Gauge theories amplitudes in multi-Regge limits

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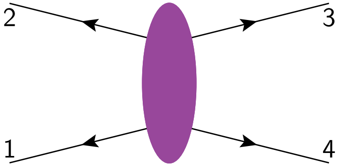
The high energy limit



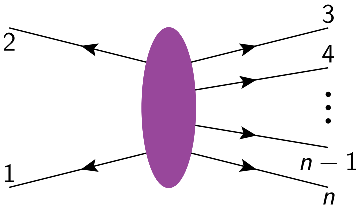
- Strong ordering in Mandelstam invariants $s_{ij} = (p_i + p_j)^2$

$$s_{12} \gg -s_{14}.$$

- Large rapidity separation $y_4 \gg y_3$.

$$\begin{aligned} p_k^{\perp} &= p_k^x + i\, p_k^y, \\ p_k^{+} &= (p_k^0 + p_k^z) = m_k^{\perp}\, e^{y_k}, \qquad (m_k^{\perp})^2 = |p_k^{\perp}|^2 + m_k^2 \\ p_k^{-} &= (p_k^0 - p_k^z) = m_k^{\perp}\, e^{-y_k}. \end{aligned}$$

The Multi-Regge Kinematics (MRK)



- Hierarchy of the Mandelstam invariants

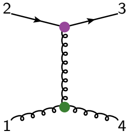
$$s_{12} \gg s_{34} \sim s_{45} \sim \cdots \sim s_{n-1\,n} \gg -s_{23} \sim -s_{234} \sim \cdots \sim -s_{1\,n}$$

- Rapidity ordering

$$y_n \gg y_{n-1} \gg \cdots \gg y_4 \gg y_3$$

The $2 \to 2$ amplitudes at leading order

$$\mathcal{M}^{\text{tree}}_{\{a_i\},\{\lambda_i\}}\left(s_{12}, s_{14}, \alpha_s(\mu^2)\right) \;=$$



Universal behaviour of the amplitudes

$$\mathcal{M}^{\text{tree}}_{\{a_i\},\{\lambda_i\}} = \Big(g\,\mathbf{T}^{c}_{a_4\,a_1}\underbrace{\delta_{\lambda_1\,\lambda_4}C_i^{(0)}}_{\textbf{impact factor}}\Big)\frac{2s_{12}}{s_{14}}\Big(g\,\mathbf{T}^{c}_{a_3\,a_2}\underbrace{\delta_{\lambda_2\,\lambda_3}C_j^{(0)}}_{\textbf{impact factor}}\Big) = \mathcal{M}_0^{[8]}\,\mathbf{c}_{[8]}$$

- $\mathbf{c}_{[i]}$ colour tensor in terms of the generators $\mathbf{T}^{c}_{a\,b}$

$$\begin{aligned}
\mathbf{T}^{c}_{a\,b} &= i\,f^{acb}, && \text{gluon,}\\
\mathbf{T}^{c}_{a\,b} &= t^{c}_{ab}, && \text{quark,}\\
\mathbf{T}^{c}_{a\,b} &= -t^{c}_{ba}, && \text{antiquark.}
\end{aligned}$$

Colour in the t-channel

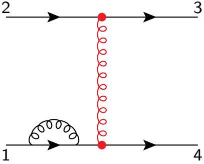
The colour charge in the t-channel is measured by the operator

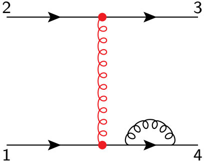
$$\mathbf{T}_t^2 = (\mathbf{T}_1^a + \mathbf{T}_4^a) \cdot (\mathbf{T}_1^a + \mathbf{T}_4^a) = (\mathbf{T}_2^a + \mathbf{T}_3^a) \cdot (\mathbf{T}_2^a + \mathbf{T}_3^a).$$

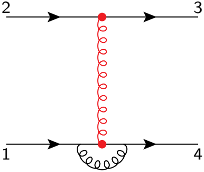
For gg, qg and qq scattering processes

$$\mathbf{T}_t^2 \, \mathbf{c}_{[8]} = C_A \, \mathbf{c}_{[8]}$$

Diagrammatic language

$\mathbf{T}_t^2 \, \mathbf{c}_{[8]}^{qq} =$ 

$+$ 

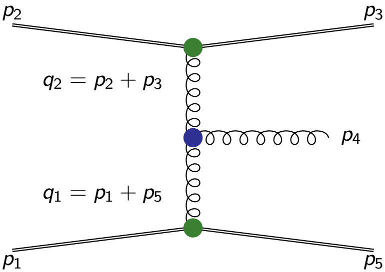
$- 2 \cdot$ 

The leading order amplitudes in MRK

Factorised QCD amplitudes in the MRK [Fadin,Kuraev,Lipatov 1976]

$$\mathcal{M}^{\text{tree}}_{ij\to i'gj'} = 2\, s_{12} \left(gC_i^{(0)}(p_1,p_5)\right) \frac{1}{s_{15}} \cdot gV^{(0)}(q_1,p_4,q_2) \cdot \frac{1}{s_{23}}$$
$$\times \left(gC_j^{(0)}(p_2,p_3)\right) \mathbf{c}_{[8,8]}, \qquad \mathbf{c}_{[8,8]} = if^{a_4xy}\,\mathbf{T}^x_{a_5a_1}\mathbf{T}^y_{a_3a_2}$$

- **Impact factors**: $C_i^{(0)}$
  Peripheral emissions
- **Lipatov vertex**: $V^{(0)}$
  Gluon emission at central rapidity



All-order structure of the amplitudes

The amplitudes admit a double expansion

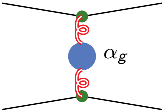
- Loop counting $a_s = \frac{g^2}{16\pi^2}$
- High-energy logarithms: for a **2 → 2 amplitude**

$$L = \frac{1}{2}\left(\log\frac{s_{12}}{s_{14}} + \log\frac{s_{13}}{s_{14}}\right) \equiv \frac{1}{2}\left(\log\frac{s}{t} + \log\frac{u}{t}\right) = \log\frac{s}{-t} - i\frac{\pi}{2}$$

$$\mathcal{M} = \mathcal{M}^{\text{tree}} + \underbrace{a_s L\,\mathcal{M}^{(1,1)} + a_s^2 L^2\,\mathcal{M}^{(2,2)}}_{\text{LL}} + \underbrace{a_s\,\mathcal{M}^{(1,0)} + a_s^2 L\,\mathcal{M}^{(2,1)}}_{\text{NLL}} + \underbrace{a_s^2\,\mathcal{M}^{(2,0)}}_{\text{NNLL}}$$

The gluon reggeization [Lipatov;Kuraev,Fadin,Lipatov 1976]

$$\mathcal{M}^{\mathrm{LL}}_{ij\to ij} = \mathcal{M}^{\mathrm{tree}}_{ij\to ij} \left(\frac{s}{-t}\right)^{\alpha_g(t)} =$$



The **gluon Regge trajectory** α_g has a perturbative expansion

$$\alpha_g(t) = \sum_{n=1}^{\infty} \alpha_g^{(n)}(t,\mu^2) \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n =$$


$+$

$+ \ldots$

The high-energy logarithms are obtained by exponentiation

$$\mathcal{M}^{\mathrm{LL}} = \mathcal{M}^{\mathrm{tree}} \left[1 + a_s \alpha_g^{(1)} L + \frac{\left(a_s \alpha_g^{(1)} L\right)^2}{2} + \ldots\right.$$

The Regge pole

Relation with Regge theory e.g. from the integral representation [Caron-Huot,Gardi,Vernazza 2017]

$$\mathcal{M}^{(-)} = \frac{\mathcal{M}(s,t) - \mathcal{M}(u,t)}{2} = \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{dj}{\sin(\pi j)} \sin\left(\frac{\pi j}{2}\right) a_j(t)\, e^{j\,L},$$

where $a_j(t)$ is related to the Mellin transform of the **discontinuities** in the s- and u-channels. The simplest **model** has only a **pole** in j

$$a_j(t) = \frac{1}{j - 1 - \alpha(t)}$$

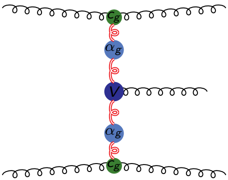
The integral reproduces the dependence $s^{\alpha(t)}$

$$\mathcal{M}^{(-)} = \frac{\pi}{\sin\frac{\pi\alpha(t)}{2}} \frac{s}{t} \times e^{L\,\alpha(t)} \simeq \mathcal{M}^{\text{tree}} \times e^{L\,\alpha(t)}$$

Multileg amplitudes at LL

The emission of n gluons in MRK is also expressed as the exchange of a reggeized gluon in each t-channel [Balitsky,Fadin,Lipatov 1979].
A $2 \to 3$ amplitude is given by

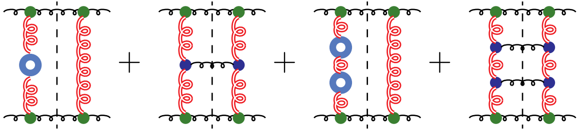
$$\mathcal{M}^{\rm LL}_{ij\to i'gj'} = \left(\frac{s_{45}}{\tau}\right)^{\alpha_g(q_1^2)} \left(\frac{s_{34}}{\tau}\right)^{\alpha_g(q_2^2)} \mathcal{M}^{\rm tree}_{ij\to i'gj'}$$



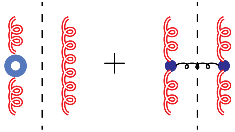
- Each t-channel gluon exchange Reggeizes with the associated trajectory.
- The factorisation scale τ is *arbitrary* at LL.

The consequences of Reggeization

- Insight into the all-order structure of the amplitudes
  - All the terms of the form $\alpha_s^n L^n$ are *known*
- Phenomenological impact: construct LL resummed cross sections

$\sigma_{\mathrm{LL}} \simeq$  $+ \ldots$

Iterations of the BFKL kernel [Balitsky,Fadin,Kuraev,Lipatov 1976-1978]

$\mathcal{K}_{\mathrm{LO}} =$ 

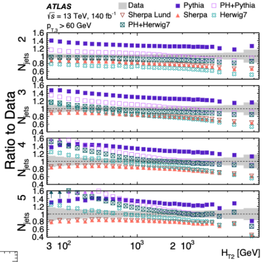
Energy logs at the LHC

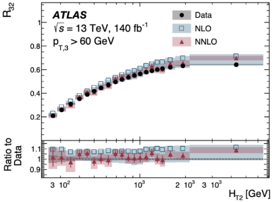
arXiv:2405.20206

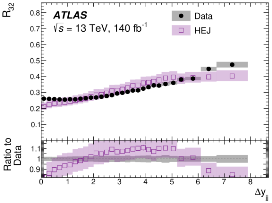
JETS PHYSICS: R3/2 RATIO

$$\frac{d\sigma_{3j}/dx}{d\sigma_{2j}/dx}$$ where $x = H_{T2}$, $m_{jj,max}$, $|\Delta y_{jj,max}|$, m_{jj}, $|\Delta y_{jj}|$

- **Differential cross sections in different jet multiplicity bins**
  - ➡**Several p_{T3} thresholds for H_{T2} measurements: explore sensitivity to resummation effects**
  - ➡ **Ratios for better sensitivity to α_s (smaller uncertainties)**
  - ➡ **NNLO needed for a good description of the data.**
  - ➡**HEJ description is better in regions with large contributions of $\log(p_{Tjet}/\sqrt{s})$**







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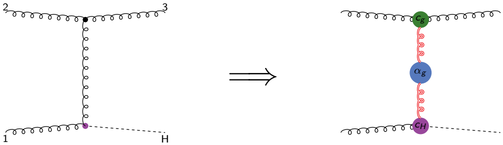
[Ana Rosario Cueto Gòmez, HP2 2024]

Regge universality

The gluon reggeization captures the high energy limit of t-channel gluon exchanges, e.g. the Higgs+jet amplitudes

$$\mathcal{M}^{(0)}_{ij\to iH} = \left(\frac{\lambda}{\sqrt{2}} C^{H(0)}(p_1, p_{\rm H})\delta^{a_1 c}\right) \frac{s}{t} \left(g\, \mathbf{T}^c_{a_3 a_2} C^{(0)}_i(p_2, p_3)\right)$$



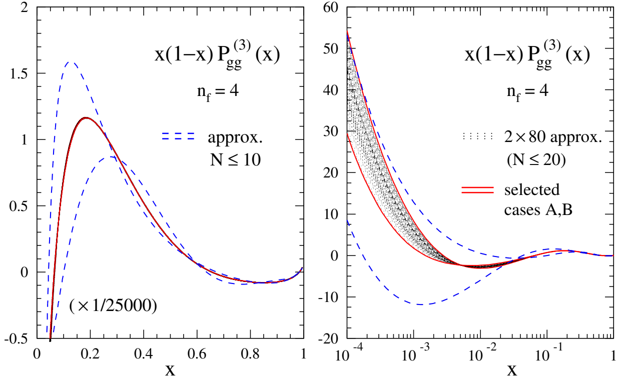
$$\mathcal{M}^{\rm LL}_{ig\to iH} = \left(\frac{s}{\tau}\right)^{\alpha_g(t)} \mathcal{M}^{\rm tree}_{ig\to iH}$$

See phenomenology! [Xiao,Yuan 2018; Celiberto,Ivanov,Mohammed,Papa 2020; Andersen,Hassan,Maier,Paltrinieri,Papaefstathiou,Smillie 2022].

General questions

- What is the structure of the **subleading logarithms**?

The high-energy limit provides **boundary data** and **cross checks** for the multiloop amplitudes ... and crucial physical quantities



$$x = \frac{Q^2}{(p+Q)^2}$$

$$\underbrace{\mathcal{O}\left(a_s^4 \log(x)\right)}_{\text{NNLL}} = ?$$

The $2 \to 2$ amplitudes at NLL and beyond

The Regge pole ansatz $\mathcal{M} \simeq s^{\alpha(t)}$ does **not** describe the $2 \to 2$ amplitudes at NLL. Going back to the language of Regge theory

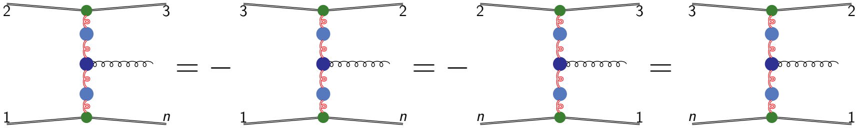
- The amplitudes are not only poles in the complex (angular momentum) plane. **Regge cuts** enter the picture.

What are the features of the Regge pole and of the cuts?

- The symmetries provide a first guidance.

Signature

The reggeized gluon (or reggeon) has **odd signature** under the exchanges of the particles $1 \leftrightarrow n$ and $2 \leftrightarrow 3$.



For each colour structure we construct the signatured amplitudes

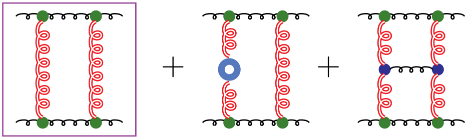
$$\mathcal{M}_{[i]}^{(\sigma_1,\sigma_2)} = \frac{1}{4}\Big[\mathcal{M}_{[i]}(p_1, p_2, p_3, \ldots, p_n) + \sigma_1 \mathcal{M}_{[i]}(p_n, p_2, p_3, \ldots, p_1) \\ + \sigma_2 \mathcal{M}_{[i]}(p_1, p_3, p_2, \ldots, p_n) + \sigma_1\sigma_2 \mathcal{M}_{[i]}(p_n, p_3, p_2, \ldots, p_1)\Big]$$

This talk focuses on the odd-odd contribution $\mathcal{M}^{(-,-)}$. If $n = 4$, $2 \leftrightarrow 3$ is equivalent to $1 \leftrightarrow 4$, so the odd signature is labeled $\mathcal{M}^{(-)}$.

The even amplitudes at NLL

The even amplitude has at least **two** Reggeon exchanges $\rightarrow$ **CUT**! [Caron-Huot 2013]

$\mathcal{M}^{(+)} =$



$+ \ldots$

- The leading-order contribution starts at $\mathcal{O}(a_s)$ with no logarithms. At n loops, the even amplitudes are of $\mathcal{O}(a_s^n\, L^{n-1})$.
- Purely **imaginary** $\mathcal{M}^{(+)} \propto (i\pi)$ [Caron-Huot,Gardi Vernazza 2017].
- The **IR poles** of the even amplitudes are known to **all orders** [Caron-Huot,Gardi,Reichel,Vernazza 2017].
- The **IR finite** terms have been computed through 13 loops [Caron-Huot,Gardi,Reichel,Vernazza 2020].

The odd amplitudes at NLL

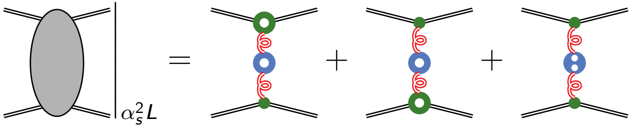
At NLL, $\mathcal{M}^{(-)}$ is a **Regge pole** [Fadin,Kozlov,Renzichenko 2015]

$$\mathcal{M}^{(-)} = \frac{s}{t}\, C_i(p_1, p_4, \tau) \left[\left(\frac{s}{\tau}\right)^{\alpha_g(t)} + \left(\frac{-s}{\tau}\right)^{\alpha_g(t)} \right] C_j(p_2, p_3, \tau)\, \mathbf{c}_{[8]}$$

The **impact factors** are now required at one-loop [Fadin,Fiore 1992]

$$C_i(p_1, p_4, \tau) = C_i^{(0)}(p_1, p_4) \cdot c_i \left(\alpha_s, \frac{q_1^2}{\tau}\right), \quad c_i \left(\alpha_s, \frac{q_1^2}{\tau}\right) = 1 + \frac{\alpha_s}{\pi}\, c_i^{(1)} \left(\frac{q_1^2}{\tau}\right) + \ldots,$$

and the gluon Regge trajectory enters at two loops [Fadin, Fiore, Kotsky 1995; Blümlein, Ravindran, van Neerven 1998]



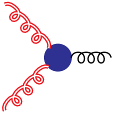
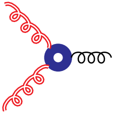
The odd-odd amplitudes at NLL

The same holds for emissions in MRK [Fadin,Kozlov,Reznichenko 2015]

$$\mathcal{M}^{(-,-)} = \frac{2s}{q_1^2 q_2^2}\, C_i(p_1, p_5, \tau)\, e^{\alpha_g(q_1^2)\left[\log\left(\frac{s_{45}}{\tau}\right) - i\frac{\pi}{4}\right]} \times V(q_1, q_2, p_4^\perp, \tau)$$
$$\times\, e^{\alpha_g(q_2^2)\left[\log\left(\frac{s_{34}}{\tau}\right) - i\frac{\pi}{4}\right]}\, C_j(p_2, p_3, \tau)\, \mathbf{c}_{[8,8]}$$

Enters one new building block: the Lipatov vertex v at NLO [Fadin,Lipatov 1993;Del Duca,Schmidt 1998;Fadin,Fucilla,Papa 2023].

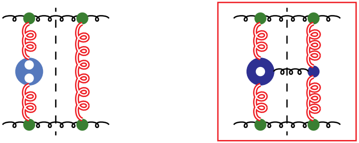
$$V(q_1, q_2, p_4^\perp, \tau) = V_0 + \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n v^{(n)}(|q_1|^2, |q_2|^2, |p_4^\perp|^2, \tau)$$

$= $  $+$  $+ \ldots$

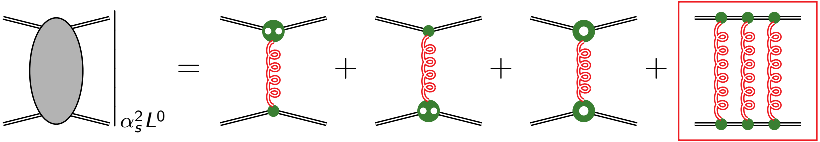
Beyond NLL

The **gluon reggeization** describes **all** the **odd** amplitudes

- Resummation of the NLL via the NLO BFKL kernel.



New structures beyond the Regge pole enter the odd amplitudes at **NNLL** [Del Duca,Glover 2001]: **three Reggeon cuts**.



Going towards NNLL

The analysis of high energy factorisation at NNLL is not complete yet.

A NNLO version of the BFKL equation may not be possible, or may describe only part of the amplitude.

Several **tools** have emerged

- Diagrammatic approach [Fadin,Lipatov 2017; Fadin 2023-2024]
  - Analysis of $2 \to 2$ scattering through 4 loops.
- Glauber SCET [Rothstein,Stewart 2016; Moult,Raman,Ridgeway,Stewart 2022; (Gao,Moult,Raman,Ridgway,Stewart 2024)^2]
  - One colour structure $(10 \oplus \overline{10})$ in $gg \to gg$ to all orders!

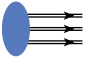
The appoach of this talk is based on the scattering of Wilson lines.

See more in the coming talk by G. Gambuti!

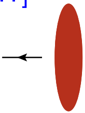
OPE for high-energy scattering

Scattering of Wilson lines

[Mueller,Patel,Balitsky,Korchemskaya,Korchemsky,Kovchegov,Jalilian-Marian,Kovner,McLerran,Weigert,Iancu,Leonidov,. . .]



$$\Psi_{\text{proj}} \sim \mathrm{T}\left[U(z_1)U(z_2)\dots\right]$$



$$U_\eta(z) = \mathcal{P}\exp\left[ig\int_{-\infty}^{+\infty} dx^+\, A_+(x^+, x^- = e^{-2\eta}x^+, x^\perp = z)\right] \equiv \mathcal{P}\exp\left[ig\, W(z)\right]_\eta$$

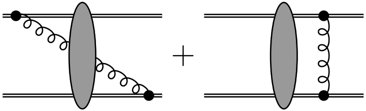
- **Rapidity divergence**: infinite Wilson lines on the lightcone
- Regulator: finite rapidity $\to$ **dependence** on $\log(\mathbf{s})$

$$\eta = \frac{1}{2}\log\frac{dx^+}{dx^-} = \frac{1}{2}\log\frac{s}{-t}$$

Evolution in rapidity

Evolution of the projectile in the background of the target
[Balitsky,Jalilian-Marian,Iancu,McLerran,Weigert,Leonidov,Kovner: B-JIMWLK, 1995-2000; Caron-Huot 2013]

$$\frac{d}{d\eta}\mathrm{T}\left[U(z_1)\dots U(z_n)\right] = -\sum_{i,j}\int d^2z\, H_{ij}[U_{\mathrm{adj}}(z)] \otimes \mathrm{T}\left[U(z_1)\dots U(z_n)\right]$$

$H_{ij}^{\mathrm{LO}} \sim$  $= \mathcal{O}(a_s)$

Two terms in the L.O. Hamiltonian

- **Add** one Wilson line in the adjoint rep. (gluon emission)
- **Colour rotation** of the dipole.

The dilute regime [Caron-Huot 2013]

$$U(z) = \exp\Big[i\, g\, \mathbf{T}_i^a\, W^a(z)\Big] \sim 1$$

This **defines** the field $W^a(z)$, with the following features

- **Odd Signature**
  $W^a(z) \to -W^a(z)$ under **initial** $\leftrightarrow$ **final** states
- **Evolution** changes the number of W-fields

$$H = g^2\, H_{n\to n} + g^4\, H_{n\to n\pm 2} + g^6\, H_{n\to n\pm 4} + \dots$$

- **Contractions** as a free field

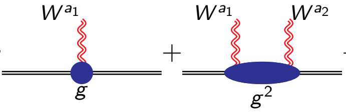
$$\langle W^{a_1}(p_1)|W^{a_2}(p_2)\rangle = \frac{i}{p_1^2}\delta^{a_1\, a_2}\, \delta^{(d-2)}(p_1 + p_2)$$

- $W^{(a)}(z) \simeq$ Reggeon.

Partonic Amplitudes [Caron-Huot 2013;+Gardi,Vernazza 2017]

$$\mathcal{M} \sim \underbrace{\langle \Psi_j |}_{\text{target}} e^{-H\eta} \underbrace{| \Psi_i \rangle}_{\text{projectile}}$$

- $|\Psi_i\rangle$ **dilute**: $U(z) = exp\left[ig\, W^a(z)\, \mathbf{T^a}\right] \sim 1,\ W^a(z)$ **Reggeon**

$|\psi_i\rangle \sim$ 

$+ \cdots \sim \begin{pmatrix} W^{a_1} \\ W^{a_1} W^{a_2} \\ \cdots \end{pmatrix}$

- Iteratively act with H and generate powers of $\eta \sim \frac{1}{2}\log\frac{s}{-t}$.

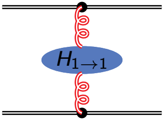
$$H = \begin{pmatrix} H_{1\to1} & 0 & H_{3\to1} & \cdots \\ 0 & H_{2\to2} & 0 & \cdots \\ H_{1\to3} & 0 & H_{3\to3} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \qquad \begin{aligned} H_{k\to k} &= \mathcal{O}(a_s) \\ H_{1\to3}, H_{3\to1} &= \mathcal{O}(a_s^2) \end{aligned}$$

Single Reggeon vs multiple Reggeons

$$\mathcal{M}^{(-)}_{ij\to ij} = \mathcal{M}^{\mathrm{SR}}_{ij\to ij} + \mathcal{M}^{\mathrm{MR}}_{ij\to ij}$$

The contribution of a single Reggeon has the LL structure

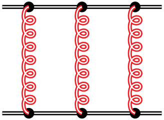
$$\mathcal{M}^{\mathrm{SR}}_{ij\to ij} =$$



$$\simeq \frac{1}{t}\, e^{H_{1\to 1}\, L}$$

Multi-Reggeon exchanges are related to **Regge cuts**. E.g. $\mathcal{O}(a_s^2 L^0)$

$$\mathcal{M}^{\mathrm{MR},(2)}_{ij\to ij} = \langle W^{a_1} W^{a_2} W^{a_3}(z) | W^{b_1} W^{b_2} W^{b_3}(x)\rangle =$$



A colourfull guide to disentangle the Regge cut

$$\mathcal{M}^{\mathrm{MR},(2)}_{ij\to ij} = \pi^2\, S(\epsilon)\left[\frac{N_c^2}{6} + \underbrace{\left((\mathbf{T}^2_{s-u})^2 - \frac{C_A^2}{4}\right)}_{N_c-\text{sublead}}\right]\mathcal{M}^{\mathrm{tree}}_{ij\to ij},$$

- **Leading colour** $\rightarrow \propto \mathcal{M}^{\mathrm{tree}}$ and **universal** $\rightarrow$ **FACTORISED**
- **Subleading colour** $\rightarrow$ factorisation breaking $\mathbf{T}^2_{s-u}$ $\rightarrow$ **CUT**

A scheme for the **three-loop gluon Regge trajectory** [GF, Gardi, Maher,Milloy,Vernazza 2021;Chakraborty,Gambuti,von Manteuffel,Tancredi 2021]

$$\mathcal{M}^{(-)}_{ij\to ij} = \frac{c_i(t,\tau)}{2}\left[\left(\frac{s}{\tau}\right)^{\alpha_g(t)} + \left(\frac{-s}{\tau}\right)^{\alpha_g(t)}\right] c_j(t,\tau)\mathcal{M}^{\mathrm{tree}}_{ij\to ij} + \mathcal{M}^{\mathrm{MR}}_{ij\to ij}\Big|_{N_c\text{-sublead.}}$$

The two-loop amplitudes

$$\mathcal{M}^{(-)}_{ij\to ij}\Big|_{\mathcal{O}(a_s^2 L^0)} = c_i^{(2)} + c_j^{(2)} + c_i^{(1)} \cdot c_j^{(1)} - \frac{\pi^2 (\alpha_g^{(1)})^2}{8} + \mathcal{M}^{\text{MR},(2)}_{ij\to ij}\Big|_{N_c\text{ sub.}}$$

$$\mathcal{M}^{\text{MR},(2)}_{ij\to ij}\Big|_{N_c\text{ sublead.}} = \pi^2 S(\epsilon)\left(\left(\mathbf{T}^2_{s-u}\right)^2 - \frac{C_A^2}{4}\right)\mathcal{M}^{\text{tree}}_{ij\to ij}$$

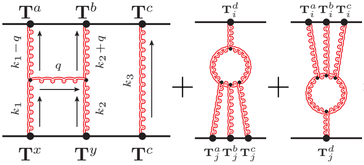
The multi-Reggeon diagrams are **not** the complete amplitudes.

- They give a prescription to **disentangle** the contributions of the **Regge cuts**, once the two-loop amplitudes are known.
- Consistency with the analysis of the IR singularities [Del Duca,GF,Magnea,Vernazza 2013-2015].

The remaining terms contribute to the **Regge pole**, defining the **two-loop impact factors**, $c_i^{(2)}$.

Three Reggeons at three loops

By computing the multi-Reggeon exchanges at three loops one gets



$$= -\pi^2 r_\Gamma^3\, L \left[S_C^{(3)}(\epsilon)\, N_c^3 + \underbrace{S_A^{(3)}(\epsilon)\mathbf{T}_{s-u}^2[\mathbf{T}_{s-u}^2, \mathbf{T}_t^2] + S_B^{(3)}(\epsilon)[\mathbf{T}_{s-u}^2, \mathbf{T}_t^2]}_{\text{cut}} \right] \mathcal{M}_{ij\to ij}^{\text{tree}}$$

- The leading term in N_c is **universal**: factorised as a **pole**.
- The commutators of the colour channel operators are **non-planar** contributions to the **Regge cut**.

Poles and cuts at three loops

In our prescription, the three-loop odd amplitudes at NNLL are

$$\mathcal{M}^{(-)}_{ij\to ij}\Big|_{\mathcal{O}(a_s^3 L)} = \Big[\boldsymbol{\alpha_g^{(3)}} + \alpha_g^{(2)}(c_i^{(1)} + c_j^{(1)}) + \alpha_g^{(1)}(c_i^{(2)} + c_j^{(2)} + c_i^{(1)} c_j^{(1)}) - \frac{\pi^2}{8}(\alpha_g^{(1)})^3\Big]\, \mathcal{M}^{\rm tree}_{ij\to ij} + \mathcal{M}^{(3),{\rm cut}}_{ij\to ij}\Big|_{\mathcal{O}(a_s^3 L)}$$

The Regge pole has one new parameter: the three-loop Regge trajectory. The required input to determine it are $\mathcal{M}^{(-)}_{ij\to ij}|_{\mathcal{O}(a_s^3 L)}$ [Caola,Chakraborty,Gambuti,von Manteuffel,Tancredi 2021] and

$$\mathcal{M}^{(3),{\rm cut}}_{ij\to ij}\Big|_{\mathcal{O}(a_s^3 L)} = \alpha_g^{(1)} \mathbf{T}_t^2 \mathcal{M}^{(2),{\rm cut}}_{ij\to ij} - \pi^2\, r_\Gamma^3 \Big(S_A^{(3)} \mathbf{T}^2_{s-u}[\mathbf{T}^2_{s-u}, \mathbf{T}^2_t] + S_B^{(3)} [\mathbf{T}^2_{s-u}, \mathbf{T}^2_t] \mathbf{T}^2_{s-u} \Big)\, \mathcal{M}^{\rm tree}_{ij\to ij}$$

The three-loop Regge trajectory

$$\alpha_g(t) = -\frac{1}{2}\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2}\,\Gamma_{\mathrm{A}}^{\mathsf{cusp}}(a_s(\lambda^2)) + C_A\,\hat{\alpha}_g(t)$$

- The IR poles are controlled by the **cusp anomalous dimension**.
- The finite parts at three loops are [GF,Gardi,Maher,Milloy,Vernazza 2021; Caola,Chakraborty,Gambuti,von Manteuffel,Tancredi 2021]

$$\begin{aligned}\hat{\alpha}_g^{(3)} = \; & C_A^2\left(\frac{297029}{93312} - \frac{799\zeta_2}{1296} - \frac{833\zeta_3}{216} - \frac{77\zeta_4}{192} + \frac{5}{24}\zeta_2\zeta_3 + \frac{\zeta_5}{4}\right) \\ & + C_A n_f\left(\frac{103\zeta_2}{1296} + \frac{139\zeta_3}{144} - \frac{5\zeta_4}{96} - \frac{31313}{46656}\right) \\ & + C_F n_f\left(\frac{19\zeta_3}{72} + \frac{\zeta_4}{8} - \frac{1711}{3456}\right) + n_f^2\left(\frac{29}{1458} - \frac{2\zeta_3}{27}\right) + O(\epsilon)\end{aligned}$$

A note regarding planarity

In the **planar** limit, the Regge cut is **absent**.

- The planar amplitudes factorise as a pure **pole**.

The Regge pole is *mostly planar*

- $\alpha_g^{(3)}$ doesn't have N_c-subleading terms.
  - In $\mathcal{N}=4$ the trajectory is completely planar

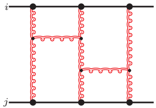
$$\alpha_g^{(3)}(t)\Big|_{\text{sYM}} = N_c^2\left(\frac{11}{48}\frac{\zeta_4}{\epsilon}+\frac{5}{24}\zeta_2\zeta_3+\frac{\zeta_5}{4}+\mathcal{O}(\epsilon)\right)$$

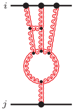
    Possible *eikonal* origin of the trajectory.
  - The trajectory in QCD (at $n_f=0$) is identical to the **planar YM** [Del Duca,Marzucca,Verbeek 2021].

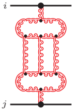
Consistency checks

The Regge pole ansatz at NNLL does **not** have more parameters to absorb the planar contributions of the multi-Reggeon exchanges.

- At 4 loops, the multi-Reggeon diagrams must be **non-planar**.



$+$


$+$


$$= \frac{\pi^2 r_\Gamma^4}{2}\left[\frac{1}{\epsilon^4}\mathbf{K}^{(4)} + \left(\frac{\zeta_3}{\epsilon} + \frac{3}{2}\zeta_4\right)\mathbf{K}^{(1)}\right]\mathcal{M}^{\text{tree}}_{ij\to ij}$$

$\mathbf{K}^{(4)}$ and $\mathbf{K}^{(1)}$ are colour tensors involving **commutators** and

$$\frac{d_{AA}}{N_A} - \frac{C_A^4}{24} \sim N_c^2, \qquad \frac{d_{AA}}{N_A} = \frac{N_c^2(N_c^2+36)}{24}$$

Beyond $2 \to 2$ scattering

Let's adopt the same scheme to analyse more general processes in the high energy limit through NNLL accuracy. I will focus on

- The amplitudes for Higgs + jet: $gg \to gH$ and $qg \to qH$
- The five-parton scattering amplitudes: $gg \to ggg$, $qg \to qgg$ and $qQ \to qQg$.

Higgs + jet at NNLL

Reggeization established at NLL [Fucilla,Nefedov,Papa 2024]. At NNLL

$$\mathcal{M}_{ig\to iH} = \frac{c_i(t,\tau)}{2}\left[\left(\frac{s}{\tau}\right)^{\alpha_g(t)} + \left(\frac{-s}{\tau}\right)^{\alpha_g(t)}\right] c_H(t,\tau)\, \mathcal{M}^{\text{tree}}_{ig\to iH} + \mathcal{M}^{\text{cut}}_{ig\to iH}$$

- Already known $c_i^{(L)}$, $i = q, g$ and $L \leq 2$ and $\alpha_g^{(L)}$ for $L \leq 3$ [GF,Gardi,Maher,Milloy,Vernazza 2021; Caola,Chakraborty,Gambuti,von Manteuffel,Tancredi 2021]
- $c_H^{(2)}$ unknown term in the **Regge pole** component
- $\mathcal{M}^{\text{cut}}_{ig\to iH}$ must be disentangled from the rest.

Regge cuts in Higgs + jet?

IR poles consistent with **no cuts** through 4 loops [Del Duca,GF 2025]

$$\mathcal{M}^{\text{cut}}_{ig\to iH} = \mathcal{M}^{\text{MR}}_{ig\to iH}\Big|_{N_c\text{-sublead.}} \text{ with } \mathcal{M}^{\text{MR}}_{ig\to iH} = P(\mathbf{T}_t^2, \mathbf{T}_{s-u}^2)\,\mathcal{M}^{\text{tree}}_{ig\to iH}$$

P polynomial involves also Casimir in adjoint representation. **Colour conservation** for 3 particle imposes

$$\mathbf{T}_t^2 = C_A \mathbf{1}, \quad \mathbf{T}_{s-u}^2 = 0.$$

The multi-Reggeon diagrams are

$$\mathcal{M}^{\text{MR}\,(L)}_{ig\to iH} = a_s^L\, f_L(\epsilon)\, C_A^L\, \mathcal{M}^{\text{tree}}_{ig\to iH}, \quad L \le 3$$

Hence

$$\mathcal{M}^{\text{cut}}_{ig\to iH} = \mathcal{O}(a_s^4)$$

Input from fixed order

$$\frac{\mathcal{M}^{(2)}_{ig\to iH}}{\mathcal{M}^{\text{tree}}_{ig\to iH}} = c_H^{(2)} + c_i^{(2)} + c_i^{(1)} c_H^{(1)} - \frac{\pi^2}{8}\left(\alpha^{(1)}\right)^2 + L\left(\alpha^{(2)} + \alpha^{(1)}(c_i^{(1)} + c_H^{(1)})\right)$$
$$+ \frac{\left(\alpha^{(1)}\right)^2}{2} L^2, \text{ where } L = \log\frac{s}{\tau} - i\frac{\pi}{2}$$

- Two-loop amplitudes known in general kinematics through $\mathcal{O}(\epsilon^2)$ [Gehrmann,Jakubčík,Mella,Syrrakos,Tancredi 2023]
- Results can be written in terms of HPLs in $v = \frac{m_H^2}{s}$ and 2dHPLs with letters $\{0, 1, -v, 1-v\}$ and argument $u = u\left(\frac{-t}{m_H^2}, v\right)$.
- Asymptotic expansion as $v \to 0$ from the differential eqs. for 2dHPLs [Del Duca,GF 2025]

The two-loop Higgs impact factor

Factorisation of the IR poles

$$c_H(t, m_H^2, \tau) = \frac{Z_{\mathrm{col}\, gH}\left(\frac{m_{H\perp}^2}{\mu^2}, a_s\right)}{\cos\left(\frac{\pi\alpha_g(t)}{2}\right)} \left(\frac{\tau}{m_{H\perp}^2}\right)^{\frac{\alpha_g(t)}{2}} \bar{D}_H(a_s, x, \mu^2),$$

where $x = \frac{-t}{m_H^2}$, $Z_{\mathrm{col}\, gH} = \exp[-\int \frac{d\lambda^2}{2}\Gamma_{gH}(\lambda^2)]$, $\bar{D}$ **finite** as $\epsilon \to 0$.

The result is expanded as $\bar{D} = \sum a_s^L \, \bar{D}^{(L)}$ and organised in weight

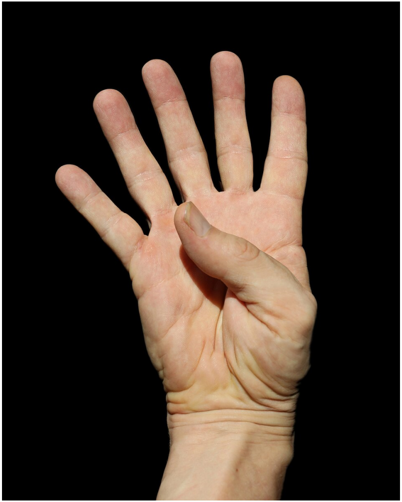
$$\bar{D}_H^{(2)}(x, \mu^2 = -t) = \bar{D}_{H,w=4}^{(2)}(x) + \underbrace{\bar{D}_{H,\beta_0}^{(2)}(x)}_{\text{weight 3}} + \sum_{i\leq 2} \bar{D}_{H,w=i}^{(2)}(x)$$

Result continued

$$\bar{D}^{(2)}_{H,w=4}(x) = 8N_c^2 \Bigg\{ \mathrm{Li}_4\left(\frac{x}{1+x}\right) - \frac{1}{2}\mathrm{Li}_4(-x) + \frac{1}{2}\mathrm{Li}_3(-x)\log(x) - \frac{1}{4}\mathrm{Li}_2(-x)\log^2(x)$$
$$+ \frac{\log^4(x)}{16} + \frac{\log^4(1+x)}{24} + \frac{1}{4}\log^2(x)\log^2(1+x) - \frac{1}{4}\log^3(x)\log(1+x)$$
$$- \frac{1}{6}\log(x)\log^3(1+x) + \zeta_2\left(\frac{15}{8}\mathrm{Li}_2(-x) - \frac{31}{16}\log^2(x) + \frac{31}{8}\log(x)\log(1+x)\right.$$
$$\left. - \log^2(1+x)\right) + \frac{\zeta_3}{8}\log\left(\frac{x}{1+x}\right) + \frac{277}{128}\zeta_4 + i\pi\Bigg[\frac{1}{2}\mathrm{Li}_3(-x) - \frac{1}{2}\mathrm{Li}_2(-x)\log(x)$$
$$+ \frac{1}{4}\log^3(x) - \frac{3}{4}\log^2(x)\log(1+x) + \frac{1}{2}\log(x)\log^2(1+x) - \frac{1}{6}\log^3(1+x)$$
$$- \frac{7}{8}\zeta_2\log\left(\frac{x}{1+x}\right)\Bigg]\Bigg\}. \tag{1}$$

Weight 4 terms written as classical polylogarithms [Duhr 2012].
Universality: same impact factor in $qg \to qH$ and $gg \to gH$.

The five-point amplitudes



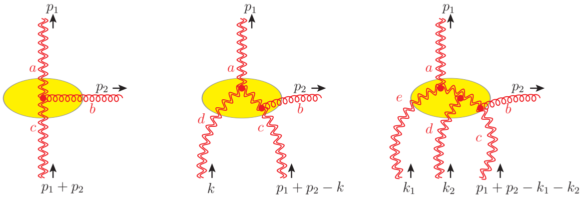
Basile Morin - Wikimedia Commons

Gluon emissions

To LO in the shockwave formalism [Caron-Huot 2013]

$$U(z)a^a(p) = -2g \int [đq][dx] \frac{q.\epsilon(p)}{q^2} e^{iq.(x-z)} e^{-ip.x} \left(U_{\text{ad}}^{ab}(x) T_{1,R}^{b} - T_{1,L}^{a}\right) U(z)$$

Expansion in W-fields [Abreu,De Laurentis,GF,Gardi,Milloy,Vernazza 2025]

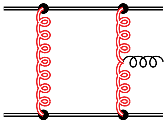


$$W^a(p_1)a^b(p_2) = -2i\, g_s f^{abc} W^c(p_1+p_2) \left[\frac{p_1^\mu}{p_1^2} + \frac{p_2^\mu}{p_2^2}\right] + \mathcal{O}(g_s^2)$$

Multi-Regge amplitudes

The emission vertices allow to predict several components of the $2 \to 3$ amplitudes [Caron-Huot,Chicherin,Henn,Zhang,Zoia 2020; Buccioni, Caola,Devoto,Gambuti 2024], see also [Abreu,De Laurentis,GF,Gardi, Milloy,Vernazza 2025]. Multi-Reggeon exchanges enter the **Regge cut**.

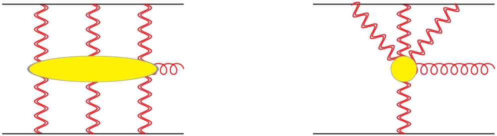
$$\mathcal{M}^{(+,+)}_{1\text{ loop}} = i\pi \frac{\alpha_s}{4\pi}\left[\frac{1}{\epsilon} + \log\left(\frac{\mu^2 |p_4^\perp|^2}{|p_3^\perp|^2 |p_5^\perp|^2}\right) + \mathcal{O}(\epsilon)\right] \mathbf{T}_{(--)} \mathcal{M}^{\text{tree}}_{ij\to i'gj'}$$

$=$ 

The component of signature **odd-odd** has both **pole** and **cut**.

Multi-Reggeon cuts in the odd-odd amplitudes

The cut is the subleading colour of the multi-Reggeon diagrams below



(a) (b)

Using $|p_3^{\perp}|^2 = -r\, z\bar{z}$, $|p_5^{\perp}|^2 = -r(1-z)(1-\bar{z})$ and $r = \frac{s_{34}s_{45}}{s_{12}}$

$$\mathcal{M}^{(-,-)}_{gg\to ggg}\Big|_{\text{cut}} = \frac{(i\pi)^2}{72}\left(\frac{\alpha_s}{\pi}\right)^2 \Bigg[\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\log\left(z\bar{z}(1-z)(1-\bar{z})\right) + 3\,D_2(z,\bar{z}) - \zeta_2$$
$$+\frac{5}{4}\log^2(z\bar{z}) + \frac{5}{4}\log^2\left((1-z)(1-\bar{z})\right) - \frac{1}{2}\log(z\bar{z})\log\left((1-z)(1-\bar{z})\right)\Bigg]\, c^{\mathrm{a}}_{[8,8]} + \ldots$$

All the channels computed in [Abreu,De Laurentis,GF,Gardi,Milloy,Vernazza 2024-25], in agreement with [Buccioni,Caola,Devoto,Gambuti 2024].

The odd-odd amplitude at two loops

$$\left.\mathcal{M}^{(-,-)}_{ij\to igj}\right|_{\alpha_s^2 L^0} = \Big[c_i^{(2)} + c_j^{(2)} + c_i^{(1)} c_j^{(1)}$$

$$+ \left(c_i^{(1)} + c_j^{(1)}\right) v^{(1)} + v^{(2)}\Big]\, \mathcal{M}^{\rm tree}_{ij\to igj} + \left.\mathcal{M}^{(-,-)}_{ij\to igj}\right|_{\rm cut}.$$

At the level of the *finite remainder* $\mathcal{H}$

$$\mathcal{H} = \sum_{j,k} r_k\, M_{mj}\, h_m, \qquad M_{mj} = \text{matrix of rational numbers}$$

- h_m **pentagon functions** [Chicherin,Sotnikov 2020] expanded to $\mathcal{O}(x^2)$ in MRK: $s_{12} = \frac{s}{x^2}$, $s_{45} = \frac{s_1}{x}$, $s_{34} = \frac{s_2}{x}$ (see [Caron-Huot,Chicherin,Henn,Zhang,Zoia 2020])
- r_k **rational coefficients** reconstructed to $\mathcal{O}(x^0)$ using `pyadic` [G. De Laurentis 2023].

The Lipatov vertex at two loops

Universality: same vertex in gg, qg and qq amplitudes.

Function basis: **generalized single-valued hyperlogarithms** $\mathcal{G}$ [O. Schnetz 2021] with alphabet $\{z,\bar{z},1-z,1-\bar{z},z-\bar{z},1-z-\bar{z}\}$, by using `HyperlogProcedures` [O. Schnetz 2021].

- No **branch cuts**: dynamics in the (euclidean) **transverse** plane.

Decomposition in the $\mathcal{N}=4$ sYM part and lower weight

- The vertex in $\mathcal{N}=4$ is **planar**.

$$v^{(2)}_{\text{sYM, Absorp}} = \frac{N_c^2}{32\epsilon^3} + \frac{N_c^2}{\epsilon}\left(\frac{1}{32}\,\mathcal{G}(0,0,z) - \frac{1}{16}\,\mathcal{G}(0,1,z) + \frac{1}{32}\,\mathcal{G}(1,1,z) - \frac{\pi^2}{96}\right) + \frac{N_c^2}{16}\,\Big(\mathcal{G}(0,1-\bar{z},0,z)$$
$$- \mathcal{G}(0,1-\bar{z},1,z) - \mathcal{G}(\bar{z},0,1,z) + \mathcal{G}(\bar{z},1,0,z) - \frac{\pi^2}{12}\,\mathcal{G}(0,z) - \frac{\pi^2}{12}\,\mathcal{G}(1,z) - \mathcal{G}(0,0,0,z) + \mathcal{G}(0,0,1,z)$$
$$- \mathcal{G}(0,1,0,z) + \mathcal{G}(0,1,1,z) + \mathcal{G}(1,0,1,z) - \mathcal{G}(1,1,1,z) - \frac{17\zeta(3)}{6}\Big)$$

The Lipatov vertex at two loops (continued)

$$
\begin{aligned}
v^{(2)}_{\text{sYM, Disp}} = \frac{N_c^2}{32\epsilon^4} &+ \frac{N_c^2}{48\epsilon^2}\Big(3\,\mathcal{G}(0,0,z) - 3\,\mathcal{G}(0,1,z) - 3\,\mathcal{G}(1,0,z) + 3\,\mathcal{G}(1,1,z) - \pi^2\Big) \\
&+ \frac{N_c^2}{192\epsilon}\Big(24\,\mathcal{G}(0,1-\bar{z},0,z) - 24\,\mathcal{G}(0,1-\bar{z},1,z) - 5\pi^2\,\mathcal{G}(0,z) - 5\pi^2\,\mathcal{G}(1,z) \\
&-24\,\mathcal{G}(0,0,0,z) + 12\,\mathcal{G}(0,0,1,z) - 12\,\mathcal{G}(0,1,0,z) + 12\,\mathcal{G}(0,1,1,z) \\
&+12\,\mathcal{G}(1,0,0,z) + 12\,\mathcal{G}(1,0,1,z) + 12\,\mathcal{G}(1,1,0,z) - 24\,\mathcal{G}(1,1,1,z) - 46\zeta(3)) \\
&+ \frac{N_c^2}{11520}\Big(-480\pi^2\,\mathcal{G}(\bar{z},1,z) - 1440\,\mathcal{G}(0,0,1-\bar{z},0,z) \\
&+1440\,\mathcal{G}(0,0,1-\bar{z},1,z) - 1440\,\mathcal{G}(0,1,1-\bar{z},0,z) + 1440\,\mathcal{G}(0,1,1-\bar{z},1,z) \\
&-1440\,\mathcal{G}(0,1-\bar{z},0,0,z) + 1440\,\mathcal{G}(0,1-\bar{z},1,1,z) - 1440\,\mathcal{G}(1,0,1-\bar{z},0,z) \\
&+1440\,\mathcal{G}(1,0,1-\bar{z},1,z) - 1440\,\mathcal{G}(\bar{z},0,1,0,z) + 1440\,\mathcal{G}(\bar{z},0,1-\bar{z},0,z) \\
&-1440\,\mathcal{G}(\bar{z},0,1-\bar{z},1,z) + 1440\,\mathcal{G}(\bar{z},1,0,1,z) + 1440\,\mathcal{G}(\bar{z},1,1-\bar{z},0,z) \\
&-1440\,\mathcal{G}(\bar{z},1,1-\bar{z},1,z) + 720\zeta(3)\,\mathcal{G}(0,z) + 720\zeta(3)\,\mathcal{G}(1,z) + 1140\pi^2\,\mathcal{G}(0,1,z) \\
&-60\pi^2\,\mathcal{G}(1,0,z) + 4320\,\mathcal{G}(0,0,0,0,z) - 2880\,\mathcal{G}(0,0,0,1,z) - 1440\,\mathcal{G}(0,0,1,0,z) \\
&+1440\,\mathcal{G}(0,0,1,1,z) - 1440\,\mathcal{G}(0,1,0,0,z) + 1440\,\mathcal{G}(0,1,0,1,z) \\
&+2880\,\mathcal{G}(0,1,1,0,z) - 2880\,\mathcal{G}(0,1,1,1,z) - 2880\,\mathcal{G}(1,0,0,0,z) \\
&+1440\,\mathcal{G}(1,0,0,1,z) + 2880\,\mathcal{G}(1,0,1,0,z) - 2880\,\mathcal{G}(1,0,1,1,z) \\
&+1440\,\mathcal{G}(1,1,0,0,z) - 2880\,\mathcal{G}(1,1,0,1,z) - 2880\,\mathcal{G}(1,1,1,0,z) \\
&+4320\,\mathcal{G}(1,1,1,1,z) - 53\pi^4\Big)
\end{aligned}
$$

Conclusion and Outlook

The pattern of the high energy logarithms has new features at NNLL

- Exchange of both a reggeized gluon and a Regge cut.

We determine the contribution of the cut using

- the consistency with the evolution in **rapidity**
- the **scheme** choice of including only the N_c-**subleading** part of the multi-Regge diagram in the cut.
  - Universality of the leading-colour contributions.

This scheme was constructed in the context of $2 \to 2$ scattering of quarks and gluons. We applied the same approach to factorise

- the five-point amplitudes in MRK
- the amplitudes for H + jet

Conclusion and Outlook

We obtain new building blocks of high energy factorisation at NNLL

- The Lipatov vertex at two loops
- The Higgs impact factor at two loops

Different partonic channels agree with each other.

Colour plays a major role in the scheme definition.

- **Absence** of cuts in the **planar** limit.

In $\mathcal{N} = 4$ sYM this is even more important

- The three-loop gluon Regge trajectory **agrees** with the planar theory [Henn,Mistlberger 2016; Del Duca,Marzucca,Verbeek 2021]
- The Lipatov vertex at two loops **agrees** with the planar theory.

Now available: 3-loop Lipatov vertex in planar $\mathcal{N} = 4$ sYM.

Thank you for listening!