Timelike-spacelike correspondence and high-energy evolution in QCD

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Partly based on 2508.03794 with Giacomo Brunello, Giulio Crisanti, Mathieu Giroux & Sid Smith

Factorization in QCD and Beyond woskhop, May 7 2026

why / what ?

- Nonlinear small-x evolution (Balitsky-Kovchegov) is a cornerstone of saturation physics

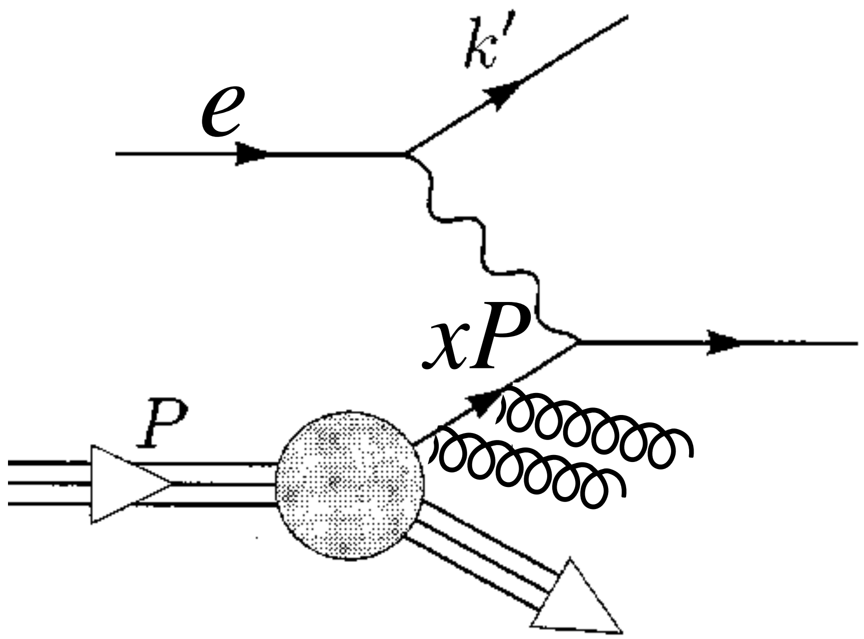
- Some effects first appear at NLO: running coupling, large collinear logs

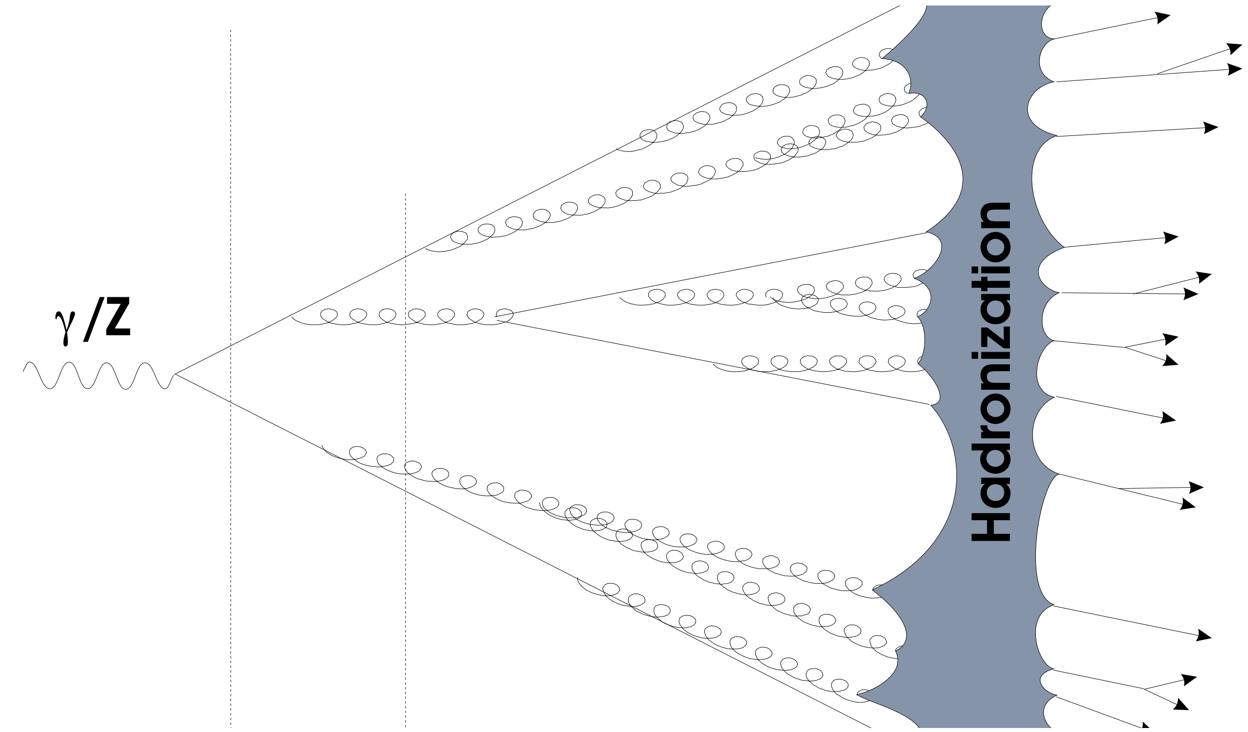
- →NNLO BK tests whether resummations are controlled

- Confirm overall consistency of formalism

Plan:
- Spacelike-timelike correspondence for DGLAP
- Recent+ongoing computation: planar BK @3-loops

I want to first discuss a different relation:





Parton distribution functions | Fragmentation functions

$f_{i/h}(x, \mu)$ | $D_{h/i}(x, \mu)$

⇒ Why are μ evolution equations related (but not equal!) ?

"spacelike" | "timelike"

Claim: consequence of conformal symmetry of pQCD in $d = 4 - 2\epsilon_*$! [unpublished interpretation of Basso+Korchemsky '06]

Wait. QCD is not conformal!

- In MS schemes, β-functions and scaling dimensions in $d = 4 - 2\varepsilon$ only depend on ε at tree-level:

$$\frac{1}{g_s}\beta(g_s, \varepsilon) = -\varepsilon - (ab_0 + a^2 b_1 + \dots) \qquad a \equiv \frac{\alpha_s N_c}{4\pi}$$

$$b_0 = \frac{11}{3} - \frac{2}{3}n^f_{\rm adj} - \frac{1}{6}n^s_{\rm adj}$$

- Better known in statistical physics context
  (why Wilson-Fisher fixed point can be tracked from d=4 to d=3)

- Conformal invariance of pQCD in $d = 4 - 2\epsilon_*(\alpha_s)$ dimensions constrains MS anomalous dimensions

Quantum numbers of DIS

- Recall DGLAP equation:

$$\mu\frac{\partial}{\partial\mu}f_{i/h}(x,\mu) = \int_x^1 \frac{d\xi}{\xi} P^S_{i/j}(x/\xi, \alpha_s(\mu)) f_{j/h}(x,\mu)$$

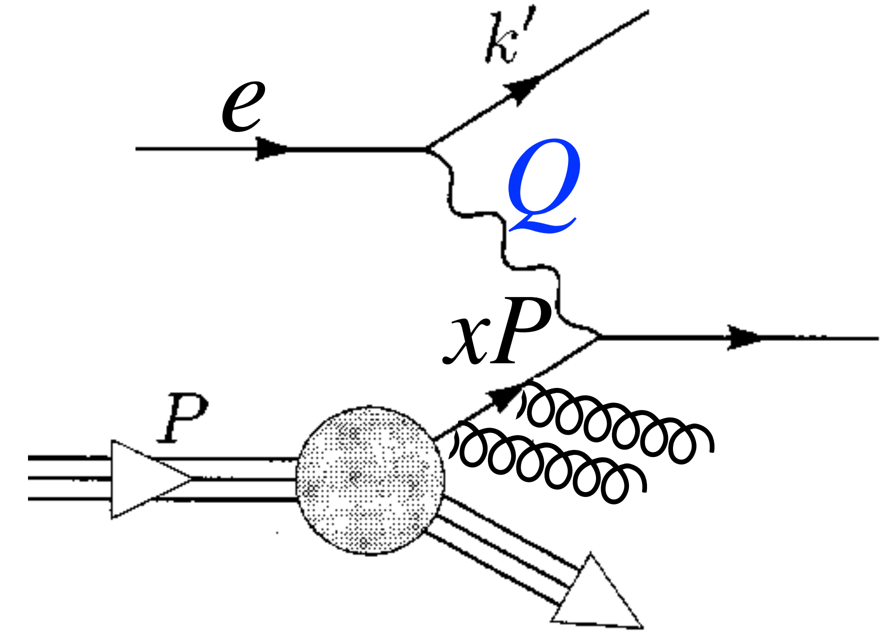
- Diagonalized by moments: $\tilde{f}(N) = \int_0^1 dx\, x^N f(x)$

$$\mu\frac{\partial}{\partial\mu}\tilde{f}_{i/h}(N,\mu) = \gamma^S_{i/j}(N,\alpha_s(\mu))\tilde{f}_{j/h}(N,\mu)$$

- Since $x \propto p^+$, Nth moment has boost eigenvalue $J = N+1$

(ex: momentum sum rule $\int dx\, x f$ has $J=2$)

Quantum numbers of DIS (II)

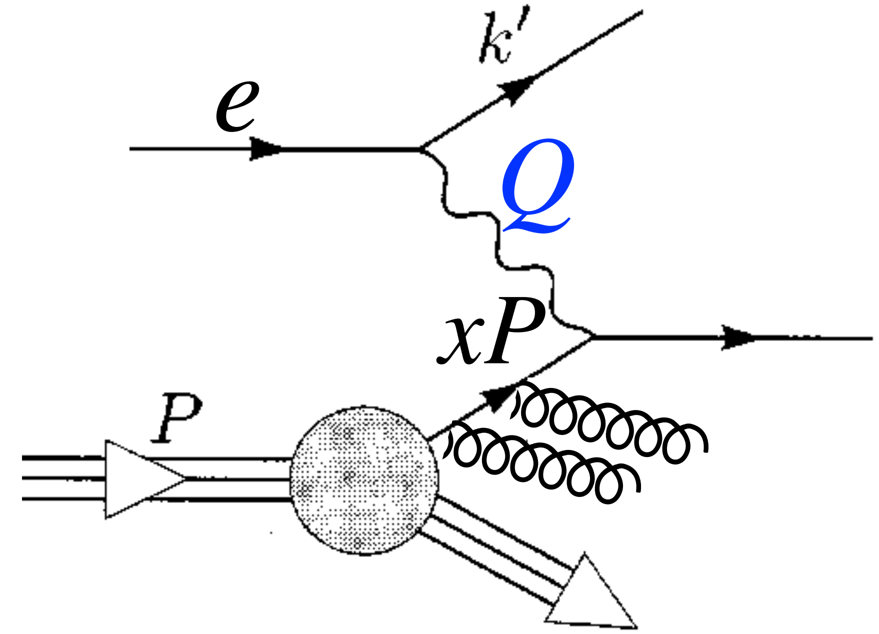


Δ = Scaling dimension: rescale Q^μ

J = spin (boost): rescale Q^+ ⟷ conjugate to energy fraction x

τ = twist : rescale just $Q^\perp$ ⟷ conjugate to transverse density

Quantum numbers of DIS (II)



Δ = Scaling dimension: rescale Q^μ

J = spin (boost): rescale Q^+ ⟷ conjugate to energy fraction x

τ = twist : rescale just $Q^\perp$ ⟷ conjugate to transverse density

- Thus the DGLAP eigenvalue parametrizes scaling dimension of lowest twist (≈ 2) operators as:

$$\gamma^S(N, \alpha_s) = \Delta(J, \alpha_s) - J - (d-2) \qquad \text{with } J = N+1$$

free theory

Fragmentation functions

$$D_{h/i} = \langle 0 | \mathcal{J}_i a_h^\dagger a_h \mathcal{J}_i^\dagger | 0 \rangle$$

- Can't apply standard OPE to 'jet operators' $\mathcal{J}_i^\dagger$ (eg. $\overline{\psi} W$) (hadron number operator stands between them)
- To relate evolution to pQCD operators, zoom in and count partons before hadronization

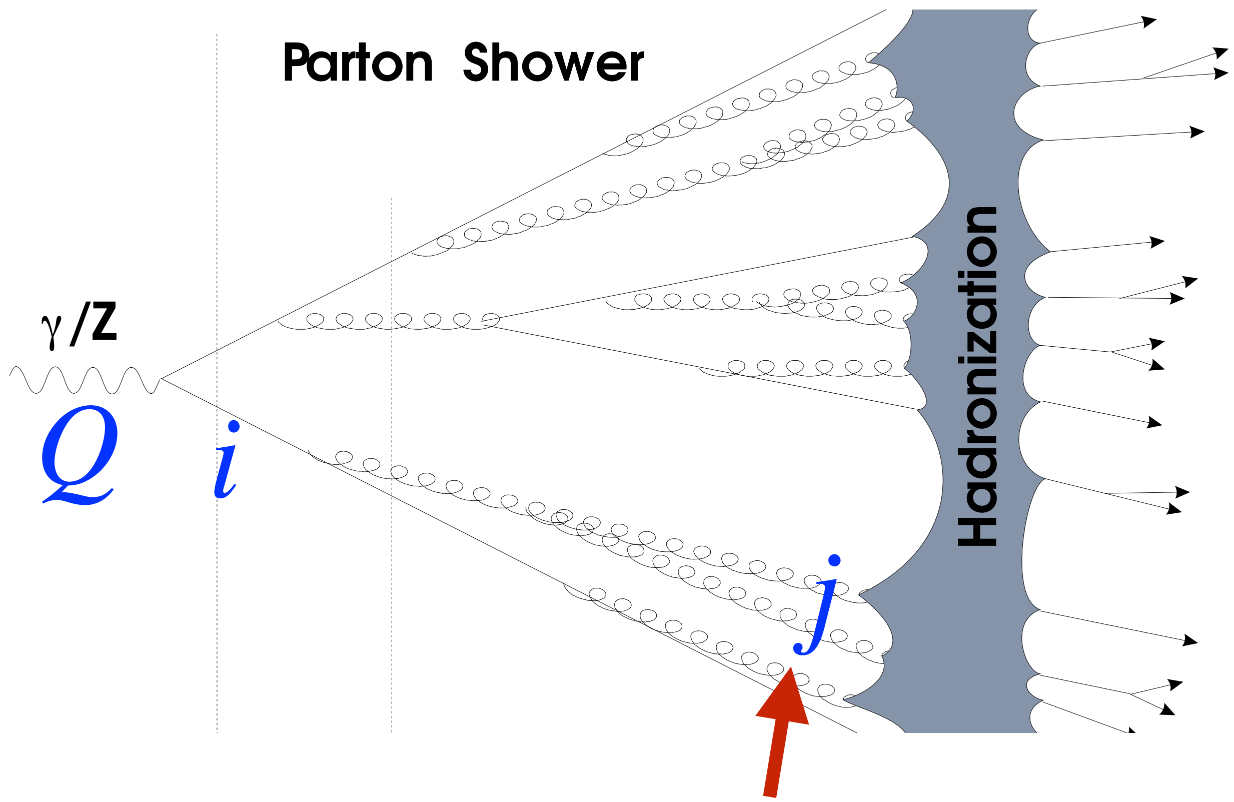
$$D_{j/i}^{\mathrm{pQCD}} = \langle 0 | \mathcal{J}_i N_j \mathcal{J}_i^\dagger | 0 \rangle$$

- Evolution of $\mathcal{J}_i(\cdots)\mathcal{J}_i^\dagger$ thus related to that of parton number N_j operator *inserted at large distances*

Fragmentation: quantum numbers

$$D^{\mathrm{PQCD}}_{j/i} = \langle 0 | \mathcal{J}_i N_j \mathcal{J}^\dagger_i | 0 \rangle$$

Insert N_j at infinity (say along z) to diagonalize dilation and boost



null infinity in pQCD

Again Q is conjugate to twist; momentum fraction ↔ spin

$(N^T)^{\mathrm{th}}$ moment transforms like angular density with extra energy $\sim N^T$

$$J_L = -N^T - (d-2)$$

[image credit: Padilla Aranda PhD thesis '95]

Null infinity in CFT

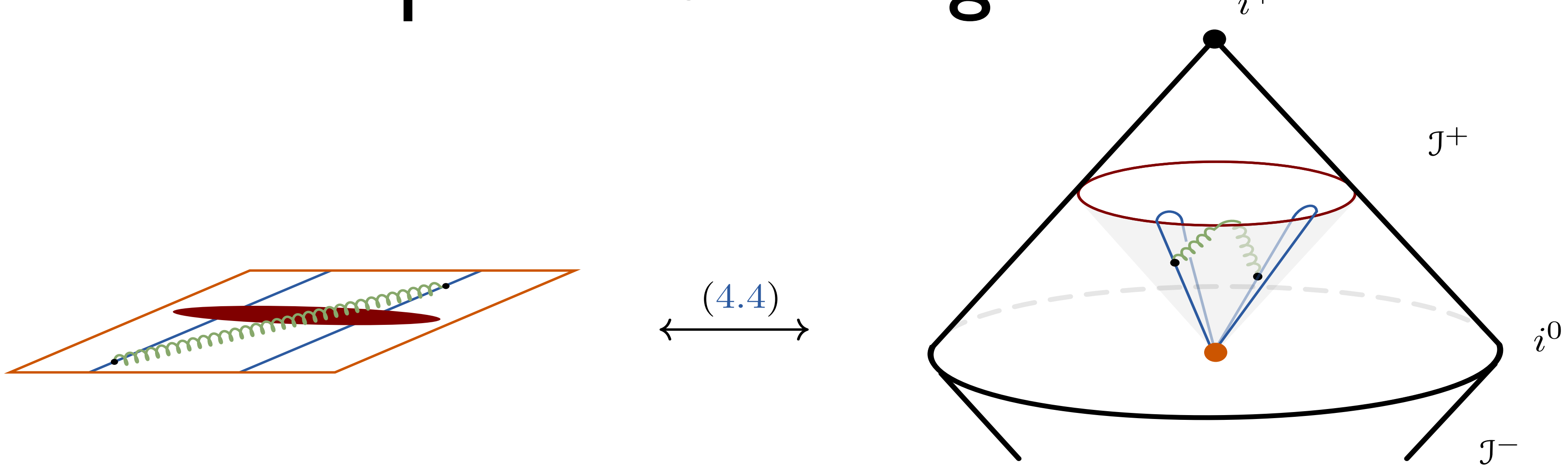
- In a conformal theory, can swap infinity and origin:

$$x^+ \leftrightarrow \ell^2/y^+$$

$$x^- = y^- - y_\perp^2/y^+ \qquad x^\perp = \ell y^\perp/y^+$$

$$dy_\mu dy^\mu = (\ell/x^+)^2 dx_\mu dx^\mu \propto dx_\mu dx^\mu$$

- Transverse plane $x^\perp$ in DIS stereographically maps to 'celestial sphere' θ in fragmentation



[Weigert '03; Hatta '08-…, Hofman& Maldacena '08]

Null infinity in CFT (II)

- Transverse plane $x^\perp$ in DIS stereographically maps to 'celestial sphere' θ in fragmentation
- Boost and dilation quantum numbers get swapped:

$$\Delta_L = 1 - J \qquad J_L = 1 - \Delta$$

[Simmons-Duffin & Kravchuk '18]

- Thus timelike anomalous dimension means:

$$\gamma^T(N^T, \alpha_s) = \Delta(J, \alpha_s) - J - (d-2) \quad \text{when } \Delta + 1 - d = N^T$$

- The fixed quantum number, boost J_L on celestial sphere, = rescaling Δ in inverted frame

Relating γ^S and γ^T

$\gamma^S(N^S, \alpha_s) = \Delta(J, \alpha_s) - J - (d-2)$ when $J - 1 = N^S$

$\gamma^T(N^T, \alpha_s) = \Delta(J, \alpha_s) - J - (d-2)$ when $\Delta + 1 - d = N^T$

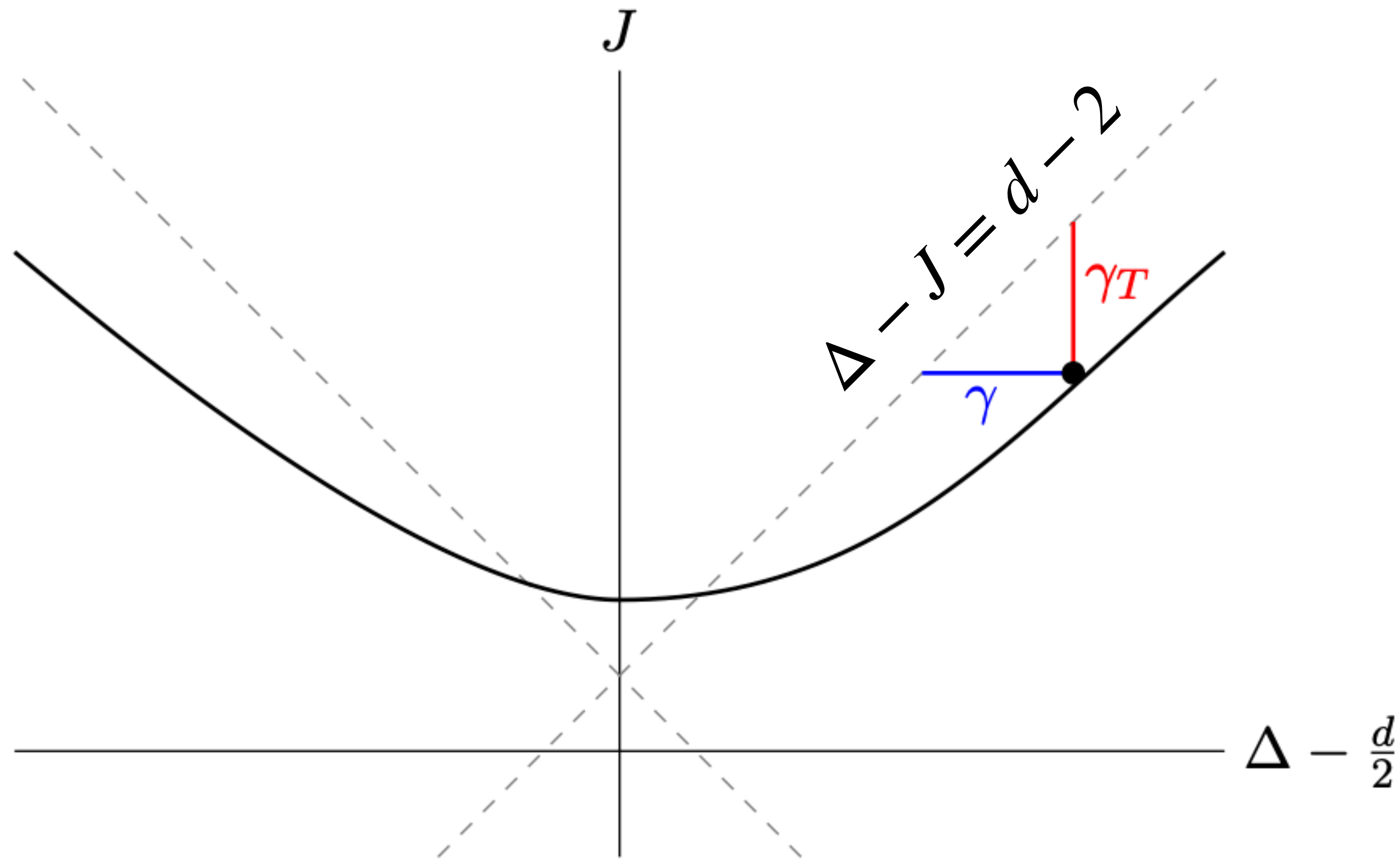
- One is inserted at origin, the other at null infinity; must be same Δ vs J curve at CFT point $\varepsilon = \varepsilon_*(\alpha_s)$!
- Since γ's actually ε-dependent, must agree for any ε!

$$\gamma^T(N^T) = \gamma^S(N^S)\,|_{\Delta(N^S)+1-d=N^T}$$
$$= \gamma^S(N^S)\,|_{\gamma+J-1=N^T}$$

$$\boxed{\gamma^T(N^T) = \gamma^S(N^T - \gamma^T(N^T))}$$

Relating γ^S and γ^T (II)

$$\gamma^T(N) = \gamma^S(N - \gamma^T(N))$$



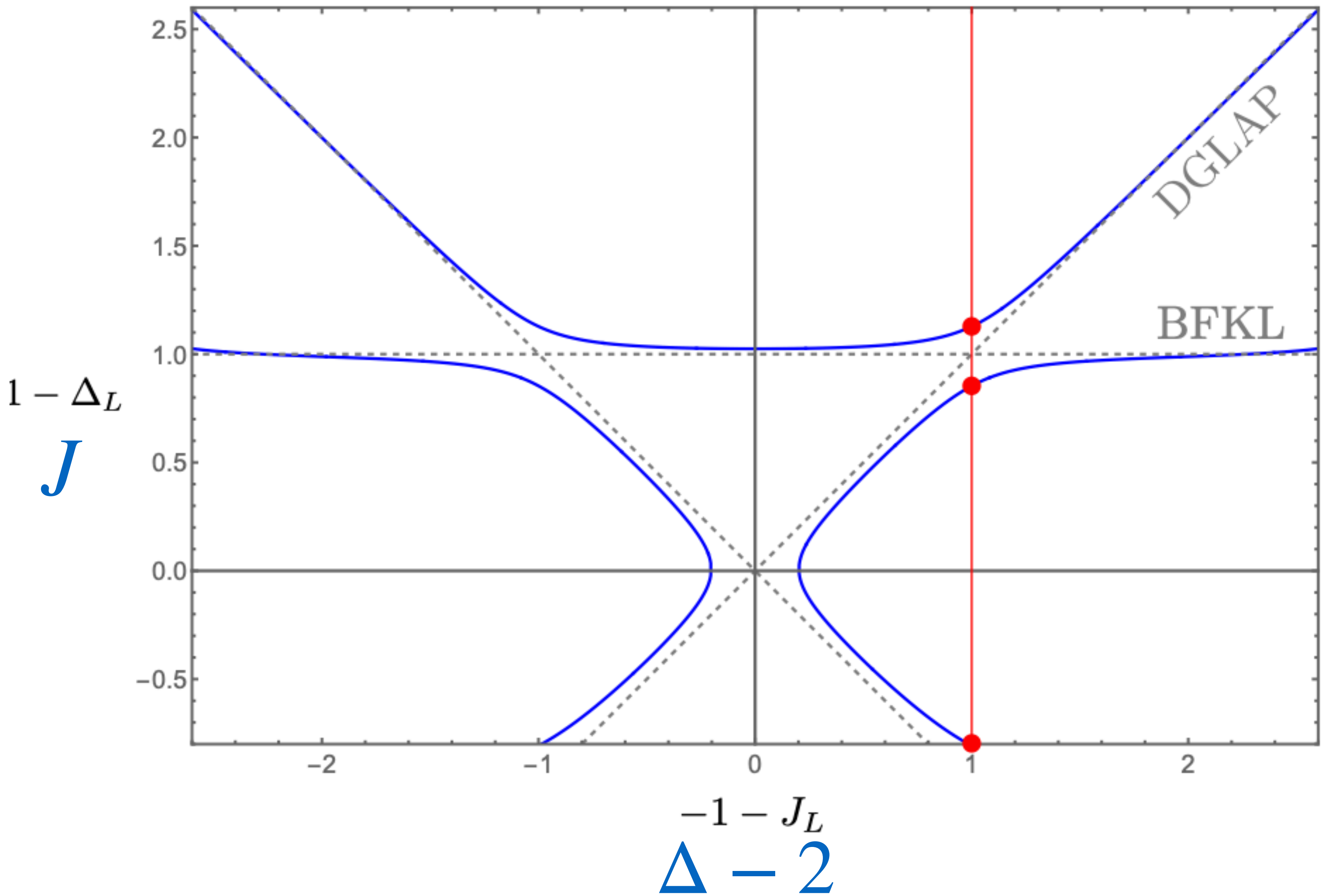
- Verified for QCD eigenvalues (matrix structure?)

$$\gamma^T(N) = \gamma^S(N) - \gamma^S(N)\partial_N \gamma^S(N) + \ldots$$

[Basso+Korchemsky '06]

- We discussed it recently in ϕ^4 theory

[SCH, Kologlu, Kravchuk, Meltzer, Simmons-Duffin '22]



- In gauge theories, curve unites BFKL and DGLAP.
- Soft-collinear limit: $J = 1 \approx (\Delta - 3 \pm \sqrt{(\Delta-3)^2 + 8\alpha_s N_c})/2$

  Includes multiplicity of jets: $\gamma^T(N^T = 1) \propto \sqrt{\alpha_s}$ [Mueller 81]
- Curve predicts other scaling exponents in fragmentation

[see Chang, Chen, Simmons-Duffin, Zhu '25, …]

Comments on ‘reciprocity’

- Besides $\gamma^S \leftrightarrow \gamma^T$, ‘Gribov-Lipatov reciprocity’ also refers to:

$$\gamma(h = (\Delta + J)/2) \text{ admits series in } \frac{1}{h(h-1)}$$

[Dokshitzer+Marchesini ’06]
[Basso+Korchemsky ’06]

- Consequence of CFT crossing equation at large spin; manifested by Lorentzian inversion formula

[Alday+Bissi ’13; SCH ‘17]

- Removes roughly half of harmonic sums in a γ ansatz

Recent+ongoing: planar limit of 3-loop BK

[2508.03794 with Giacomo Brunello,
Giulio Crisanti, Mathieu Giroux & Sid Smith;
+ in progress with Mathieu Giroux]

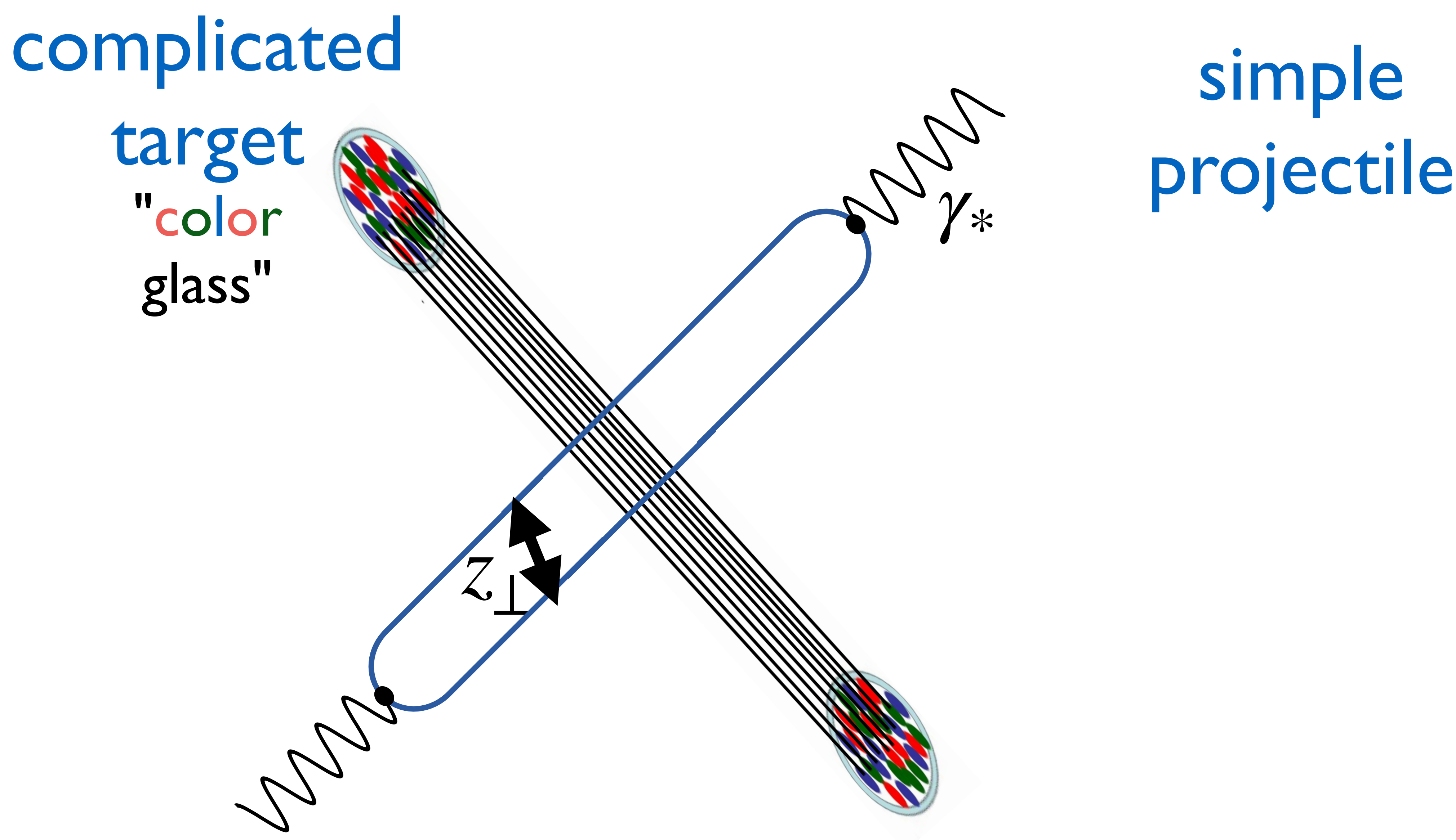
small-x evolution equations

	planar (large N_c) →	Full colors
Linear ↓	planar BFKL	BFKL (/BKP)
Nonlinear	BK	B-JIMWLK

↘ finite x, DGLAP

what to aim for?

recall rapidity factorization (leading power)

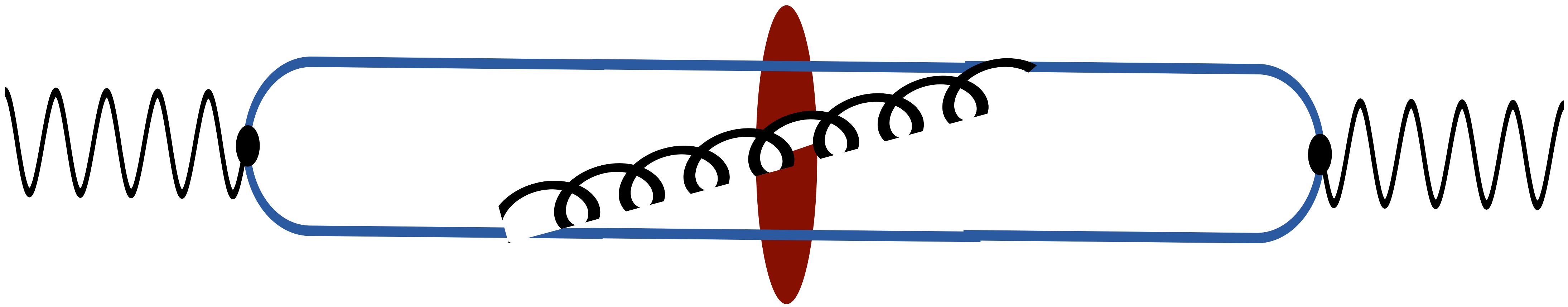


@ leading order in α_s , target → dipole expectation value:

$$\mathcal{M} \propto \langle \text{target} \,|\, U(z_1, z_2) \,|\, \text{target} \rangle$$

Color vs planar

- beyond leading order in α_s , "dipole" mixes with multi-color Wilson line correlators:



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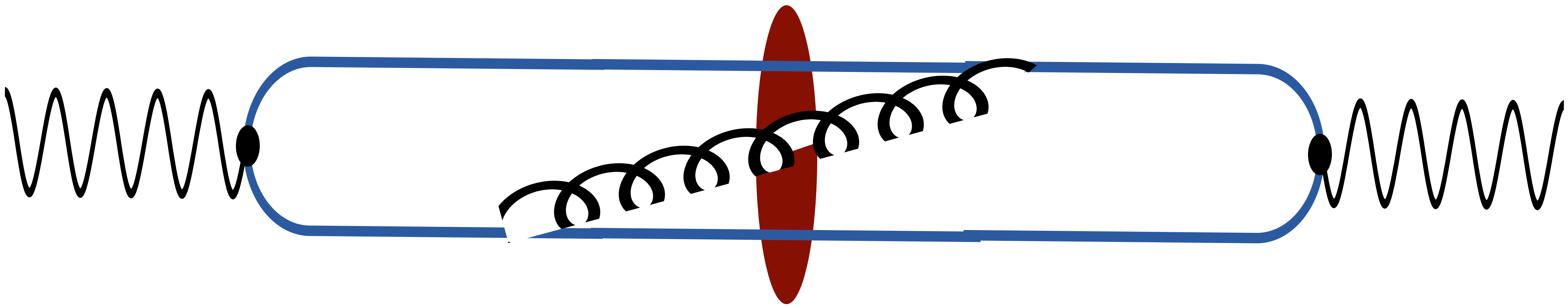
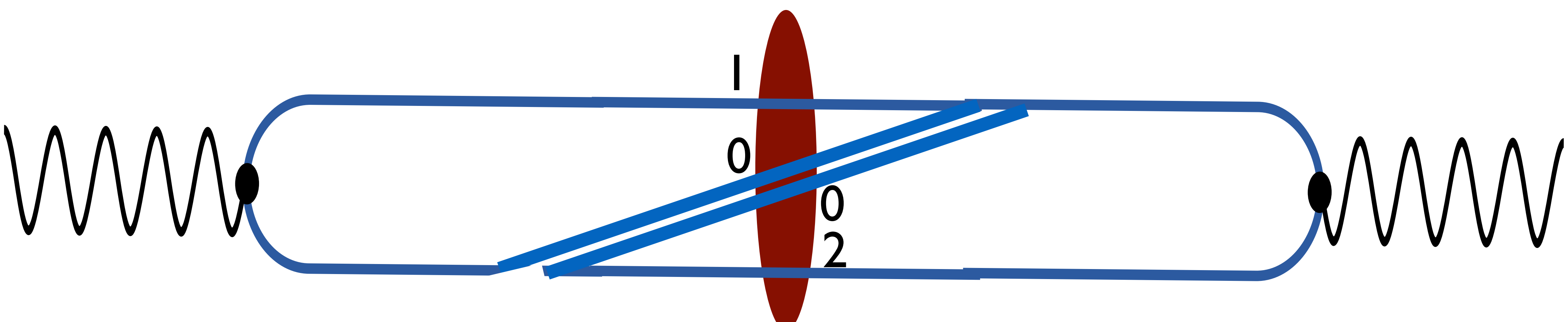


- Planar (large N_c) limit: only products of dipoles!



$$\frac{d}{d\eta}U_{12} = \frac{\alpha_s N_c}{2\pi}\int \frac{d^2 z_0}{\pi}\frac{z_{12}^2}{z_{10}^2 z_{02}^2}\left(U_{10}U_{02} - U_{12}\right)$$

Comments

- Full-color evolution (B-JIMWLK) hard to use.
  NLO ~2013, not simulated yet (?)

- Planar evolution (BK) more tractable.
  NLO ~2007, simulations ~2015 (Balitsky+Chirilli 2007) (Lappi+Mäntysaari 2015)

- Linearized evolution (BFKL): NLO ~1998;
  NNLO gluon trajectory known in QCD;
  NNLO Pomeron (linearized dipole) only in N=4 SYM
  (Gromov+Levkovich-Maslyuk,+Sizov 2015)
  (SCH+Herranen 2016)

- Nonlinearities are interesting: saturation!
  Linear & nonlinear evolution = same shockwave diagrams

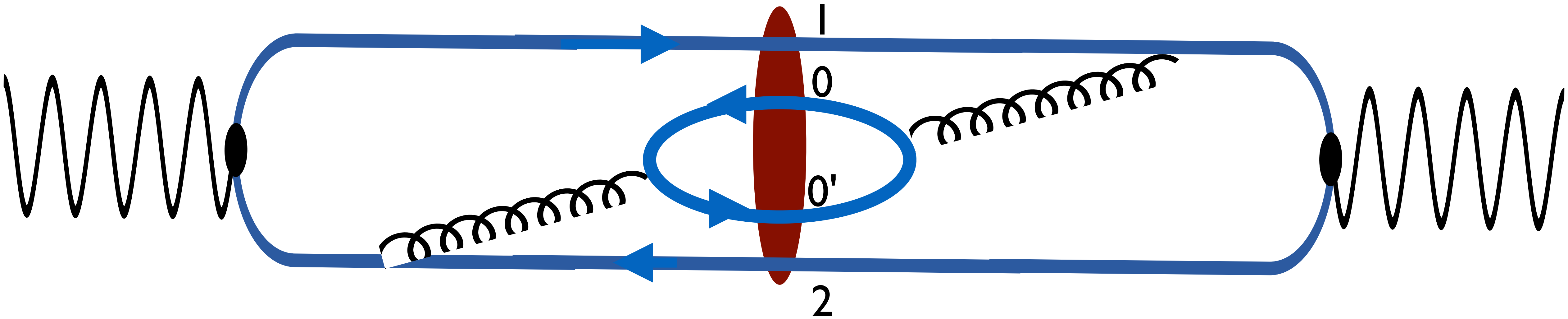
small-x evolution equations

	planar (large N_c) →	Full colors
Linear ↓	planar BFKL simplifications not worthwhile	BFKL (/BKP)
Nonlinear	BK	B-JIMWLK

what we'll consider @ NNLO!

limited use case

"Planar QCD": keep quarks



$$\sim U_{10} U_{0'2}$$

- 't Hooft-Veneziano large-N limit with n_F / N_c fixed
- Same as treating quarks as adjoint $\sim U_{10} U_{00'} U_{0'2}$ then dropping $U_{00'}$ dipole in the end
- Our paper: the $\mathcal{O}(\epsilon)$ part of NLO BK in planar QCD

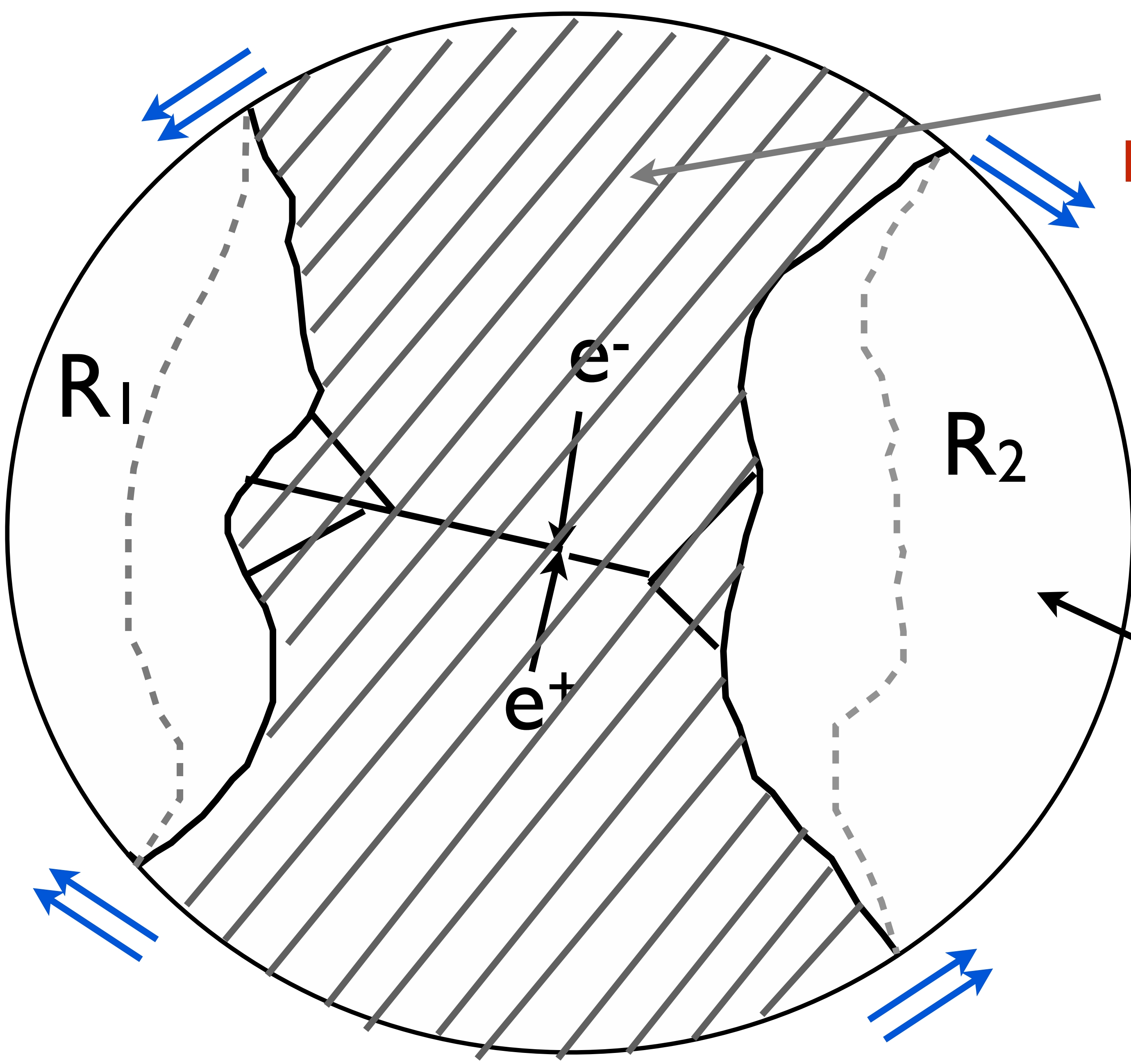
"Timelike-Spacelike": [Mueller '18]
a surprising equivalence

Rapidity evolution (small x amplitude)	⇔	Soft radiation (exclusive cross-section)
Transparent		Allowed region
Opaque		Vetoed region
Rapidity Y		Veto energy
saturation: opaque region grows		saturation: Vetoed region grows

[Weigert '03; Hatta '08-…, Hofman& Maldacena '08]

"Non-global logs"

Q: Cross-section for $e^+e^- \to X$,
w/ less than μ_0 energy outside some region(s) R



'opaque'
radiation vetoed

'transparent'
allow radiation
here

- **Quantitative** equivalence:

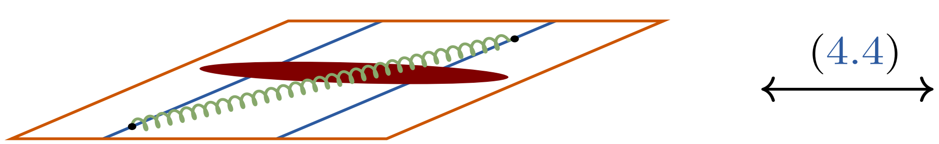
BK: $$\frac{d}{d\eta}U_{12} = \frac{\lambda}{8\pi^2}\int \frac{d^2 z_0}{\pi}\,\frac{z_{12}^2}{z_{10}^2 z_{02}^2}\left(U_{10}U_{02} - U_{12}\right)$$ Rapidity evolution

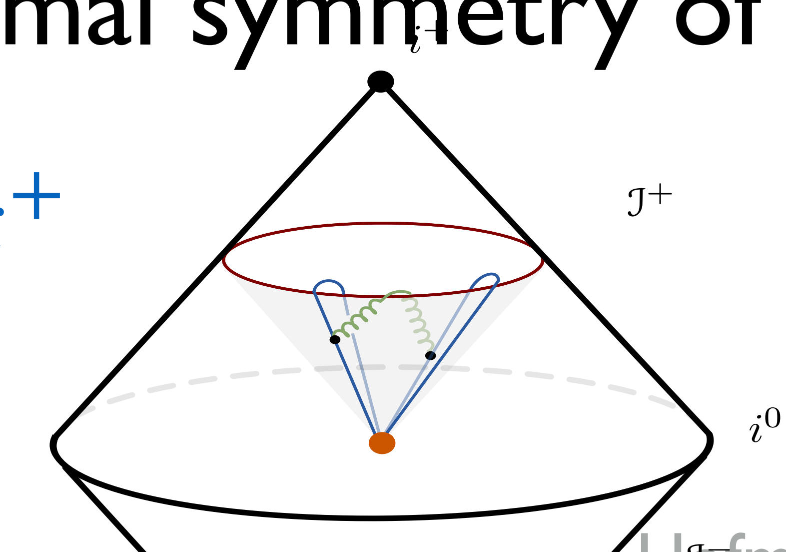
BMS: $$E\frac{d}{dE}U_{12} = \frac{\lambda}{8\pi^2}\int \frac{d^2\Omega_0}{4\pi}\,\frac{\alpha_{12}}{\alpha_{10}\alpha_{02}}\left(U_{10}U_{02} - U_{12}\right)$$ Soft evolution

$$\alpha_{ij} \equiv \frac{1-\cos\theta_{ij}}{2} \to z_{ij}^2 \equiv (z_i - z_j)^2, \qquad \frac{d\Omega}{4\pi} \to \frac{d^2 z}{\pi}$$

- Consequence of conformal symmetry of pQCD:

$x^+ \leftrightarrow 1/x^+$





[Weigert '03; Hatta '08-…, Hofman& Maldacena '08]

Rapidity vs Soft divergences

$H_{\rm NGL}^{\overline{MS}}$ does not depend on ε

$H_{\rm BK}(\epsilon)$ does

$$H_{\rm BK}(\epsilon_* = -\bar{\beta}(a)) = H_{\rm NGL}^{\overline{\rm MS}}$$

(up to scheme)

$$\Rightarrow \quad H_{\rm BK}^{(2)}\Big|_{\epsilon^0} = H_{\rm NGL}^{(2)} + b_0 H_{\rm BK}^{(1)}\Big|_{\epsilon^1}$$

2508.03794

$$H_{\rm BK}^{(3)}\Big|_{\epsilon^0} = H_{\rm NGL}^{(3)} + b_0 H_{\rm BK}^{(2)}\Big|_{\epsilon^1} + b_1 H_{\rm BK}^{(1)}\Big|_{\epsilon^1} - b_0^2 H_{\rm BK}^{(1)}\Big|_{\epsilon^2}$$

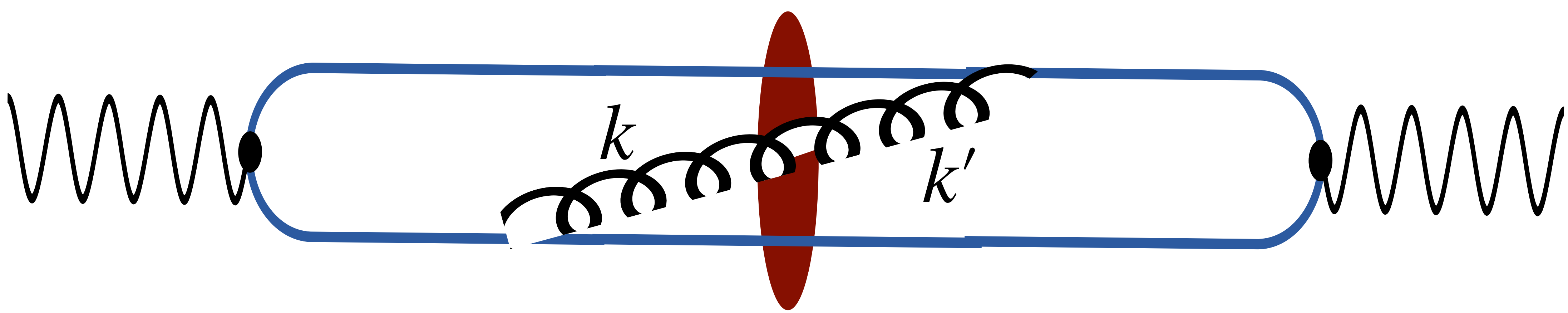
in progress

[for semi-infinite Wilson lines: Vladimirov '16]

- We calculated NLO BK at $\mathcal{O}(\epsilon)$:

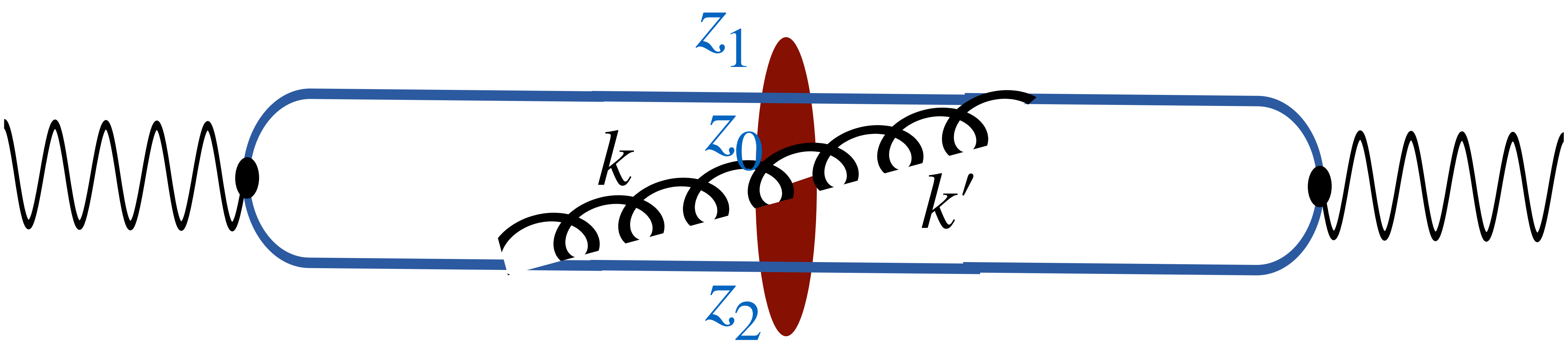
$$H^{(3)}_{\rm BK}\Big|_{\epsilon^0} = H^{(3)}_{\rm NGL} + b_0 H^{(2)}_{\rm BK}\Big|_{\epsilon^1} + b_1 H^{(1)}_{\rm BK}\Big|_{\epsilon^1} - b_0^2 H^{(1)}_{\rm BK}\Big|_{\epsilon^2}$$

- Let me illustrate techniques, starting with LO



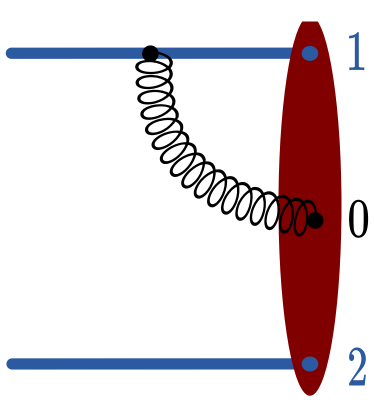
  - Two virtuality integrals dk^-, dk'^-: residues ✔
  - Two Fourier transforms $d^{2-2\epsilon}k_\perp, d^{2-2\epsilon}k'_\perp$
  - One overall dk^+/k^+ (→rapidity log, don't do)

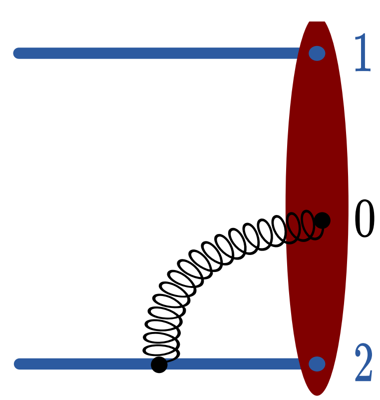
Sample calculations from 2508.03794



- In BK, we fix the transverse position z_i of each particle which crosses the shock

- Can do all Fourier transforms at "amplitude" level

$\mathcal{A}^{(g,0)i}_{[102]}(z_0) =$ 

$+$ 

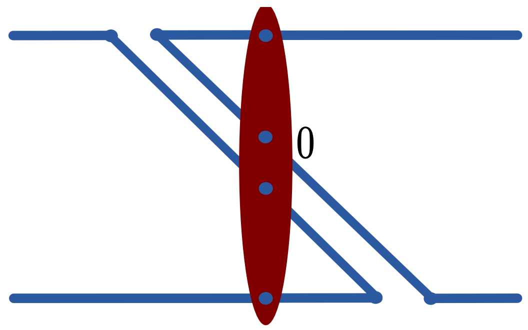
$$= \int \frac{d^{2-2\epsilon}k_\perp}{(2\pi)^{2-2\epsilon}} \frac{2ik^i_\perp}{k_\perp^2}\left(e^{ik_\perp \cdot z_{01}} - e^{ik_\perp \cdot z_{02}}\right)$$

$$= C_\varepsilon \left[\frac{z^i_{10}}{(z^2_{10})^{1-\varepsilon}} - \frac{z^i_{20}}{(z^2_{20})^{1-\varepsilon}}\right]$$

[Brunello,SCH,Crisanti,Giroux&Smith '25]

1-loop BK in any dimension [known previously]

$$\nu\partial_\nu \mathcal{U}_{12} = C_\varepsilon \int_{z_0} \left(\frac{z_{12}^2}{z_{10}^2 z_{02}^2}\right)^{1-\varepsilon} (\mathcal{U}_{10}\mathcal{U}_{02} - \mathcal{U}_{12})\, \tilde{H}_{[102]}$$



- Expand in ϵ:

$$\left(\tilde{H} = \frac{\alpha_s N_c}{4\pi} H^{(1)} + \ldots\right)$$

$$L_\mu \equiv \log\left[\frac{\overline{\mu}^2 z_{12}^2}{4\mathrm{e}^{-2\gamma_\mathrm{E}}}\right]$$

$$\tilde{H}^{(1)}_{[102]} = 2 + 2\varepsilon\left(L_\mu - \frac{z_{01}^2 - z_{02}^2}{z_{12}^2}\log\frac{z_{01}^2}{z_{02}^2}\right) + \varepsilon^2\left(\frac{\pi^2}{6} + L_\mu^2 - 2L_\mu\frac{z_{01}^2 - z_{02}^2}{z_{12}^2}\log\frac{z_{01}^2}{z_{02}^2} + \frac{z_{01}^2 + z_{02}^2}{z_{12}^2}\log^2\frac{z_{01}^2}{z_{02}^2}\right) + \mathcal{O}(\varepsilon^3)$$

- Note that transcendental weight increases with ϵ

correspondence @ NLO

$$H^{(2)}_{\rm BK}\Big|_{\epsilon^0} = H^{(2)}_{\rm NGL} + b_0 H^{(1)}_{\rm BK}\Big|_{\epsilon^1}$$

- Explicitly verified recently [Balitsky& Chirilli '24]

The NLO BK equation in d=4 QCD reads [28, 29]

$$\frac{d}{d\eta}\left[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\right] \overset{d=4}{=} \frac{\alpha_s}{2\pi^2}\int d^2 z_3\, \frac{z_{12}^2}{z_{13}^2 z_{23}^2}\Big\{1 + \frac{\alpha_s}{4\pi} b_0\Big[\ln\frac{z_{12}^2\mu^2}{4} + 2\gamma_E - \frac{z_{13}^2 - z_{23}^2}{z_{12}^2}\ln\frac{z_{13}^2}{z_{23}^2}\Big]$$
$$+ \frac{\alpha_s N_c}{4\pi}\Big(\frac{67}{9} - \frac{\pi^2}{3} - \frac{10 n_f}{9 N_c}\Big)\Big\}\left[\mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_3}\}\mathrm{tr}\{\hat U^\eta_{z_3}\hat U^{\dagger\eta}_{z_2}\} - N_c \mathrm{tr}\{\hat U^\eta_{z_1}\hat U^{\dagger\eta}_{z_2}\}\right] + \frac{\alpha_s^2 N_c}{16\pi^2} K_{\rm conf}$$

In $d = d_*$, circled part canceled by $b_0 H^{(1)}_{\rm BK}\Big|_{\epsilon^1}$!
Rest matches two-loop NGL kernel $H^{(2)}_{\rm NGL}$ [SCH `15]

2-loop Fourier transforms

- Main:

$$\frac{1}{{C_\varepsilon}^2}\int_{k_1,k_2}\frac{\mathrm{e}^{ik_1\cdot z_{01}+ik_2\cdot z_{0'1}}}{(k_1^2+\tau k_2^2)k_2^2}=\frac{\tau^\varepsilon(z_{10}^2)^{2\varepsilon}}{32\varepsilon^2}\tilde{F}\left[\frac{z_{10'}^2}{\tau z_{10}^2}\right]$$

hypergeometric $_2F_1(1-2\varepsilon,1-\varepsilon;2-\varepsilon;-x)$ +...

- Can be obtained using Schwinger parameters → Gaussians → Mathematica Integrate[]

- Better: differential equation $\frac{\mathrm{d}}{\mathrm{d}x}\begin{bmatrix}F_1\\F_2\end{bmatrix}=\varepsilon\begin{bmatrix}\frac{2}{1+x} & 0\\ \frac{1}{1+x} & \frac{1}{x}\end{bmatrix}\cdot\begin{bmatrix}F_1\\F_2\end{bmatrix}$

- Canonical form simplifies $\epsilon\to 0$: [Brunello et al 2311.14432]

$$F_2[x]=\varepsilon\log(1+x)+\varepsilon^2(\log^2(1+x)-\mathrm{Li}_2(-x))+\mathcal{O}(\varepsilon^3)$$

[Henn 2013]

- We need to take derivatives anyway (numerators).

Sample result: double real

$$\tilde{H}^{(2)}_{[100'2]} \equiv \begin{bmatrix} (n^s_{\text{adj}} - 2n^f_{\text{adj}} + 2 - 2\varepsilon(1-\delta))\tilde{H}^{(2)\mathcal{N}=0}_{[100'2]} \\ +(n^f_{\text{adj}} - 4)\tilde{H}^{(2)\mathcal{N}=1}_{[100'2]} \\ +\tilde{H}^{(2)\mathcal{N}=4}_{[100'2]} \end{bmatrix}$$

organize by number of fermions/scalars

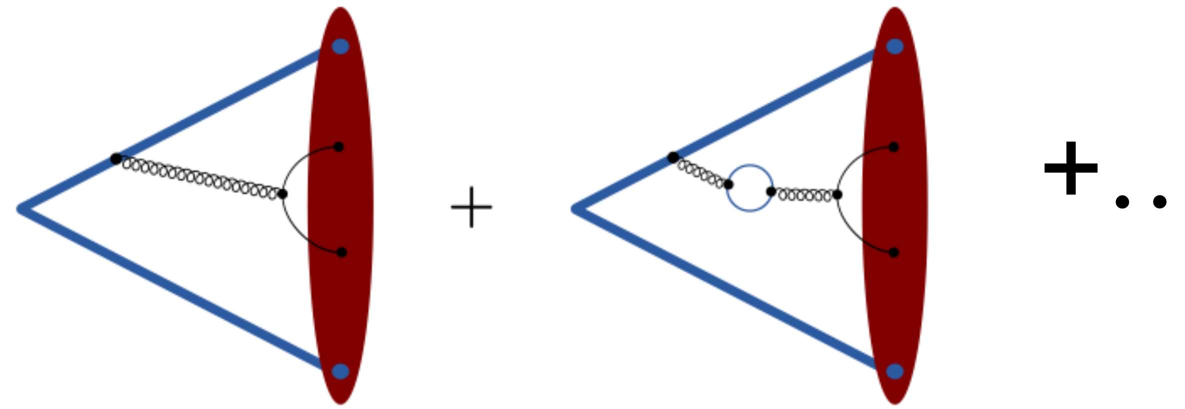
$$\begin{aligned} \frac{z_{12}^2 z_{00'}^2}{z_{10}^2 z_{20'}^2}\tilde{H}^{(2)\mathcal{N}=0}_{[100'2]} &= \left(\frac{v+1}{v-1}\log v - 2 - 8\varepsilon\right)(1-\varepsilon\log uv + 2\varepsilon L_\mu) + \frac{8\varepsilon}{1-v}\mathrm{Li}_2(1-v) \\ &+\varepsilon(1+\mathcal{P})\Bigg\{ z_{20}^2 \frac{z_{20'}^2(z_{10}^2 + z_{10'}^2 - z_{00'}^2) - 2z_{10'}^2 z_{20}^2}{(z_{10}z_{20'}^2 - z_{10'}z_{20}^2)^2} F_1^{\text{long}} \\ &+ \frac{z_{20'}^2(z_{00'}^2(z_{10}^2 + z_{10'}^2) - (z_{10}^2 - z_{10'}^2)^2) - 2z_{02}^2 z_{10'}^2 z_{00'}^2}{z_{10'}^2 z_{20'}^2 z_{00'}^2} B_2\left(\frac{z_{10}^2}{z_{10'}^2}, \frac{z_{00'}^2}{z_{10'}^2}\right) \\ &+ \frac{z_{00'}^2 + z_{10'}^2 - z_{10}^2}{z_{00'}^2}\left[2\mathrm{Li}_2\left(1 - \frac{z_{10}^2}{z_{10'}^2}\right) + \log\frac{z_{10}^2}{z_{10'}^2}\log\frac{z_{00'}^2}{z_{10'}^2}\right] \\ &+ \log\frac{z_{10}^2}{z_{10'}^2}\left[\log\frac{z_{10}^2 z_{00'}^4}{z_{10'}^2 z_{20}^4} - 2\right]\Bigg\} + \mathcal{O}(\varepsilon^2), \end{aligned}$$

...

all explicit logs and Li_2 functions, attached to arXiv submission

three-loop-BK_non-conformal.wl

Large n_f test



- NGL side: self-energy can be resumed to all orders

$$\tilde{H}_{[100'2],\mathrm{NGL}} = \frac{a^2}{|1-\Pi^{(1)}_{\mathrm{ren}}|^2}\left[(n^s_{\mathrm{adj}} - 2n^f_{\mathrm{adj}})\frac{v}{u}\left(\frac{(v+1)\log v}{v-1} - 2\right) + n^f_{\mathrm{adj}}\frac{2v\log v}{v-1}\right]$$

- BK side: two-loop ε^1 and three-loop ε^0 very nasty… yet:

$$\tilde{H}^{(3,0)}_{[100'2]}\Big|_n - b_0\big|_n \times \tilde{H}^{(2,1)]}_{[100'2]}\Big|_n = \tilde{H}^{\mathrm{NGL}(3,0)}_{[100'2]}\Big|_n \quad ✔$$

- Confirms natural identification of conformal invariant and Lorentz invariant cutoffs:

$$\frac{x_{10}^2 x_{02}^2}{x_{12}^2}(k^+)^2 < \nu^2 \quad \longleftrightarrow \quad \frac{s_{01}s_{02}}{s_{12}} < \mu^2$$

What was most *time-consuming* in this project?

- 25%: doing the integrals [essentially done in 2311.14432]
- 75%: making sure we're integrating the right thing! (factors of 2, signs, conventions,…)
  [thanks Balitsky& Chirilli 2007 for including so many details!]

- Ongoing $H^{(3)}_{\mathrm{NGL}}$ calculation:
  - much more standard: integrand = on-shell $|\mathscr{M}|^2, \ldots$
  - can automate integrate over energies & angles
  [SCH+Giroux, in progress]

Conclusions

- Conformal invariance of pQCD anomalous dimension relates DGLAP vs frag. (eigenvalues; matrix structure?)

- Relates NNLO BK to $\mathcal{O}(\epsilon)$ at NLO to NGLs:

$$H_{\rm BK}^{(3)}\Big|_{\epsilon^0} = \underbrace{H_{\rm NGL}^{(3)}}_{\text{in progress}} \underbrace{+ b_0 H_{\rm BK}^{(2)}\Big|_{\epsilon^1} + b_1 H_{\rm BK}^{(1)}\Big|_{\epsilon^1} - b_0^2 H_{\rm BK}^{(1)}\Big|_{\epsilon^2}}_{\text{done}}$$

- Tested for large n_F ($\alpha_s^3 n_F^2$ terms)

- Anticipated further tests:
  - -Collinear limit of linearized eigenvalue $\leftrightarrow$ DGLAP
  - -3-loop gluon Regge trajectory [SCH & Giroux, in progress]

Sample result: single real

$$\tilde{H}^{(2,1)}_{[102]} = 8\frac{94-n^f_{\text{adj}}-13n^s_{\text{adj}}-12\delta}{27} - 24\zeta_3 + c^{(2,0)}\tilde{H}^{(1,1)}_{[102]} + b_0(2\tilde{H}^{(1,2)}_{[102]}+L^2_\mu - \zeta_2) + (L_1-2b_0)L_2$$
$$+ \left(2\zeta_2+\frac{4}{9}(n^s_{\text{adj}}-2n^f_{\text{adj}}+2)\right)(\tilde{H}^{(1,1)}_{[102]}+2L_1-2L_\mu) + \left(\frac{1}{2}\tilde{H}^{(1,1)}_{[102]} - L_\mu\right)(b_0(2L_\mu-L_1) + L_2)$$
$$+ 2\frac{z^2_{12}+z^2_{20}-z^2_{10}}{z^2_{12}}\mathcal{L}_{1,0,1}(\zeta,\bar{\zeta}) + 2\frac{z^2_{12}+z^2_{10}-z^2_{20}}{z^2_{12}}\mathcal{L}_{1,0,1}(1-\zeta,1-\bar{\zeta})\,. \qquad (3.58\text{b})$$

most complicated function is $\text{Li}_3(1-\zeta)$