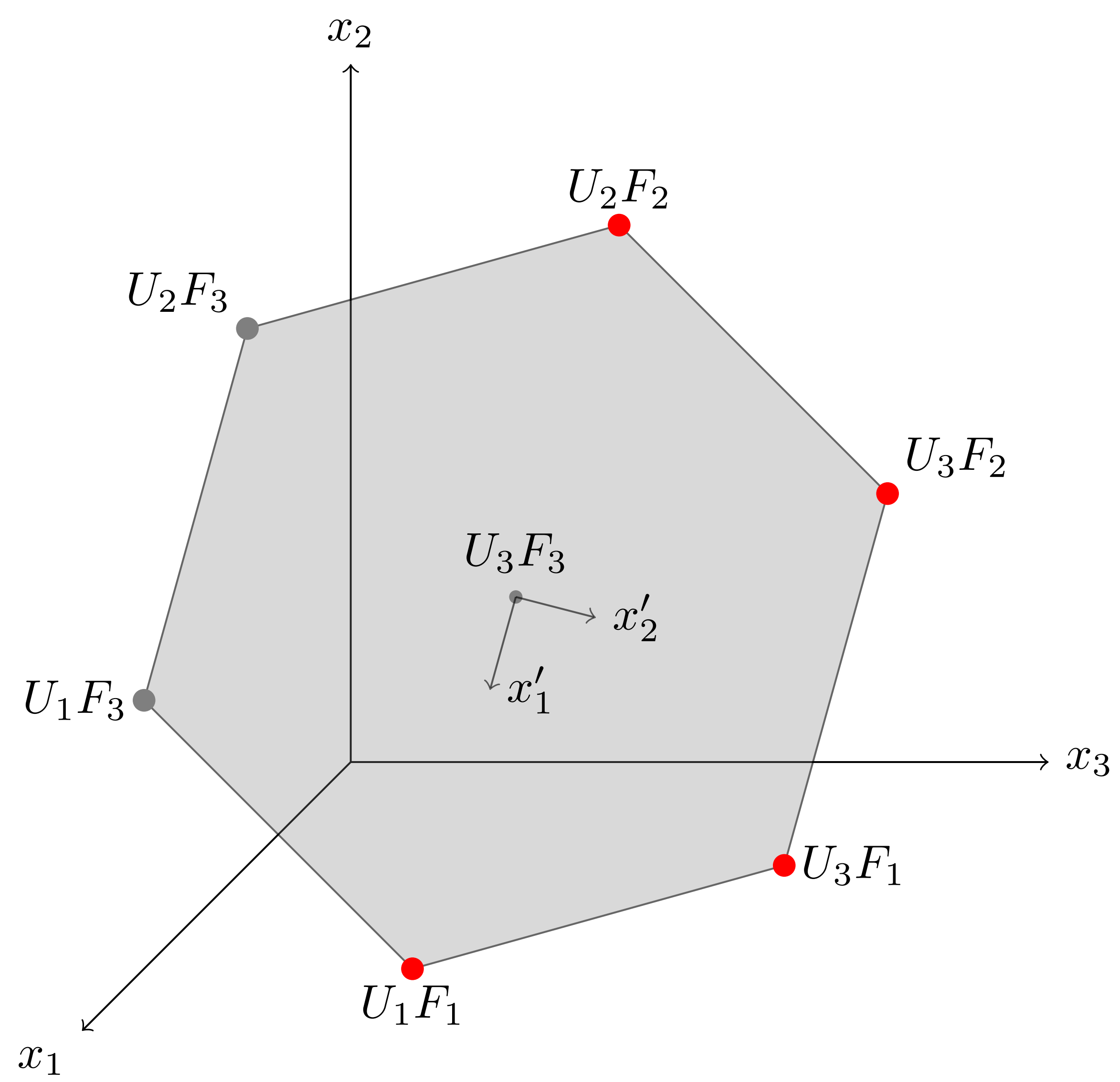
$\mathcal{F}(\mathbf{x}, \mathbf{s}) = (-p_1^2)\,\lambda x_1 x_3 + (-p_2)^2\,\lambda x_2 x_3 + (-q_1)^2 x_1 x_2$

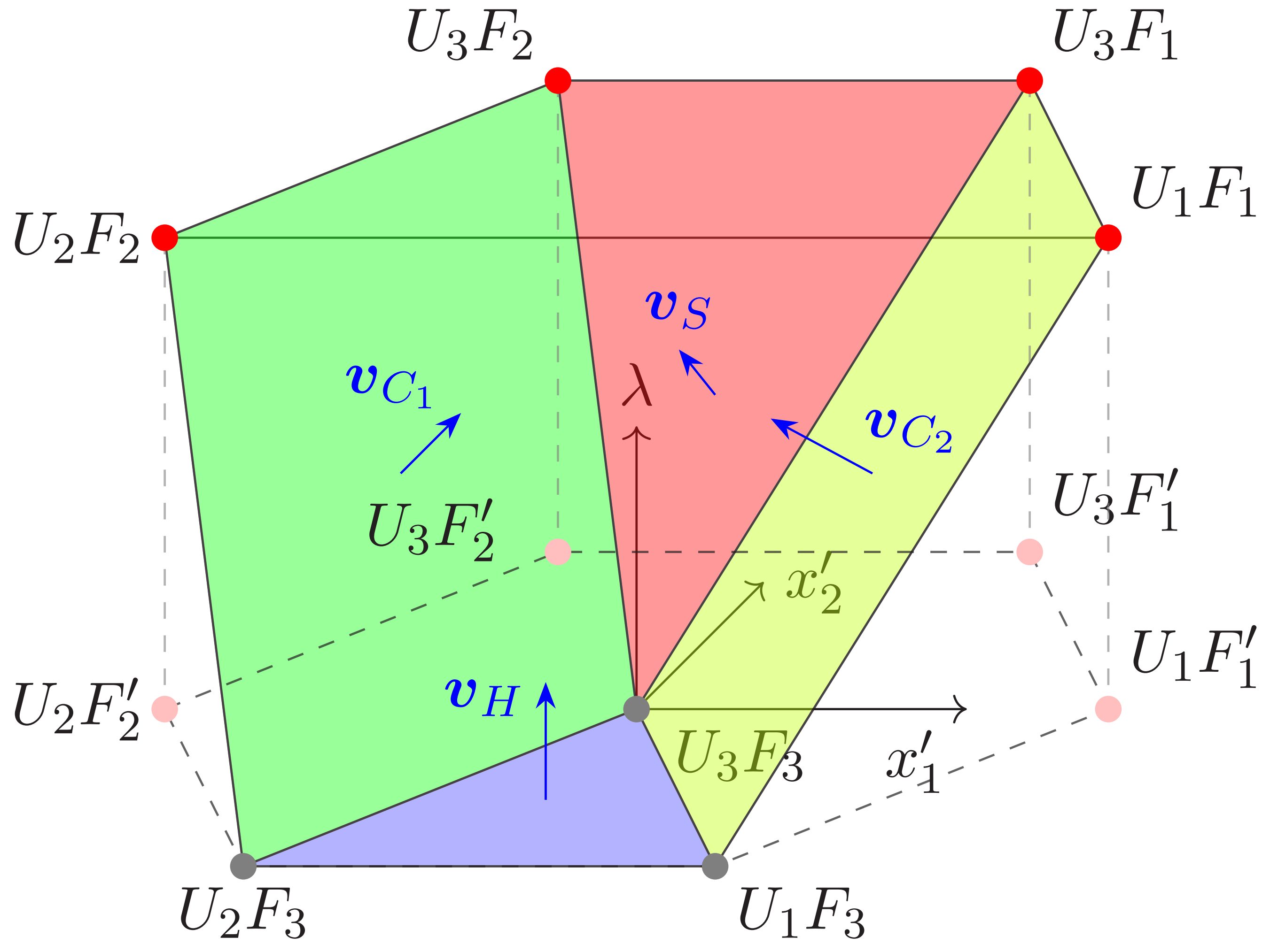


Newton Polytope

Facet Regions







Singularities as $\lambda \to 0$ encoded by Newton Polytope

The regions: normal vectors of the lower facets

$$\boldsymbol{v}_H = (0,0,0;1), \qquad \boldsymbol{v}_S = (-1,-1,-2;1),$$

$$\boldsymbol{v}_{C_1} = (-1,0,-1;1), \qquad \boldsymbol{v}_{C_2} = (0,-1,-1;1),$$

Examining these facets we can see the IR structure of our integrand explicitly

hard, **soft**, **collinear-1**, **collinear-2**

Interpreting Facet Regions

The region vectors in momentum space and Lee-Pomeransky space are related, we can see this using Schwinger parameters $\hat{\alpha}_i$ and Lee-Pomeransky parameters $\hat{x}_i$

$$\frac{1}{D_n^{\nu_j}} = \frac{(-i)^{\nu_j}}{\Gamma(\nu_j)} \int_0^\infty \mathrm{d}\hat{\alpha}_j \hat{\alpha}_j^{\nu_j - 1} e^{i\hat{\alpha}_j D_j} \text{, with } \hat{x}_j \propto \hat{\alpha}_j$$

$$(D_1^{-1}, \ldots, D_N^{-1}) \sim (\hat{x}_1, \ldots, \hat{x}_N)$$

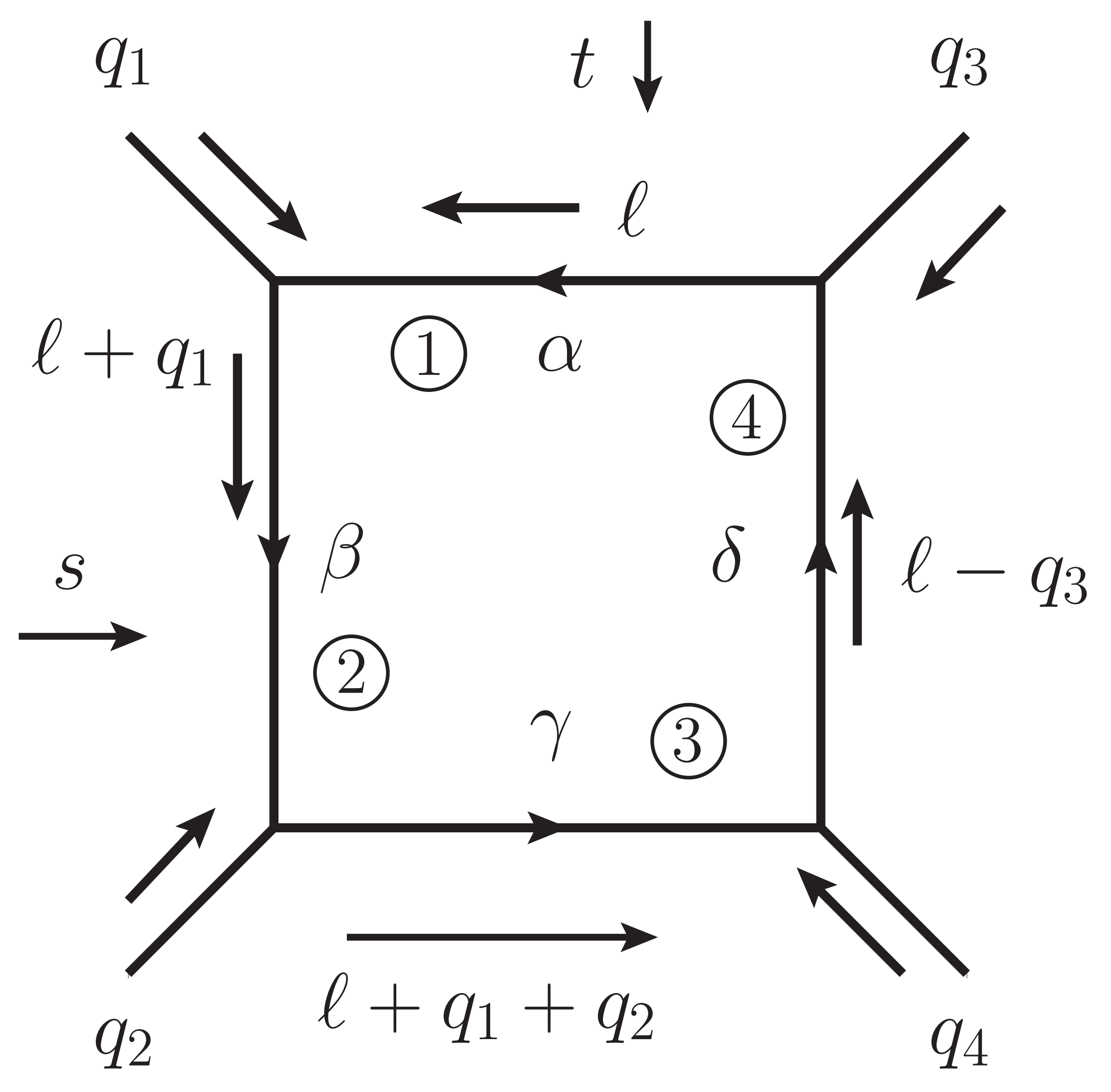
Triangle example

$$\begin{aligned}
\text{Hard}: \quad & (D_1^{-1}, D_2^{-1}, D_3^{-1}) \sim (\lambda^0, \lambda^0, \lambda^0), & (\hat{x}_1, \hat{x}_2, \hat{x}_3) \sim (\lambda^0, \lambda^0, \lambda^0) \\
\text{Collinear to } \mathrm{p}_1: \quad & (D_1^{-1}, D_2^{-1}, D_3^{-1}) \sim (\lambda^{-1}, \lambda^0, \lambda^{-1}), & (\hat{x}_1, \hat{x}_2, \hat{x}_3) \sim (\lambda^{-1}, \lambda^0, \lambda^{-1}) \\
\text{Collinear to } \mathrm{p}_2: \quad & (D_1^{-1}, D_2^{-1}, D_3^{-1}) \sim (\lambda^0, \lambda^{-1}, \lambda^{-1}), & (\hat{x}_1, \hat{x}_2, \hat{x}_3) \sim (\lambda^0, \lambda^{-1}, \lambda^{-1}) \\
\text{Soft}: \quad & (D_1^{-1}, D_2^{-1}, D_3^{-1}) \sim (\lambda^{-1}, \lambda^{-1}, \lambda^{-2}), & (\hat{x}_1, \hat{x}_2, \hat{x}_3) \sim (\lambda^{-1}, \lambda^{-1}, \lambda^{-2})
\end{aligned}$$

This allows us to connect the regions we saw in momentum space with those we can determine geometrically

Example: Interpreting Facet Regions

Limit: $s, |t|, |u| \gg m_t^2 \gg m_H^2\,, \quad m_H^2 \to 0$ and $\lambda \sim m_t/Q$



Automatically find remaining regions in parameter space

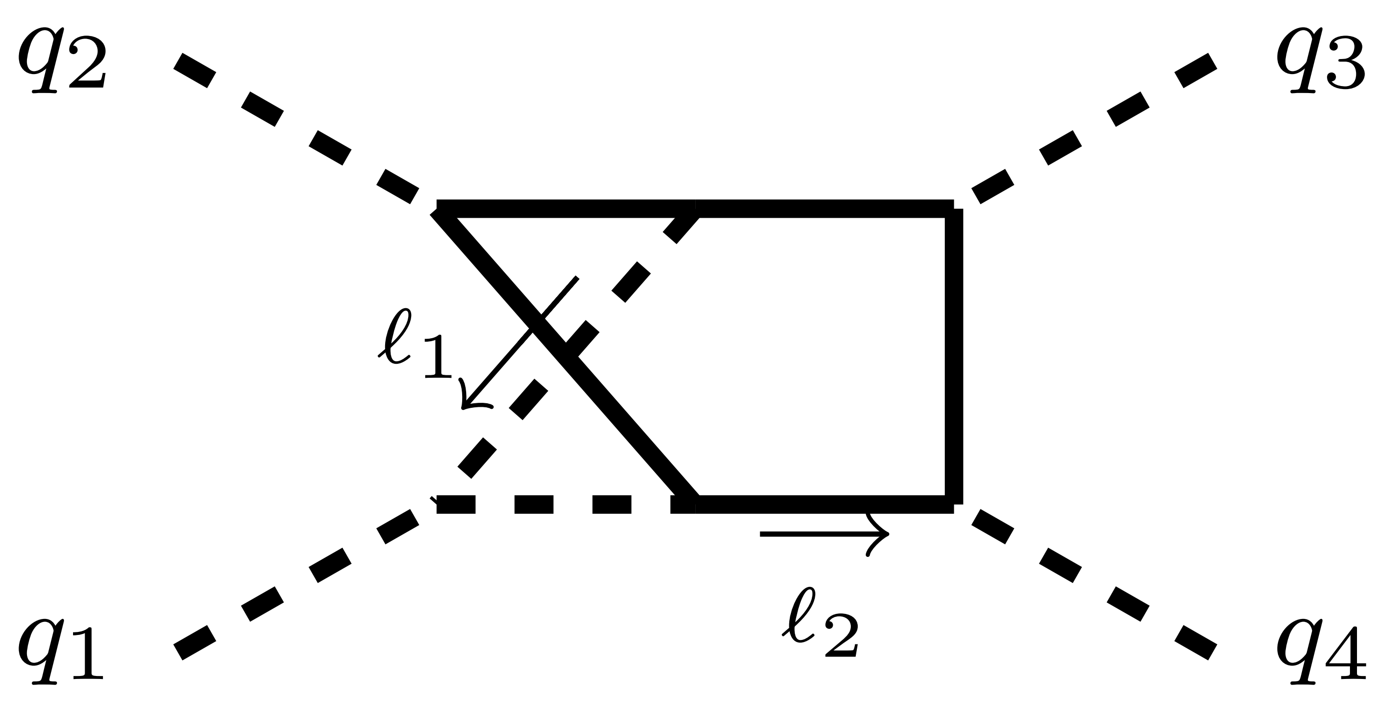
$\mathbf{u}^R$	order	interpretation	routing
(-2, -2, 0, 0)	$4 - 2(\epsilon + \alpha + \beta)$	c_1	ℓ
(0, -2, -2, 0)	$4 - 2(\epsilon + \beta + \gamma)$	c_2	$\ell - q_1$
(-2, 0, 0, -2)	$4 - 2(\epsilon + \alpha + \delta)$	c_3	$\ell + q_3$
(0, 0, -2, -2)	$4 - 2(\epsilon + \gamma + \delta)$	c_4	$\ell - q_1 - q_2$
(0, 0, 0, 0)	0	h	n/a

Using a set of possible loop momenta modes can **systematically search** for momentum routing to give a momentum space interpretation

Implemented in pySecDec by Y. Ulrich (TBA)

Example: Interpreting Facet Regions

Jaskiewicz, SPJ, Szafron, Ulrich 25



New features:

1. Soft modes appear $l_S^\mu = Q(\lambda, \lambda, \lambda)$
2. Soft regions are power enhanced at level of scalar integral

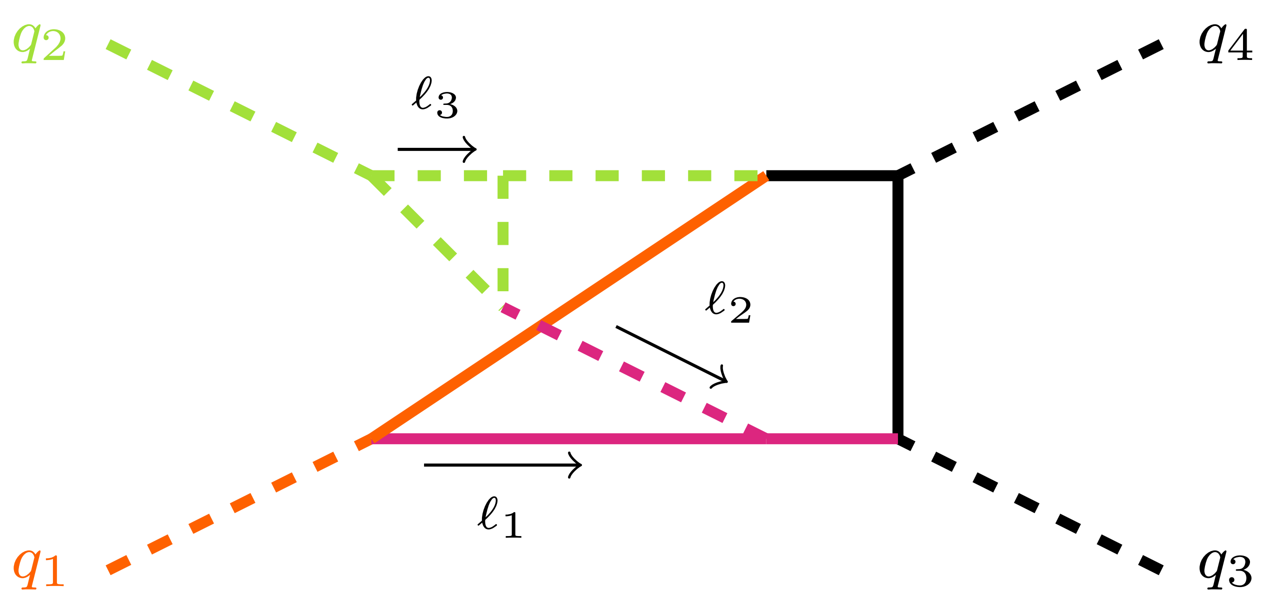
$\mathbf{u}^R$	order	interpretation	routing
(-2, -2, -2, 0, 0, 0, -2)	-4ϵ	c_1c_1	$\ell_1,\ \ell_2$
(-2, -2, 0, 0, -2, -2, 0)	-4ϵ	c_1c_1	$\ell_1,\ \ell_2 - q_3 - q_4$
(-2, -1, 0, -1, -2, -2, -1)	$-1-4\epsilon$	ss	$\ell_1,\ \ell_2 - q_3 - q_4$
(-2, 0, 0, -2, -2, -2, 0)	-4ϵ	c_3c_3	$\ell_1,\ \ell_2 - q_4$
(-2, 0, 0, 0, -2, -2, -2)	-4ϵ	c_2c_2	$\ell_1,\ \ell_2 - q_3 - q_4$
(-1, -2, -2, -1, 0, -1, -2)	$-1-4\epsilon$	ss	$\ell_1 - q_1,\ \ell_2$
(0, -2, -2, -2, 0, 0, -2)	-4ϵ	c_4c_4	$\ell_1 - q_1,\ \ell_2$
(0, -2, -2, 0, 0, -2, -2)	-4ϵ	c_2c_2	$\ell_1 - q_1,\ \ell_2$
(0, 0, -2, -2, 0, -2, -2)	-4ϵ	$c_4\bar{c}_2$	$\ell_1 - \ell_2 + q_3 + q_4,\ \ell_1$
(0, 0, -2, -2, 0, 0, 0)	-2ϵ	c_4h	$\ell_1 - \ell_2 + q_3 + q_4,\ \ell_1$
(0, 0, 0, -2, -2, -2, -2)	-4ϵ	$c_3\bar{c}_2$	$\ell_1 - \ell_2 + q_3,\ \ell_1 - q_4$
(0, 0, 0, -2, -2, 0, 0)	-2ϵ	c_3h	$\ell_1 - \ell_2 + q_3,\ \ell_1 - q_4$
(0, 0, 0, 0, 0, -2, -2)	-2ϵ	hc_2	$\ell_1,\ \ell_1 + \ell_2 - q_3 - q_4$
(0, 0, 0, 0, 0, 0, 0)	0	hh	n/a

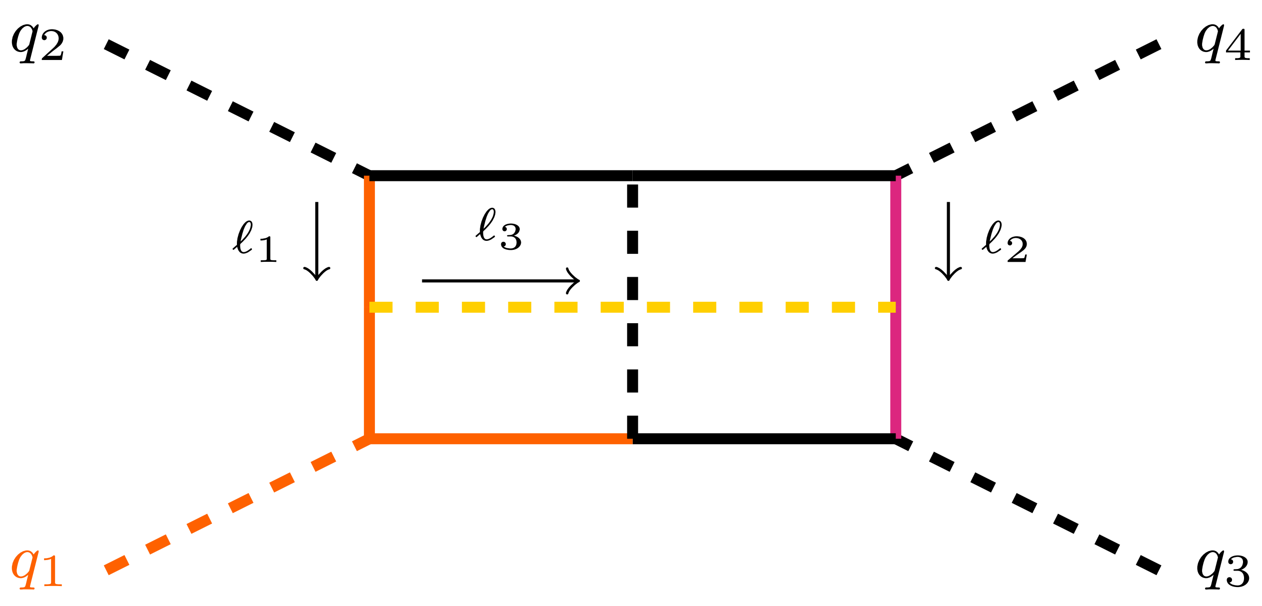
Can again find momentum space interpretation

Example: Interpreting Facet Regions

Jaskiewicz, SPJ, Szafron, Ulrich 25

At 3-loops we systematically checked for new loop momenta modes





Indeed find new modes entering

Hard-collinear $l^\mu_{HC_i} = Q(1, \lambda, \lambda^{\frac{1}{2}})$

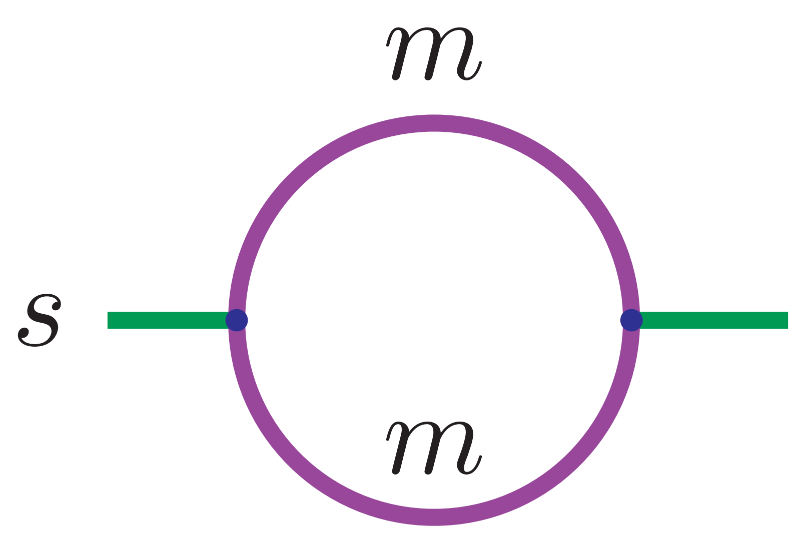
Soft-collinear $l^\mu_{SC_i} = Q(\lambda, \lambda^2, \lambda^{\frac{3}{2}})$

Ultra-soft $l^\mu_{US} = Q(\lambda^2, \lambda^2, \lambda^2)$

Expect new modes entering at each loop order, consistent with results in the literature

Ma 23

Thresholds



Beyond Euclidean Region

$$f_j(\mathbf{x}) = \sum_i^m c_{ji}(\mathbf{s})\, \mathbf{x}^{\mathbf{v}_{ji}}$$

Allow ``Minkowski region"

What happens to the polytope picture if we relax the assumption $c_{ij}(\mathbf{s}) \in \mathbb{R}^N_{>0}$?

Solutions of $\mathscr{F}(\mathbf{x};\mathbf{s}) = 0$ or $x_j \dfrac{\partial \mathscr{F}(\boldsymbol{x};\mathbf{s})}{\partial x_j} = 0$ not all fully characterised by some $x_j \to 0$

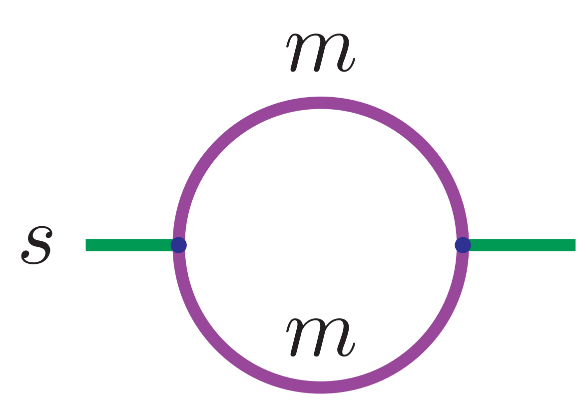
Example:

$$x_1 - x_2$$

Start by considering solutions of just the 1st Landau Equation

1 $$\mathscr{F}(\mathbf{x};\mathbf{s}) = 0$$

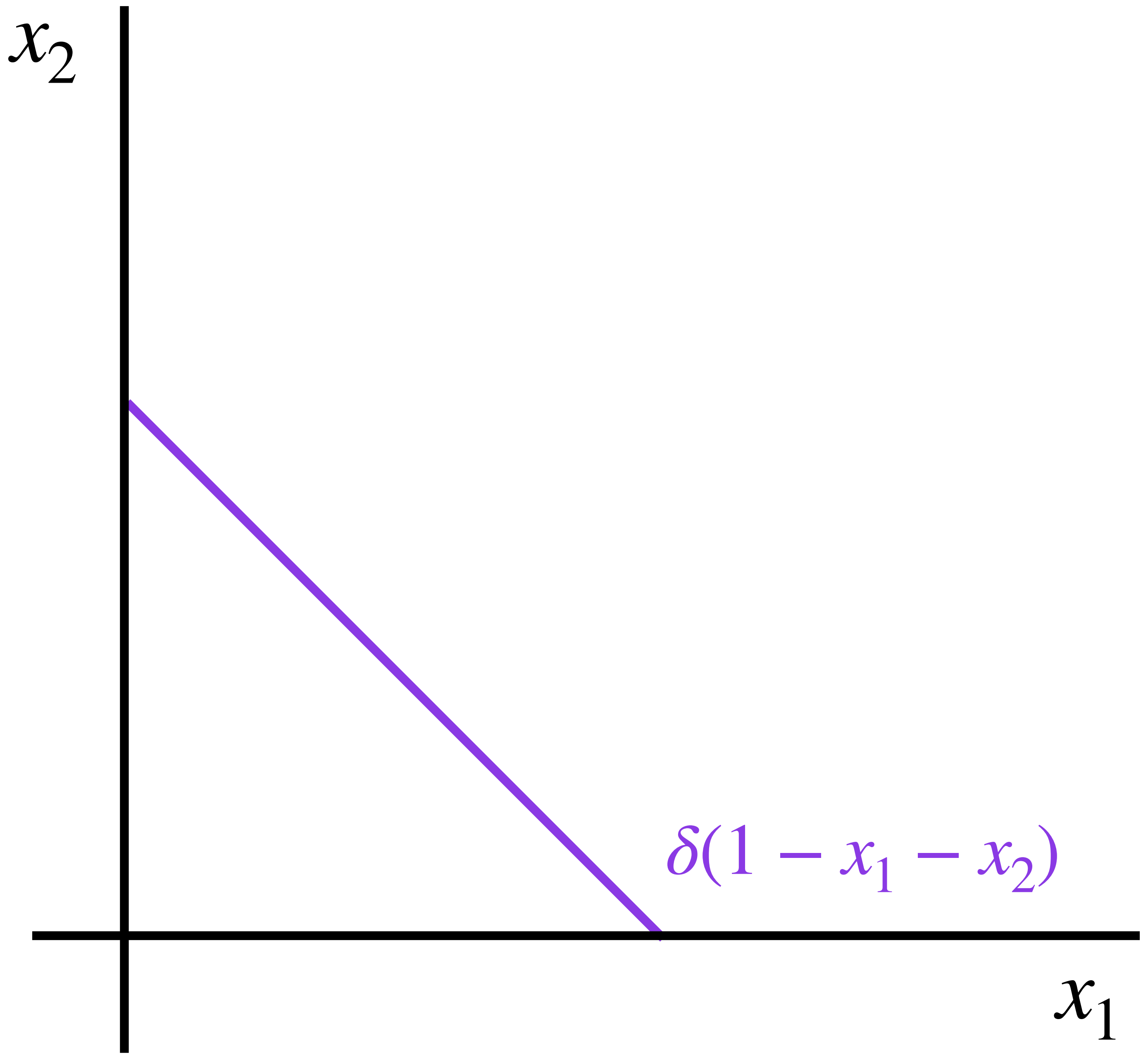
Thresholds

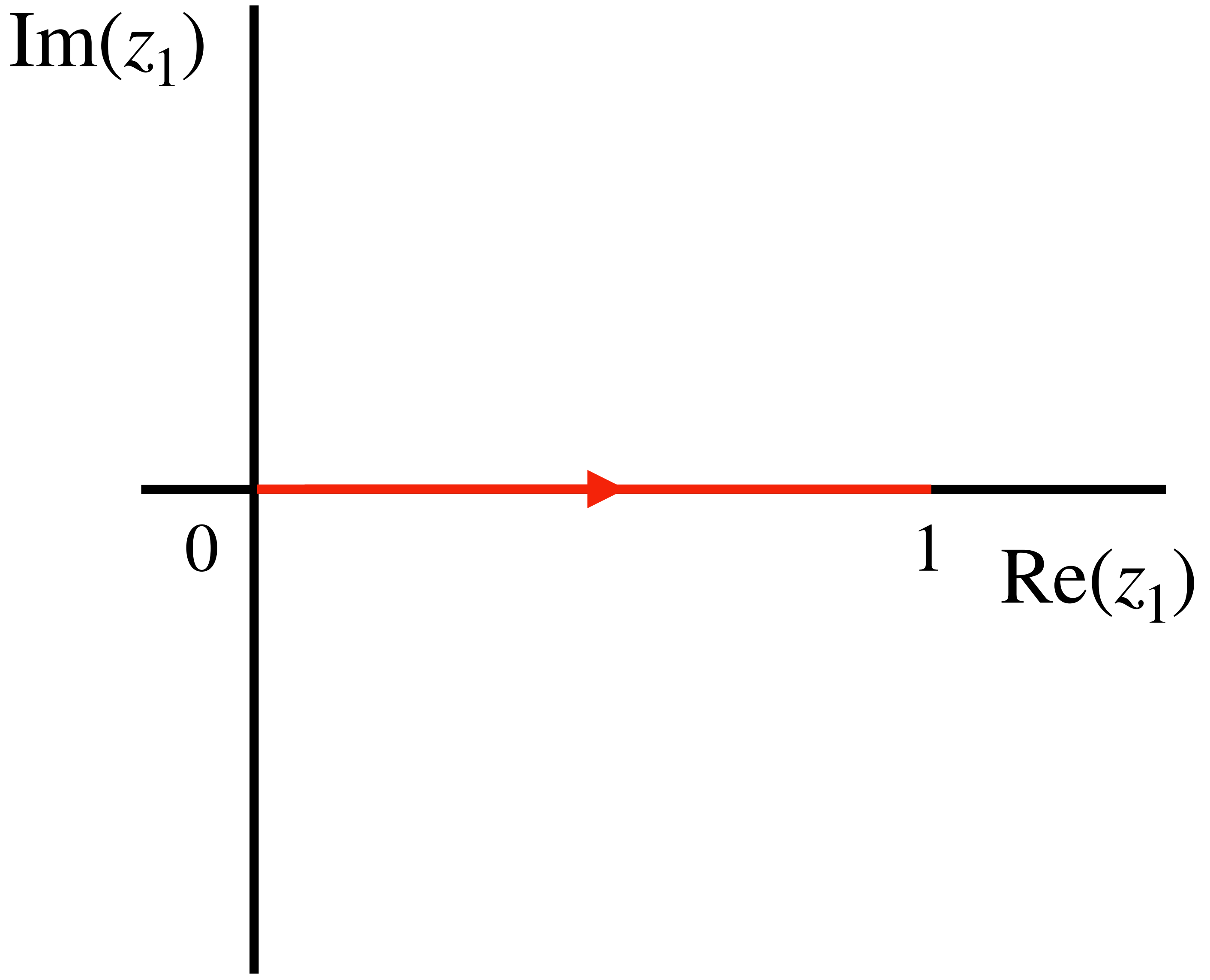


$$= \int_0^\infty \mathrm{d}x_1\, \mathrm{d}x_2 \frac{\mathcal{U}(\mathbf{x})^{-2+2\epsilon}}{(\mathcal{F}(\mathbf{x};\mathbf{s}) - i\delta)^{\epsilon}} \delta(1 - x_1 - x_2) \to \int \mathrm{d}z_1 \frac{\mathcal{U}(z_1)^{-2+2\epsilon}}{\mathcal{F}(z_1; s, m)^{\epsilon}} = \int_0^1 \mathrm{d}x\, |J_z| \frac{\mathcal{U}(z_1(x))^{-2+2\epsilon}}{\mathcal{F}(z_1(x); s, m)^{\epsilon}}$$

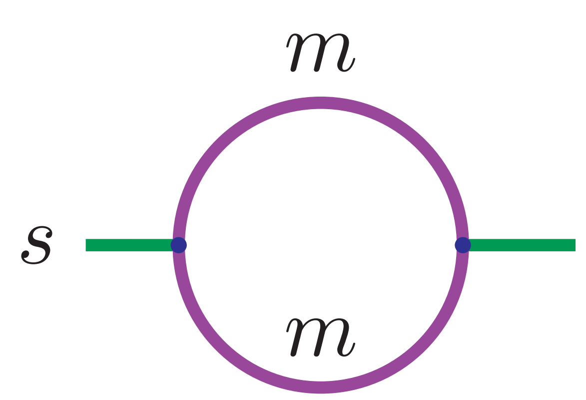
$$\mathcal{F}(\mathbf{x};\mathbf{s}) = -s x_1 x_2 + (m^2 x_1 + m^2 x_2)(x_1 + x_2)$$

Fix $m^2 = 1$ and $s = 0$





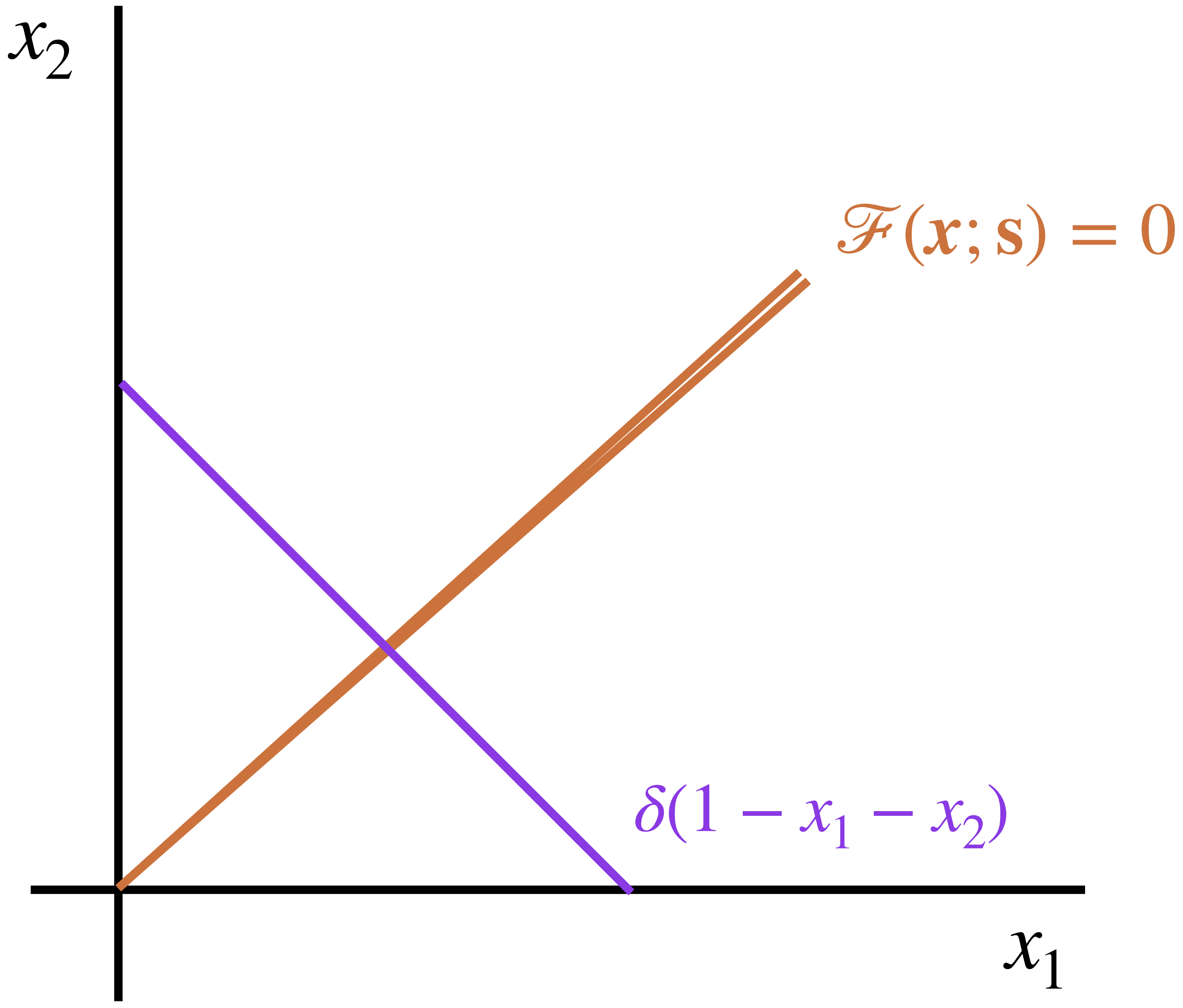
Thresholds

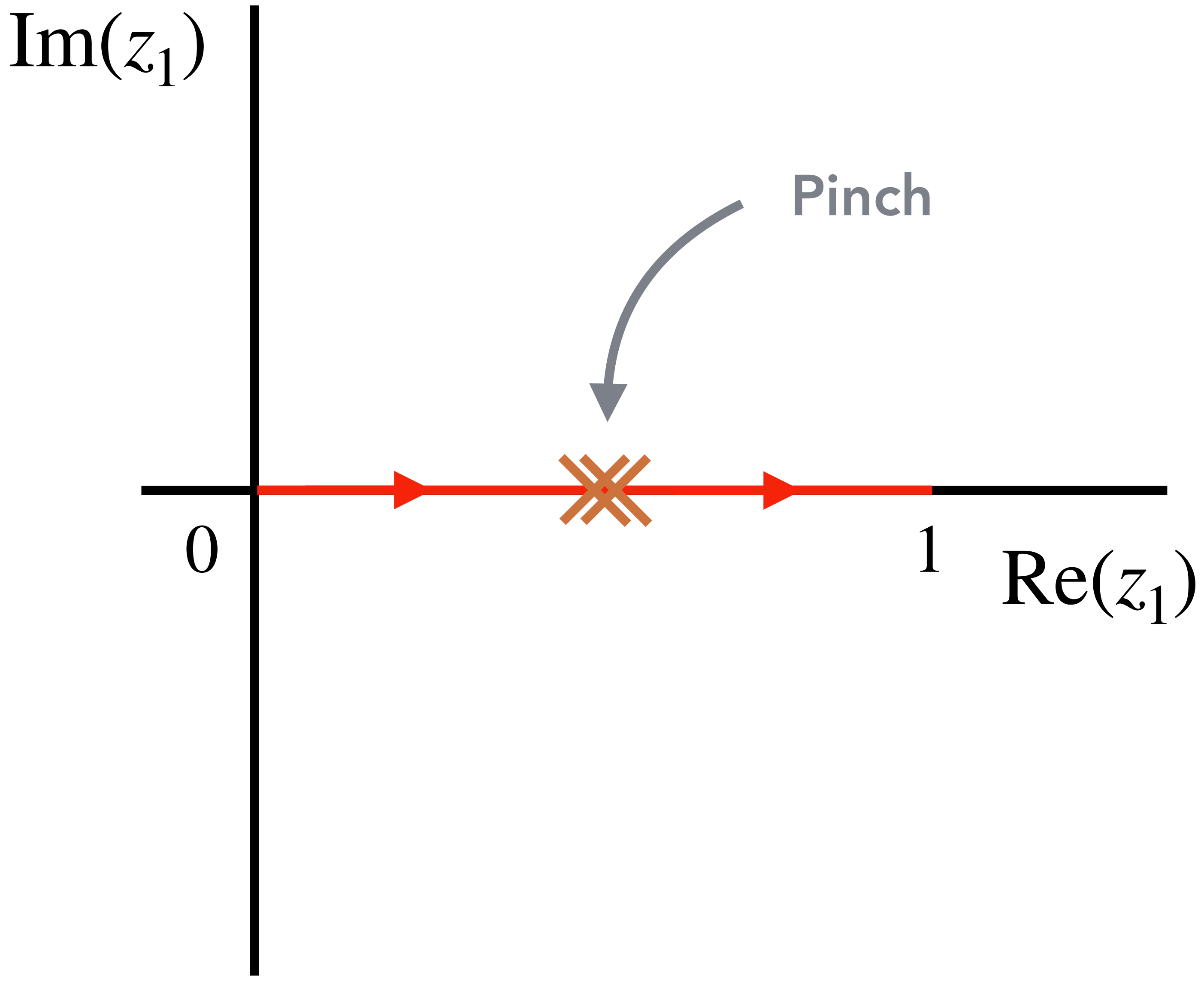


$$= \int_0^\infty \mathrm{d}x_1\,\mathrm{d}x_2 \frac{\mathcal{U}(\mathbf{x})^{-2+2\epsilon}}{(\mathcal{F}(\mathbf{x};\mathbf{s}) - i\delta)^\epsilon}\delta(1-x_1-x_2) \to \int \mathrm{d}z_1 \frac{\mathcal{U}(z_1)^{-2+2\epsilon}}{\mathcal{F}(z_1;s,m)^\epsilon} = \int_0^1 \mathrm{d}x\,|J_z|\frac{\mathcal{U}(z_1(x))^{-2+2\epsilon}}{\mathcal{F}(z_1(x);s,m)^\epsilon}$$

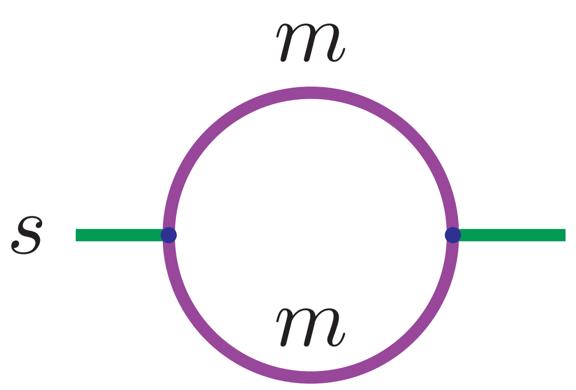
$$\mathcal{F}(\mathbf{x};\mathbf{s}) = -s x_1 x_2 + (m^2 x_1 + m^2 x_2)(x_1 + x_2)$$

Fix $m^2 = 1$ and $s \gtrsim 4$ ← Threshold





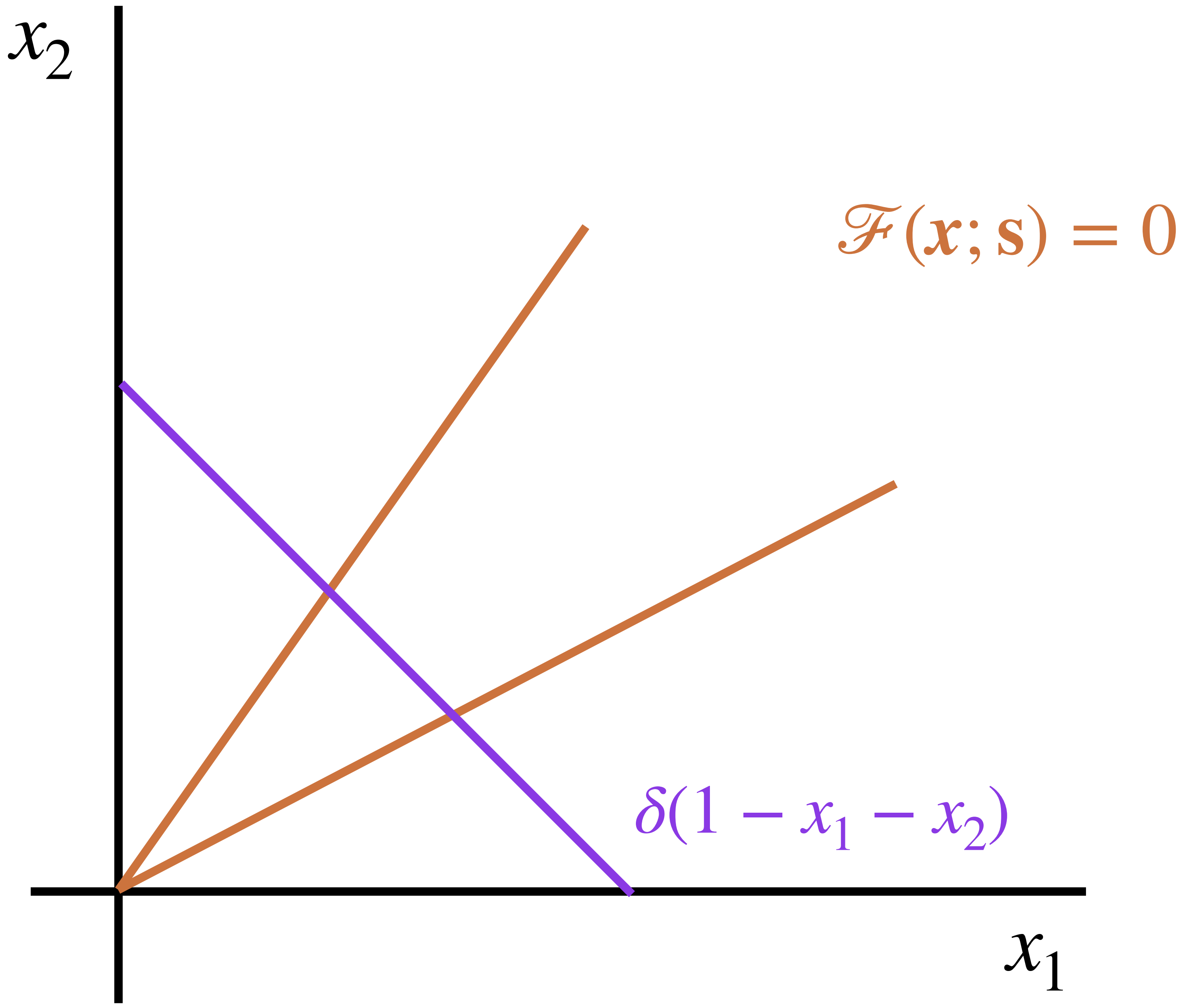
Thresholds

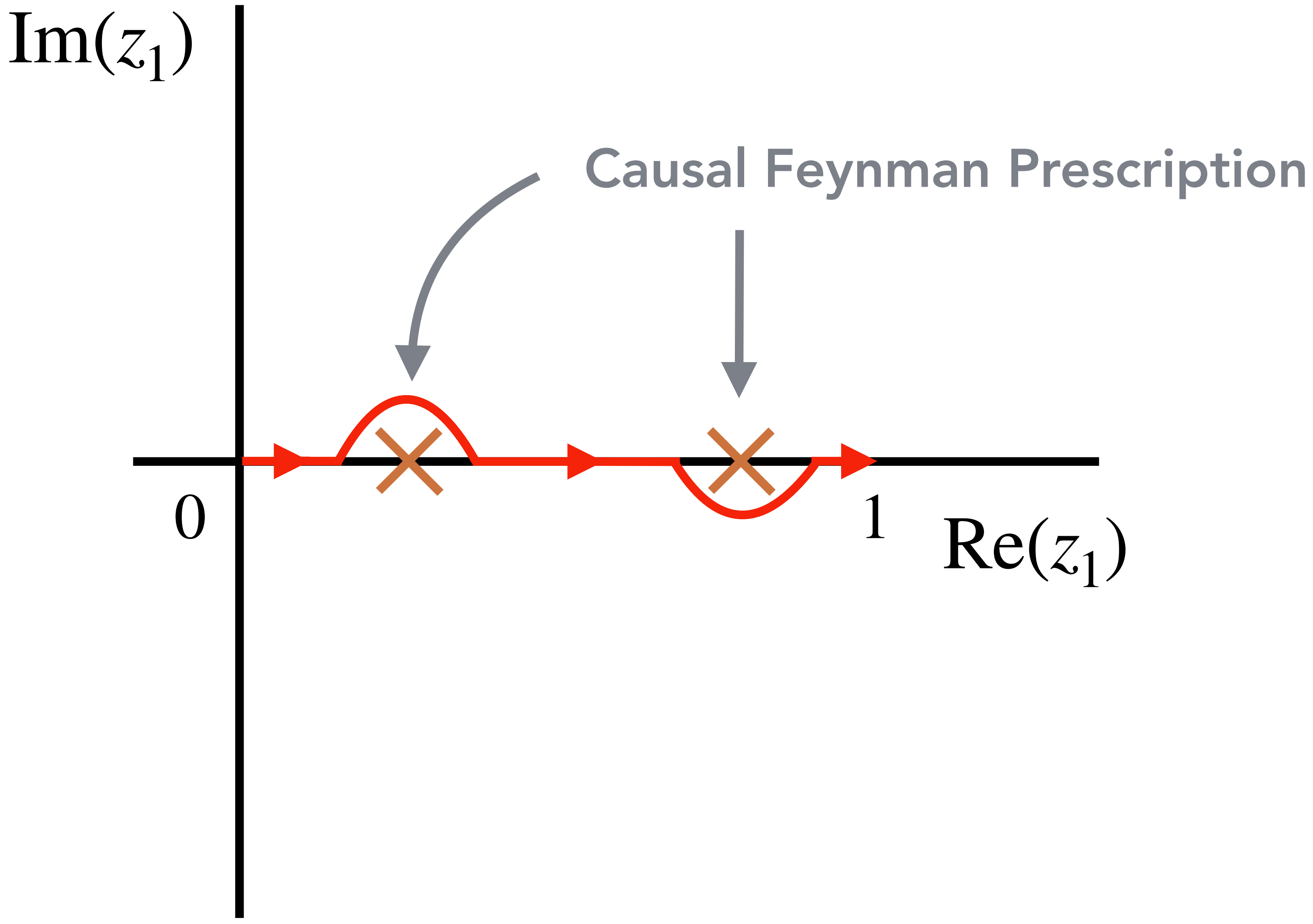


$$= \int_0^\infty \mathrm{d}x_1\,\mathrm{d}x_2 \frac{\mathcal{U}(\mathbf{x})^{-2+2\epsilon}}{(\mathcal{F}(\mathbf{x};\mathbf{s}) - i\delta)^\epsilon}\delta(1 - x_1 - x_2) \to \int \mathrm{d}z_1 \frac{\mathcal{U}(z_1)^{-2+2\epsilon}}{\mathcal{F}(z_1;s,m)^\epsilon} = \int_0^1 \mathrm{d}x\,|J_z|\,\frac{\mathcal{U}(z_1(x))^{-2+2\epsilon}}{\mathcal{F}(z_1(x);s,m)^\epsilon}$$

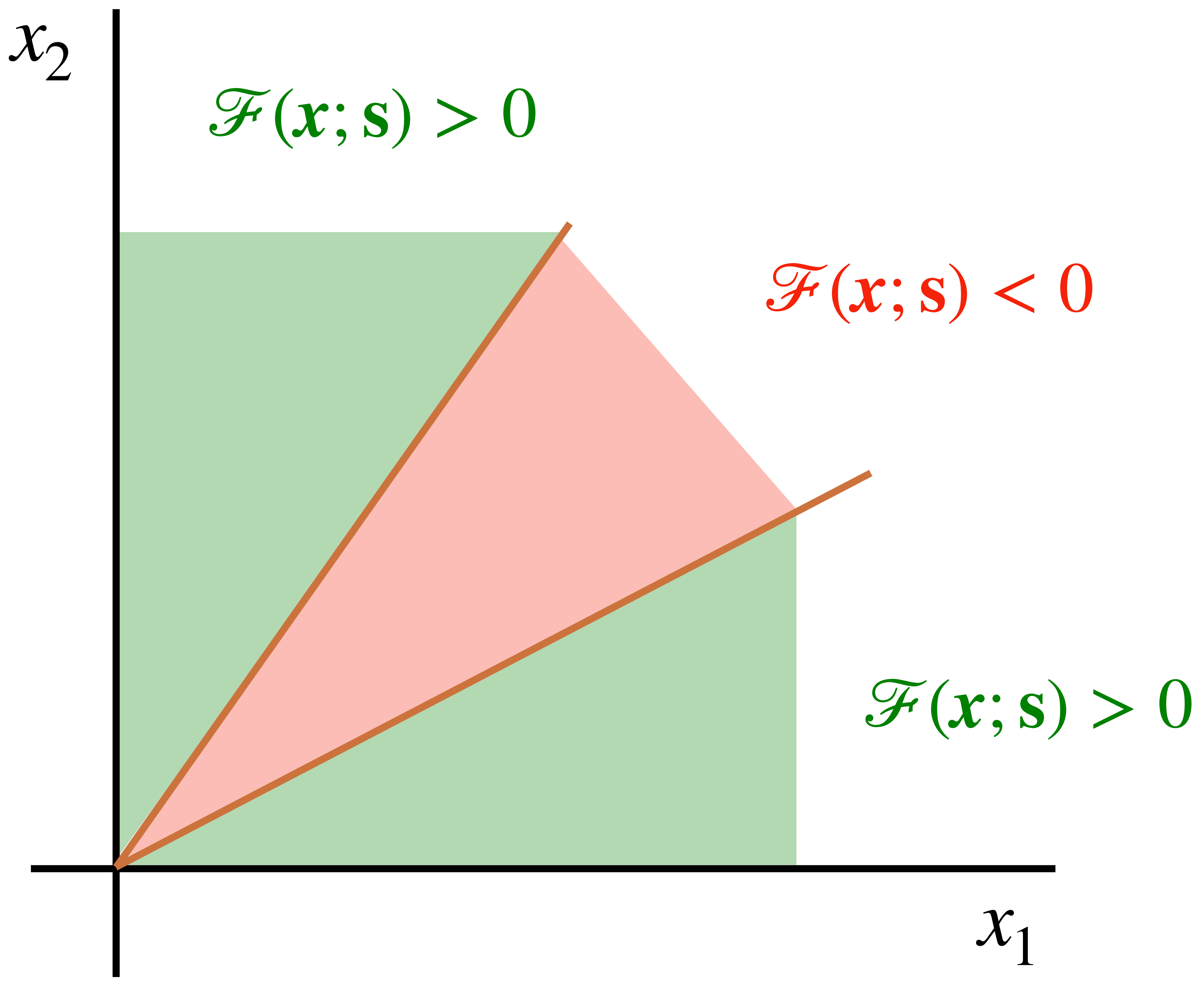
$$\mathcal{F}(\mathbf{x};\mathbf{s}) = -s x_1 x_2 + (m^2 x_1 + m^2 x_2)\,(x_1 + x_2)$$

Fix $m^2 = 1$ and $s \gg 4$ ← Above threshold



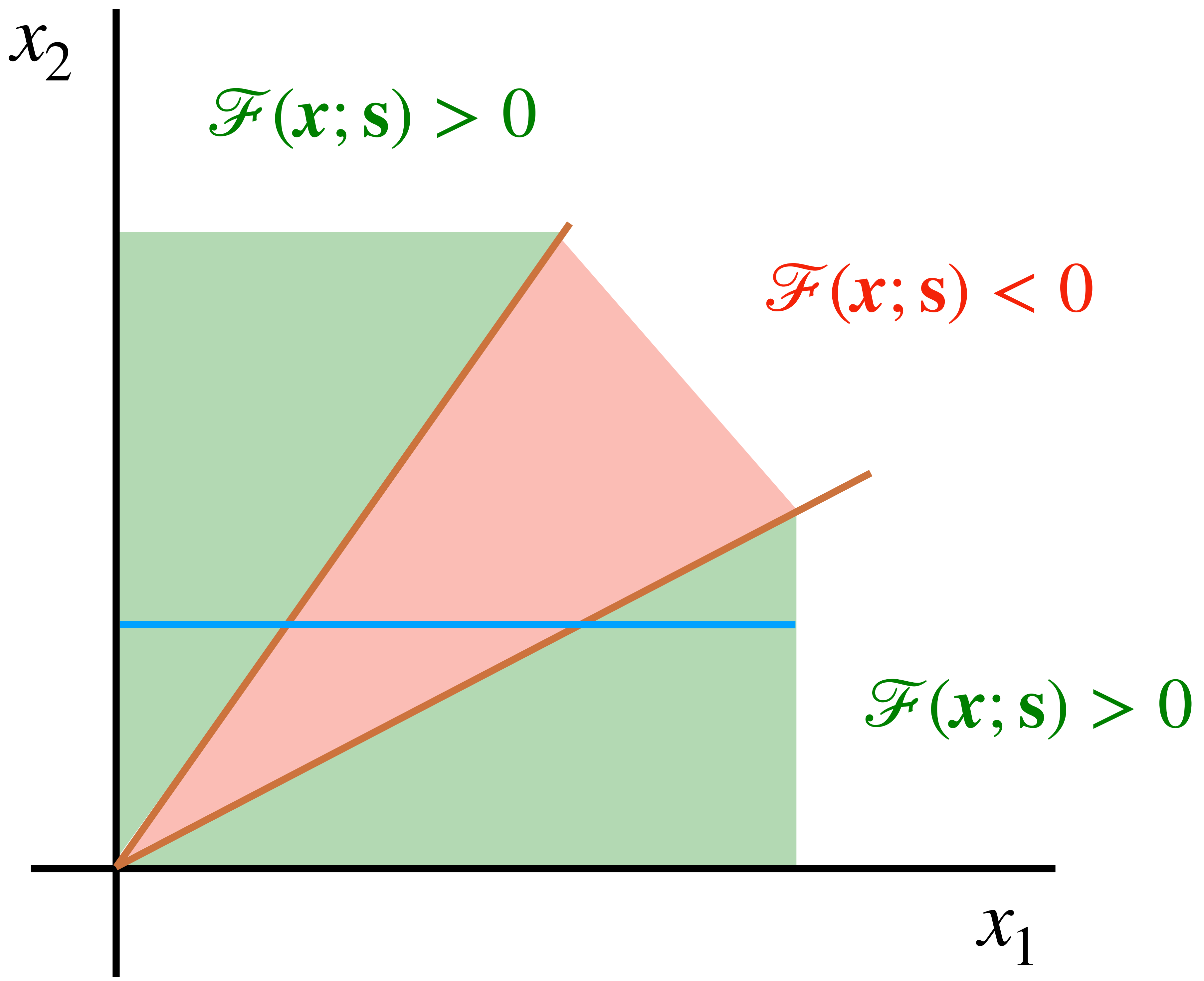


Thresholds



x_2
$\mathscr{F}(\boldsymbol{x};\mathbf{s}) > 0$
$\mathscr{F}(\boldsymbol{x};\mathbf{s}) < 0$
$\mathscr{F}(\boldsymbol{x};\mathbf{s}) > 0$
x_1

Thresholds



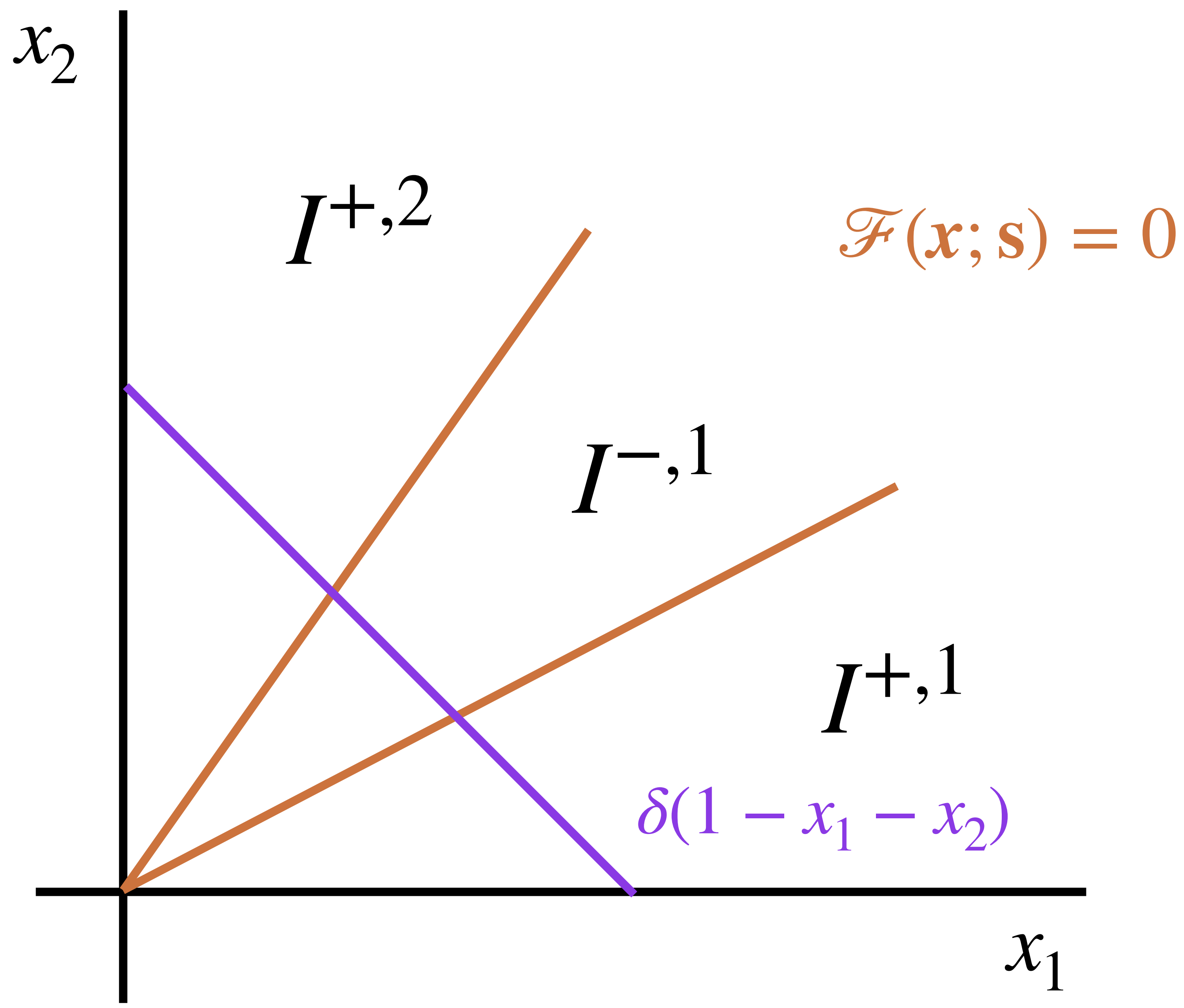
Think about $\mathscr{F}(x_i)$ with fixed $x_{j\neq i}$ and $\mathbf{s}$

Will intersect $\mathscr{F}(\mathbf{x};\mathbf{s}) = 0$ only 0, 1 or 2 times ($\mathscr{F}(x_i)$ is at most quadratic)

Thresholds and Discontinuities

SPJ, Olsson Stone 25

Why not just remap the $\mathscr{F}(\mathbf{x};\mathbf{s}) = 0$ hypersurface to a boundary of integration?



Manifestly non-negative integrands

$$I(\mathbf{s}) = \sum_{n_+=1}^{N_+} I^{+,n_+}(\mathbf{s}) + \lim_{\delta\to 0^+} (-1 - i\delta)^{-(N - LD/2)} \sum_{n_-=1}^{N_-} I^{-,n_-}(\mathbf{s})$$

Generates imaginary part of integral

Corresponds to considering discontinuities of the integral, allows to define ``Euclidean'' like integrals even above physical thresholds

Britto 23; Britto, Hannesdottir 25

Relation to Complete Monotonicity? Positive Geometry? E.g. Henn, Arkani-Hamed, Sturmfels, Trnka

Decomposition

w/ Bennett, Chargeishvili, Magerya, Olsson, Stone

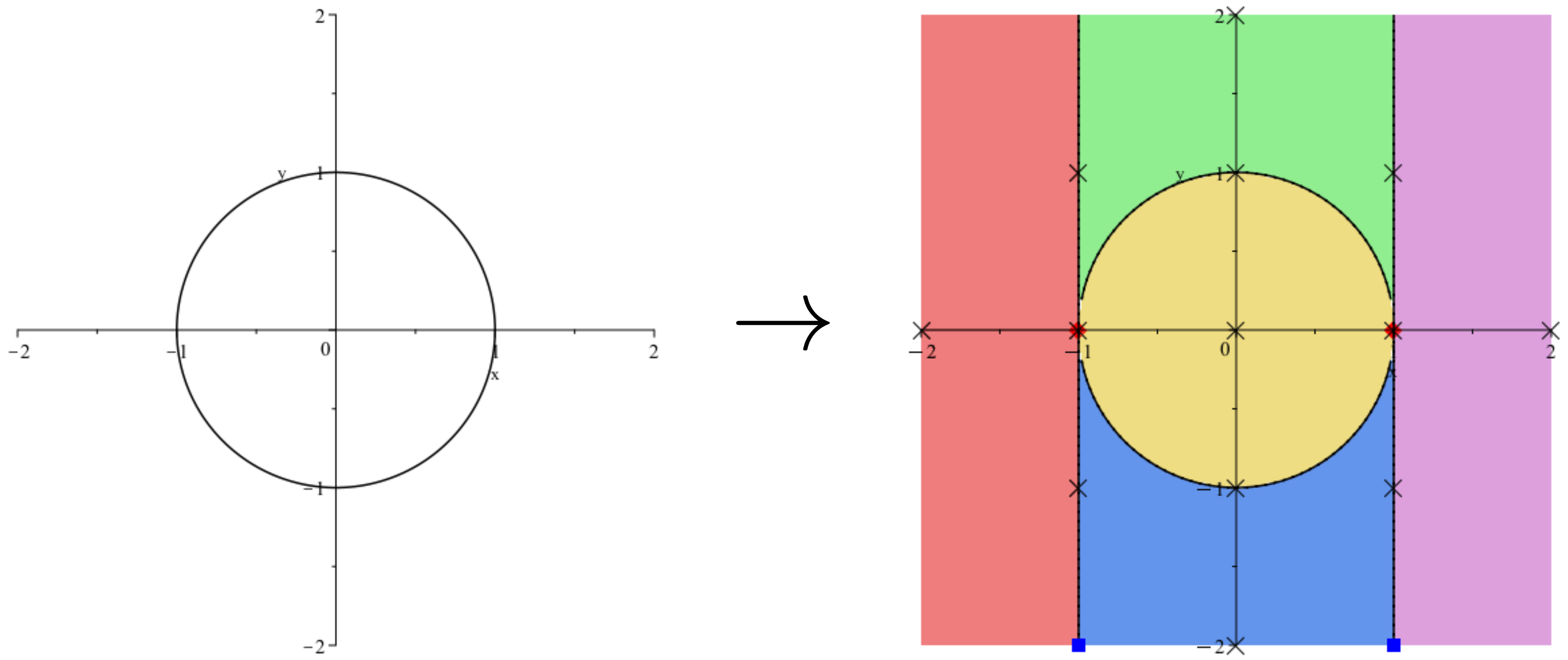
How to achieve this in practice?

Trying to find sign-invariant decomposition of
$\{\mathscr{F}(\mathbf{x};\mathbf{s}) < 0\} \cup \{0 < \mathbf{x}\} \cup \mathbf{s}_R,$

Borrow technique from real algebraic geometry
Generic **C**ylindrical **A**lgebraic **D**ecomposition

codim 0	projection of cells either disjoint or identical	cell boundaries are roots of polynomials	union of disjoint cells is $\mathbb{R}^N$

Collins 75; Davenport, Heintz 88; Lazard 94; McCallum 19



Lee, del Río, Rahkooy 25

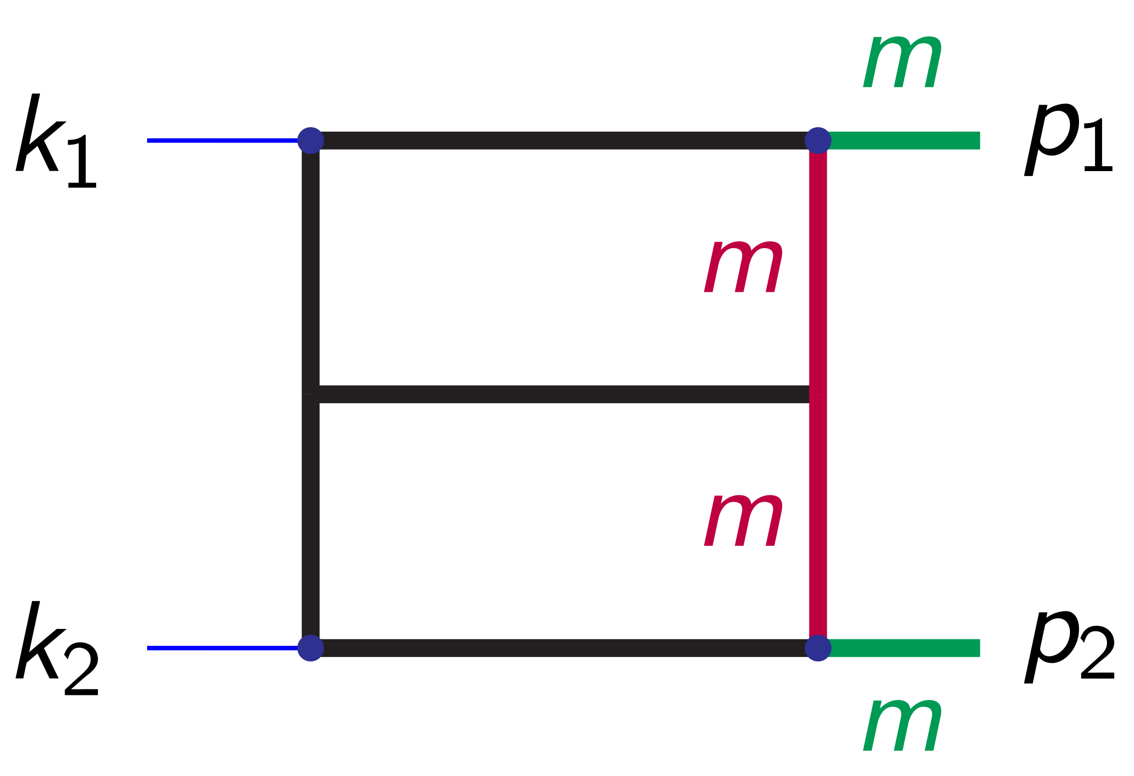
Projection operators consist of Discriminants and Resultants

Similar to projections used in Landau literature

Correia, Giroux, Mizera 25

$$\begin{array}{ccc}
\mathcal{F}_n & & \mathrm{CAD}(\mathcal{F}_n) \\
\downarrow \text{proj} & & \uparrow \text{Lift} \\
\mathcal{F}_{n-1} & & \mathrm{CAD}(\mathcal{F}_{n-1}) \\
\downarrow \text{proj} & & \uparrow \text{Lift} \\
\vdots & & \vdots \\
\downarrow \text{proj} & & \uparrow \text{Lift} \\
\mathcal{F}_2 & & \mathrm{CAD}(\mathcal{F}_2) \\
\downarrow \text{proj} & & \uparrow \text{Lift} \\
\mathcal{F}_1 & \xrightarrow[\text{split into cells}]{\text{find roots}} & \mathrm{CAD}(\mathcal{F}_1)
\end{array}$$

Example



$\mathbf{s}^*$: $s = 4,\ t = -1,\ u = -1$ (with $m = 1$)

$$\mathcal{U}(\mathbf{x}) = x_1x_3 + x_2x_3 + x_1x_4 + x_2x_4 + x_1x_5 + x_2x_5 + x_3x_5 + x_4x_5 + x_1x_6 + x_2x_6 + x_5x_6 + x_3x_7 + x_4x_7 + x_5x_7 + x_6x_7$$

$$\mathcal{F}(\mathbf{x};\mathbf{s}) = -4x_1x_4x_5 + 2\left[x_1x_3x_6 + x_2x_3x_6 + x_2x_5x_6 + x_3x_5x_6 + x_2x_3x_7 + x_2x_4x_7 + x_2x_5x_7 + x_3x_5x_7 + x_2x_6x_7 + x_3x_6x_7 + x_5x_6x_7\right] + (x_1 + x_2 + x_5 + x_7)\,x_6^2 + (x_3 + x_4 + x_5 + x_6)\,x_7^2$$

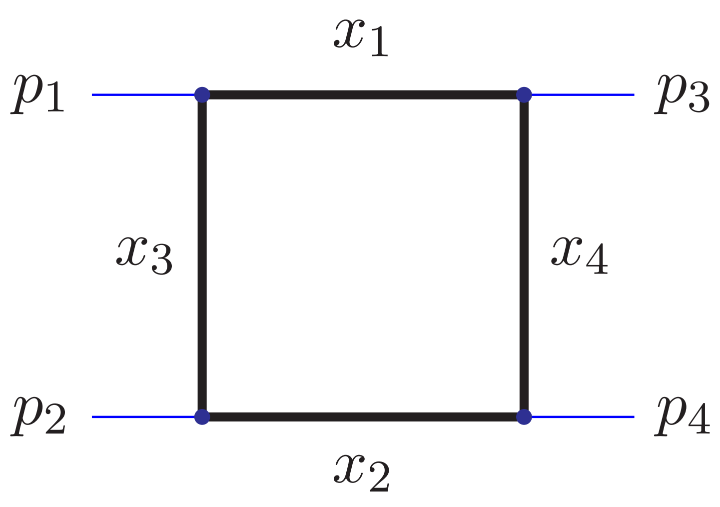
$$\{\mathcal{F}(\mathbf{x}) < 0\} \ \wedge\ \{0 < \mathbf{x}\}$$

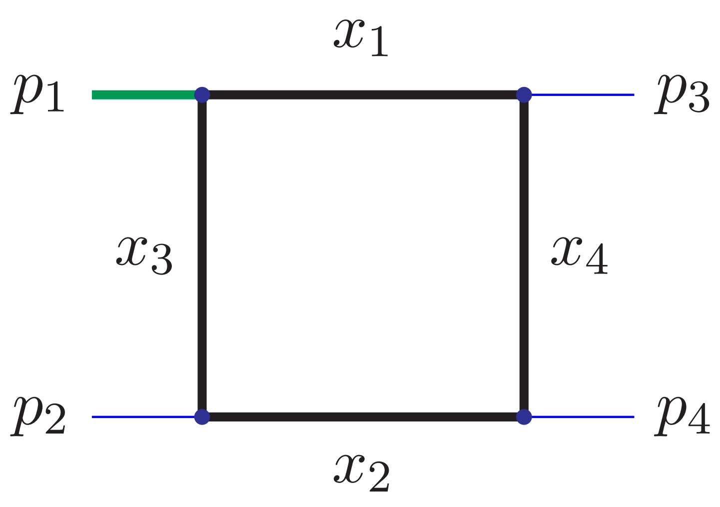
GCAD with projection ordering $x_6, x_7, x_2, x_3, x_5, x_1, x_4$

$$0 < x_6 \wedge 0 < x_7 \wedge 0 < x_2 \wedge 0 < x_3 \wedge x_5 < 0 \wedge \frac{(2x_2 + x_7)\,x_7}{4x_5} < x_1 \wedge \frac{\mathcal{N}(\mathbf{x}_{\neq 4})}{4x_1x_5 - (2x_2 + x_7)\,x_7} < x_4$$

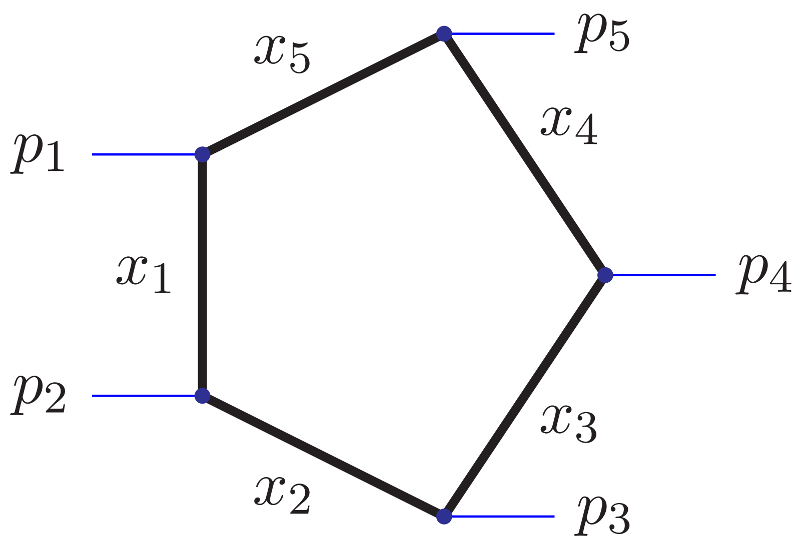
$$x_4 \to x_4 + \frac{\mathcal{N}(\mathbf{x}_{\neq 4})}{4x_1x_5 - (2x_2 + x_7)\,x_7} \quad \textbf{THEN} \quad x_1 \to x_1 + \frac{(2x_2 + x_7)x_7}{4x_5}$$

Further Examples

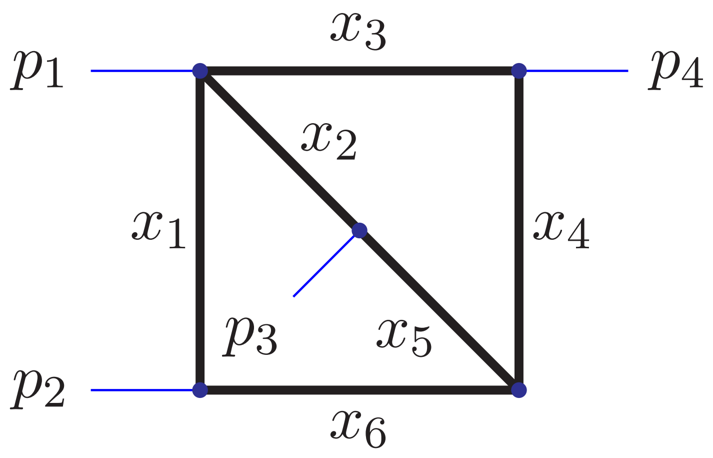


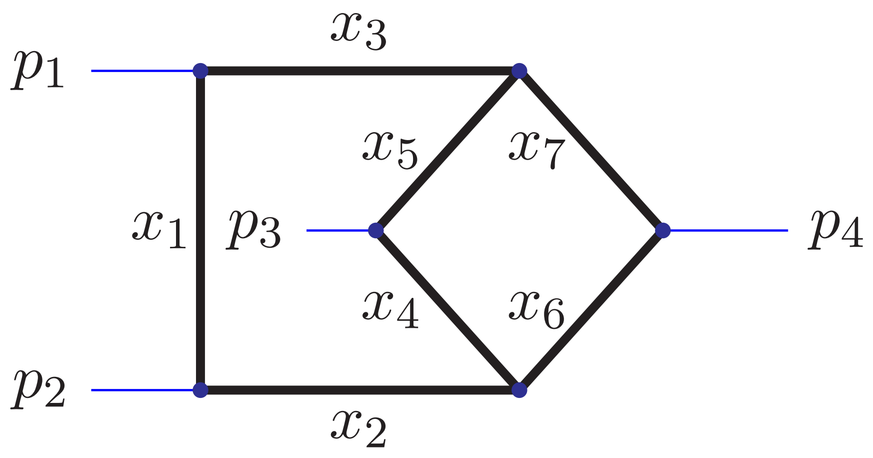


$\mathbf{s}_{p_1^2>0} = \{0 < p_1^2 < \infty,$
$0 < s_{12} < \infty, -s_{12} < s_{13} < 0\}$



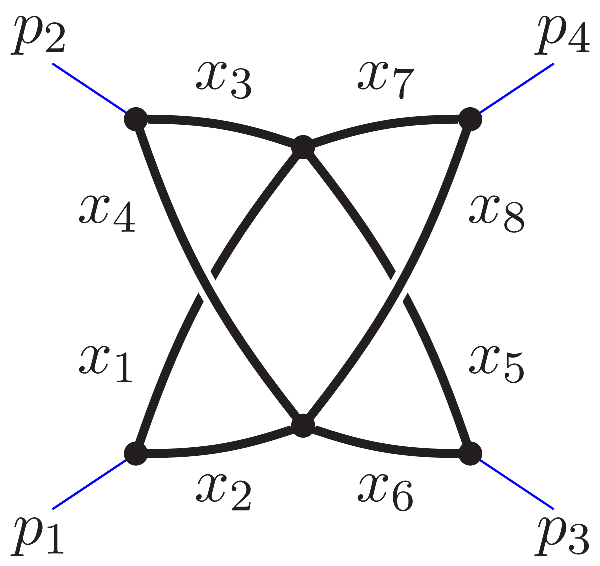
$\mathbf{s}_R = \{0 < s_{12}, s_{34}, s_{51} < \infty,$
$-\infty < s_{23}, s_{45} < 0\}$





$\mathbf{s}_{\text{phys}} = \{0 < s_{12} < \infty, \; -s_{12} < s_{23} < 0\}$

$\mathbf{s}_{\text{phys}} = \{0 < s_{12} < \infty, -s_{12} < s_{13} < 0\}$



Univariate algorithm works for many cases

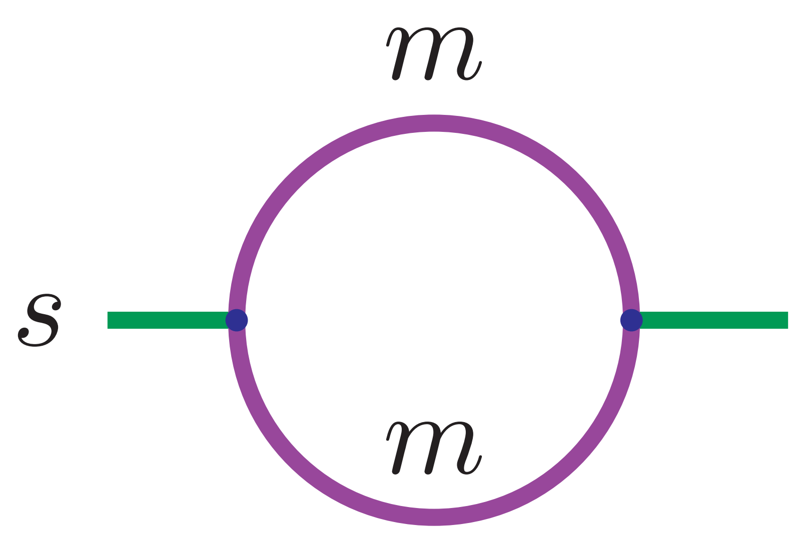
SPJ, Olsson Stone 25

General algorithm: Generic Cylindrical Algebraic Decomposition

SPJ, pySecDec collaboration (WIP)

Speeds up numerical evaluation of integrals by factor of $10^3 - 10^4$ in difficult cases

Method of Regions: Beyond Facet Regions



Beyond Facet Regions

To find regions, simultaneously solve the Landau Equations

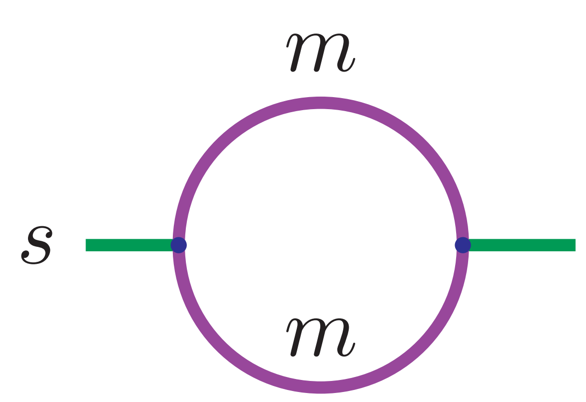
Focus on leading Landau singularities ($\Longrightarrow$ ignore solutions with some $x_j = 0$)

1 $$\mathcal{F}(\mathbf{x};\mathbf{s}) = 0$$ ← At most quadratic in x_i

2 $$\frac{\partial \mathcal{F}(\mathbf{x};\mathbf{s})}{\partial x_j} = 0 \quad \forall j$$ ← At most linear in x_i

Can we use Generic Cylindrical Algebraic Decomposition to solve this problem?

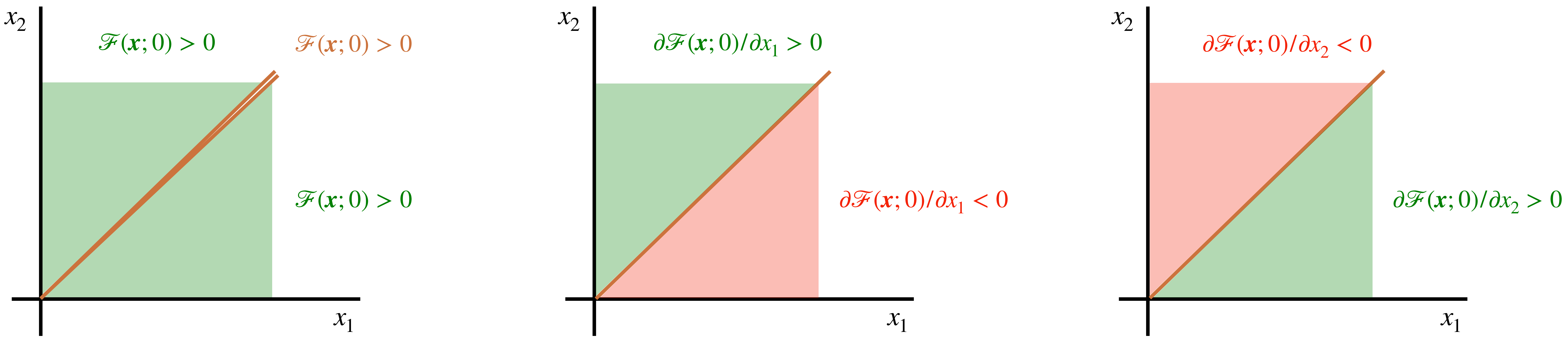
Example 1: Beyond Facet Regions



$$\mathscr{F}(\mathbf{x};\mathbf{s}) = -s x_1 x_2 + (m^2 x_1 + m^2 x_2)(x_1 + x_2)$$

Suppose we are interested in expanding in $y = s - 4m^2 \sim 0$, consider the limit $y \to \lambda y$ ($m^2 = 1$ and $s = 4$)

$$\mathscr{F}(\mathbf{x};0) = (x_1 - x_2)^2, \qquad \frac{\partial \mathscr{F}(\mathbf{x};0)}{\partial x_1} = 2(x_1 - x_2), \qquad \frac{\partial \mathscr{F}(\mathbf{x};0)}{\partial x_2} = -2(x_1 - x_2)$$



Example 1: Beyond Facet Regions

Focus on leading Landau singularities ($\implies$ ignore solutions with some $x_j = 0$)

We want to characterise the solution of $\mathcal{F}(\mathbf{x};0) = 0,\ \partial\mathcal{F}(\mathbf{x};0)/\partial x_1 = 0,\ \partial\mathcal{F}(\mathbf{x};0)/\partial x_2 = 0$

1 $$\mathcal{F}(\mathbf{x};\mathbf{s}) = 0$$ ← **At most quadratic in x_i**

2 $$\frac{\partial\mathcal{F}(\mathbf{x};\mathbf{s})}{\partial x_j} = 0 \quad \forall j$$ ← **At most linear in x_i**

Can we use Generic Cylindrical Algebraic Decomposition (GCAD) to solve this problem?

Example 1: Beyond Facet Regions

Let's compute a GCAD of the following systems

$$\mathcal{F}(\mathbf{x};0) > 0 \ \wedge \ \frac{\partial \mathcal{F}(\mathbf{x};0)}{\partial x_1} > 0 \ \wedge \ \frac{\partial \mathcal{F}(\mathbf{x};0)}{\partial x_2} < 0 \ \wedge \ x_1 > 0 \ \wedge \ x_2 > 0 \qquad \Longrightarrow \ x_2 < x_1$$

$$\mathcal{F}(\mathbf{x};0) > 0 \ \wedge \ \frac{\partial \mathcal{F}(\mathbf{x};0)}{\partial x_1} < 0 \ \wedge \ \frac{\partial \mathcal{F}(\mathbf{x};0)}{\partial x_2} > 0 \ \wedge \ x_1 > 0 \ \wedge \ x_2 > 0 \qquad \Longrightarrow \ x_1 < x_2$$

Can dissect integral into two integrals that have solutions of the Landau Equations only on the boundary

Examining the facet regions of the new integrals we obtain

This dissection can be found with Asy2

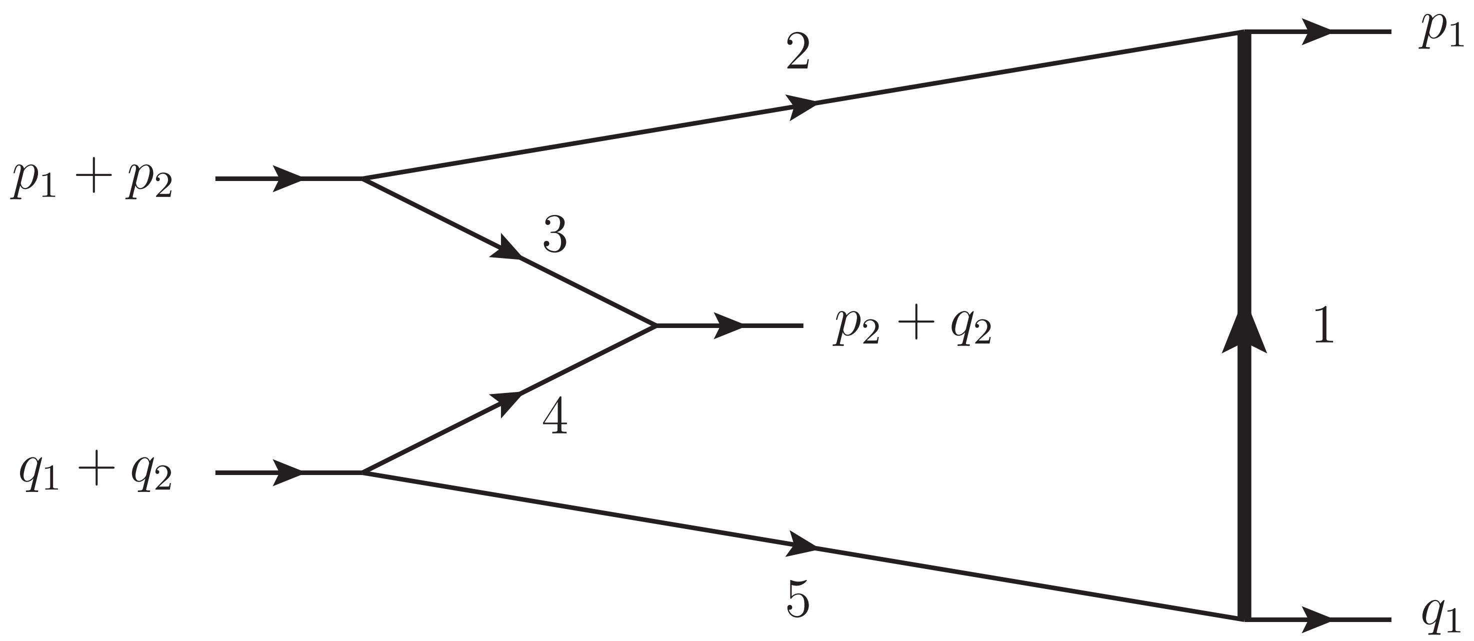
Jantzen, A. Smirnov, V. Smirnov 12

$$\{\lambda^0, \lambda^0\} \ \& \ \{\lambda^0, \lambda^{\frac{1}{2}}\}$$

Hard region:
Facet of original integral

Potential region:
``Hidden" region, facet of dissected integral

Example 2: Beyond Facet Regions



Jantzen, A. Smirnov, V. Smirnov 12

$$\mathcal{F}(\mathbf{x};\mathbf{s}) = x_1(x_1+x_2+x_3+x_4+x_5)\, m^2 + (x_2-x_3)(x_4-x_5)\, q^2$$

Expansion $m^2 \to \lambda m^2$

$$\mathcal{F}(\mathbf{x};0) = (x_2-x_3)(x_4-x_5)\, q^2$$

GCAD system

$$\mathcal{F}(\mathbf{x};0) > 0 \ \wedge\ \frac{\partial \mathcal{F}(\mathbf{x};0)}{\partial x_2} > 0 \ \wedge\ \frac{\partial \mathcal{F}(\mathbf{x};0)}{\partial x_3} < 0 \ \wedge\ \frac{\partial \mathcal{F}(\mathbf{x};0)}{\partial x_4} > 0 \ \wedge\ \frac{\partial \mathcal{F}(\mathbf{x};0)}{\partial x_5} < 0 \ \wedge\ x_i > 0$$

$$x_3 < x_2 \ \wedge\ x_5 < x_4$$

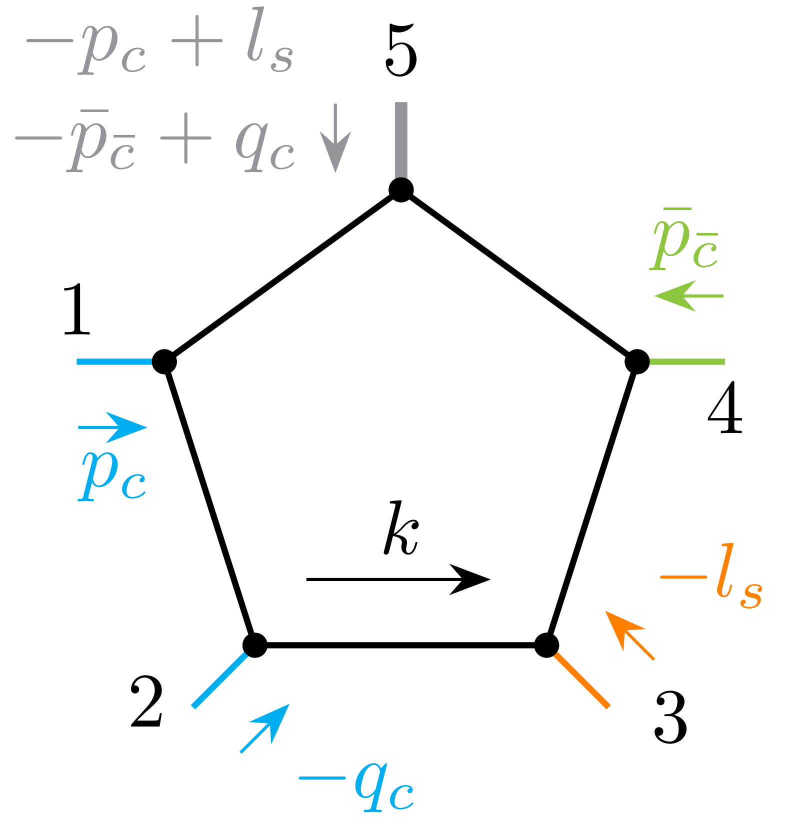
The new facet introduced by this dissection corresponds to a Glauber region

Jantzen, A. Smirnov, V. Smirnov 12

This dissection can be found with Asy2

Jantzen, A. Smirnov, V. Smirnov 12

Example 3: Beyond Facet Regions



Becher, Hager, Jaskiewicz, Neubert, Schwienbacher 24

$$\mathscr{F}(\mathbf{x};\mathbf{s}) = -x_1x_3\, s_{23} - x_1x_4\, s_{51} - x_3x_5\, s_{45} - x_4x_5\, m^2 - x_2x_4\, s_{34} - x_2x_5\, s_{12}$$

Expansion $s_{34} \to \lambda s_{34},\ s_{12} \to \lambda s_{12},$

$$\mathscr{F}(\mathbf{x};0) = -x_1x_3\, s_{23} - x_1x_4\, s_{51} - x_3x_5\, s_{45} - x_4x_5\, m^2$$

← Does not factor in these variables

One of the GCAD systems

(Apologies to the authors if this is not the relevant kinematic region…)

$$\mathscr{F}(\mathbf{x};0) > 0 \ \wedge\ \frac{\partial \mathscr{F}(\mathbf{x};0)}{\partial x_1} < 0 \ \wedge\ \frac{\partial \mathscr{F}(\mathbf{x};0)}{\partial x_3} > 0 \ \wedge\ \frac{\partial \mathscr{F}(\mathbf{x};0)}{\partial x_4} < 0 \ \wedge\ \frac{\partial \mathscr{F}(\mathbf{x};0)}{\partial x_5} < 0 \ \wedge\ x_i > 0 \ \wedge\ s_{23} > 0 \ \wedge\ s_{45} < 0 \ \wedge\ s_{51} < 0$$

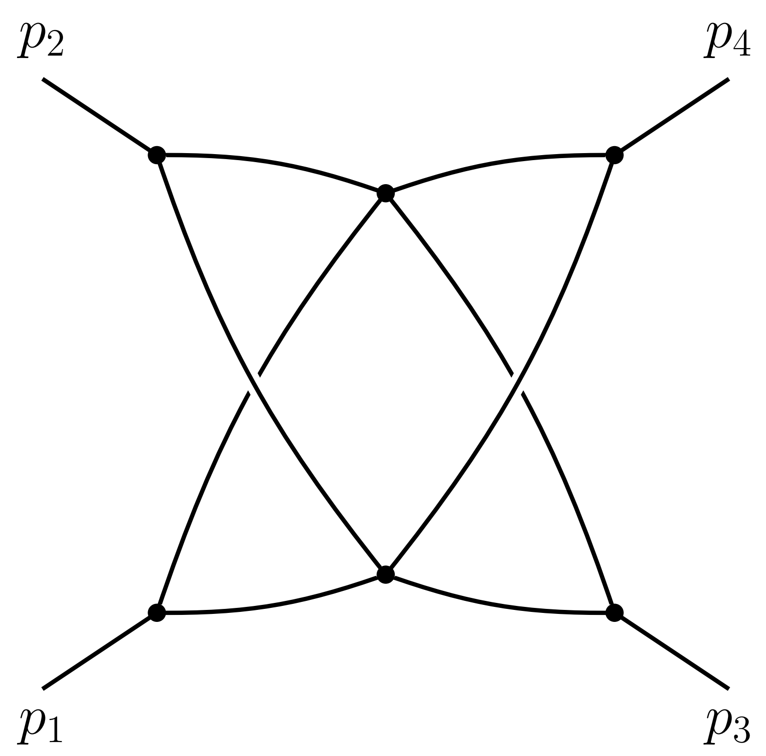
$$\frac{m^2 s_{23}}{s_{45}} < s_{51} \ \wedge\ \frac{s_{23}x_1}{-s_{45}} < x_5 \ \wedge\ \frac{-s_{45}x_3}{m^2} < x_4 < \frac{s_{23}x_3}{-s_{51}}$$

The new facet introduced by this dissection corresponds to a Glauber region

Not found with Asy2

Becher, Hager, Jaskiewicz, Neubert, Schwienbacher 24

Example 4: Beyond Facet Regions



Gardi, Herzog, Jones, Ma 24

$$\mathscr{F}(\mathbf{x};\mathbf{s}) = -s_{12}\,(x_1x_4 - x_0x_5)\,(x_3x_6 - x_2x_7) - s_{13}\,(x_1x_2 - x_0x_3)\,(x_5x_6 - x_4x_7) + \; p_i^2\;(\ldots)$$

Consider $p_i^2 \to \lambda p_i^2$ (and insert $s_{12} = 1,\ s_{13} = -1$ as inessential simplification)

$$\mathscr{F}(\mathbf{x};0) = -(x_3x_4 - x_2x_5)(x_1x_6 - x_0x_7)$$

One of the GCAD systems

$$\mathscr{F}(\mathbf{x};0) < 0 \;\wedge\; \frac{\partial\mathscr{F}(\mathbf{x};0)}{\partial x_0} < 0 \;\wedge\; \frac{\partial\mathscr{F}(\mathbf{x};0)}{\partial x_1} > 0 \wedge \frac{\partial\mathscr{F}(\mathbf{x};0)}{\partial x_2} < 0 \wedge \frac{\partial\mathscr{F}(\mathbf{x};0)}{\partial x_3} > 0 \wedge \frac{\partial\mathscr{F}(\mathbf{x};0)}{\partial x_4} > 0 \wedge \frac{\partial\mathscr{F}(\mathbf{x};0)}{\partial x_5} < 0 \wedge \frac{\partial\mathscr{F}(\mathbf{x};0)}{\partial x_6} > 0 \wedge \frac{\partial\mathscr{F}(\mathbf{x};0)}{\partial x_7} < 0 \wedge x_i > 0$$

$$x_5 > \frac{x_3x_4}{x_2} \wedge x_7 > \frac{x_1x_6}{x_0}$$

The new facet introduced by this dissection corresponds to a Landshoff scattering region

Non-linear
Not found with Asy2

Existing Tools

Various tools attempt to find re-mappings using **linear** changes of variables

ASY/FIESTA Jantzen, A. Smirnov, V. Smirnov 12

Check all pairs of variables (α_1, α_2) which are part of monomials of opposite sign

For each pair, try to build linear combination $x_1 \to bx_1', x_2 \to x_2' + bx_1'$ s.t negative monomial vanishes

Repeat until all negative monomials vanish **or** warn user

ASPIRE Ananthanarayan, Pal, Ramanan, Sarkar 18; B. Ananthanarayan, Das, Sarkar 20

Consider Gröbner basis of $\left\{\mathcal{F}, \partial\mathcal{F}/\alpha_1, \partial\mathcal{F}/\alpha_2, \ldots\right\}$ (i.e. $\mathcal{F}$ and Landau equations)

Eliminate negative monomials with linear transformations $x_1 \to bx_1', x_2 \to x_2' + bx_1'$

This is not enough to expose all regions in parameter space

Outlook

Calculating integrals

Improving numerical methods by avoiding contour deformation

Classifying Facet and Hidden Regions

Wide angle $2 \to 2$ - **done** Gardi, Herzog, Jones, Ma 24

Forward $2 \to 2$ scattering - **in progress**

Multi-Regge limits

Want to make connection to SCET or Glauber SCET communities!

Constructing Graphical Algorithms

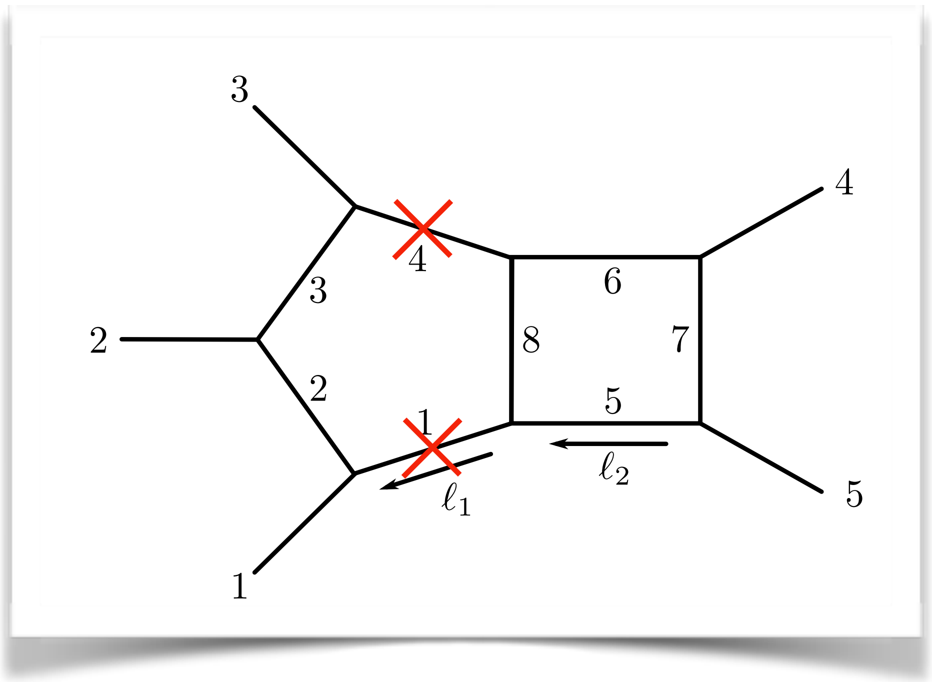
Simpler/alternative way to obtain regions

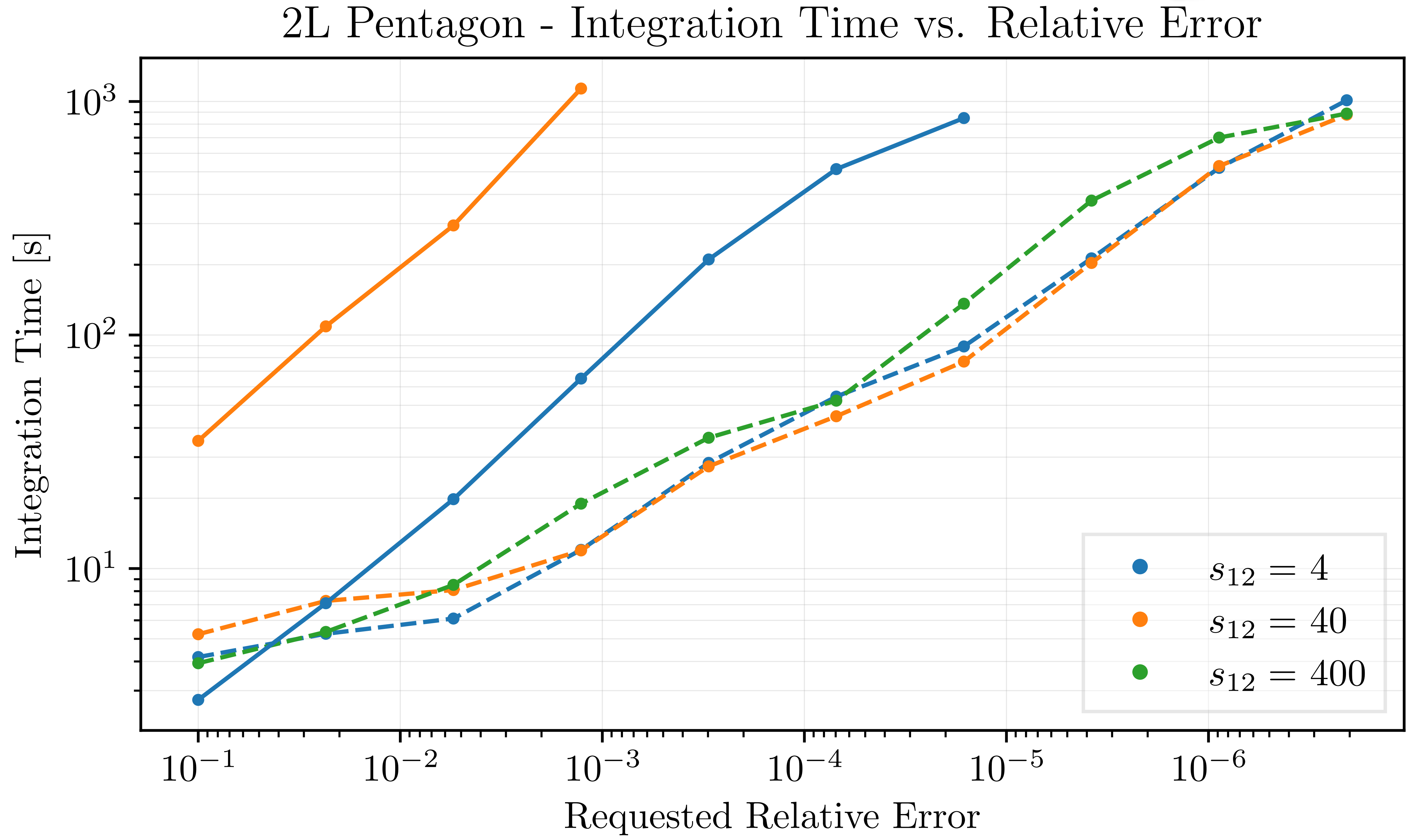
Simpler way to obtain GCAD output

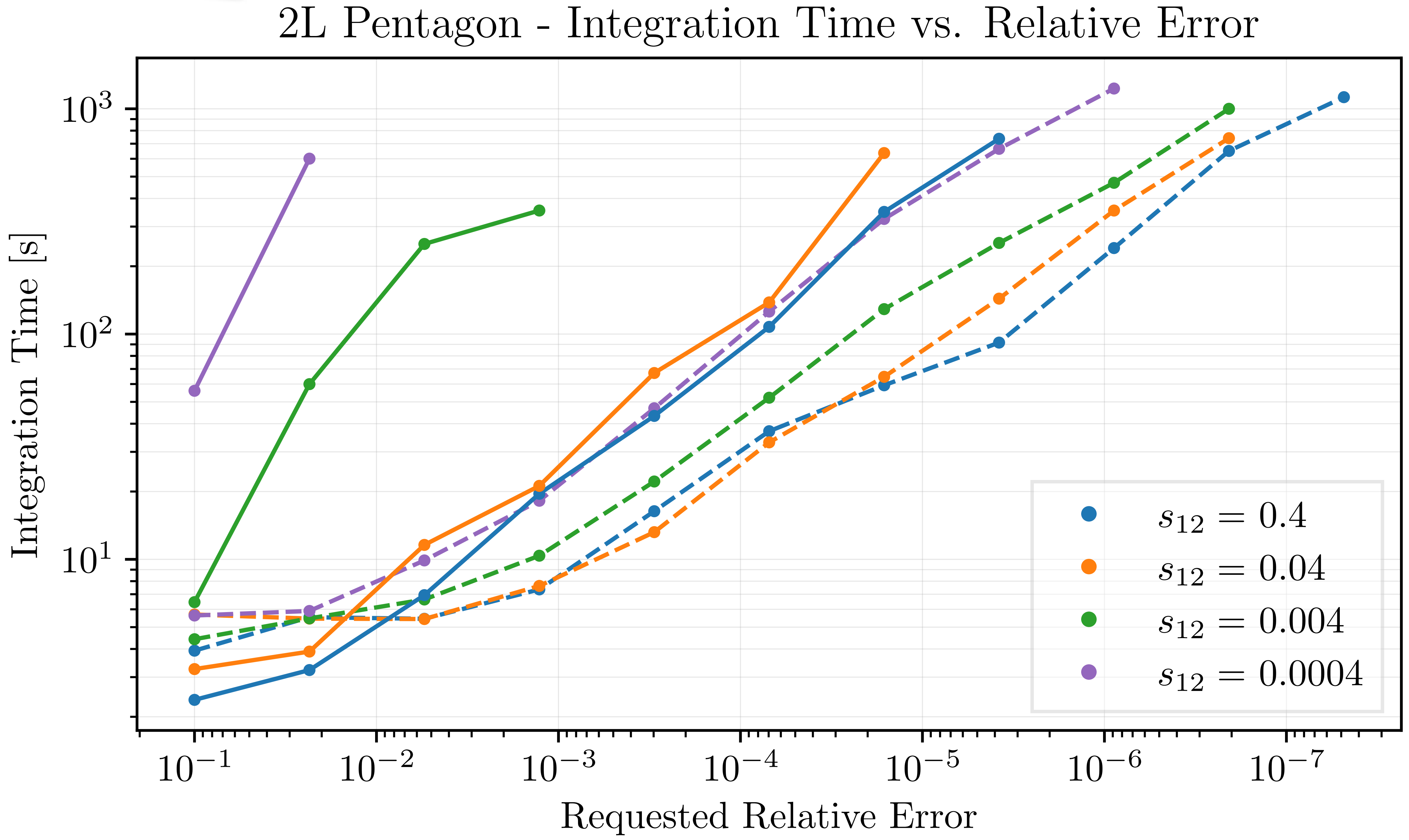
Thank You For Listening!

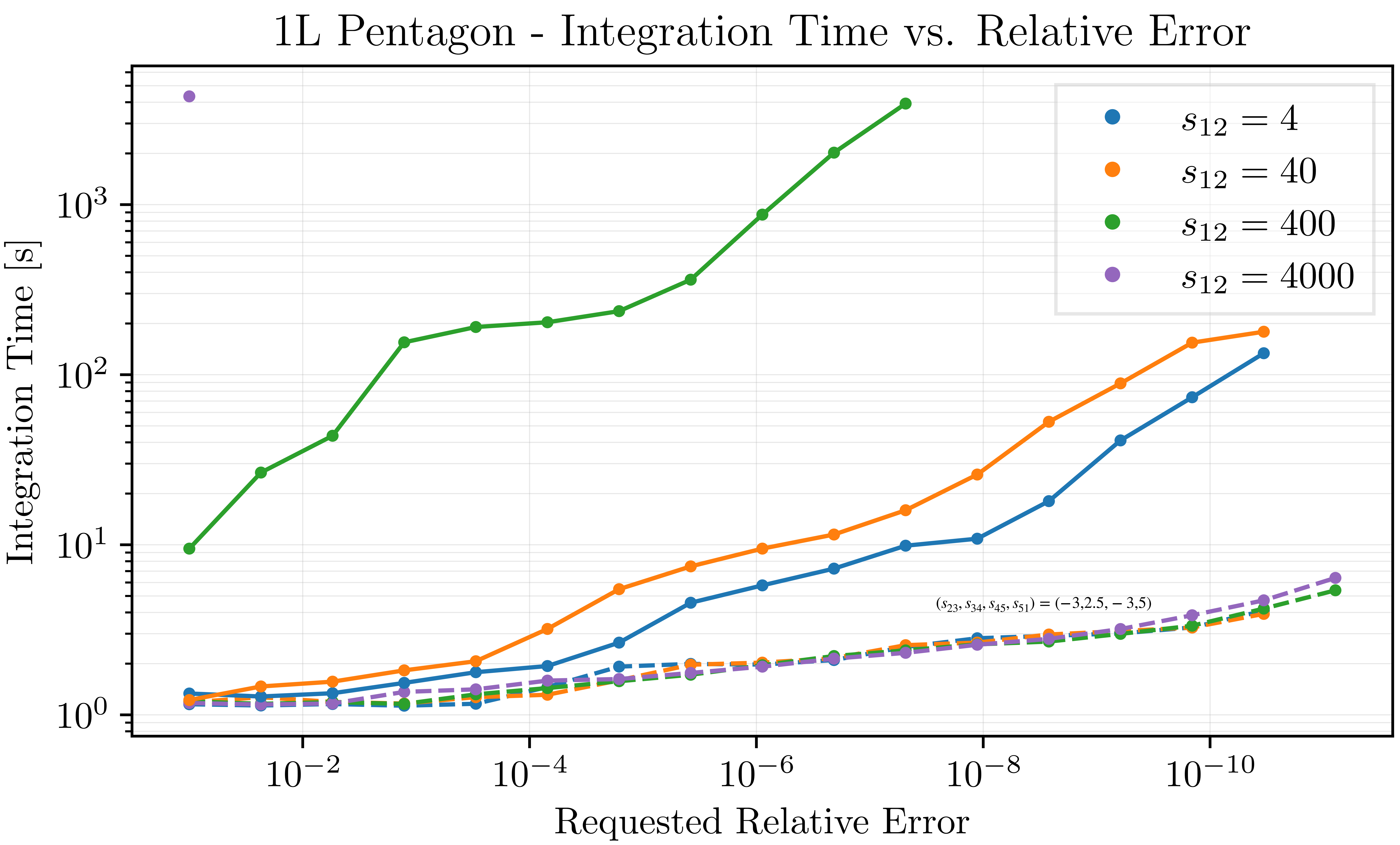
Backup

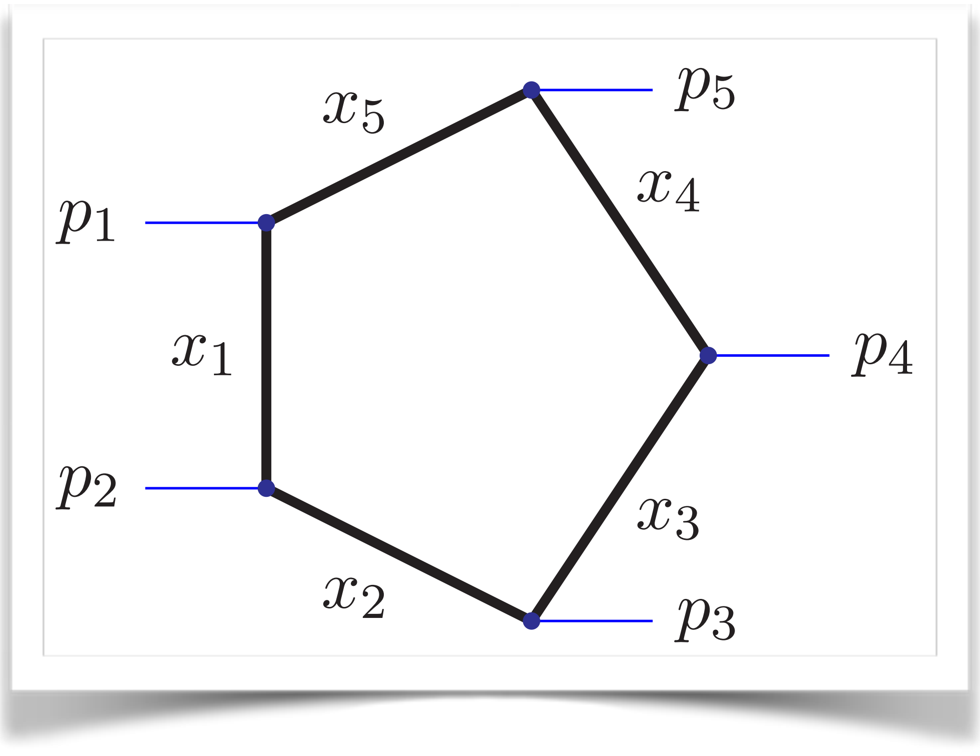
Avoiding Contour Deformation Example

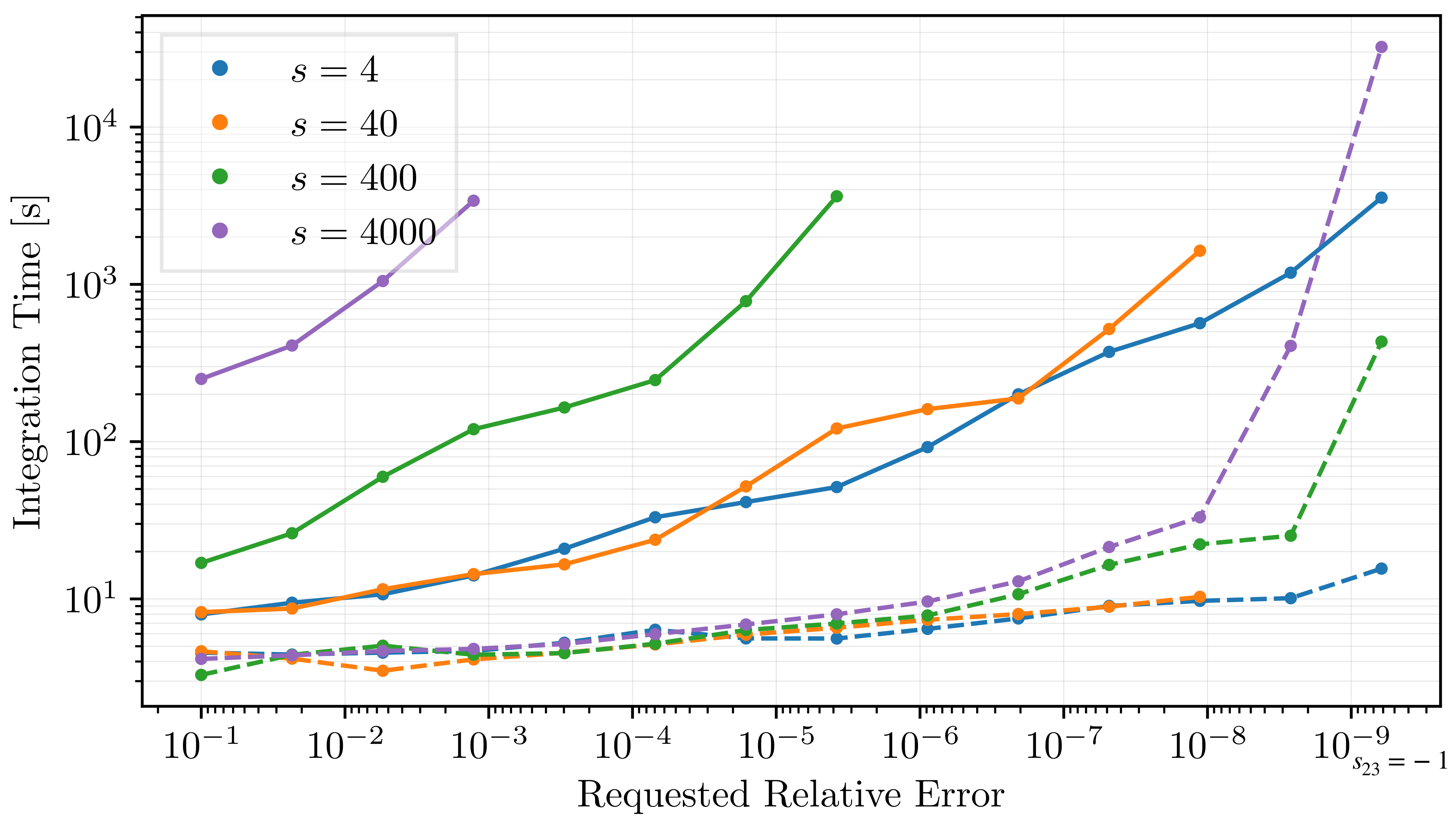




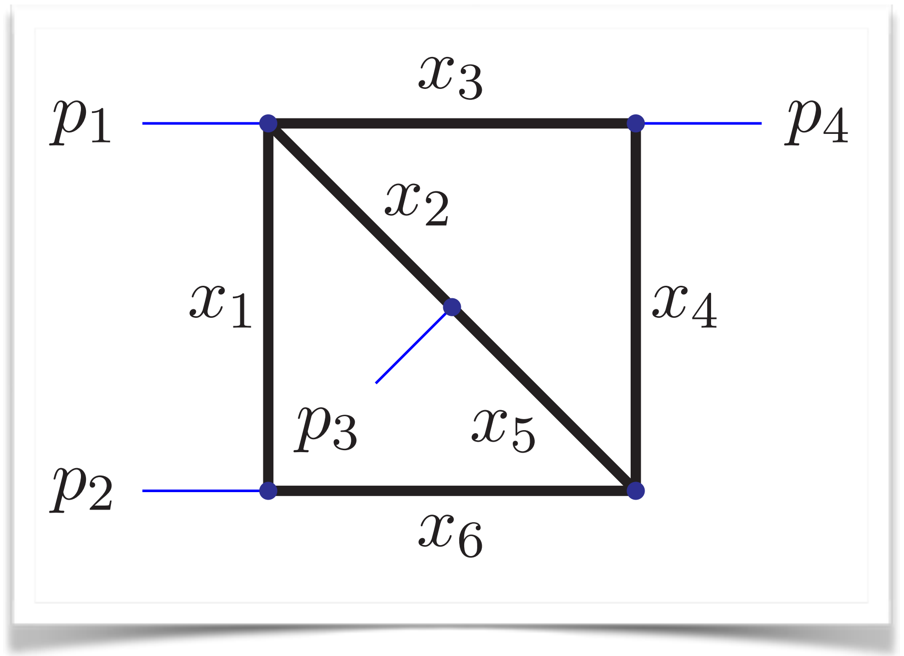



1L Pentagon - Integration Time vs. Relative Error
Integration Time [s]
Requested Relative Error
10^0
10^1
10^2
10^3
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
$s_{12} = 4$
$s_{12} = 40$
$s_{12} = 400$
$s_{12} = 4000$
$(s_{23}, s_{34}, s_{45}, s_{51}) = (-3, 2.5, -3, 5)$

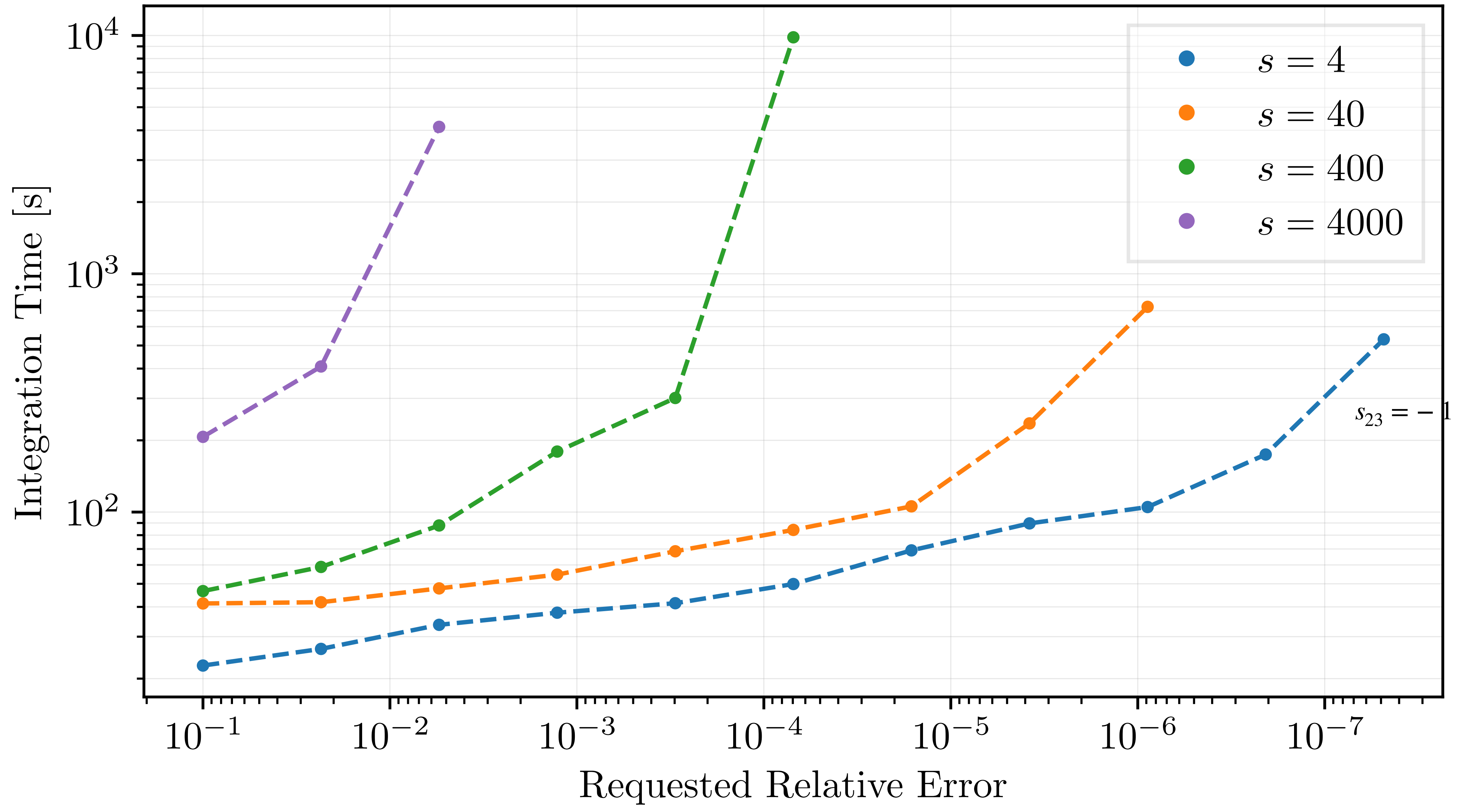

p_1
p_2
p_3
p_4
p_5
x_1
x_2
x_3
x_4
x_5



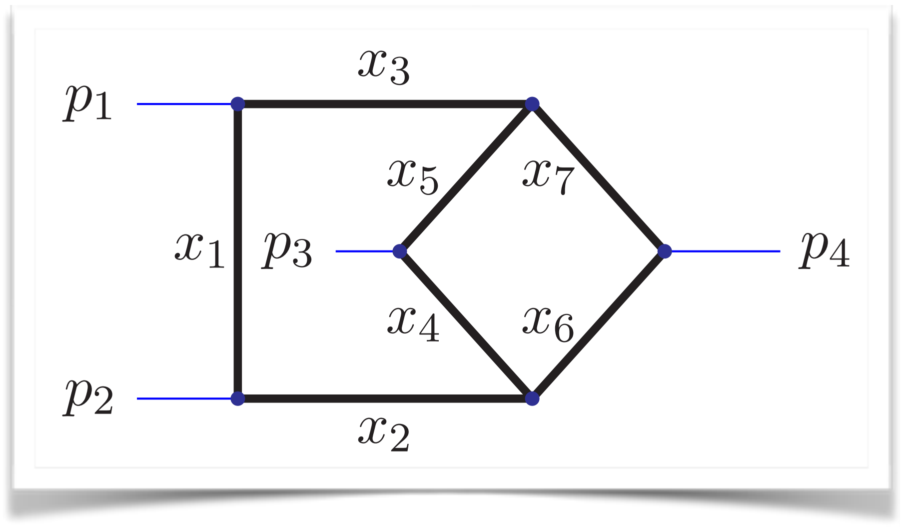
$s = 4$
$s = 40$
$s = 400$
$s = 4000$
Integration Time [s]
Requested Relative Error
$s_{23} = -1$



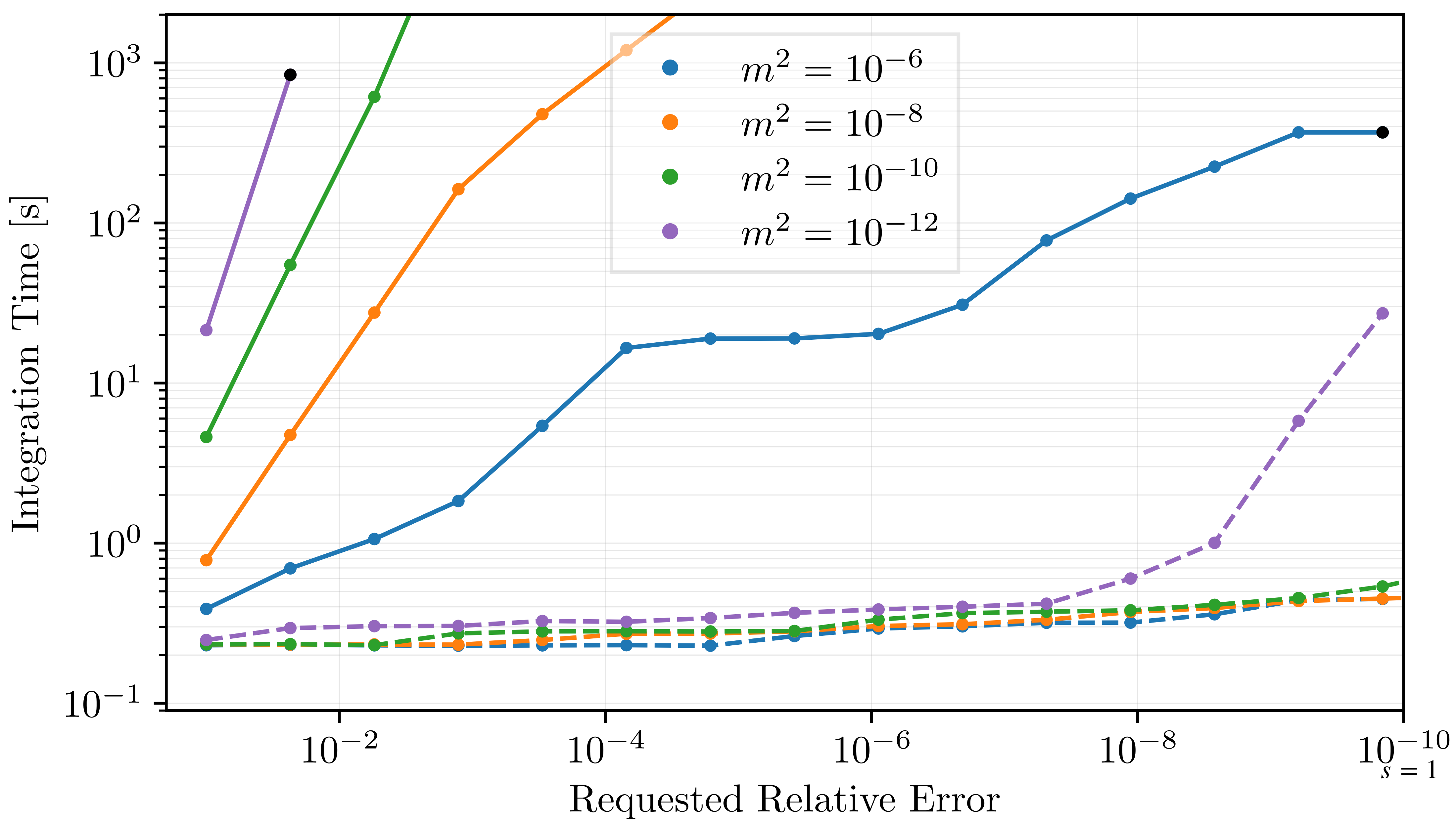
p_1
p_2
p_3
p_4
x_1
x_2
x_3
x_4
x_5
x_6

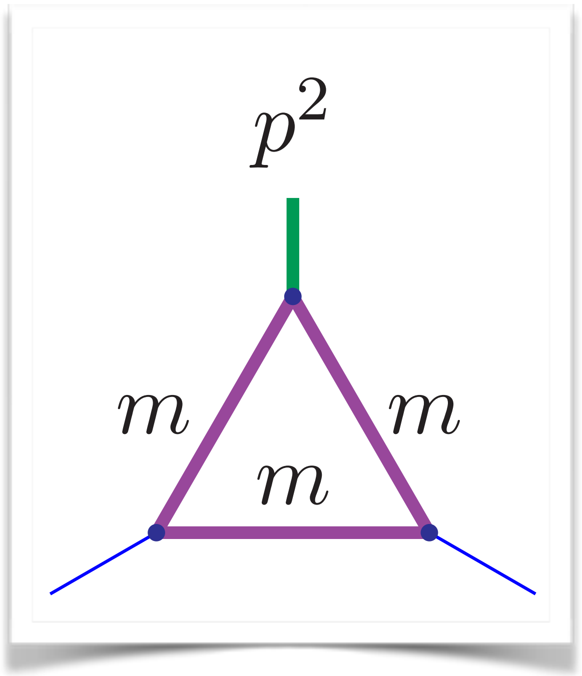


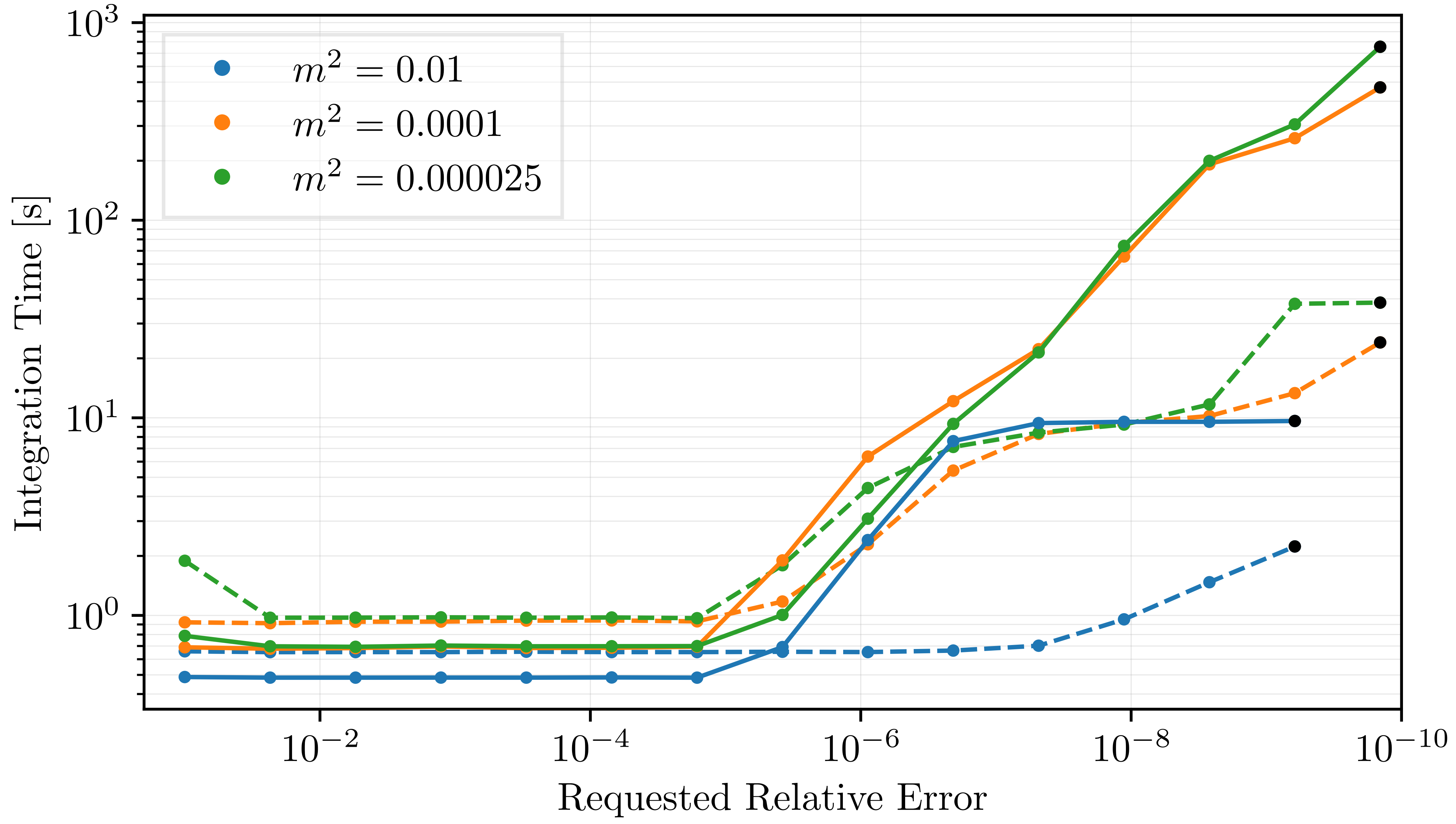
$s = 4$
$s = 40$
$s = 400$
$s = 4000$
Integration Time [s]
Requested Relative Error
$s_{23} = -1$

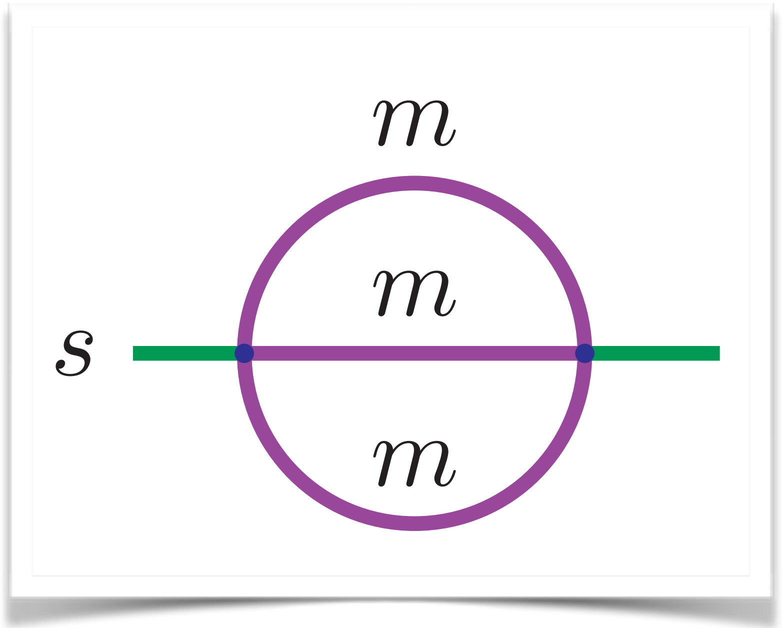


p_1
p_2
p_3
p_4
x_1
x_2
x_3
x_4
x_5
x_6
x_7


$m^2 = 10^{-6}$
$m^2 = 10^{-8}$
$m^2 = 10^{-10}$
$m^2 = 10^{-12}$
Integration Time [s]
Requested Relative Error
$s = 1$


p^2
m
m
m


$m^2 = 0.01$
$m^2 = 0.0001$
$m^2 = 0.000025$
Integration Time [s]
Requested Relative Error


m
m
s
m