Resumming anti collinear DGLAP logs in JIMWLK evolution

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JIMWLK

JIMWLK gives evolution of the S-matrix. One way to interpret - evolution of the "Projectile" probability density

$$\frac{d}{dY}\mathcal{W}_P[S] = -H_{\rm JIMWLK}[S, J]\mathcal{W}_P[S]$$

where S- an eikonal scattering matrix of a projectile gluon.
At leading order

$$\begin{aligned} H^{\rm LO}_{\rm JIMWLK} &= \frac{\alpha_s}{2\pi^2}\int_{\mathbf{x},\,\mathbf{y},\,\mathbf{z}} \frac{(\mathbf{x}-\mathbf{z})\cdot(\mathbf{y}-\mathbf{z})}{(\mathbf{x}-\mathbf{z})^2(\mathbf{y}-\mathbf{z})^2} \\ &\times \Big[J^a_L(\mathbf{x})\,J^a_L(\mathbf{y}) + J^a_R(\mathbf{x})\,J^a_R(\mathbf{y}) - 2J^a_L(\mathbf{x})\,S^{ab}(\mathbf{z})\,J^b_R(\mathbf{y}) \Big]. \end{aligned}$$

$J_{L(R)}$ - left (right) rotation operator. Every gluon in the projectile - a factor of S. Color charge $J_{L(R)}$ produces a gluon emission vertex when acting on a projectile parton.
"Dense" target and "dilute" projectile: color fields in the target are correlated on a transverse distance scale Q_T^{-1} and in the projectile on Q_P^{-1}.
Anti collinear: $Q_T \gg Q_P$. If at saturation, $Q_{T(P)}$ are saturation momenta. Otherwise it is just the (inverse) correlation length.

JIMWLK at NLO

At NLO:

$$H^{\rm NLO}_{\rm JIMWLK} = \int_{\mathbf{x},\mathbf{y},\mathbf{z}} K_{JSJ}(\mathbf{x},\mathbf{y},\mathbf{z}) \left[J^a_L(\mathbf{x}) J^a_L(\mathbf{y}) + J^a_R(\mathbf{x}) J^a_R(\mathbf{y}) - 2 J^a_L(\mathbf{x}) S^{ab}(\mathbf{z}) J^b_R(\mathbf{y}) \right]$$
$$+ \int K_{JSSJ}(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}') \left[J^a_L(\mathbf{x}) D^{ad}(\mathbf{z},\mathbf{z}') J^d_R(\mathbf{y}) - \frac{N_c}{2} [J^a_R(\mathbf{x}) J^a_R(\mathbf{y}) + J^a_L(\mathbf{x}) J^a_L(\mathbf{y})] \right]$$
$$+ \int K_{q\bar q}(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}') \left[2\, J^a_L(\mathbf{x})\, D^{ab}_F(\mathbf{z},\mathbf{z}')\, J^b_R(\mathbf{y}) - \frac{1}{2} [J^a_R(\mathbf{x}) J^a_R(\mathbf{y}) + J^a_L(\mathbf{x}) J^a_L(\mathbf{y})] \right]$$
$$+ \ldots$$

with

$$D^{ab}(\mathbf{z}_1,\mathbf{z}_2) = {\rm Tr}[T^a S(\mathbf{z}_1) T^b S^+(\mathbf{z}_2)]\,; \quad D^{ab}_F(\mathbf{z}_1,\mathbf{z}_2) = {\rm Tr}[\tau^a V(\mathbf{z}_1) \tau^b V^+(\mathbf{z}_2)]$$

K_{JSJ} - probability to emit one gluon, K_{JSSJ} - probability to emit two gluons ...

The Kernels

The kernels K :

$$K_{JSJ}(\mathbf{x},\mathbf{y},\mathbf{z}) = \frac{\alpha_s^2(\mu) X\cdot Y}{4\pi^3 X^2 Y^2}\left[\beta_0 \left(\frac{1}{2}\ln[X^2\mu^2] + \frac{1}{2}\ln[Y^2\mu^2]\right) + \left(\frac{67}{9} - \frac{\pi^2}{3}\right) N_c - \frac{1}{9}\right.$$

$$K_{JSSJ}(\mathbf{x},\,\mathbf{y},\,\mathbf{z},\,\mathbf{z}')$$
$$= \frac{\alpha_s^2(\mu)}{16\pi^4}\left[\frac{4}{Z^4} + \left\{2\frac{X^2(Y')^2 + (X')^2Y^2 - 4(X-Y)^2Z^2}{Z^4(X^2(Y')^2 - (X')^2Y^2)} + \frac{(X-Y)^4}{X^2(Y')^2 - (X')^2Y^2}\right.\right.$$
$$\times\left(\frac{1}{X^2(Y')^2} + \frac{1}{Y^2(X')^2}\right) + \frac{(X-Y)^2}{Z^2}\left(\frac{1}{X^2(Y')^2} - \frac{1}{Y^2(X')^2}\right)\right\}\ln\left(\frac{X^2(Y')^2}{(X')^2Y^2}\right)$$
$$\left. - \frac{2I(\mathbf{x},\,\mathbf{z},\,\mathbf{z}')}{Z^2} - \frac{2I(\mathbf{y},\,\mathbf{z},\,\mathbf{z}')}{Z^2}\right] + \widetilde{K}(\mathbf{x},\,\mathbf{y},\,\mathbf{z},\,\mathbf{z}'),$$

μ^2 is the UV cutoff, $X \equiv \mathbf{x} - \mathbf{z}$, *etc.*

$$\beta_0 = \beta_0^g + \beta_0^q \equiv \frac{11N_c - 2N_f}{3}\,.$$

The Large Logarithms

There are three types of large logarithms

1. UV logarithms associated with μ^2.
2. "UV" logarithms associated with emitting two gluons (or $\bar{q}q$) at the same transverse position - a.k.a. "anti collinear logarithms".
   $|\mathbf{z}-\mathbf{z}'| \to 0, \quad K_{JSSJ} \to 1/Z^2$
3. Other logarithms not explicitly written - have to do with "kinematic constraint" or k^- ordering - nothing to say about those.

The first two are generally treated as the choice of scale in running coupling.
More specifically: one subtracts a UV divergent piece from K_{JSSJ} and adds it to K_{JSJ}. Since K_{JSJ} term is of the same form as LO, it is combined with LO to define the running coupling.
But what to subtract exactly? The UV divergent part is unambiguous, but whatever finite part one subtracts in the end defines the scale of the coupling.
E.g. - Balitsky versus Kovchegov-Weigert subtractions

$$K_{JSJ} \to K^B_{JSJ} = \frac{\alpha_s^2(\mu)\beta_0}{16\pi^3}\left\{-\frac{(X-Y)^2}{X^2Y^2}\ln(X-Y)^2\mu^2 + \frac{1}{X^2}\ln Y^2\mu^2 + \frac{1}{Y^2}\ln X^2\mu^2\right\}$$

$$K_{JSJ} \to K^{KW}_{JSJ} = \frac{\alpha_s^2(\mu)\beta_0}{8\pi^3}\frac{X\cdot Y}{X^2Y^2}\left\{\frac{X^2\ln X^2\mu^2 - Y^2\ln Y^2\mu^2}{X^2-Y^2} - \frac{X^2Y^2}{X\cdot Y}\frac{\ln\frac{X^2}{Y^2}}{X^2-Y^2}\right\}.$$

The running coupling

The UV log is of course the same, but the scale to which the coupling runs are quite different. The two expressions taken literally resum to

$$K^B_{JSJ} \to \frac{\alpha_s((X-Y)^2)}{2\pi^2}\frac{X\cdot Y}{X^2Y^2} + \frac{\alpha_s(X^2)}{4\pi^2}\frac{1}{X^2}\left(1-\frac{\alpha_s((X-Y)^2)}{\alpha_s(Y^2)}\right) + \frac{\alpha_s(Y^2)}{4\pi^2}\frac{1}{Y^2}\left(1-\frac{\alpha_s((X-Y)^2)}{\alpha_s(X)}\right),$$

$$K^{KW}_{JSJ} \to \frac{1}{2\pi^2}\frac{\alpha_s(X^2)\alpha_s(Y^2)}{\alpha_s(R^2)}\frac{X\cdot Y}{X^2Y^2},$$

where, in the latter

$$R^2 = \sqrt{X^2\,Y^2}\left(\frac{Y^2}{X^2}\right)^{\Theta/2}, \qquad \Theta = \frac{X^2+Y^2}{X^2-Y^2} - 2\frac{X^2Y^2}{X\cdot Y}\frac{1}{X^2-Y^2}.$$

There is a lot of arbitrariness in the subtraction.

Our point: the physics here is not just the scale setting, but also (and mostly) of anti collinear DGLAP.

A closer look.

Let's look closer to understand where do the UV logs ($\ln \mu^2$) come from. K_{JSJ} alone (a single gluon emission) gives

$$K_{JSJ}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \frac{\alpha_s^2(\mu) X \cdot Y}{4\pi^3 X^2 Y^2} \beta_0 \left(\ln \mu^2 + ... \right)$$

This is twice the factor needed for the renormalization of α_s in the LO term.

Why is that?

Because running coupling is the matrix element between **physical** (or dressed) gluon states, while K_{JSJ} is calculated for bare gluons.

To relate the two multiply by the wave function renormalization factor

$$A_\mu^Q(x) = \mathcal{Z}^{-1/2}(Q) A_\mu(x)\,; \quad \mathcal{Z}^{1/2}(Q^2) = 1 + \frac{\alpha_s}{8\pi} \beta_0 \ln \frac{Q^2}{\mu^2}\,.$$

Here Q^2 is the scale at which the "renormalized" field is defined in QFT as usual, or transverse "resolution scale".

The Dressed Gluon

Technically: we have to multiply K^{LO} by $\mathcal{Z}^{1/2}$ to express H_{JIMWLK} in terms of the renormalized gluon operator.

$$K_{JSJ} \rightarrow \frac{\alpha_s^2 X \cdot Y}{4\pi^3 X^2 Y^2} \left(\beta_0 \left[\frac{1}{2} \ln[X^2\mu^2] + \frac{1}{2} \ln[Y^2\mu^2] - \frac{1}{2} \ln \frac{\mu^2}{Q^2} \right] + ... \right)$$

This is the correct UV logarithm.

What about the divergence in K_{JSSJ}? It **should also disappear** if we express the H_{JIMWLK} in terms of the physical dressed gluon amplitudes.

So far we have only included the simple multiplicative wave function renormalization factor into the "redefinition" of the dressed gluon. But at NLO the physical dressed gluon state contains a two gluon (and a quark-anti quark) component due to gluon splitting.

We need to include those in the definition of the dressed gluon scattering amplitude - and then we should be good.

The Dressed Gluon S-matrix (forget about quarks)

Of course Q^2 is just the transverse resolution with which we resolve the dressed gluon.

The S-matrix of the gluon state dressed at order α_s with resolution Q then is

$$\mathbb{S}^{ab}_Q(\mathbf{z}) = S^{ab}(\mathbf{z}) + \frac{\alpha_s}{2\pi^2}\int_0^1 \frac{d\xi}{\xi_+(1-\xi)_+}\left(\xi^2+(1-\xi)^2+\xi^2(1-\xi)^2\right)$$

$$\times \int_{\mu^{-1}<|Z|<Q^{-1}} \frac{d^2Z}{Z^2}\Big[D^{ab}(\mathbf{z}+(1-\xi)Z, \mathbf{z}-\xi Z) - N_c S^{ab}(\mathbf{z})\Big]$$

$$D^{ab}(\mathbf{z}_1,\mathbf{z}_2) = \mathrm{Tr}[T^a S(\mathbf{z}_1) T^b S^+(\mathbf{z}_2)]$$

This follows from the $Z \to 0$ asymptotics of K_{JSSJ}.

For simplicity, instead of exact kinematics we can use $\xi \to 1/2$:

$$\mathbb{S}^{ab}_Q(\mathbf{z}) = [1 + \frac{\alpha_s\beta_0^g}{4\pi}\ln\frac{\mu^2}{Q^2}]S^{ab}(\mathbf{z}) - \frac{\alpha_s\beta_0^g}{4\pi^2 N_c}\int_{|Z|<Q^{-1}}\frac{d^2Z}{Z^2}D^{ab}(\mathbf{z}+Z/2, \mathbf{z}-Z/2)$$

The linear in S term - the "wave function renormalization". In DIS parlance this is the "virtual" DGLAP log. The quadratic term gives the two gluon component of the dressed gluon - the "real" DGLAP log.

The idea of resummation

We should express H_{JIMWLK} in terms of the dressed gluon amplitudes. This should resum all logarithms associated with anticollinear DGLAP splittings.

Technically: express S in terms of $\mathbb{S}_Q$ perturbatively, and substitute into H^{LO}_{JIMWLK} - the result in terms of $\mathbb{S}_Q$ is the resummation of DGLAP logs and should not contain any explicit DGLAP logarithms. And then we have to think what is Q?

One issue to understand. $\mathbb{S}_Q$ as defined does not know anything about any neighbors: just like DGLAP - gluons evolve (split) independently. But if there are sources nearby the splitting probability is modified.

I.e.: the distance to the closest source is an IR cutoff on the size of the pair (again follows from K_{JSSJ}).
Thus define cascade limited by the nearest source:

$$\bar{Q}^2 = \max\left\{Q^2, \frac{1}{X^2}, \frac{1}{Y^2}\right\}.$$

Perturbatively

$$H_{\rm JIMWLK} = H_Q^{J\mathbb{S}J} + H_Q^{J\mathbb{S}\mathbb{S}J} + \dots$$

where

$$H_Q^{J\mathbb{S}J} = \int_{\mathbf{x},\mathbf{y},\mathbf{z}} \frac{\alpha_s}{2\pi^2}\frac{X\cdot Y}{X^2Y^2}$$

$$\times\Bigg\{\left(1+\frac{\alpha_s\beta_0^g}{8\pi}\left[\ln[X^2\mu^2]+\ln[Y^2\mu^2]\right]\right)\left[J_L^a(\mathbf{x})J_L^a(\mathbf{y})+J_R^a(\mathbf{x})\,J_R^a(\mathbf{y})-2J_L^a(\mathbf{x})\mathbb{S}_{\bar{Q}}^{ab}(\mathbf{z})J_R^b(\mathbf{y})\right]$$

$$+\frac{\alpha_s\beta_0^g}{8\pi}(\ln X^2\bar{Q}^2+\ln Y^2\bar{Q}^2)\left[J_L^a(\mathbf{x})\,J_L^a(\mathbf{y})\;+\;J_R^a(\mathbf{x})\,J_R^a(\mathbf{y})-2J_L^a(\mathbf{x})\,\mathbb{S}_{\bar{Q}}^{ab}(\mathbf{z})\,J_R^b(\mathbf{y})\right]\Bigg\}.$$

$$H_Q^{J\mathbb{S}\mathbb{S}J} = \int_{\mathbf{x},\,\mathbf{y},\mathbf{z},\mathbf{z}'} K_{JSSJ}(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}')J_L^a(\mathbf{x})\mathbb{D}_{\bar{Q}}^{ad}(\mathbf{z},\mathbf{z}')J_R^d(\mathbf{y}) - \frac{\alpha_s^2\beta_0^g}{4\pi^3}\int_{\mathbf{x},\,\mathbf{y},\mathbf{z}}\frac{X\cdot Y}{X^2Y^2}\mathbb{D}_{\bar{Q}}^{ab}(\mathbf{z})J_L^a(\mathbf{x})J_R^b(\mathbf{y})$$

$$-\left[\frac{N_c}{2}\int_{\mathbf{x},\,\mathbf{y},\mathbf{z},\mathbf{z}'}K_{JSSJ}(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{z}') - \frac{\alpha_s^2\beta_0^g}{8\pi^3}\int_{\mathbf{x},\,\mathbf{y},\mathbf{z}}\frac{X\cdot Y}{X^2Y^2}\,\ln\frac{\mu^2}{\bar{Q}^2}\right][J_R^a(\mathbf{x})J_R^a(\mathbf{y})+J_L^a(\mathbf{x})J_L^a(\mathbf{y})]\,.$$

Note real-virtual cancellations: for $X\to 0$, we have $\bar{Q}=1/|X|$, $\mathbb{S}_{\bar{Q}}(\mathbf{z})=S(\mathbf{z})\simeq S(\mathbf{x})$, and the real and virtual terms cancel for points $\mathbf{z}$ such that $|X|<Q_T^{-1}$.

The Resummation - the scale of α_s

So far everything was done to one loop order - it was just rewriting H_{JIMWLK} in terms of perturbatively defined $\mathbb{S}_Q$. Our goal now is to resum the logs that enter $\mathbb{S}_Q$ - a.k.a DGLAP logs.

First we deal with the running coupling. This is easy - observe that to $O(\alpha_s^2)$ we can write

$$H_Q^{J\mathbb{S}J} = \int_{\mathbf{x},\mathbf{y},\mathbf{z}} \frac{\alpha_s(\Lambda^2)}{2\pi^2} \frac{X \cdot Y}{X^2 Y^2} \Big[J_L^a(\mathbf{x})\, J_L^a(\mathbf{y}) + J_R^a(\mathbf{x})\, J_R^a(\mathbf{y}) - 2 J_L^a(\mathbf{x})\, \mathbb{S}_{\bar{Q}}^{ab}(\mathbf{z})\, J_R^b(\mathbf{y}) \Big] .$$

with

$$\Lambda^2 = \Big[X^2 Y^2 \bar{Q}^2 \Big]^{-1}$$

This clearly suggests that Λ is the appropriate scale choice for the coupling constant.

Jumping the gun - with our eventual choice of Q this is $\Lambda = min\{1/X^2, 1/Y^2\}$

And now for the DGLAP logs.

The Resummation - what is Q?

In principle Q is an arbitrary scale - akin to factorization scale.

Varying Q does not change physics, but it does change the place where the large logarithms live. Choosing a convenient scale does a lot for simplifying a problem.

E.g. say we choose $Q = Q_T$. Since Q_T is the target correlation length, $S(\mathbf{z}_1) = S(\mathbf{z}_2)$ for $|\mathbf{z}_1 - \mathbf{z}_2| < Q_T^{-1}$ and therefore $\mathbb{S}_{Q_T} = S$.
So at $Q = Q_T$ there is very little dressing of the gluon, but H_{JIMWLK} contains large logarithms $\ln X^2 Q_T^2$.

What if we choose $Q = Q_P$? Then the dressed gluon is very different from the bare one $\mathbb{S}_{Q_P} \neq S$. But the typical distance between the charges in the projectile is Q_P^{-1}, so there are no large logs in H_{JIMWLK}: $\ln X^2 Q_P^2 \sim 1$!

H_{JIMWLK} is like PDF in DGLAP: at coarse resolution it does not contain logs, but at fine resolution it does. $\mathbb{S}_Q$ is like the "hard part" - at fine resolution it is perturbative, but at coarse resolution it needs to be resummed.

Resumming DGLAP logs I

The idea is: choose Q so that all the logs reside in $\mathbb{S}_Q$, while H_{JIMWLK} is log free.

We start with H_{JIMLWK} with $Q = Q_P$. It has no large logs, so $H_{JIMWLK}[\mathbb{S}_{Q_P}] \approx H^{LO}_{JIMWLK}[\mathbb{S}_{Q_P}]$.
We solve for $\mathbb{S}_{Q_P}$ in terms of $\mathbb{S}_{Q_T} = S$, and substitute the solution into H^{LO}_{JIMWLK}.

The DGLAP-like evolution for $\mathbb{S}_Q$ is the perturbative definition promoted to differential equation:

$$\mathbb{S}^{ab}_Q(\mathbf{z}) = [1 + \frac{\alpha_s \beta^g_0}{4\pi} \ln \frac{\mu^2}{Q^2}] S^{ab}(\mathbf{z}) - \frac{\alpha_s \beta^g_0}{4\pi^2 N_c} \int_{|Z|<Q^{-1}} \frac{d^2 Z}{Z^2} D^{ab}(\mathbf{z} + Z/2, \mathbf{z} - Z/2)$$

becomes

$$\frac{\partial}{\partial \ln Q^2} \mathbb{S}^{ab}_Q(\mathbf{z}) = -\frac{\alpha_s \beta^g_0}{4\pi} \left[\mathbb{S}^{ab}_Q(\mathbf{z}) - \frac{1}{N_c} \int \frac{d\phi}{2\pi} \left(\mathbb{D}^{ab}_Q(\mathbf{z} + \frac{1}{2} Q^{-1} \mathbf{e}_\phi, \mathbf{z} - \frac{1}{2} Q^{-1} \mathbf{e}_\phi) \right) \right]$$

where $\mathbf{e}_\phi$ is the radial unit vector at ϕ.
The procedure is equivalent to solving RG equation - independence of Q^2

$$\frac{d}{d \ln Q^2} H_{\rm JIMLWK} = \frac{\partial H}{\partial \ln Q^2} + \int_u \left[\frac{\delta H}{\delta \mathbb{S}_Q(u)} \frac{\partial \mathbb{S}_Q(u)}{\partial \ln Q^2} \right] = 0 \, .$$

Resumming DGLAP logs II

The upshot is:

1. We evolve $\mathbb{S}_Q$ via the "DGLAP" equation

$$\frac{\partial \mathbb{S}_Q^{ab}(\mathbf{z})}{\partial \ln Q^2} = -\frac{\alpha_s \beta_0^g}{4\pi}\left[\mathbb{S}_Q^{ab}(\mathbf{z}) - \frac{1}{N_c}\int \frac{d\phi}{2\pi}\left(\mathbb{D}_Q^{ab}(\mathbf{z} + \frac{1}{2}Q^{-1}\mathbf{e}_\phi, \mathbf{z} - \frac{1}{2}Q^{-1}\mathbf{e}_\phi)\right)\right]$$

2. Initial condition $\mathbb{S}_{Q_T}(\mathbf{z}) = S(\mathbf{z})$
3. Stop evolution at $Q(\mathbf{z}) = Q_P$.
4. Substitute the solution into H^{LO}_{JIMWLK}

A logical improvement - instead of Q_P, which is an average quantity which depends on the JIMWLK evolution interval Y, we use $Q^2 = max(\frac{1}{X^2}, \frac{1}{Y^2})$.

This ensures that for any given configuration the transverse size of the DGLAP cascade is cut off by the position of the source closest to the gluon which is being dressed.

What does this resummation do?

Solution in the saturation regime

The Dense (saturation) regime is easy. Here $\mathbb{S}_Q$ is small - thus we can neglect higher powers of S in the equation. Then only the virtual term in the DGLAP evolution is important.

$$\frac{\partial \mathbb{S}_Q(\mathbf{z})}{\partial \ln Q^2} = -\frac{\alpha_s \beta_0^g}{4\pi}\, \mathbb{S}_Q(\mathbf{z})$$

with the simple scaling solution

$$\mathbb{S}_Q(\mathbf{z}) = \left[\frac{Q_T^2}{Q^2}\right]^{\frac{\alpha_s \beta_0^g}{4\pi}} \mathbb{S}_{Q_T}(\mathbf{z}) \approx \left[\frac{Q_T^2}{Q^2}\right]^{\frac{\alpha_s \beta_0^g}{4\pi}} S(\mathbf{z})\,.$$

And the resummed Hamiltonian:

$$H^{\rm resummed}_{\rm JIMWLK} = \int_{\mathbf{x},\mathbf{y},\mathbf{z}} \frac{\alpha_S(\Lambda)^2}{2\pi^2}\frac{X\cdot Y}{X^2 Y^2}\Bigg\{ J_L^a(\mathbf{x})\, J_L^a(\mathbf{y}) + J_R^a(\mathbf{x})\, J_R^a(\mathbf{y})$$

$$-2J_L^a(\mathbf{x})\left[\theta\left(Q_T - Q(\mathbf{z})\right)\left[\frac{Q_T^2}{Q^2(\mathbf{z})}\right]^{\frac{\alpha_s \beta_0^g}{4\pi}} + \theta\left(Q(\mathbf{z}) - Q_T\right)\right] S^{ab}(\mathbf{z})\, J_R^b(\mathbf{y}) \Bigg\}$$

Take note.

Isn't it strange?

$$\mathbb{S}_Q(\mathbf{z}) = \left[\frac{Q_T^2}{Q^2}\right]^{\frac{\alpha_s\beta_0}{4\pi}} S(\mathbf{z})\,.$$

For smaller Q the scattering matrix is larger. Smaller Q means larger transverse size of the dressed gluon, larger S-matrix means smaller scattering amplitude. So a larger state scatters weaker?

How did this happen? Because the "splitting function" does not contain the contribution of the soft pole:

$$\mathbb{S}_Q^{ab}(\mathbf{z}) = S^{ab}(\mathbf{z}) + \frac{\alpha_s}{2\pi^2}\int_0^1 \frac{d\xi}{\xi_+(1-\xi)_+}\left(\xi^2+(1-\xi)^2+\xi^2(1-\xi)^2\right)$$
$$\times \int_{\mu^{-1}<|Z|<Q^{-1}} \frac{d^2Z}{Z^2}\left[D^{ab}(\mathbf{z}+(1-\xi)Z, \mathbf{z}-\xi Z) - N_c S^{ab}(\mathbf{z})\right]$$

And

$$\int_0^1 \frac{d\xi}{\xi_+(1-\xi)_+}\left(\xi^2+(1-\xi)^2+\xi^2(1-\xi)^2\right) = -\frac{11}{6} = -\frac{\beta_0^g}{2N_c}$$

Because we have subtracted the $1/\xi$ pole!

Is it for real?

Our DGLAP - like cascade contains negative probabilities!

Yes. This is the right thing to do - the soft pole goes into the JIMWLK evolution!

We are resumming not all DGLAP splittings but only those that are not accounted for by the JIMWLK evolution itself. Recall that JIMWLK does include the doubly logarithmic part of DGLAP.

So everything is backwards : Splitting probabilities are negative, the larger the state, the weaker it scatters.

We are subtracting some, because JIMWLK puts too much in!

So the DGLAP resummation slows down the JIMWLK evolution.

Quarks?

Adding in quarks is not a problem.

It does bring in an interesting twist: the "fundamental Wilson line" $\mathbb{V}_Q$ evolves differently from the adjoint $\mathbb{S}_Q$, since the DGLAP splitting function is different. As a result

$$\mathbb{S}^{ab}_Q(\mathbf{z}) \neq 2\,Tr[\mathbb{V}^\dagger_Q(\mathbf{z})\tau^a\mathbb{V}_Q(\mathbf{z})\tau^b]$$

Nevertheless can be handled in the same way. E.g. in the saturation regime:

$$\mathbb{S}_Q(\mathbf{z}) = \left[\frac{Q_T^2}{Q^2}\right]^{\frac{\alpha_s\beta_0}{4\pi}} S(\mathbf{z})\,.$$

$$\mathbb{V}_Q(\mathbf{z}) = \left[\frac{Q_T^2}{Q^2}\right]^{\frac{3\alpha_s(N_c^2-1)}{8\pi N_c}} V(\mathbf{z})\,.$$

Weak field limit

In the weak field limit the eikonal S-matrix:

$$S^{ab}(\mathbf{x}) = \exp[igT^c\alpha^c(\mathbf{x})]^{ab} = \hat{1} + ig(T^{c_1})^{ab}\alpha^{c_1}(\mathbf{x}) + \frac{(ig)^2}{2}(T^{c_1}T^{c_2})^{ab}\alpha^{c_1}(\mathbf{x})\alpha^{c_2}(\mathbf{x}) + O(g^3),$$

The dressed gluon S matrix can still be expanded in α:

$$\mathbb{S}^{ab}(\mathbf{x}, Q) \approx \delta^{ab} + igT^{c_1}_{ab}\int_{\mathbf{z}_1} R^{(1)}_Q(\mathbf{x}-\mathbf{z}_1)\alpha^{c_1}(\mathbf{z}_1)$$

$$+\frac{(ig)^2}{2}\sum_{k=1}^{n_{\text{adj.}}}[T^{c_1}T^{c_2}]_{ab}\int_{\mathbf{z}_1,\mathbf{z}_2} R^{(2)}_Q(\mathbf{x}-\mathbf{z}_1, \mathbf{x}-\mathbf{z}_2)\alpha^{c_1}(\mathbf{z}_1)\alpha^{c_2}(\mathbf{z}_2),$$

$R^{(1)}_Q(\mathbf{x}-\mathbf{z}_1)$ - density of “bare” gluons at point $\mathbf{z}_1$ in the dressed gluon at point $\mathbf{x}$; $R^{(2)}_Q(\mathbf{x}-\mathbf{z}_1, \mathbf{x}-\mathbf{z}_2)$ - density of bare gluon pairs at $\mathbf{z}_1$ and $\mathbf{z}_2$ in the dressed gluon at $\mathbf{x}$.

Solutions of DGLAP

Equations for $\mathbb{S}_Q$ and $\mathbb{V}_Q$ become equations for $R_Q^{(1)}$, $R_Q^{(2)}$ and $r_Q^{(1)}$, $r_Q^{(2)}$.

Those amazingly can be solved:

$$R_Q^{(2,1),\mathrm{LLA}}(\mathbf{p}) = \frac{3\beta_0 - 11N_c^3 - 11N_c - 3N_c^2\lambda_+}{3N_c^2(\lambda_- - \lambda_+)}\left(\frac{\max(Q^2, \mathbf{p}^2)}{Q^2}\right)^{a_s\lambda_-} + (\lambda_+ \leftrightarrow$$

with

$$\lambda_\pm = \frac{\beta_0}{2} - \frac{11}{3}N_c \pm \sqrt{\left(\frac{\beta_0}{2} - \frac{11}{3}N_c\right)^2 - 4C_F n_F}$$

The resummed characteristic function I

Mellin transforming the resummed equation for the "Pomeron" $[\alpha(\mathbf{x}) - \alpha(\mathbf{y})]^2$

$$\Delta\chi(\gamma) = \frac{1}{\gamma - 1} - \left(\frac{3\beta_0 - 11N_c^3 - 11N_c - 3N_c^2\lambda_+}{3N_c^2(\lambda_- - \lambda_+)(\gamma - 1 + a_s\lambda_-)} + (\lambda_+ \leftrightarrow \lambda_-) \right)$$
$$= \frac{1}{\gamma - 1} - \frac{N_c(\gamma - 1) - \frac{4}{3}a_s C_F n_F}{N_c\left[(\gamma - 1)^2 - a_s(\gamma - 1)\left(\frac{22}{3}N_c - \beta_0\right) + 4a_s^2 C_F n_F\right]}.$$

The full characteristic function:

$$\chi(n,\gamma) = \chi_0(n,\gamma) + \delta_{n,0}\Delta\chi(\gamma)\,.$$

The resummed characteristic function II

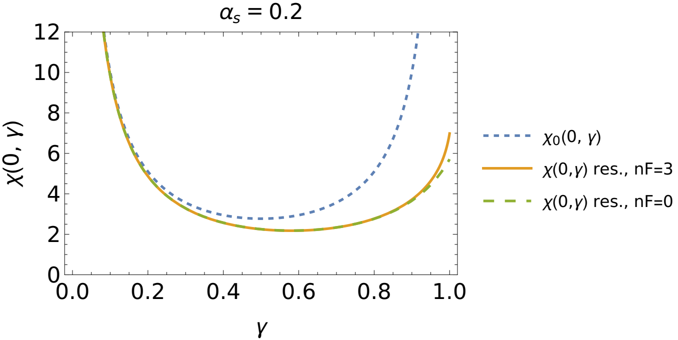

Figure: The plot of resummed characteristic function for $n = 0$.

$$\chi(0, \gamma \to 1) = \frac{\pi}{\alpha_s N_c} \begin{cases} 4/3 & \text{for } n_F \neq 0, \\ 12/11 & \text{for } n_F = 0. \end{cases} \tag{1}$$

Looking forward

Running coupling plus anti collinear resummation -in preparation.

More on $\chi(\gamma \to 1)$ - in progress.

Including collinear logarithms - thinking about it.

Can we sensibly interpolate between the weak field limit and saturation regime?

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