The Method of Regions: Fundamentals, Subtleties, and All–order Prescriptions

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Factorization in QCD and Beyond

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Asymptotic expansion of Feynman integrals

Evaluating multi-loop Feynman integrals poses significant challenges.

For Feynman integrals with multiple scales in the external kinematics, a natural idea is to consider the asymptotic expansion.

$$\mathcal{I}(\lambda,\epsilon) \sim \sum_{\mu,k,n} c_{\mu,k,n}(\epsilon)\cdot\lambda^{\mu(\epsilon)}\cdot\log^k\lambda\cdot\lambda^n,$$

with $\lambda \sim \Lambda_{\text{small}}/\Lambda_{\text{large}} \ll 1$.

The “Method of Regions”:

For multi-scale Feynman integral $\mathcal{I}$, there always exists a set of “regions” $R_1, R_2, \ldots, R_n$, such that

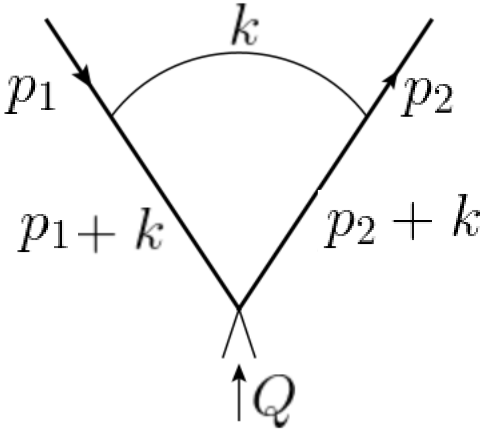
$$\mathcal{I} = \mathcal{I}^{(R_1)} + \mathcal{I}^{(R_2)} + \cdots + \mathcal{I}^{(R_n)}.$$

(Beneke & Smirnov 1997)

The Method of Regions (MoR)

Example: one-loop massless form factor

(Becher, Broggio, Ferroglia 2014)



The "on-shell expansion"

$$p_1^\mu \sim \overset{+}{(1}, \overset{-}{\lambda}, \overset{\perp}{\sqrt{\lambda}}), \qquad p_2^\mu \sim \overset{+}{(\lambda}, \overset{-}{1}, \overset{\perp}{\sqrt{\lambda}}),$$

$$p_1^2/Q^2 \sim p_2^2/Q^2 \sim \lambda \ll 1.$$

The Feynman integral

$$\mathcal{I} = \mathcal{C} \cdot \int d^D k \frac{1}{(k^2 + i0)\left((p_1 + k)^2 + i0\right)\left((p_2 + k)^2 + i0\right)}$$

can be evaluated directly or from MoR.

The Method of Regions (MoR)

Step 1: identify **4** regions in total:

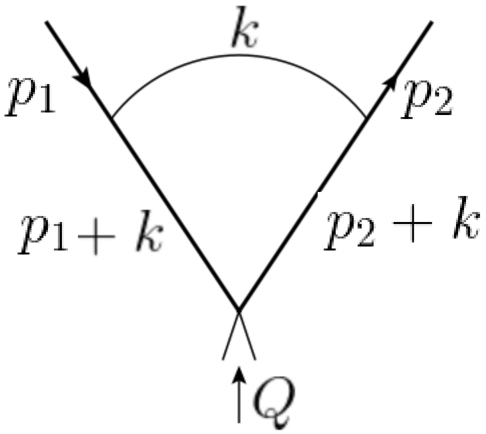
Hard region: $k^\mu \sim Q(1,1,1)$

Collinear-1 region: $k^\mu \sim Q(1,\lambda,\lambda^{1/2})$

Collinear-2 region: $k^\mu \sim Q(\lambda,1,\lambda^{1/2})$

Soft region: $k^\mu \sim Q(\lambda,\lambda,\lambda)$

(called "ultrasoft" in some other literature)



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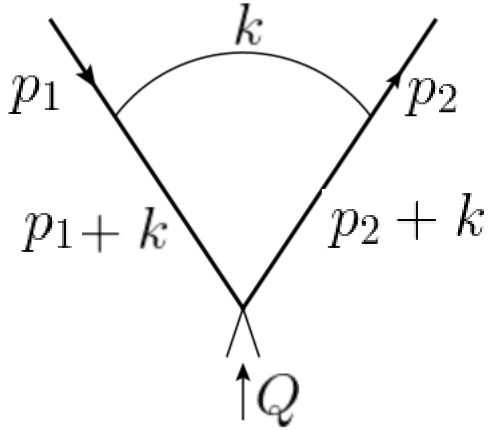

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Soft region: $k^\mu \sim Q(\lambda,\lambda,\lambda)$

(called "ultrasoft" in some other literature)

Step 2: perform expansion around each region:

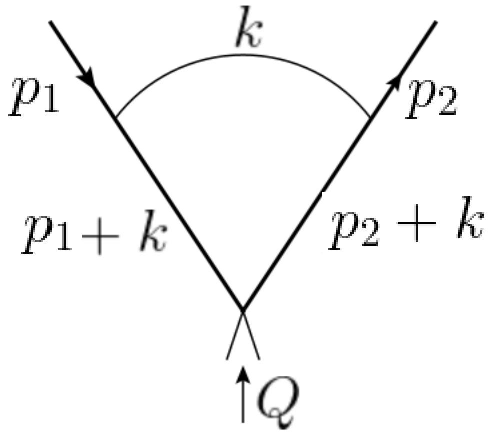
$$\mathcal{I}_H = \mathcal{C}\cdot\int d^Dk\,\frac{1}{(k^2+i0)\,(k^2+2p_1\cdot k+i0)\,(k^2+2p_2\cdot k+i0)}+\dots$$

$$\mathcal{I}_{C_1} = \mathcal{C}\cdot\int d^Dk\,\frac{1}{(k^2+i0)\,((p_1+k)^2+i0)\,(2p_2\cdot k+i0)}+\dots$$

$$\mathcal{I}_{C_2} = \mathcal{C}\cdot\int d^Dk\,\frac{1}{(k^2+i0)\,(2p_1\cdot k+i0)\,((p_2+k)^2+i0)}+\dots$$

$$\mathcal{I}_S = \mathcal{C}\cdot\int d^Dk\,\frac{1}{(k^2+i0)\,(2p_1\cdot k+p_1^2+i0)\,(2p_2\cdot k+p_2^2+i0)}+\dots$$

The Method of Regions (MoR)



Step 1: ….

Step 2: ….

Step 3: sum over their contributions, and the original integral is reproduced:

$$\mathcal{I} = \mathcal{I}_H + \mathcal{I}_{C_1} + \mathcal{I}_{C_2} + \mathcal{I}_S = \frac{1}{Q^2}\left(\ln\frac{Q^2}{(-p_1^2)}\ln\frac{Q^2}{(-p_2^2)} + \frac{\pi^2}{3} + \dots\right)$$

This equality holds to ***all orders of λ***!

The Method of Regions (MoR)

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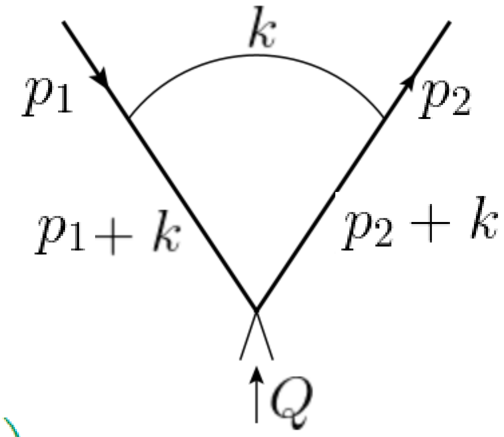

$$\mathcal{I} = \mathcal{I}_H + \mathcal{I}_{C_1} + \mathcal{I}_{C_2} + \mathcal{I}_S .$$

$$\mathcal{I}_H = c_3\ F_4\left(1, 1+\epsilon, 1+\epsilon, 1+\epsilon; \frac{p_1^2}{Q^2}, \frac{p_2^2}{Q^2}\right);$$

$$\mathcal{I}_{C_1} = -c_3\ \left(\frac{p_1^2}{Q^2}\right)^{-\epsilon}\ F_4\left(1, 1-\epsilon, 1-\epsilon, 1+\epsilon; \frac{p_1^2}{Q^2}, \frac{p_2^2}{Q^2}\right);$$

$$\mathcal{I}_{C_2} = -c_3\ \left(\frac{p_2^2}{Q^2}\right)^{-\epsilon}\ F_4\left(1, 1-\epsilon, 1+\epsilon, 1-\epsilon; \frac{p_1^2}{Q^2}, \frac{p_2^2}{Q^2}\right);$$

$$\mathcal{I}_S = c_3 c_s\ \left(\frac{p_1^2 p_2^2}{(Q^2)^2}\right)^{-\epsilon}\ F_4\left(1-2\epsilon, 1-\epsilon, 1-\epsilon, 1-\epsilon; \frac{p_1^2}{Q^2}, \frac{p_2^2}{Q^2}\right).$$

matching ***(Boos & Davydychev 1991)***.

This examples shows the power of MoR: it can **approximate, and even restore** multi-scale Feynman integrals.

Two fundamental questions

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1. How to prove MoR? (Is this technique always valid, and why?)
2. How to identify the regions? (How to use this technique?)

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(Jantzen 2011) If the regions satisfy all the following conditions, then MoR reproduces the Feynman integral:

1, the regions cover the entire integration domain;

2, commutativity of the expansions;

3, convergence of the expansions;

4, fulfill certain cancellation patterns.

Although it is an important progress, it is still far from a rigorous proof.

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1. How to prove MoR? (Is this technique always valid, and why?)

2. **How to identify the regions? (How to use this technique?)**

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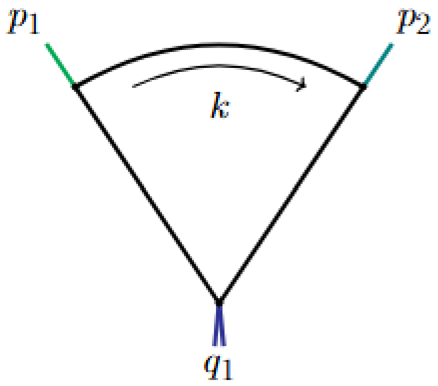
Actually, it has changed Problem 1 into Problem 2.

Subtleties of the region structure

Determining the region structure can be subtle.



Subtlety 1: the region structure depends on the external kinematics.

Expansion 1:

$$p_1^2/q_1^2 \sim \lambda,$$
$$p_2^2/q_1^2 \sim \lambda.$$

Four regions:

$$\begin{array}{llcccc} & & & + & - & \perp \\ H: & k^\mu & \sim & (1, & 1, & 1); \\ C_1: & k^\mu & \sim & (1, & \lambda, & \sqrt{\lambda}); \\ C_2: & k^\mu & \sim & (\lambda, & 1, & \sqrt{\lambda}); \\ S: & k^\mu & \sim & (\lambda, & \lambda, & \lambda). \end{array}$$

Subtleties of the region structure

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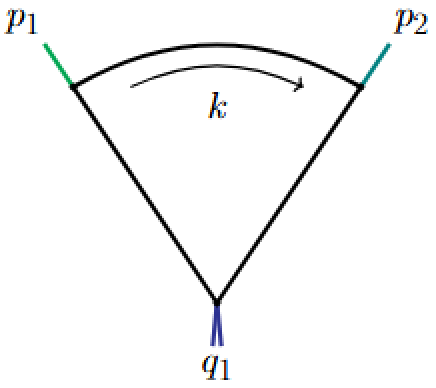

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Expansion 1:

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Four regions:

H

C_1

C_2

S

Expansion 2:

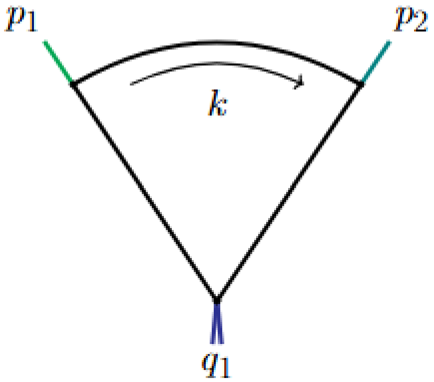
$$p_1^2/q_1^2 \sim \lambda,$$
$$p_2^2/q_1^2 \sim \lambda^2.$$

Four regions:

$$H: \quad k^\mu \sim (1, 1, 1);$$
$$C_1: \ k^\mu \sim (1, \lambda, \sqrt{\lambda});$$
$$C_2^2: \ k^\mu \sim (\lambda^2, 1, \lambda);$$
$$SC_2: \ k^\mu \sim (\lambda^2, \lambda, \lambda^{3/2}).$$

Subtleties of the region structure

Identifying regions can be subtle.



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Four regions:

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Expansion 2:

$$p_1^2/q_1^2 \sim \lambda,$$
$$p_2^2/q_1^2 \sim \lambda^2.$$

Four regions:

H

C_1

C_2^2

SC_2

Expansion 3:

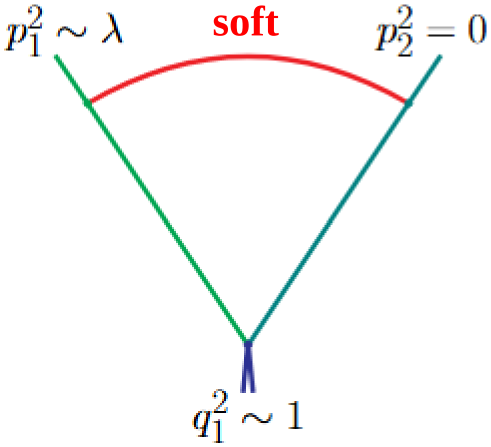
$$p_1^2/q_1^2 \sim \lambda,$$
$$p_2^2 = 0.$$

Two regions:

$$H: \quad k^\mu \sim (1,1,1);$$
$$C_1: \; k^\mu \sim (1,\lambda,\sqrt{\lambda}).$$

Subtleties of the region structure

<u>Subtlety 2</u>: dependence on graph topologies.



scaleless, =0

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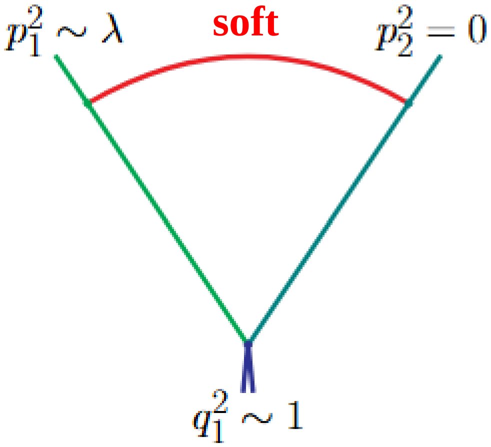
Subtleties of the region structure

Subtlety 2: dependence on graph topologies.



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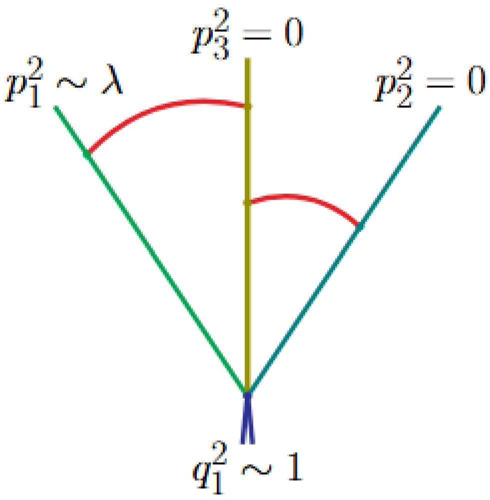

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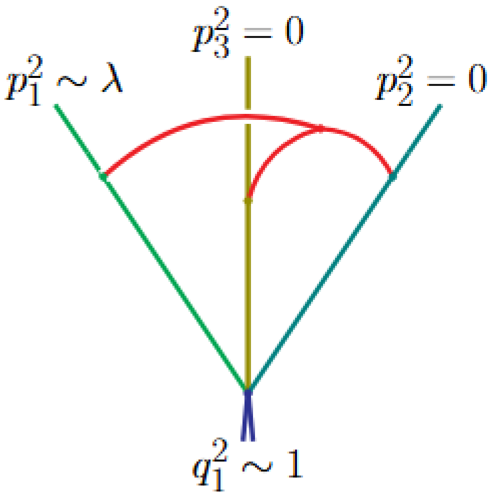

scaleful

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Identifying regions systematically

Progress from 2010: a geometric approach.

(Pak & Smirnov 2010)

- Consider ***parametric representation*** of Feynman integrals (Feynman, Lee-Pomeransky, etc.).
- An *N*-propagator multiscale Feynman integral corresponds to an ***(N+1)-dim polytope***.
- The regions corresponds to certain types of boundaries (called ***lower facets***) of this polytope.
- Computer codes: `Asy2`, `ASPIRE`, `pySecDec`, etc.

→ See Stephen's talk for more detail.

Identifying regions systematically

However, this is not yet the end of the story.

Limitations

- All-loop-order results not available.

  Note that **dim(polytope) = #(propagators) + 1**.

- The output of this approach describes regions in parameter space, while physically we are interested in their momentum-space interpretation.

- This geometric approach may miss regions that are "hidden" inside the polytope.

  This has been noticed in ***(Jantzen, A.Smirnov, V.Smirnov 2012)***, and the geometric approach has been modified accordingly (`Asy` → `Asy2`).

  However, such modifications work only for some simplest cases.

Identifying regions systematically

Classifying regions into "***facet regions***" & "***hidden regions***":

Facet regions: those on the boundary of the polytope, thus identifiable by the geometric approach.

They are NOT sensitive to the signs in $\mathcal{F}$ (Symanzik polynomial), corresponding to "endpoint-type" singularities in parameter space.

Hidden regions: those within the polytope, thus NOT identifiable by the geometric approach.

They are due to cancellations within $\mathcal{F}$, corresponding to "pinch-type" singularities in parameter space.

In what follows, we shall present their general configurations.

Facet regions

Wide–angle kinematics

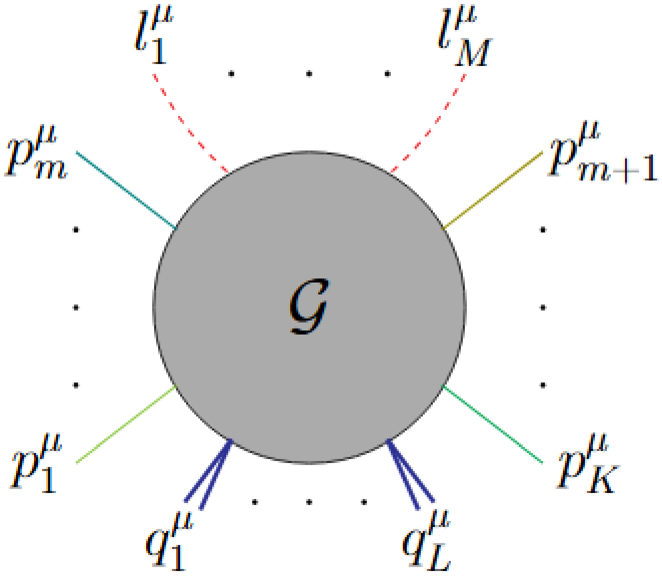
(YM, 2601.22144)

Spacelike–collinear kinematics

Wide–angle kinematics

YM, 2601.22144

- A general expansion form

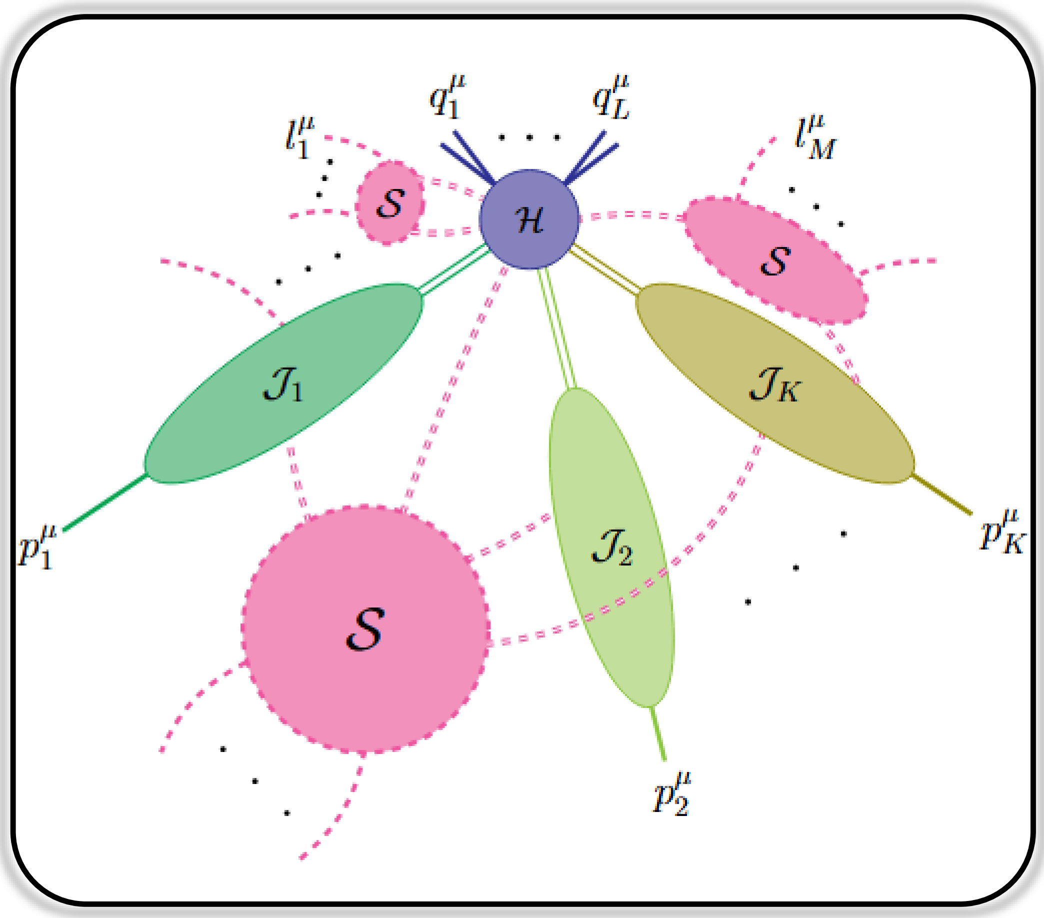


$$\begin{array}{rlcl}
 & \quad\; + \quad\; - \quad\; \perp & & \\
p_i^\mu \sim & (1,\lambda,\sqrt{\lambda}) & \to & C_i \\
 & (1,\lambda^2,\lambda) & \to & C_i^2 \\
 & \cdots & \to & \cdots \\
 & (1,0,0) & \to & C_i^\infty \\
q_j^\mu \sim & (1,1,1) & \to & H \\
l_k^\mu \sim & (\lambda,\lambda,\lambda) & \to & S \\
 & (\lambda^2,\lambda^2,\lambda^2) & \to & S^2 \\
 & (\lambda,\lambda^2,\lambda^{3/2}) & \to & SC_i \\
 & \cdots & \to & \cdots
\end{array}$$

For the expansions above, facet regions constitute the majority (mostly, the complete list) of regions for a given graph, and we have understood their structures to all loop orders!

Fundamental pattern

YM, 2601.22144



There can be various possible modes in this pattern.

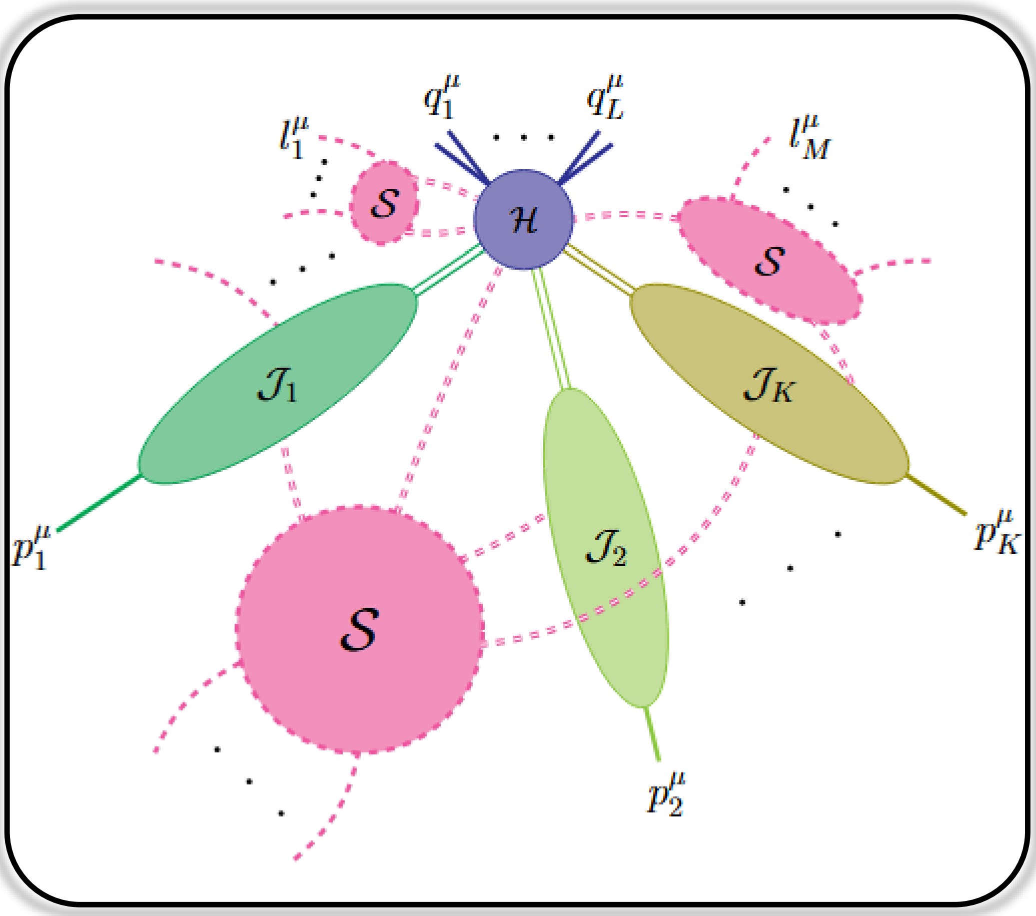
Modes in $\mathcal{J}_i$ **:** $C_i,\ C_i^2,\ \ldots;$

Modes in $\mathcal{S}$ **:** $S,\ SC_i,\ S^2,\ \ldots.$

There are infinitely many configurations that are consistent with this pattern.

Fundamental pattern

YM, 2601.22144

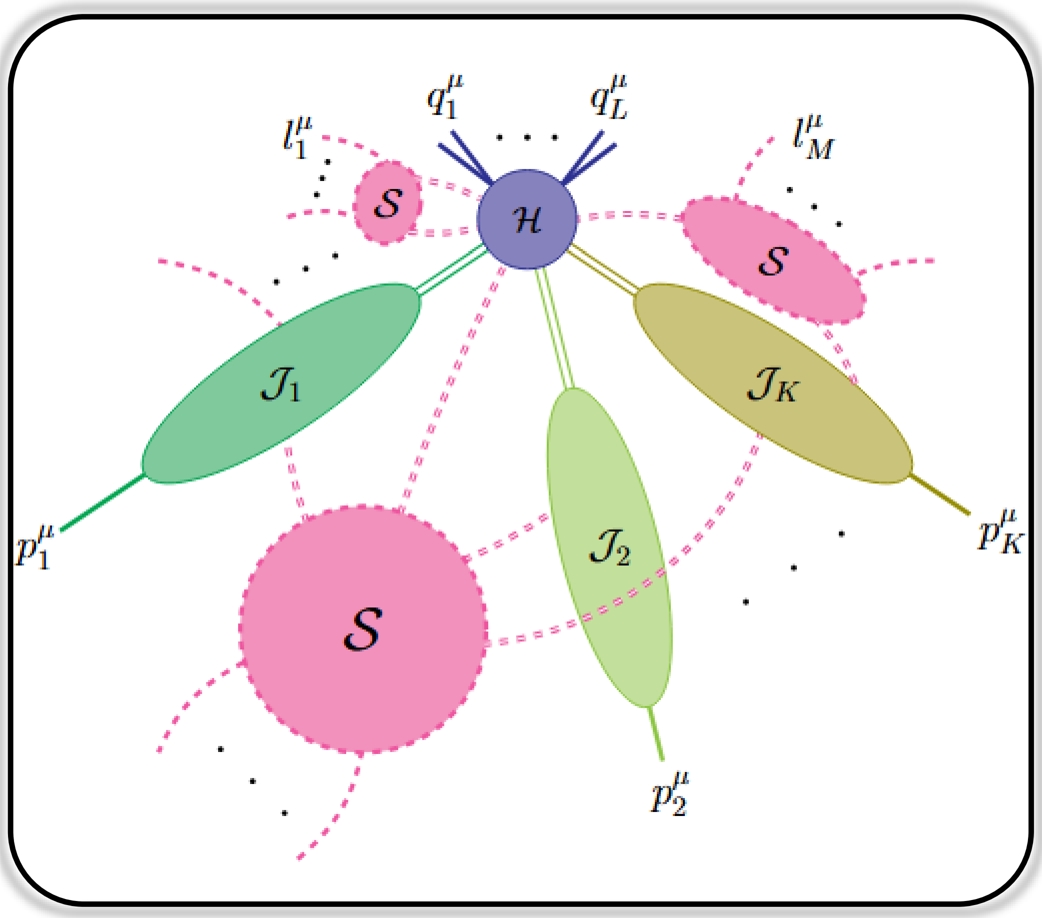


<u>Question:</u>

In order to exclude the scaleless configurations, how should the subgraphs and their modes be constrained?

Fundamental pattern

YM, 2601.22144



Question:

In order to exclude the scaleless configurations, how should the subgraphs and their modes be constrained?

Answer:

Surprisingly concise!

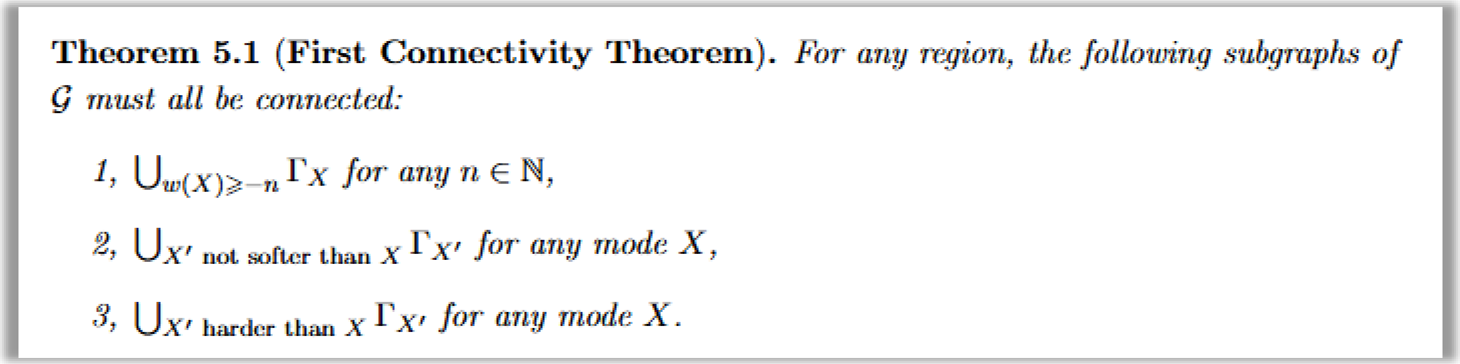
Two requirements:

- ***connectivity***
- ***"infrared compatibility"***

Subgraph requirements

YM, 2601.22144

- **The connectivity requirement**



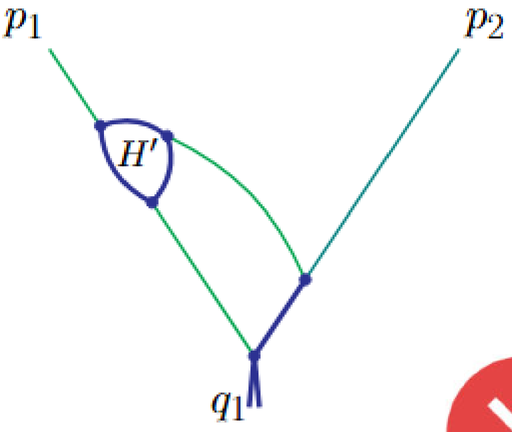
Theorem 5.1 (First Connectivity Theorem). *For any region, the following subgraphs of $\mathcal{G}$ must all be connected:*

1, $\bigcup_{w(X)\geqslant -n}\Gamma_X$ *for any* $n\in\mathbb{N}$,

2, $\bigcup_{X' \text{ not softer than } X}\Gamma_{X'}$ *for any mode* X,

3, $\bigcup_{X' \text{ harder than } X}\Gamma_{X'}$ *for any mode* X.

Except the hard subgraph $\mathcal{H}$, there cannot be a second subgraph whose mode is harder than its neighbors.



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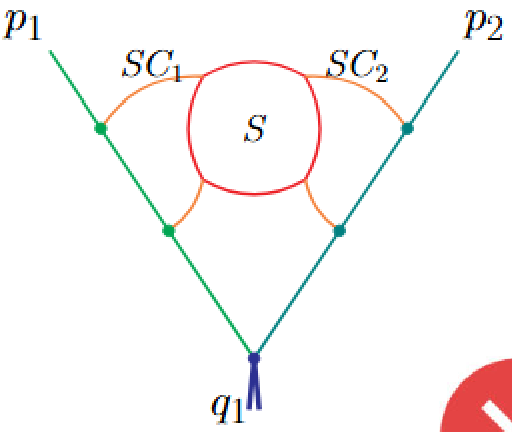

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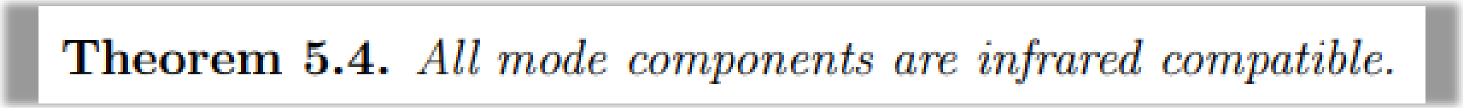
Subgraph requirements

YM, 2601.22144

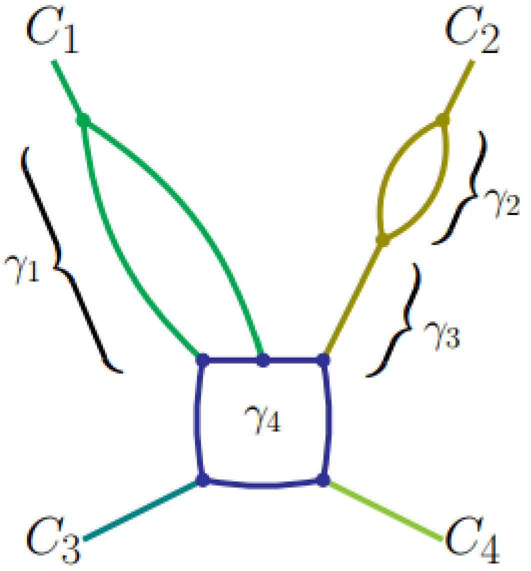
- **The infrared-compatibility requirement**



Theorem 5.4. *All mode components are infrared compatible.*

Basic idea: "delivery of infrared message".

Roughly speaking, every "fundamental ingredient" of the region must receive its corresponding infrared scaling, *either from the external kinematics directly, or delivered by some internal subgraphs*.



$\gamma_1 : C_1$

$\gamma_2, \gamma_3 : C_2$

$\gamma_4 : H$

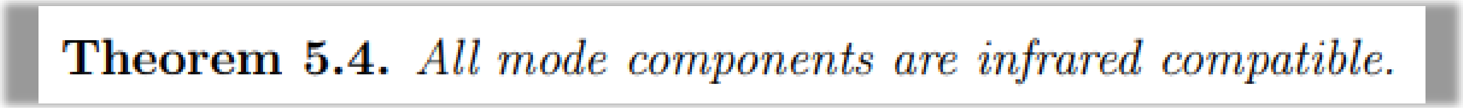
scaleful

✓

Subgraph requirements

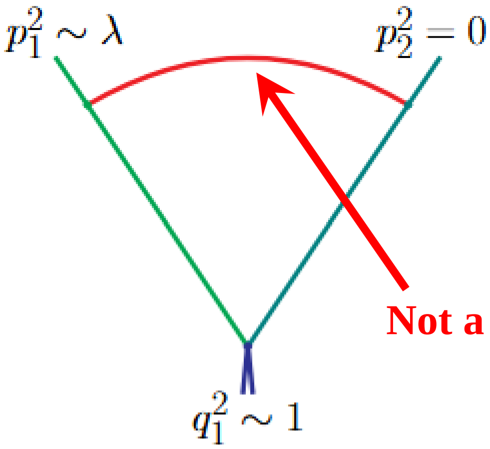
YM, 2601.22144

- **The infrared-compatibility requirement**



Theorem 5.4. *All mode components are infrared compatible.*

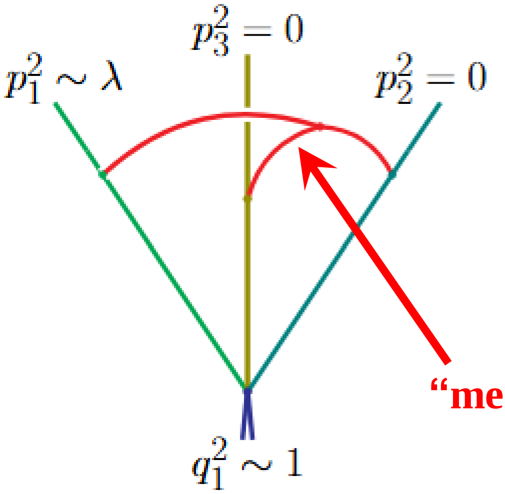
Delivery of infrared scalings via “messengers”:



Not a “messenger”

scaleless

✗



“messenger”

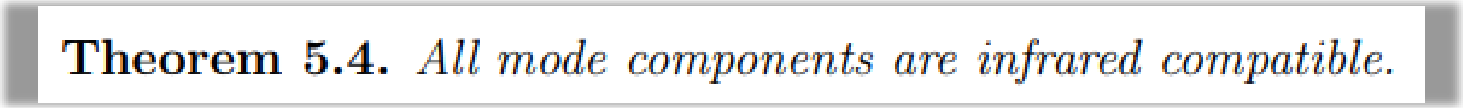
scaleful

✓

Subgraph requirements

YM, 2601.22144

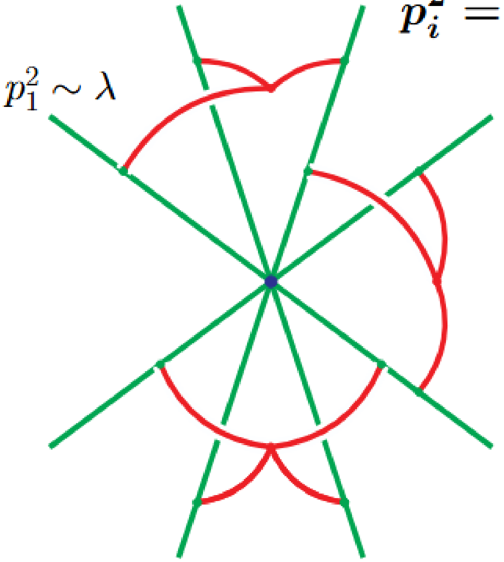
- **The infrared-compatibility requirement**



Theorem 5.4. *All mode components are infrared compatible.*

More complicated examples

$$p_i^2 = 0 \quad (i = 2, 3, \ldots)$$



scaleful

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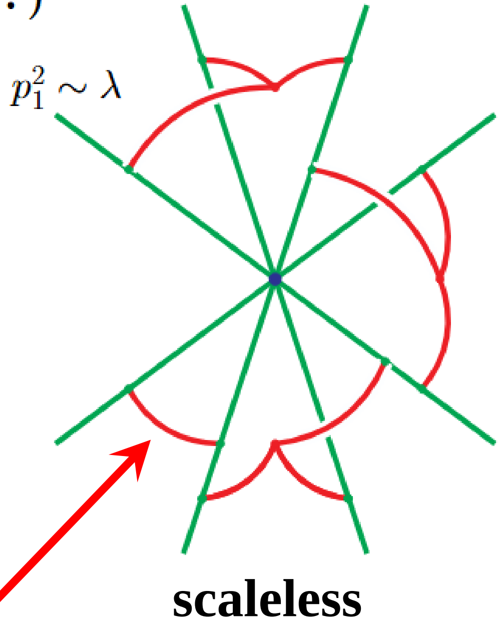


scaleless

Not a “messenger”

✗

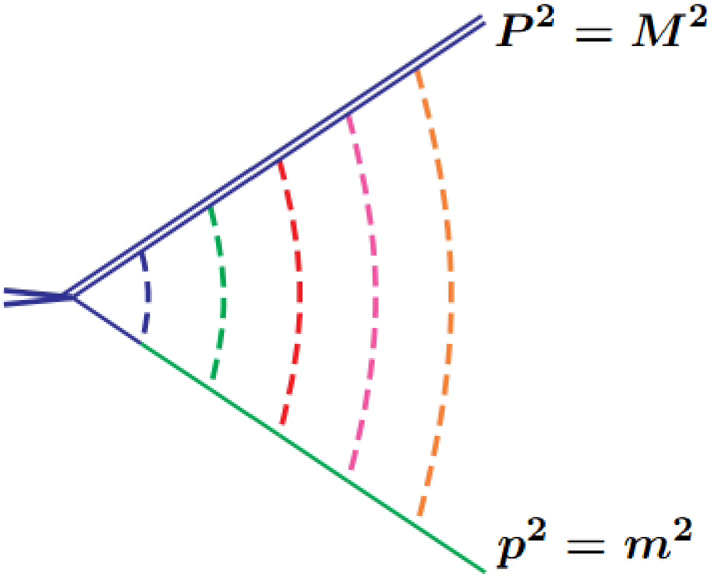
Absence of cascading modes

YM, 2601.22144

- **Cascading modes: the number of modes increases to infinity as the loop number increases.**

No cascading modes in massless wide-angle kinematics!

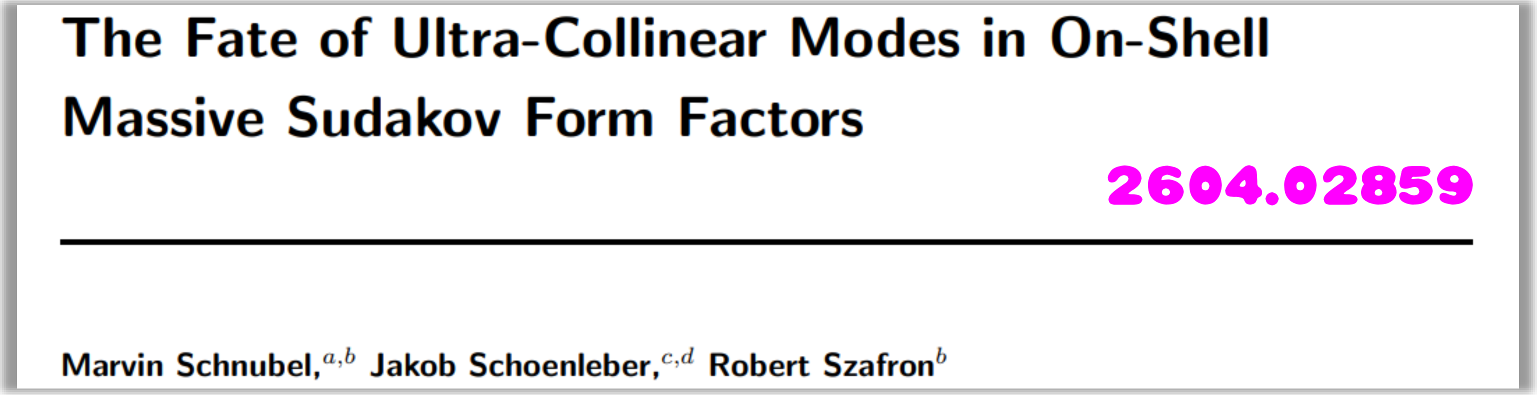
This is different from some other expansions, e.g.,



$$\frac{m^2}{M^2} \sim \lambda.$$

Intermezzo – another recent work

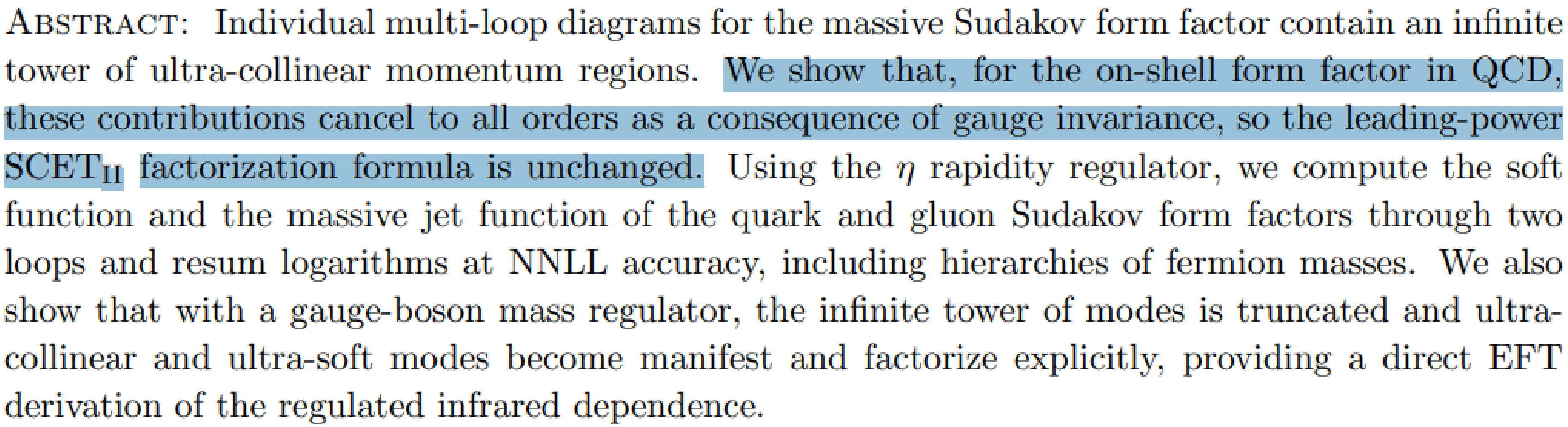
- Cascading modes, if present, ***may cancel*** at the amplitude level.



The Fate of Ultra-Collinear Modes in On-Shell Massive Sudakov Form Factors

2604.02859

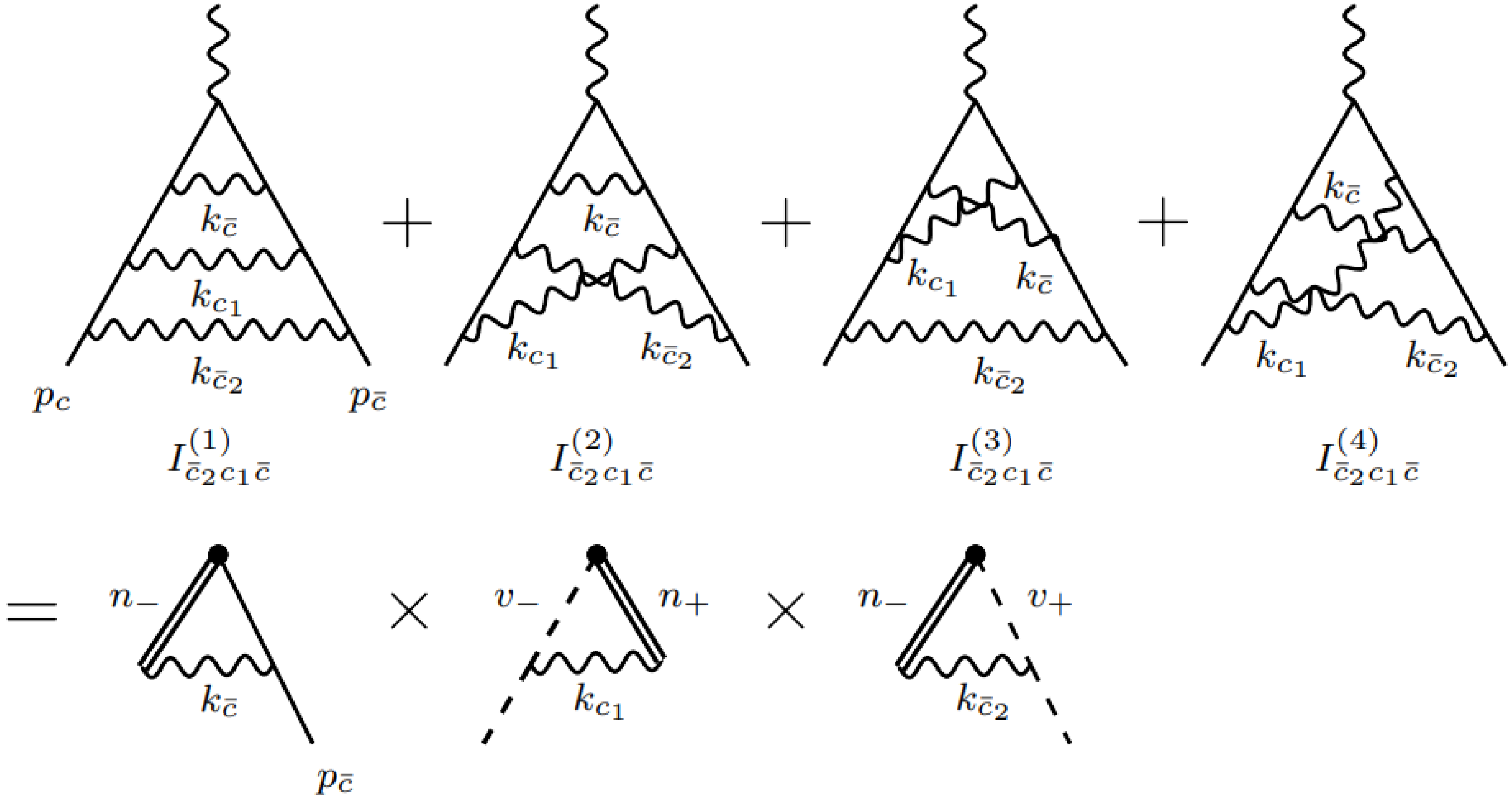
Marvin Schnubel,[a,b] **Jakob Schoenleber,**[c,d] **Robert Szafron**[b]



ABSTRACT: Individual multi-loop diagrams for the massive Sudakov form factor contain an infinite tower of ultra-collinear momentum regions. We show that, for the on-shell form factor in QCD, these contributions cancel to all orders as a consequence of gauge invariance, so the leading-power $\mathrm{SCET}_{\mathrm{II}}$ factorization formula is unchanged. Using the η rapidity regulator, we compute the soft function and the massive jet function of the quark and gluon Sudakov form factors through two loops and resum logarithms at NNLL accuracy, including hierarchies of fermion masses. We also show that with a gauge-boson mass regulator, the infinite tower of modes is truncated and ultra-collinear and ultra-soft modes become manifest and factorize explicitly, providing a direct EFT derivation of the regulated infrared dependence.

Intermezzo – another recent work

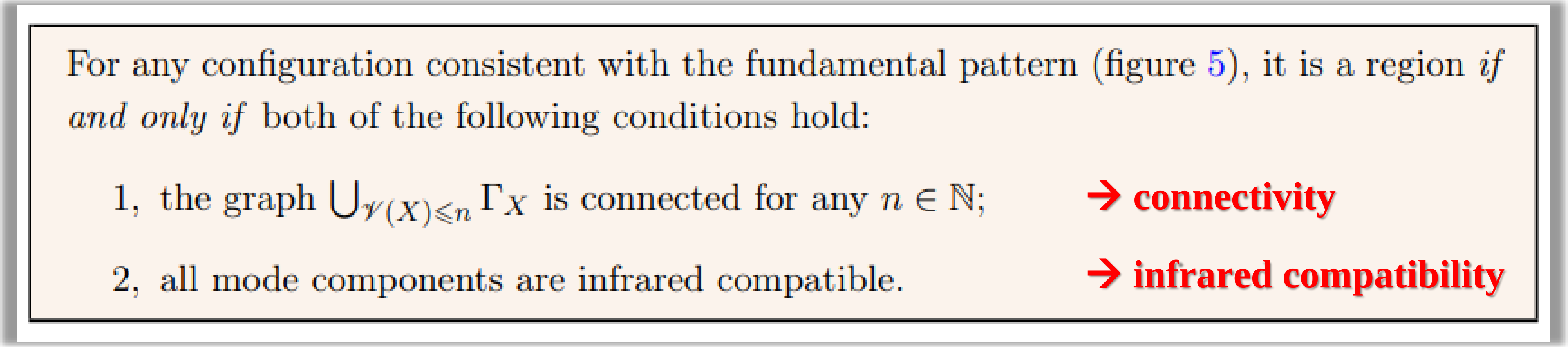
- Cancellation of ultra-collinear modes:



$$F_1(Q^2,m^2) = C(Q^2,\mu)\, Z_c^{1/2}(m^2,\mu,\nu)\, Z_{\bar c}^{1/2}(m^2,\mu,\nu)\, S(m^2,\mu,\nu) + \mathcal{O}(\lambda^2)\,,$$

A brief summary of above

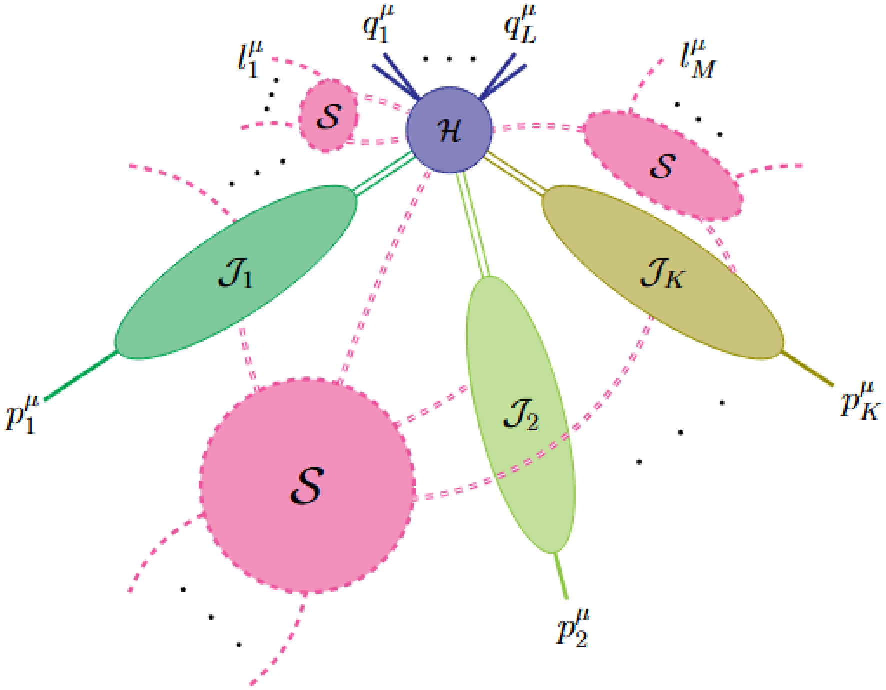
YM, 2601.22144



For any configuration consistent with the fundamental pattern (figure 5), it is a region *if and only if* both of the following conditions hold:

1, the graph $\bigcup_{\mathcal{V}(X)\leqslant n}\Gamma_X$ is connected for any $n\in\mathbb{N}$; → **connectivity**

2, all mode components are infrared compatible. → **infrared compatibility**



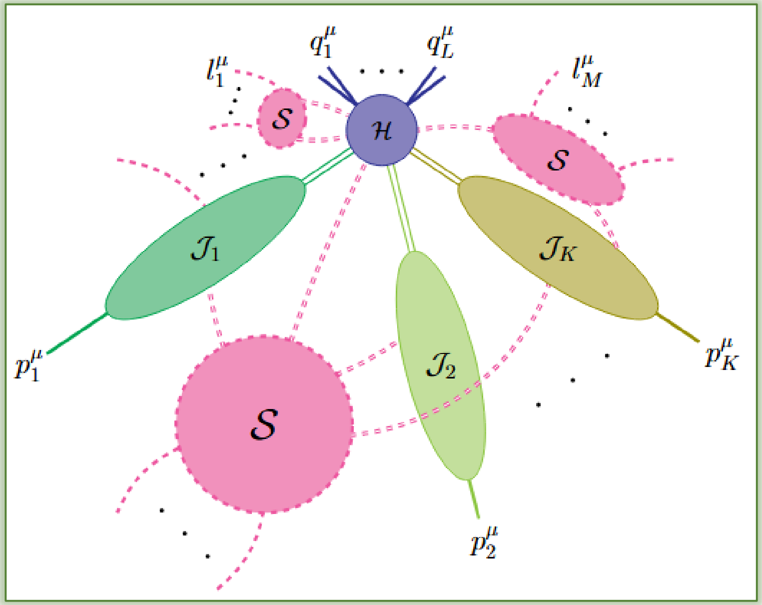
Remark on factorization

Relating our MoR results and the factorization formula:

Soft (matrix in colour flow space)

Jets (colour singlet)

$$\mathcal{M}_N\left(p_i/\mu,\epsilon\right) = \sum_L \mathcal{S}_{NL}\left(\beta_i\cdot\beta_j,\epsilon\right)\, H_L\left(\frac{2p_i\cdot p_j}{\mu^2},\frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2}\right)\prod_{i=1}^{n}\frac{J_i\left(\frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2},\epsilon\right)}{\mathcal{J}_i\left(\frac{2(\beta_i\cdot n_i)^2}{n_i^2},\epsilon\right)}$$



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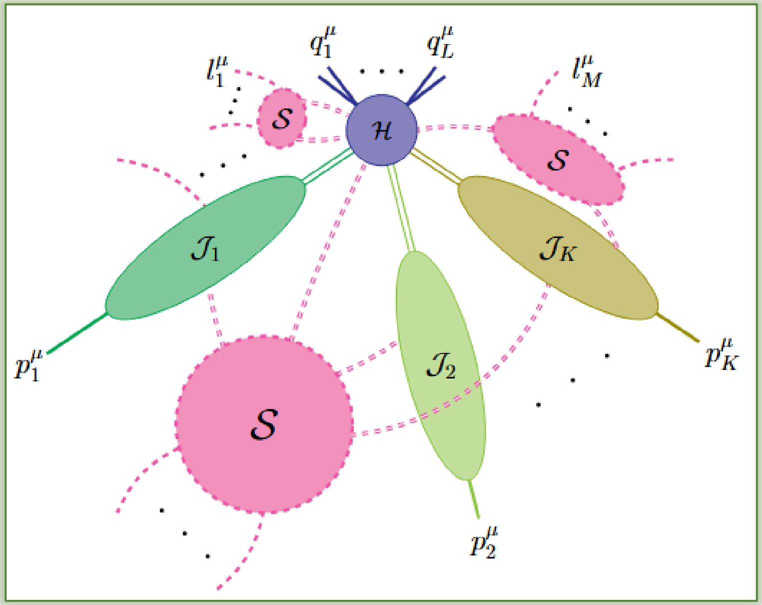
local (integrand)



global (integral)

Facet regions

Wide–angle kinematics

Spacelike–collinear kinematics

(Gardi, Herzog, Jones, YM in preparation)

Spacelike–collinear external momenta

Given two collinear momenta p_1 and p_2, we call them

timelike collinear: if they are both in the initial or final state;

spacelike collinear: if one is initial and the other is final.

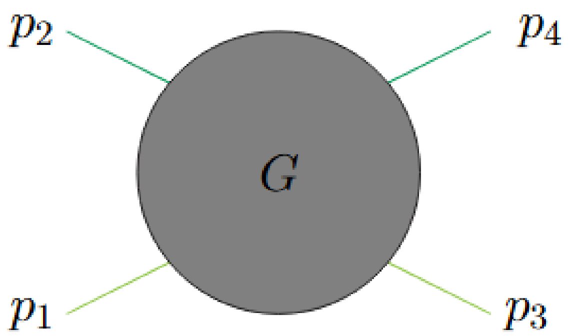
Feynman graphs with spacelike collinear external momenta possibly feature ***Glauber singularities***, which are responsible for factorization breaking. ***(Catani, de Florian & Rodrigo 2011; Forshaw, Seymour & Siodmok 2012; ...)***

→ See Federico's and Thomas' talks

Note that *Glauber singularities are absent in wide-angle kinematics*: one can always deform the momentum-space integration contour to avoid them. ***(Collins & Sterman 1981)***

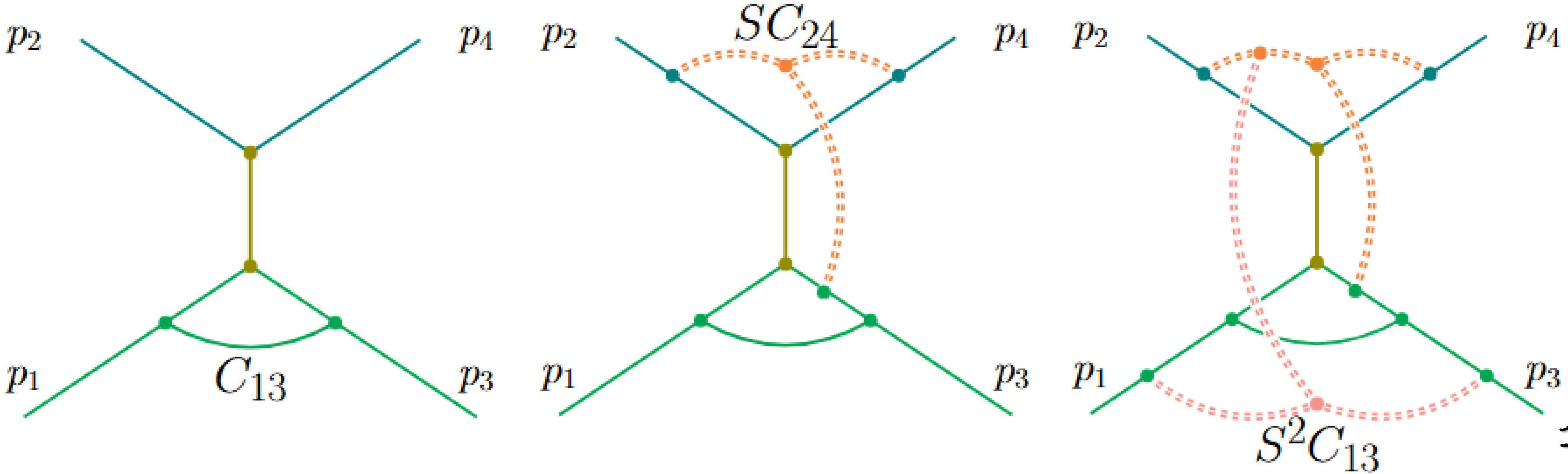
Two–to–two forward scattering

- The 2-to-2 forward scattering as an example:



$$p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0,$$
$$\frac{t}{s} \sim \frac{t}{u} \sim \lambda.$$

Feature 1: cascading modes exist.



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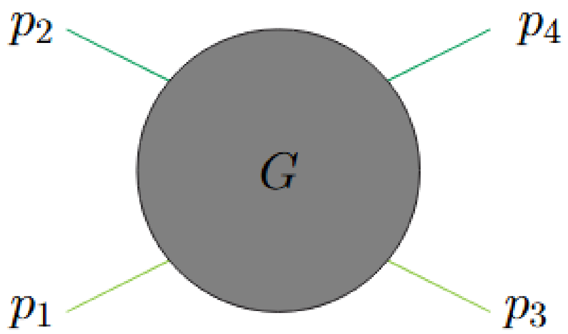

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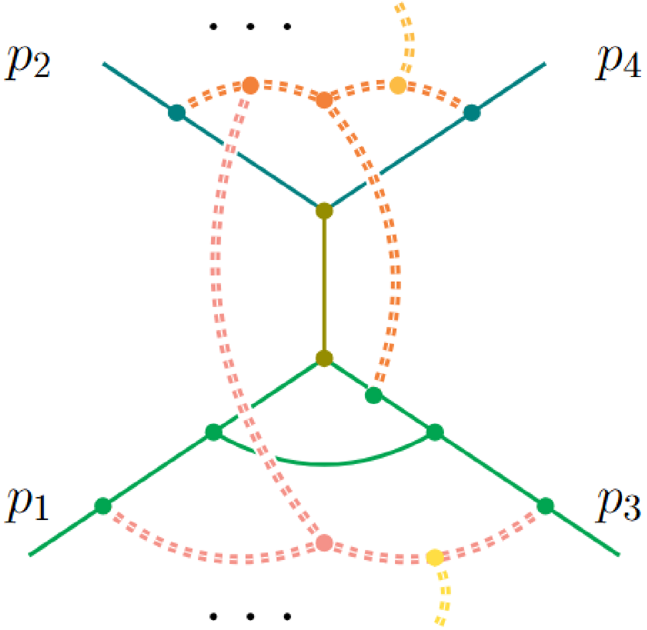

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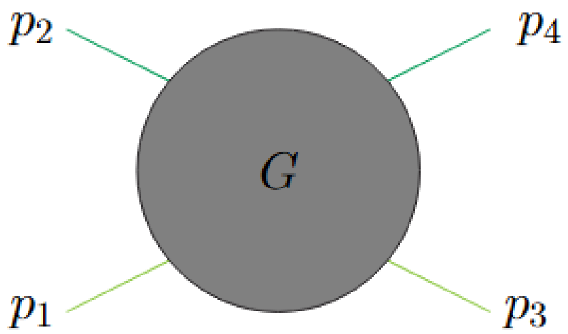

$$p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0,$$
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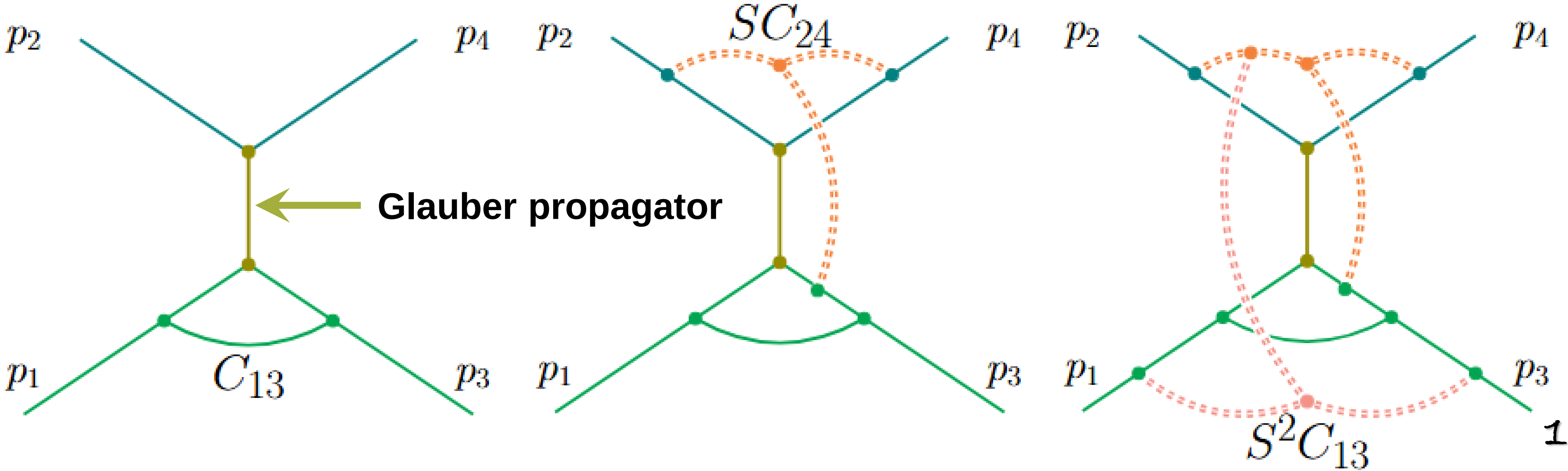
Feature 2: no Glauber loops (although Glauber propagators can exist).



Glauber propagator

Hidden regions

(Gardi, Herzog, Jones, YM, 2024)

Wide–angle kinematics

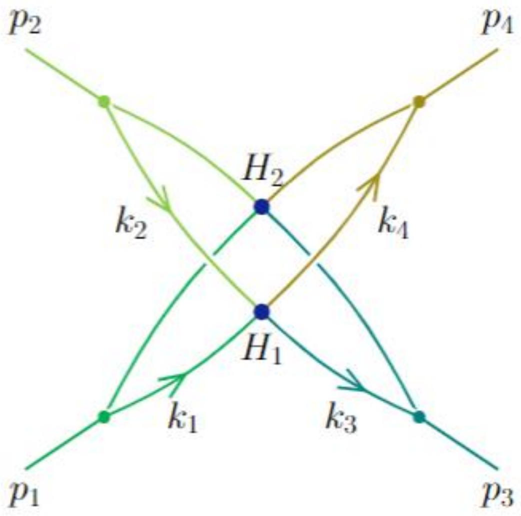
Spacelike–collinear kinematics

Wide–angle kinematics

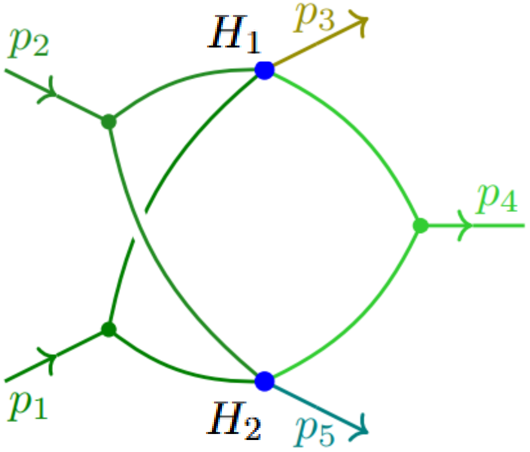
- For wide-angle kinematics, hidden regions feature the pattern of ***Landshoff scattering*** (multiple hard subgraphs):

  (Landshoff 1974)

For 1+2→3+4, this pattern occurs from three loops.



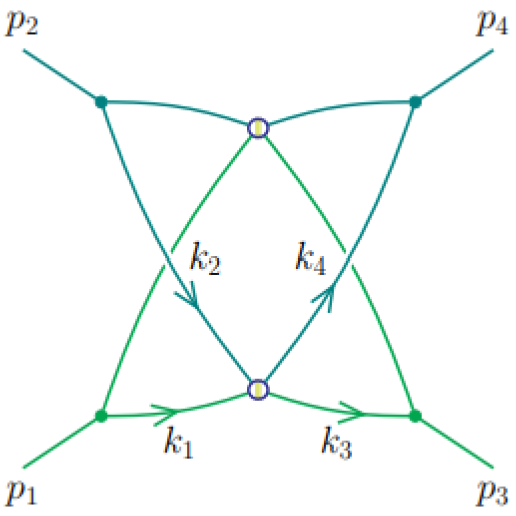
For 1+2→3+4+5, this pattern occurs from two loops.



Spacelike–collinear kinematics

- For spacelike-collinear kinematics, hidden regions feature some ***Glauber loop momenta***:

For 1+2→3+4, this pattern occurs from three loops.



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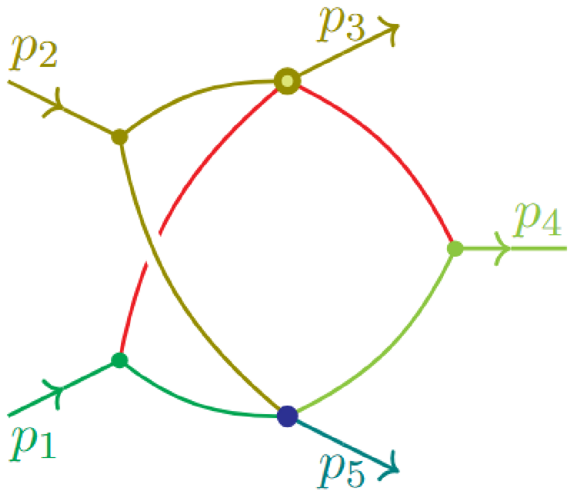

Spacelike–collinear kinematics

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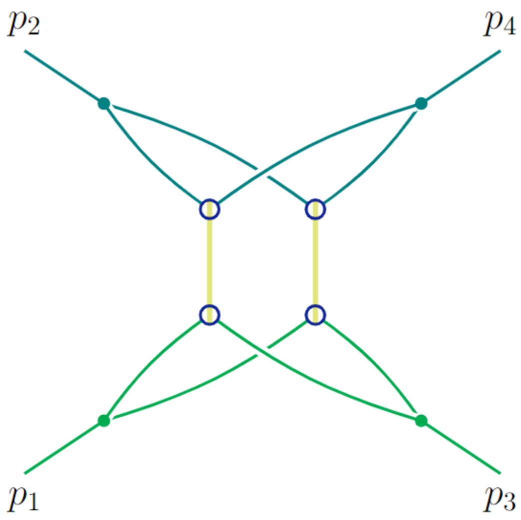

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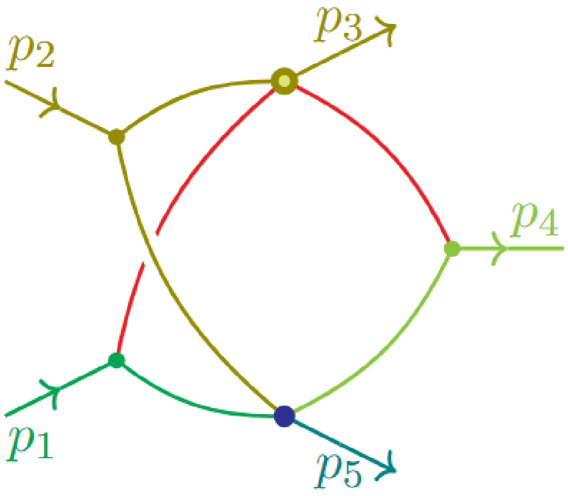
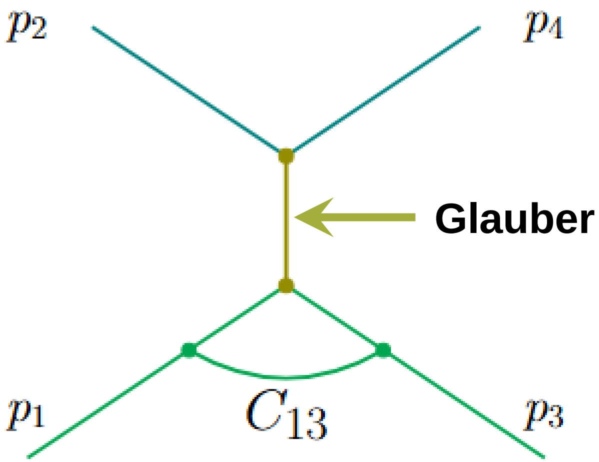


→ For more details: Zehao's talk in today's discussion session

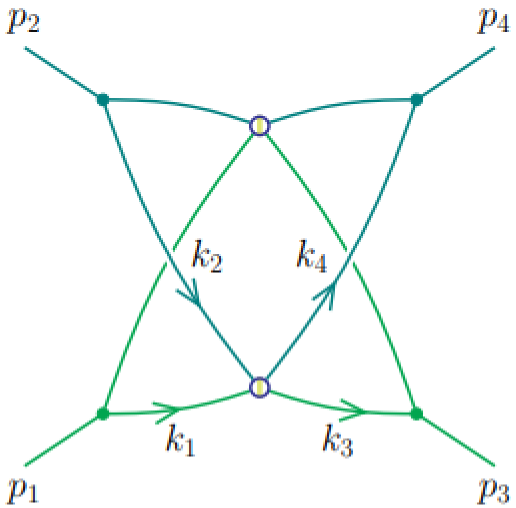
Remark on Glauber line/loop

Glauber propagator ⇔ Glauber loop momenta? No.

Region where there is a Glauber propagator but no Glauber loop:



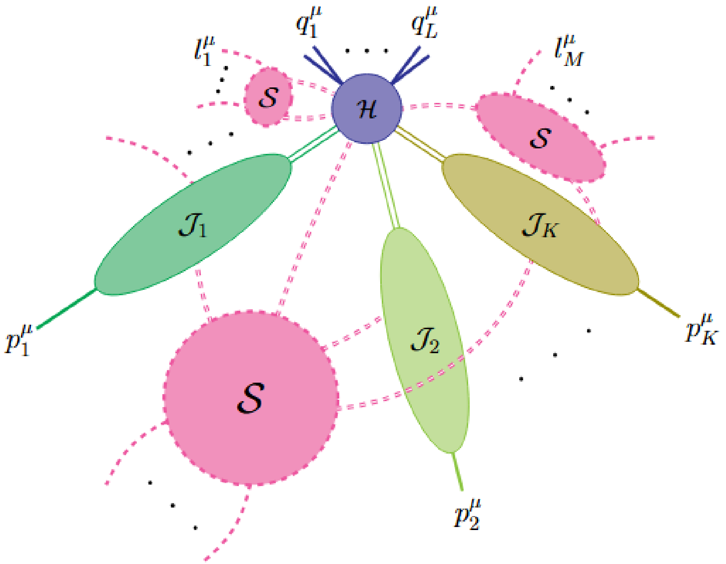
Region where there is a Glauber loop but no Glauber propagator:

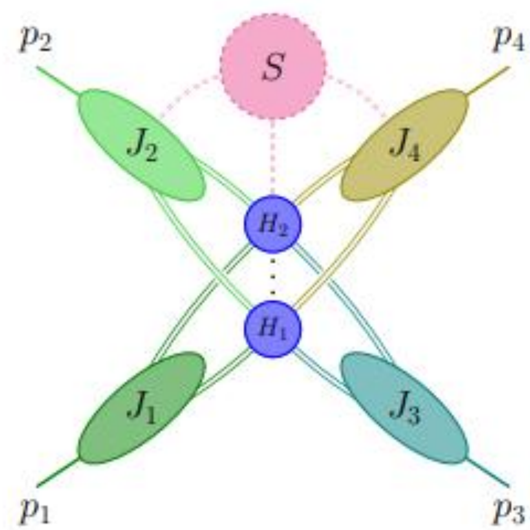


Conjectures on all–order structure

Unfortunately, our current understanding is limited to the first loop orders, where hidden regions start to arise. We conjecture the following:

Each facet region features a ***connected off-shell subgraph*** (hard or Glauber). In contrast, each hidden region features a ***disconnected off-shell subgraph*** (hard or Glauber).





Conjectures on all–order structure

Furthermore,

hidden regions in massless scattering exclusively appear as:

Landshoff scattering ***(wide–angle)***

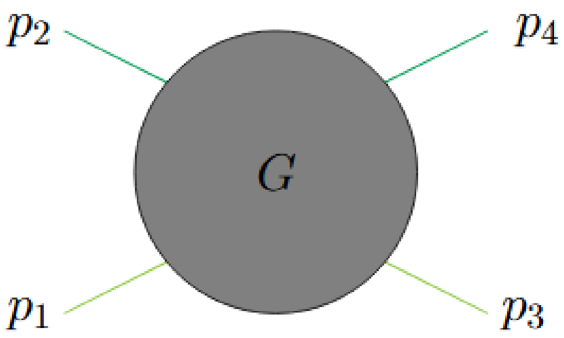
↕

Glauber singularities ***(spacelike–collinear)***

which seem to be in ***one-to-one correspondence***.

Comparison with Glauber SCET

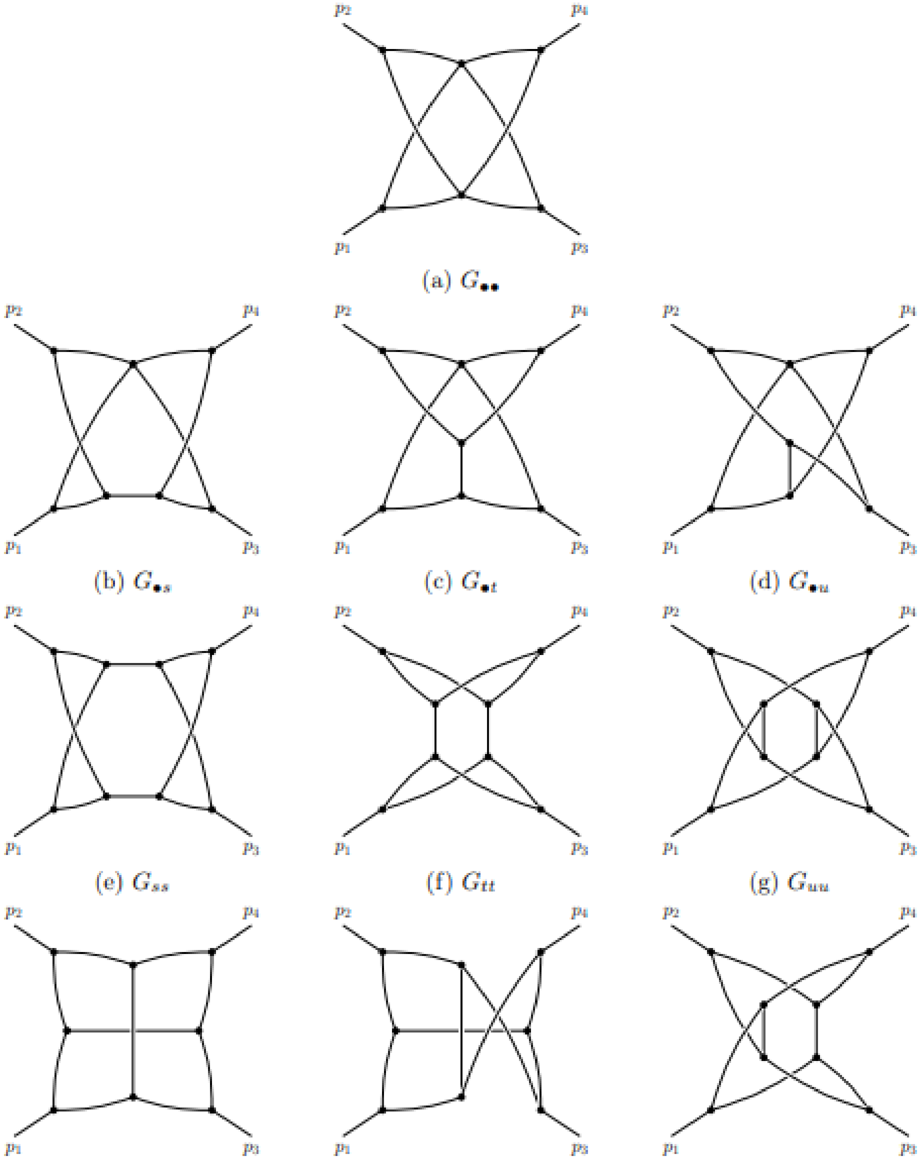
Our results for two-to-two forward scattering:



- **one-loop: no graph has Glauber region;**
- **two-loop: no graph has Glauber region;**
- **three-loop: very few graphs have Glauber regions.**

Comparison with Glauber SCET

The graphs:



(a) $G_{\bullet\bullet}$

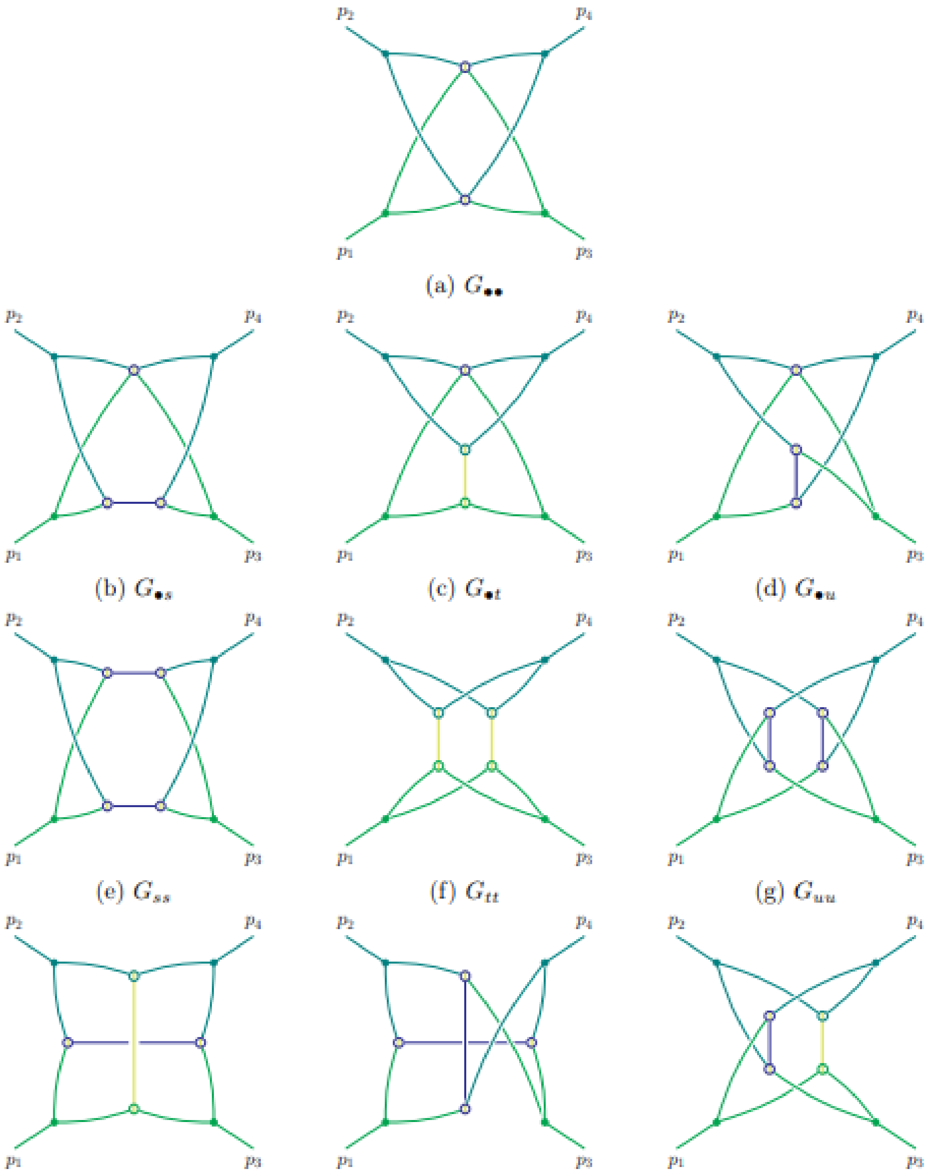
(b) $G_{\bullet s}$ (c) $G_{\bullet t}$ (d) $G_{\bullet u}$

(e) G_{ss} (f) G_{tt} (g) G_{uu}

(h) G_{st} (i) G_{su} (j) G_{tu}

Comparison with Glauber SCET

The Glauber region:



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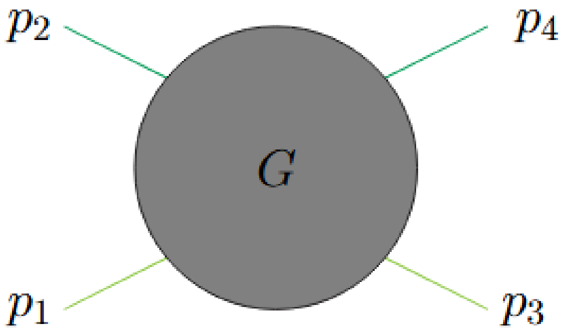

- **one-loop: no graph has Glauber region;**
- **two-loop: no graph has Glauber region;**
- **three-loop: very few graphs have Glauber regions.**

This is clearly different from Glauber SCET.

Is the difference from the non-analytic rapidity regulator? If so, by inputting the regulator in pySecDec (as a new propagator), we might obtain more graphs with Glauber regions. Should that be expected?

Some other interesting topics

(1) Generalization to massive Feynman integrals?

- the subgraph requirements need adjustment;
- near the threshold limit, more hidden regions.

(2) Generalization to phase-space integrals?

(Smirnov & Wunder 2024; Haug, Smirnov, Wunder 2025)

(3) Connections to multi-Regge kinematics?

→ Talks on Thursday and Friday

(4) Can we learn from the Landau-analysis community?

THANK YOU!