The spacelike collinear splitting amplitude at two loops

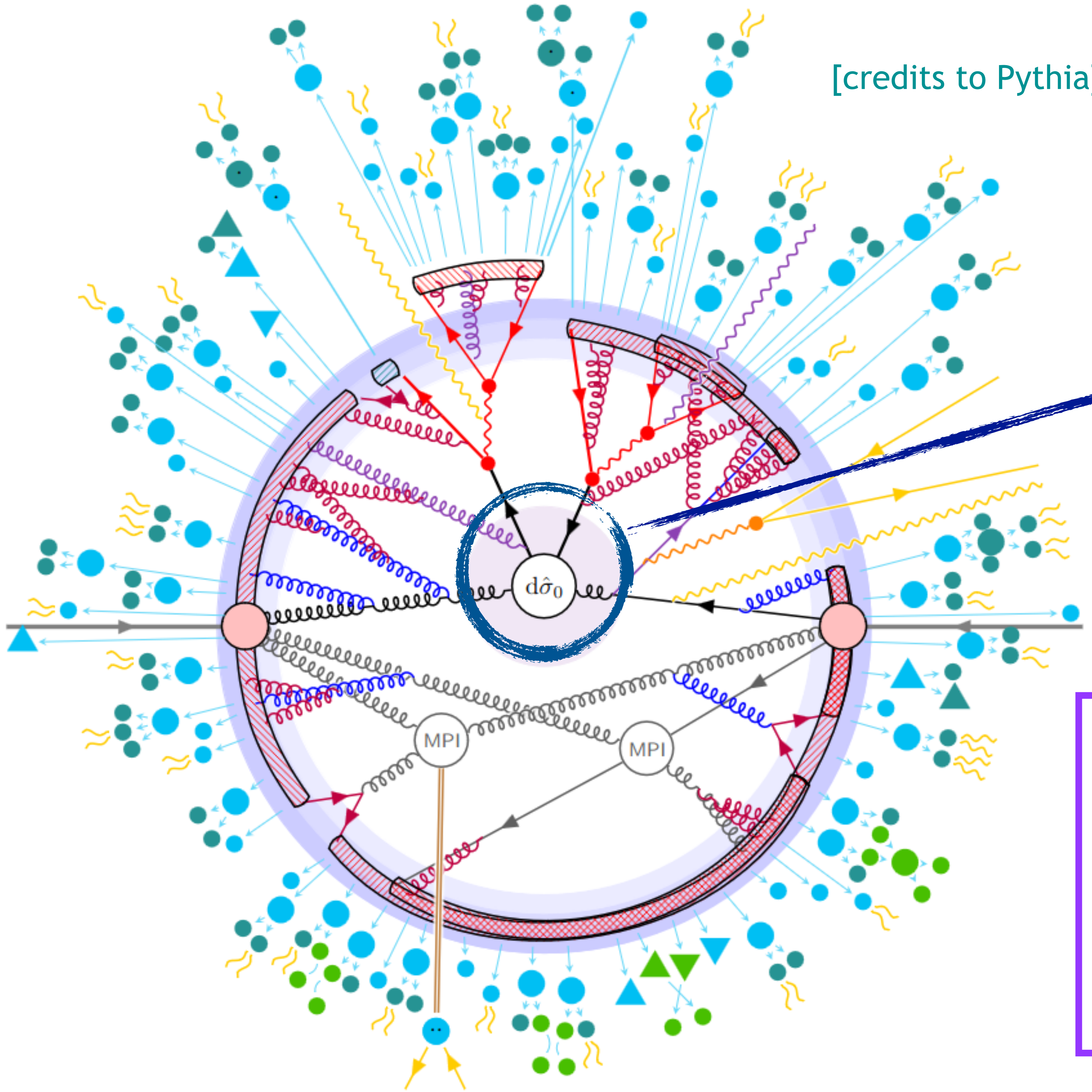
Federico Buccioni

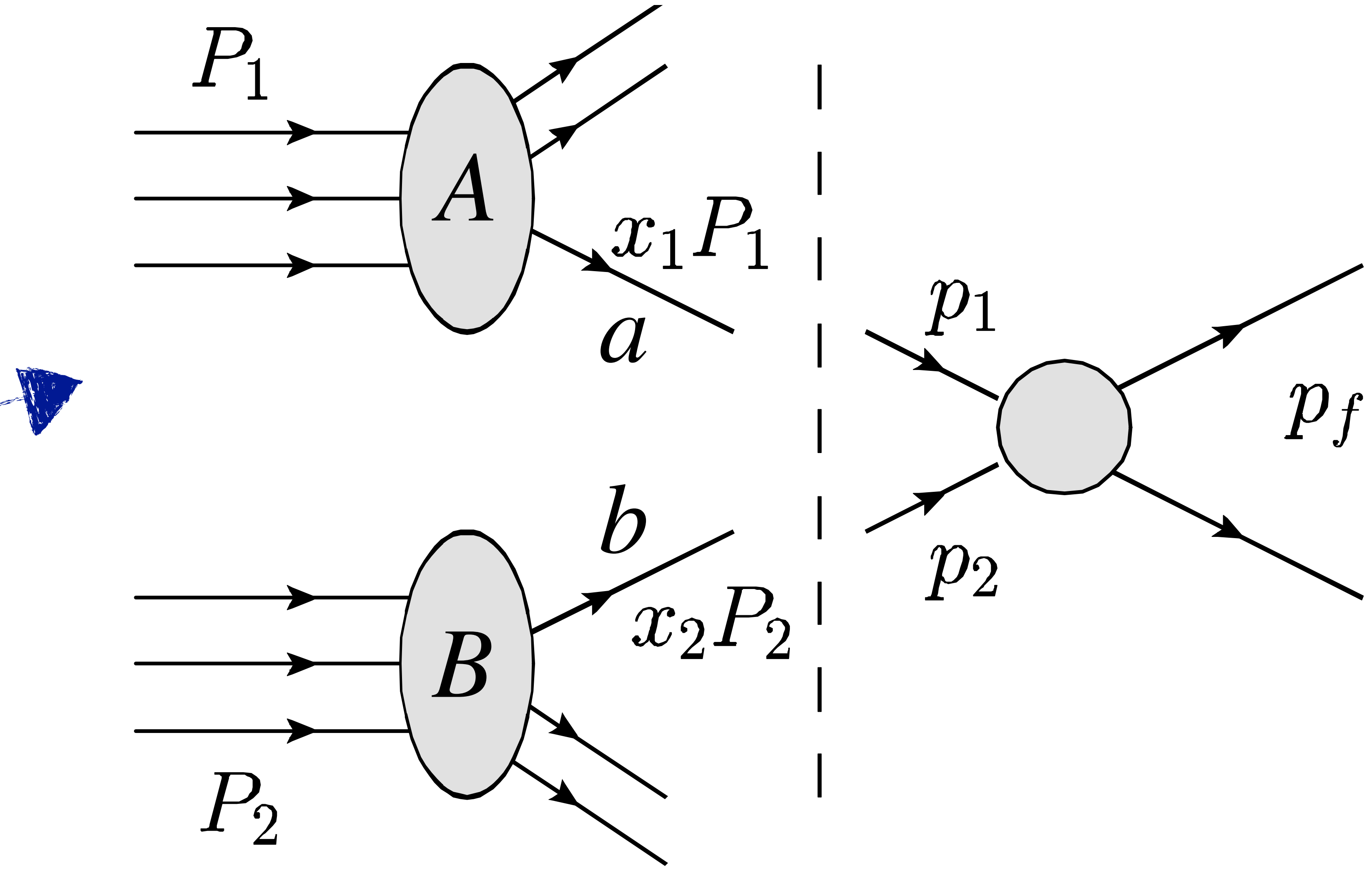
Factorisation in QCD and beyond

Edinburgh, 06/04/2026

based on [FB, H. Fang and K. Yan 2603.27123]

Hadronic cross sections and collinear factorisation





The master formula

$$\frac{d\sigma_{AB\to f}}{dO} = \sum_{ab}\int_0^1 dx_1 dx_2\, f_{a,A}(x_1,\mu) f_{b,B}(x_2,\mu)\, \frac{d\hat{\sigma}_{ab\to f}(x_1,x_2,\mu)}{dO} + \mathcal{O}\left(\frac{\Lambda^p_{\mathrm{QCD}}}{Q^p}\right)$$

Fully-differential hadronic cross section

fundamental property (**assumption**):

universality of collinear factorisation: separation of long- and short-distance dynamics independent of scattering process

Fully differential jet cross sections

Consider a **fully differential** m-jets cross section in pQCD ($ab \to f \Rightarrow 2 \to m$)

need a definition of jets (jet algorithm + $p_\perp$ cut, rapidity intervals,…); $\Lambda \to$ fiducial volume

$$\int_\Lambda \equiv \int F(\vec{p}) \quad \to \text{eg} \to \quad F(\vec{p}) := \prod_i \theta(p_{i,\perp} > 50\,\mathrm{GeV}) \times \theta(2.5 - y_i) \times \theta(y_i + 2.5)$$

in principle: this allows the investigation of **any (multi-)jet observable** (provided it is *IR safe*) at fixed order in pQCD

$$\mathrm{d}\hat{\sigma}_m^{\mathrm{LO}} = \int_\Lambda \mathrm{d}\Phi_m \, |\mathscr{M}_{2\to m}^{(0l)}|^2$$

$$\mathrm{d}\hat{\sigma}_m^{\delta\mathrm{NLO}} = \int_\Lambda \mathrm{d}\Phi_m \, |\mathscr{M}_{2\to m}^{(1l)}|^2 + \int_\Lambda \mathrm{d}\Phi_{m+1} \, |\mathscr{M}_{2\to m+1}^{(0l)}|^2$$

after complete sum of V+R: only **collinear singularities** left

↓

PDF renormalisation
(universal?)

…

$$\mathrm{d}\hat{\sigma}_m^{\delta\mathrm{N^3LO}} = \int_\Lambda \mathrm{d}\Phi_m \, |\mathscr{M}_{2\to m}^{(3l)}|^2 + \int_\Lambda \mathrm{d}\Phi_{m+1} \, |\mathscr{M}_{2\to m+1}^{(2l)}|^2 + \int_\Lambda \mathrm{d}\Phi_{m+2} \, |\mathscr{M}_{2\to m+2}^{(1l)}|^2 + \int_\Lambda \mathrm{d}\Phi_{m+3} \, |\mathscr{M}_{2\to m+3}^{(0l)}|^2$$

Hadronic cross sections and collinear factorisation

a rigorous proof (hadronic XS) exists only for inclusive colour-singlet production [Collins, Soper, Sterman '85, '88]

many have questioned/investigated the validity/limitations in more involved cases: **differential jet cross sections**

some (illustrative) examples:

- *"factorisation is violated in production of high-transverse-momentum particles in hh collisions"* [Collins and Qiu 0705.2141]
- *"Glauber Gluons and Multiple Parton interactions"* [Gaunt 1405.2080]
- *"Drell-Yan with jet vetoes: breaking of generalized factorization"* [Zeng 1507.01652]
- *"Coloumb gluons will generally destroy coherence"* [Forshaw and Holguin 2109.03665]

and many more…

Clearer picture: breakdown of collinear factorisation in spacelike collinear splittings: indication of PDF factorisation breaking?

- seminal paper *"Space-like (vs time-like) collinear limits in QCD: Is factorization violated?"* [Catani, de Florian, Rodrigo 1112.4405]
- shortly followed by *"On the Breaking of Collinear Factorization in QCD"* [Forshaw, Seymour, Siodmok 1206.6363]
- more recently *"Catani's generalization of collinear factorization breaking"* [Cieri, Dhani, Rodrigo 2402.14749]

Is factorisation violated?

Owing to their absorptive ('imaginary') origin, strict-factorization breaking effects partly cancel at the level of *squared amplitudes* and, hence, in order-by-order perturbative calculations of physical observables. Indeed, we find that such a cancellation is complete up to the next-to-leading order (NLO). Nonetheless, strict factorization is violated at higher orders. For instance, the simplest subprocess in which strict collinear factorization is definitely violated at the squared amplitude level is $2 \to 3$ parton scattering, in the kinematical configurations where one of the three final-state partons is collinear or almost collinear to one of the two initial-state partons. In this subprocess we find non-abelian factorization breaking effects that first occur at the two-loop level. Therefore, these effects contribute to hard-scattering processes in hadron–hadron collisions: they produce next-to-next-to-leading order (NNLO) logarithmic contributions to three jet production with one low-p_T jet (the low-p_T jet is originated by the final-state parton that is almost collinear to one of initial-state partons), and next-to-next-to-next-to-leading order (N^3LO) contributions to one-jet and di-jet inclusive production.

THIS TALK

[Catani, De Florian, Rodrigo 1112.4405]

Two important consequences

The strict factorization breaking effects uncovered in the simple example of $2 \to 3$ parton scattering have more general implications in the context of perturbative QCD computations of jet and hadron production in hadron–hadron collisions. Starting from the N^3LO in perturbation theory, these effects severely complicate the mechanism of cancellation of IR divergences that leads to the factorization theorem of mass (collinear) singularities [29]. These complications challenge the universal (process-independent) validity of mass-singularity factorization, and they are related to issues that arise in the context of factorization of transverse-momentum dependent distributions [30, 31, 32]. The perturbative resummation of large logarithmic terms produced by collinear parton evolution is also affected by the violation of strict collinear factorization: parton evolution gets tangled with the colour and kinematical structure of the hard-scattering subprocess, and this leads to the appearance of 'entangled logarithms'. An example of entangled logarithms is represented by the class of 'super-leading' non-global logarithms discovered [33] in the N^4LO computation of the dijet cross section with a large rapidity gap between the two jets. Indeed, the physical mechanism that produces those super-leading logarithms [34] is directly related to the mechanism that generates the violation of strict collinear factorization.

THIS TALK

see Jeff's and Thomas' talks

[Catani, De Florian, Rodrigo 1112.4405]

Plan of the talk

- (breaking of strict) Collinear factorisation
- Spacelike splitting from full-colour QCD amplitudes
- (two-loop) Generalised splitting amplitudes
- Cross-section level results
- Summary and outlook

see talks by Jeff and Thomas

Disclaimer:

this is a talk about fixed-order perturbation theory

I won't discuss all-orders effects (neither resummation)

I will work in QCD (not in an EFT)

see talk by Thomas

Beyond this talk

(in the context of **QCD observables**)

Critical developments and results from many participants in this workshop

should definitely have a discussion/comparison

orthogonal directions to a fairly large degree

- An EFT for forward scattering and factorization violation (Glauber SCET) [Rothstein and Stewart 1601.04695]
- 2-loop spacelike splitting amplitude in $\mathcal{N} = 4$ sYM [Henn, Ma, Xu, Yan, Zhang, Zhu 2406.14604]
- Factorisation restoration through Glauber gluons [Becher, Hager, Jaskiewicz, Neubert, Schwienbacher 2408.10308]
- Low-energy theory of jet processes and PDF factorization [Becher, Hager, Jaskiewicz, Neubert, Schwienbacher 2509.07082]

I will superficially discuss connections to Multi-Regge Kinematics (MRK)

Reggeon dynamics vs Glauber dynamics

- Double spacelike collinear from MRK (in $\mathcal{N} = 4$ sYM) [Duhr, Venkata, Zhang 2507.05355]

see tomorrow's session on QCD in the high-energy limit

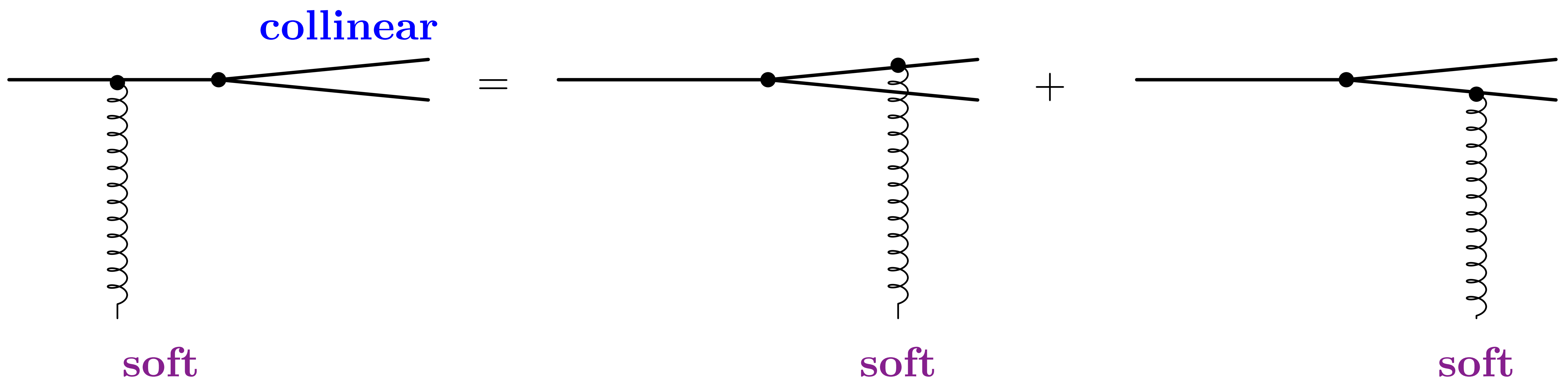
(breaking of strict)

Collinear factorisation

Colour coherence and strict collinear factorisation

- **Colour coherence:**

a soft gluon cannot resolve the internal structure of a bunch of collinear partons: only sensitive to the net colour charge



- **Strict collinear factorisation:**

when a collinear factorisation formula depends **only** on the momenta and quantum numbers (flavour, colour, spin) of collinear partons

↓

strict collinear factorisation

↓

violated when

↓

there is a non-trivial dependence on quantum numbers of non-collinear (spectator) partons

Amplitude level collinear factorisation

Consider an $(n+1)$-point scattering amplitude in QCD $\mathscr{A}_{n+1}$

convention: all particles with momenta p_i are *outgoing* → $p_i^0 > 0$ final state, $p_i^0 < 0$ initial state

single collinear limit: pair of particles (a, b) with helicities (λ_a, λ_b) become collinear → form A with helicity λ_A

$$\mathscr{A}_{n+1}(a^{\lambda_a}, b^{\lambda_b}, \ldots) \simeq \sum_{\lambda_A=\pm} \mathbf{Sp}_{\lambda_A}(a^{\lambda_a}, b^{\lambda_b}) \mathscr{A}_n(A^{\lambda_A}, \ldots)$$

operator in colour space

in pQCD, both $\mathbf{Sp}$ and $\mathscr{A}$ admit an expansion in α_s

$$\mathbf{Sp}_{\lambda_A}(a^{\lambda_a}, b^{\lambda_b}) = \sum_{\ell=0} \left(\frac{\alpha_s}{4\pi}\right)^{\ell} \mathbf{Sp}^{(\ell)}_{\lambda_A}(a^{\lambda_a}, b^{\lambda_b})$$

At two-loop order:

$$\mathscr{A}^{(2)}_{n+1}(a^{\lambda_a}, b^{\lambda_b}, \ldots) \simeq \sum_{\lambda_A=\pm} \left[\mathbf{Sp}^{(0)}_{\lambda_A} \times \mathscr{A}^{(2)}_n + \mathbf{Sp}^{(1)}_{\lambda_A} \times \mathscr{A}^{(1)}_n + \mathbf{Sp}^{(2)}_{\lambda_A} \times \mathscr{A}^{(0)}_n\right]$$

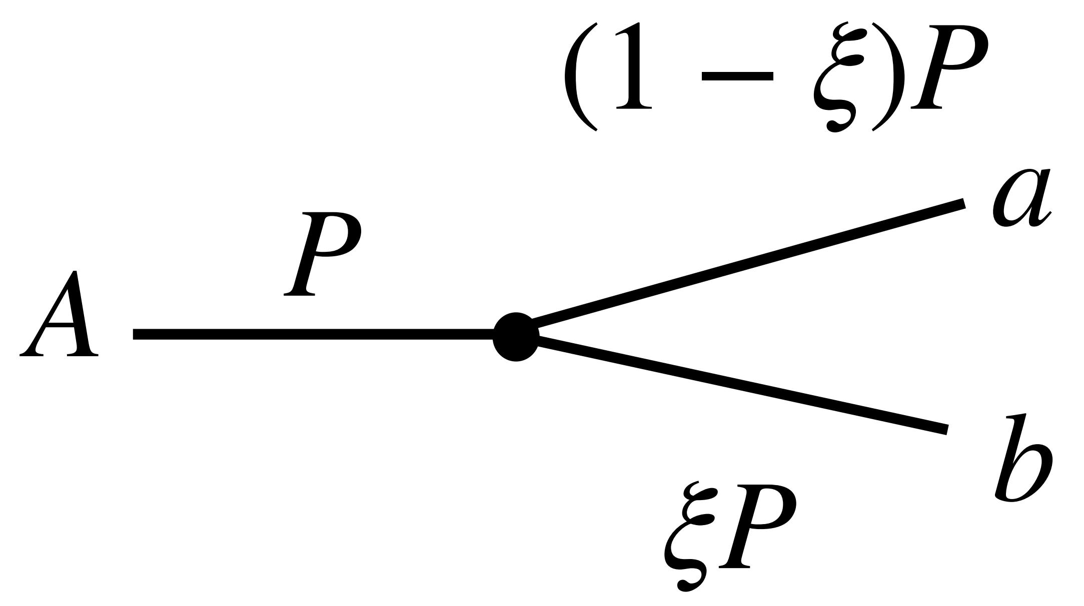
this talk

- spacelike
- all partonic config.
- all helicity config.

Spacelike vs timelike kinematics

Consider the splitting:

$$A(P) \to a((1-\xi)P) + b(\xi P), \qquad \text{with} \qquad 0 < \xi < 1$$



Timelike splitting:

$(p_a^0 p_b^0 > 0,\, P^0 > 0)$, i.e. either both (a, b) in the final state or both in the initial state

Results available in the literature:

- one-loop (d-dimensions) [Bern, Del Duca, Kilgore, Schmidt hep-ph/9903516] [Kosower, Uwer hep-ph/9903515]
- two-loop [Bern, Dixon, Kosher hep-ph/0404293] [Badger, Glover hep-ph/0405236] + higher-orders in the dimensional regulator [Duhr, Gehrmann, Jacquier 1411.3587]
- three-loop (d=4) [Guan, Herzog, Ma, Mistlberger, Suresh 2408.03019]

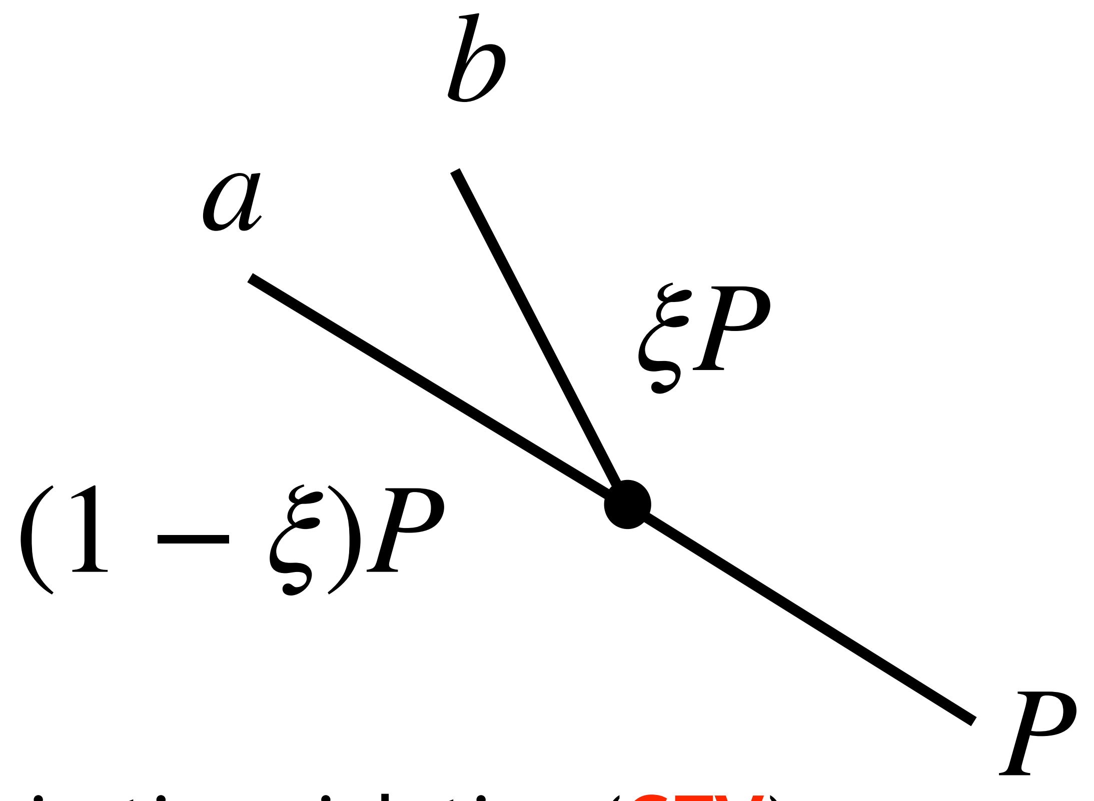
In the TL case: **no violation of strict collinear factorisation**

universal splitting amplitudes, depend only on quantum numbers of collinear partons

Spacelike vs timelike kinematics

Spacelike splitting:

$(p_a^0 p_b^0 < 0,\, P^0 < 0)$, i.e. a in the initial state and b in the final state (or viceversa)



$$a((1-\xi)P) \to A(P) + b(\xi P), \qquad \text{with} \qquad \xi < 0$$

loop corrections induce a non-trivial dependence on quantum numbers of spectators: strict collinear factorisation violation (**CFV**)

<u>"*mathematical issue*" in extracting the SL from TL:</u>

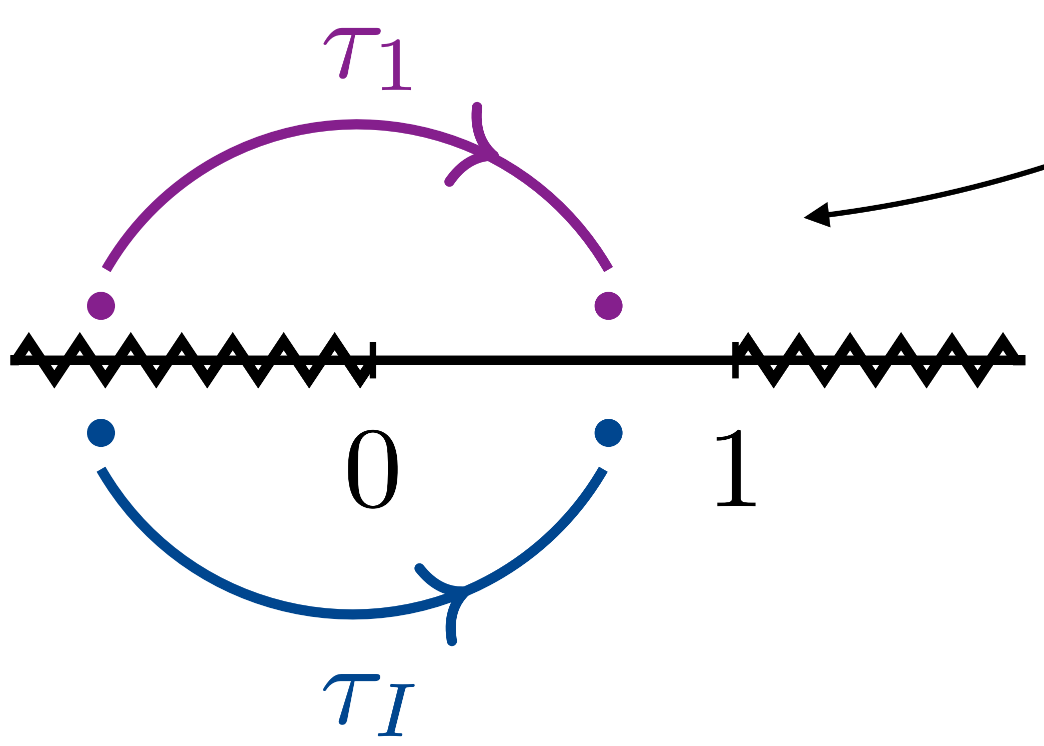
ambiguity in the sign of the infinitesimal imaginary part of energy fraction ξ: **prevents a direct analytic continuation from TL**

consider a set of variables τ_I , where I is a spectator parton (all reduce to ξ in coll. limit)

$$\tau_I = \frac{s_{Ib}}{s_{Ib} + s_{Ia}} \xrightarrow{a \| b} |\,\xi\,| \exp[-i\pi(\eta_{Ib} - \eta_{IA})]$$

$\eta_{ij} = 1$ $\quad$ i, j both outgoing

$\eta_{ij} = 0$ $\quad$ otherwise



τ_I unambiguous in TL splitting

$\tau_I \to -\,|\,\xi\,| \pm i0^{\pm}$ $\quad$ SL: for incoming/outgoing

Collinear factorisation at one loop

Let's see this explicitly at 1-loop. Simplified framework: IR structure (poles) of scattering amplitudes

[Catani, De Florian, Rodrigo 1112.4405]

[Forshaw, Seymour, Siodmok 1206.6363]

Consider $(n+1)$-scattering process, with $p_{1,2}$ in the initial state i.e. $p^0_{1,2} < 0$

IR factorisation

$$|\mathscr{A}^{(1)}_{n+1}\rangle = \mathbf{I}^{(1)}_{n+1}\,|\mathscr{A}^{(0)}_{n+1}\rangle + |\mathscr{A}^{(1,\text{fin})}_{n+1}\rangle$$

coll factorisation

$$|\mathscr{A}^{(1)}_{n+1}\rangle = \mathbf{I}^{(1)}_{n+1}\cdot\mathbf{Sp}^{(0)}\,|\mathscr{A}^{(0)}_{n}\rangle + |\mathscr{A}^{(1,\text{fin})}_{n+1}\rangle$$

=

collinear factorisation

$$|\mathscr{A}^{(1)}_{n+1}\rangle = \mathbf{Sp}^{(0)}\,|\mathscr{A}^{(1)}_{n}\rangle + \mathbf{Sp}^{(1)}\,|\mathscr{A}^{(0)}_{n}\rangle$$

IR factorisation

$$|\mathscr{A}^{(1)}_{n+1}\rangle = \mathbf{Sp}^{(0)}\cdot\mathbf{I}^{(1)}_{n}\,|\mathscr{A}^{(0)}_{n}\rangle + \mathbf{Sp}^{(1)}\,|\mathscr{A}^{(0)}_{n}\rangle + \text{fin}$$

define $\mathbf{I}^{(1)}_C$ via: $\mathbf{Sp}^{(1)} = \mathbf{I}^{(1)}_C\cdot\mathbf{Sp}^{(0)}$

at poles level one can establish:

$$\mathbf{I}^{(1)}_C\cdot\mathbf{Sp}^{(0)} = \mathbf{I}^{(1)}_{n+1}\cdot\mathbf{Sp}^{(0)} - \mathbf{Sp}^{(0)}\cdot\mathbf{I}^{(1)}_{n}$$

Collinear factorisation at one loop

The infrared operator can be written as

Coloumb/Glauber phase

$$\mathbf{I}^{(1)}_{n+1} = \frac{\alpha_s}{4\pi}\left\{ -\sum_{i=1}^{n+1}\left(\frac{C_i}{\epsilon^2} + \frac{\gamma_i}{\epsilon} + i\pi\frac{C_i}{\epsilon}\right) + \frac{1}{\epsilon}\sum_{(i,j)}^{n+1}\mathbf{T}_i\cdot\mathbf{T}_j \ln\left(\frac{\mu^2}{|s_{ij}|}\right) + \frac{2i\pi}{\epsilon}\mathbf{T}^2_{12}\right\} \qquad \mathbf{T}_{12} = \mathbf{T}_1 + \mathbf{T}_2$$

an equivalent structure holds for the underlying n-point scattering process

Consider a **timelike splitting**, e.g. $p_n \parallel p_{n+1}$: $p_{\tilde{n}} \to p_n(z_n p_{\tilde{n}}) + p_{n+1}(z_{n+1} p_{\tilde{n}})$, with $z_{n,n+1} > 0$. Moreover: $\mathbf{T}_{\tilde{n}} = \mathbf{T}_n + \mathbf{T}_{n+1}$

$$\mathbf{I}^{(1)}_C \cdot \mathbf{Sp}^{(0)} = \mathbf{I}^{(1)}_{n+1}\cdot \mathbf{Sp}^{(0)} - \mathbf{Sp}^{(0)}\cdot \mathbf{I}^{(1)}_n$$

all terms that do not involve $\tilde{n}, n$ or $n+1$ commute, so vanish in the difference

$$\mathbf{I}^{(1)}_C = \frac{\alpha_s}{4\pi}\left\{(C_{\tilde{n}} - C_n - C_{n+1})\left(\frac{1}{\epsilon^2} + \frac{i\pi}{\epsilon}\right) + \frac{\gamma_{\tilde{n}} - \gamma_n - \gamma_{n+1}}{\epsilon} + \frac{2}{\epsilon}C_n \ln z_n + \frac{2}{\epsilon}C_{n+1}\ln z_{n+1} + \frac{2}{\epsilon}\mathbf{T}_n\cdot\mathbf{T}_{n+1}\ln\left(\frac{\mu^2 z_n z_{n+1}}{|s_{n+1,n}|}\right)\right\}$$

where we used $s_{jn} = z_n s_{j\tilde{n}}$, $s_{j,n+1} = z_{n+1} s_{j\tilde{n}}$

strict collinear factorisation: depends only on collinear partons

consequence of **colour coherence** (+ colour conservation)

Collinear factorisation at one loop

Consider instead a space like splitting, e.g. $p_1 \parallel p_{n+1}$: $p_{\tilde{n}} \to p_1(z_1 p_{\tilde{n}}) + p_{n+1}(z_{n+1} p_{\tilde{n}})$, with $p_{\tilde{n}}^0 < 0$, $z_1 > 0$, $z_{n+1} < 0$

$$\mathbf{I}_{n+1}^{(1)} = \frac{\alpha_s}{4\pi} \left\{ -\sum_{i=1}^{n+1} \left(\frac{C_i}{\epsilon^2} + \frac{\gamma_i}{\epsilon} + i\pi \frac{C_i}{\epsilon} \right) + \frac{1}{\epsilon} \sum_{(i,j)}^{n+1} \mathbf{T}_i \cdot \mathbf{T}_j \ln \left(\frac{\mu^2}{|s_{ij}|} \right) + \frac{2i\pi}{\epsilon} \mathbf{T}_{12}^2 \right\}$$

does not trivially vanish anymore in the difference

recall: $$\mathbf{I}_C^{(1)} \cdot \mathbf{Sp}^{(0)} = \mathbf{I}_{n+1}^{(1)} \cdot \mathbf{Sp}^{(0)} - \mathbf{Sp}^{(0)} \cdot \mathbf{I}_n^{(1)}$$

in SL kinematics we get:

$$\mathbf{I}_C^{(1)} = \frac{\alpha_s}{4\pi} \left\{ \frac{C_{\tilde{n}} - C_1 - C_{n+1}}{\epsilon^2} + \frac{2i\pi}{\epsilon}(C_1 - C_{n+1} - C_{\tilde{n}}) + \frac{\gamma_{\tilde{n}} - \gamma_1 - \gamma_{n+1}}{\epsilon} + \frac{2}{\epsilon} C_1 \ln z_1 + \frac{2}{\epsilon} C_{n+1} \ln |z_{n+1}| \right.$$
$$\left. + \frac{2}{\epsilon} \mathbf{T}_1 \cdot \mathbf{T}_{n+1} \ln \left(\frac{\mu^2 z_1 |z_{n+1}|}{s_{1,n+1}} \right) + \tilde{\Delta}_C^{(1)} \right\}$$

$$\tilde{\Delta}_C^{(1)} = -4 \frac{i\pi}{\epsilon} \mathbf{T}_2 \cdot \mathbf{T}_{n+1}$$

→ depends on colour charge of spectator 2 → **violates strict collinear factorisation**

A glimpse at two-loop amplitudes

source of strict CFV at one-loop: **dipole operator**

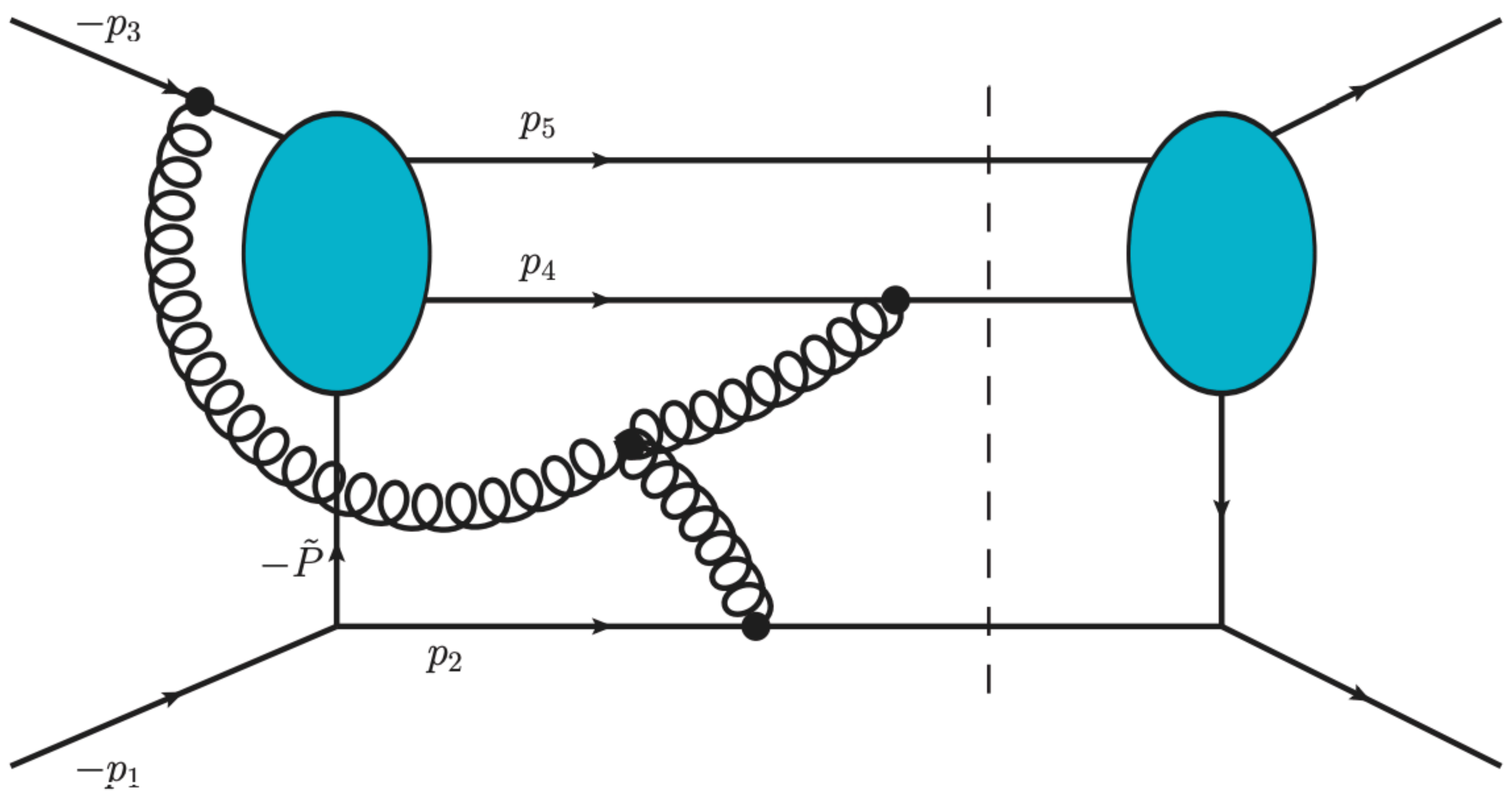
responsible: soft (Glauber) gluon exchange between incoming spectator parton + collinear outgoing parton

this will repeat at two-loops: **two dipole insertions**

responsible: double (soft) Glauber gluon exchange between incoming spectator parton + collinear outgoing parton

These contributions were argued to cancel at XS level [Catani, De Florian, Rodrigo 1112.4405]

Genuine new colour operator at 2-loop for $n \geq 5$ colour partons: violates strict collinear factorisation



[Catani, De Florian, Rodrigo 1112.4405]

Tripole operator (studied in IR poles in [Catani, De Florian, Rodrigo 1112.4405])

- pure non-abelian origin
- absorptive contribution
- relevant in the 2-loop single-collinear (RVV) of dijet cross-sections

are there more CFV contributions? YES (this talk, new)

Spacelike splitting from full-colour QCD amplitudes

Five-parton scattering in QCD at two loops

Two-loop five-point scattering amplitudes in full-colour QCD for all partonic channels computed in

[Agarwal, FB, Devoto, Gambuti, von Manteuffel, Tancredi 2311.09870], [De Laurentis, Ita, Sotnikov 2311.18752 + Klinkert 2311.10086]

We use the results from [Agarwal, FB, Devoto, Gambuti, von Manteuffel, Tancredi 2311.09870]

UV-renormalised amplitudes in the 't-Hooft-Veltman scheme:

$$\mathscr{A}_5^{(L)} = \sum_{k=-2L} \sum_m \epsilon^k R_m^{(k)}(s_{ij}; \mathrm{tr}_5, \epsilon_5) f_m^{(w_k)}(\{W\}) \qquad L = 1,2$$

- $R_m^{(k)}$ rational functions
- $f_m^{(w_k)}$ massless pentagon functions (iterated integrals) [Gehrmann et al 1807.09812, Abreu et al 1812.08941, Chicherin et al 1812.11160, Chicherin et al 2009.07803]
- s_{ij} Mandelstam invariants + parity odd invariant $\mathrm{tr}_5 = 4i\varepsilon(p_1, p_2, p_3, p_4)$ and $\epsilon_5 = i\sqrt{|\det(s_{ij})|}$
- $\{W\}$ pentagon alphabet: 31 letters
- w_K: transcendental weight of the pentagon function → we need up to $w_K = 4$

Kinematics in the collinear limit

Parametrisation of the kinematics in the $p_2 \parallel p_3$ collinear limit: $\{s_{12}, s_{23}, s_{34}, s_{45}, s_{51}\} \to \{s, x, z, y, \delta\}$

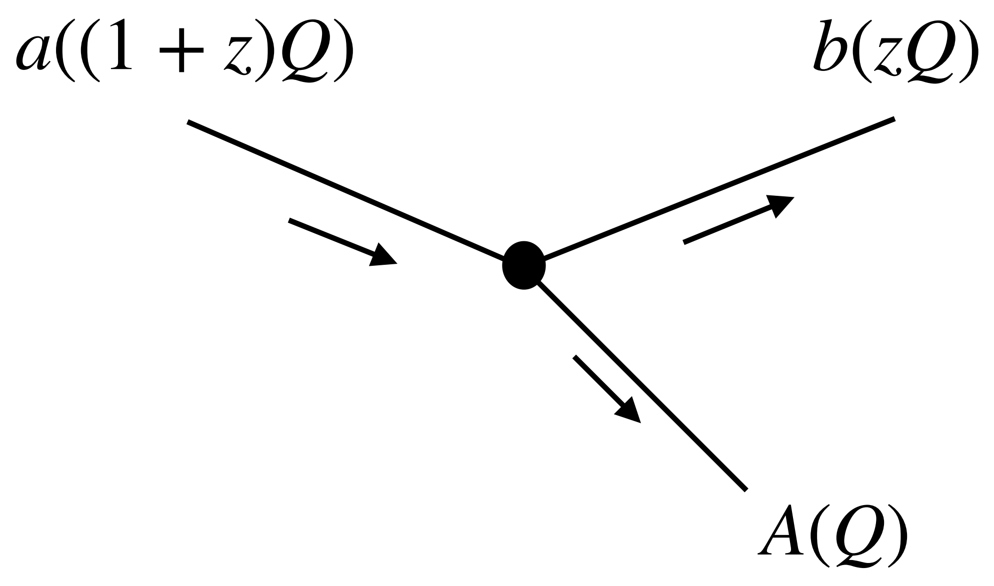
$$s_{12} = s(1+z), \quad s_{23} = -sz\delta^2, \quad s_{34} = s(1-x)z \quad s_{45} = s, \quad s_{51} = -s\left(1 - x - \frac{4y\sqrt{x(1-x)(1+z)}}{1+y^2}\delta\right)$$

expressions for tr_5 and ϵ_5 follow

- this is an **exact parametrisation**: the **collinear limit** is captured by $\delta \to 0$
- z energy fraction carried by 3 in the splitting. Here: $z > 0$

  moreover: $z \to 0$ is the soft limit and $z \to \infty$ is the high-energy limit (*small-x*)
- equivalent parametrisation with $0 < z < 1$ upon (simultaneously): $s \to zs$ and $z \to (1-z)/z$

  soft limit: $z \to 1$, small-x limit: $z \to 0$
- x related to the invariant t in the underlying $2 \to 2$ process
- y describes the transverse (azimuthal) dof



Expansion in the collinear limit

Strategy:

1. expand 1- and 2-loop scattering amplitudes for all relevant partonic channels in $\delta \to 0$
2. match result to generalised 1- and 2-loop splitting amplitudes (colour operators) acting on four point amplitudes
3. extract 2-loop corrections to splitting amplitude in spacelike kinematics

Implementation:

A. leading singularity $1/\delta$ captured by tree-level (factored out): loop corrections can only induce $\ln^{\ell}(\delta)$ corrections

B. expand rational functions: straightforward but painful

C. expand pentagon functions: **two alternative methods**

C1. systematically take **discontinuity of pentagon functions** (work at symbol level) and move from SL to TL kinematics

C2. solve **differential equations** wrt s_{ij} satisfied by pentagon functions as a **power series in** δ

agreement between the two methods

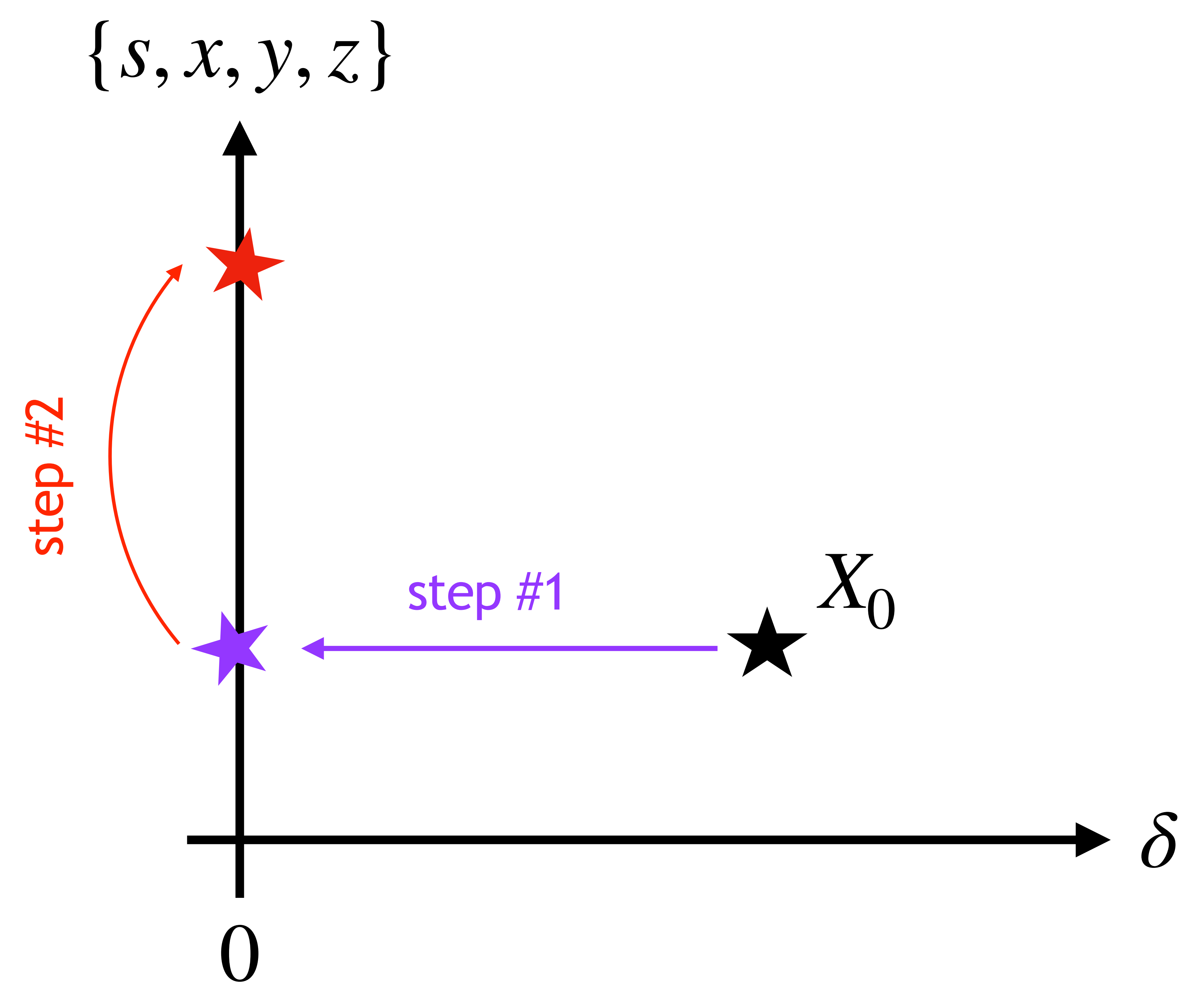
in rational functions spurious $1/\delta^3$ poles → need to expand PF up to δ^3

Expansion via differential equations

At leading power (LP) in $\delta \sim 0$ the pentagon alphabet reduces to 16 letters

$$\{\delta\},\ \{s\},\ \{1 \pm y, i \pm y\},\ \{x, 1-x, z, 1+z, 1+z-x, x+z, 1-x-xz, 1+z-xz, 1+xz, x-z+xz\}$$

Pentagon functions defined as iterated integrals with boundary point in X_0



step #1:

A. at fixed $\{s, x, y, z\}$ solve 1-dimension deq in δ

transport point to $\delta = 0$: extract LP solution in $\delta \to 0$ (δ^0)

complete tower of logarithms $\ln^\ell \delta$

step #2:

A. integrate deqs in $\{s, x, y, z\}$, actually: write (compact) ansatz in terms of GPLs and fix coefficients

B. fix undetermined numerical constants via high-precision evaluation of GPLs + PSLQ algorithm (basis of constants known)

Expansion via differential equations

solutions of PF beyond LP: write pentagon function as a generalised series around $\delta = 0$

$$f_i^{(w)} = \sum_{n=0} \sum_{\ell=0}^{w} \delta^n \ln^\ell(\delta) g_{i,n\ell}^{(w-\ell)}(\{s\}, \{y\}, \{x, z\})$$

solve 1-dimensional deq in δ up to δ^3: boundary point of the deq is the LP solution δ^0

one can obtain solutions at arbitrarily high powers of δ

comparison vs numerical evaluation in octuple precision vs PentagonFunctions++ [Chicherin, Sotnikov 2009.07803], excellent agreement

Finally:

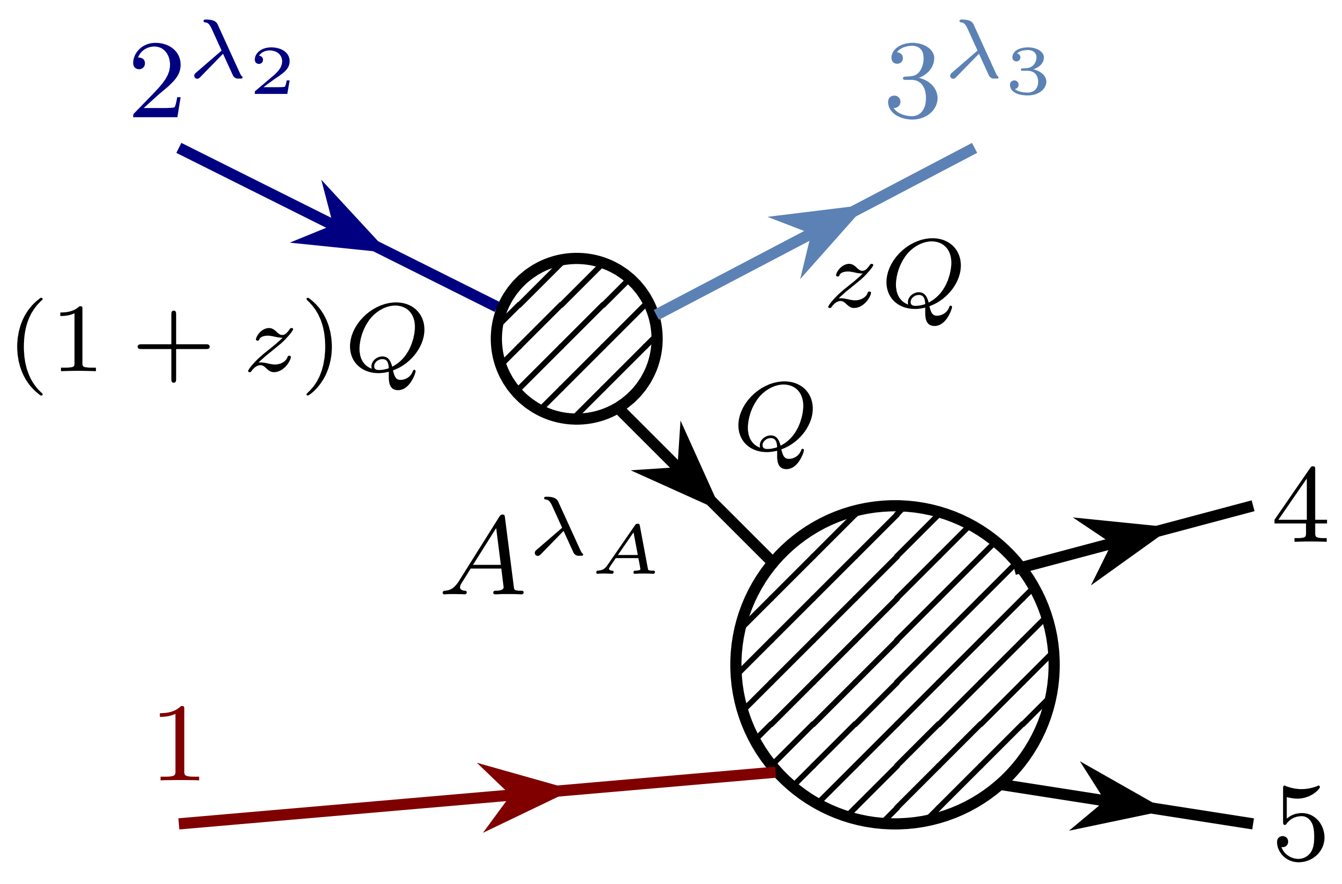
combining rational and transcendental functions: the dust settles (drastic simplification)

- scattering amplitudes expressed in terms of simple single-valued polylogarithms + $\{i\pi, \zeta_2, \zeta_3\}$
- no $\{x, z\}$ entangled dependence; y enters only through $\ln^k\left(\frac{i+y}{i-y}\right)$

(two-loop)
Generalised splitting amplitudes

General form and main ingredients

Consider the spacelike splitting: $a^{\lambda_a}((1+z)Q) \to A^{\lambda_A}(Q) + b^{\lambda_b}(zQ), \quad z > 0$



alternatively, upon performing simultaneously: $Q \to zQ, \;\; z \to (1-z)/z$

$$a^{\lambda_a}(Q) \to A^{\lambda_A}(zQ) + b^{\lambda_b}((1-z)Q), \quad 0 < z < 1$$

here, take a=2 and b=3

results presented for UV renormalised sp. amp.

$$\mathbf{Sp}_{\lambda_A}(2^{\lambda_2},3^{\lambda_3}) = i\sqrt{4\pi\alpha_s}\,\mathrm{Split}_{\lambda_A}(2^{\lambda_2},3^{\lambda_3}) \times \left\{ \exp\left[\mathscr{G}_{\vec{\lambda}}(z;\epsilon) + \mathbf{\Delta}_{\vec{\lambda}}(z_I,\bar{z}_I;\epsilon) \right] \mathbf{R}_{\vec{\lambda}}(z) \right\}$$

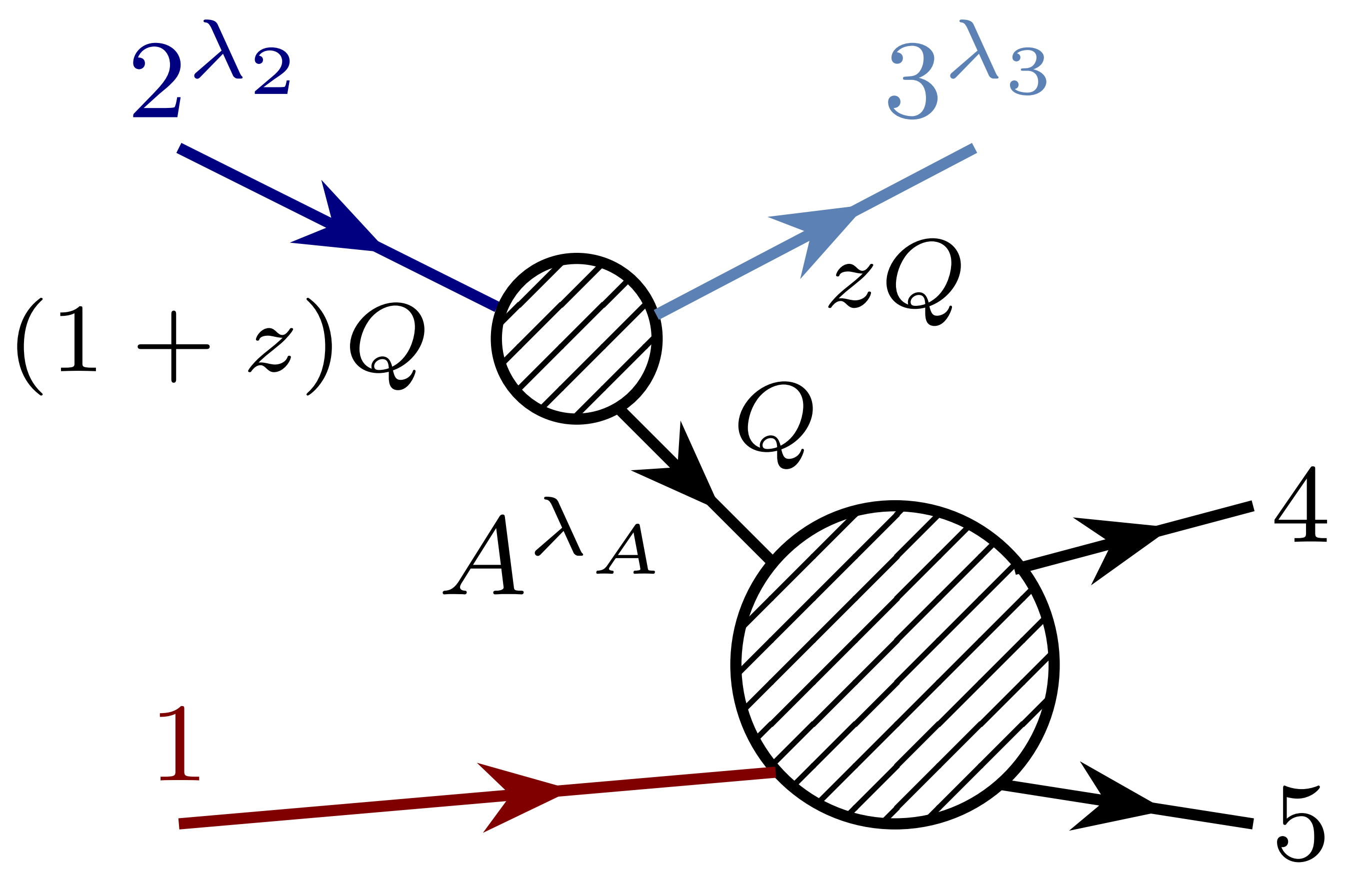
Some definitions:

$$\vec{\lambda} = \{\lambda_A; \lambda_2, \lambda_3\} \qquad z_I = \frac{\langle ba\rangle\langle \mathrm{in\,I}\rangle}{\langle \mathrm{in\,b}\rangle\langle \mathrm{aI}\rangle}, \quad \bar{z}_I = z_I^*, \;\; \text{with} \;\; a \parallel b$$

$\mathrm{Split}_{\lambda_A}$ ← captures the leading singular behaviour

General form and main ingredients

Consider the spacelike splitting: $a^{\lambda_a}((1+z)Q) \to A^{\lambda_A}(Q) + b^{\lambda_b}(zQ), \quad z > 0$



alternatively, upon performing simultaneously: $Q \to zQ, \;\; z \to (1-z)/z$

$$a^{\lambda_a}(Q) \to A^{\lambda_A}(zQ) + b^{\lambda_b}((1-z)Q), \quad 0 < z < 1$$

here, take a=2 and b=3

results presented for UV renormalised sp. amp.

$$\mathbf{Sp}_{\lambda_A}(2^{\lambda_2},3^{\lambda_3}) = i\sqrt{4\pi\alpha_s}\,\mathrm{Split}_{\lambda_A}(2^{\lambda_2},3^{\lambda_3}) \times \left\{ \exp\left[\mathscr{G}_{\vec{\lambda}}(z;\epsilon) + \mathbf{\Delta}_{\vec{\lambda}}(z_I,\bar{z}_I;\epsilon)\right]\, \mathbf{R}_{\vec{\lambda}}(z) \right\}$$

At tree-level:

$$\mathbf{T}^c_{qq} = T^c_{i_3 i_2}, \qquad \mathbf{T}^c_{qg} = T^{a_3}_{c i_2}, \qquad \mathbf{T}^c_{gq} = T^{a_2}_{i_3 c}, \qquad \mathbf{T}^c_{gg} = i f^{a_2 c a_3}$$

Singlet and dipole contributions

perturbative expansion: $\mathscr{G}_{\vec{\lambda}}(z;\epsilon) = \sum_{n=1} \left(\frac{\alpha_s}{4\pi}\right)^n \mathscr{G}^{(n)}_{\vec{\lambda}}(z;\epsilon)$

$$\bar{\mu} = \frac{\mu^2}{-s_{ab}} \frac{1+z}{z}$$

$$\bar{r}^{(n)} \equiv \bar{\mu}^{-n\epsilon} r^{(n)}(z,\epsilon)$$

$$\mathscr{G}^{(n)}(z;\epsilon) = r^{(n)}(z,\epsilon) + \mathscr{C}_{ab}\,\mathbf{T}_3 \cdot \mathbf{T}_1\, \bar{\mu}^{n\epsilon} \mathrm{disc}[\bar{r}^{(n)}(z,\epsilon)]$$

singlet — strictly factorised

dipole — anti-hermitian (breaks coll. factorisation)

all singlet functions up to two loops in all partonic channels available in the literature [Badger, Glover hep-ph/0405236]

caveat: our splitting amplitude is written in exponential form. Beyond one-loop ≠ from Badger, Glover's (@ 2L: subtract 1-loop^2)

procedure:

- $r^{\ell}(z;\epsilon)$ from [Badger, Glover, hep-ph/0405236] and in their results take $w = -z - i0^+$, where w fraction of energy in the timelike splitting
- extract discontinuity as $\mathrm{disc}[f(z)] = 2i\mathrm{Im}[f(z)]$

Tripole contributions

perturbative expansion: $\boldsymbol{\Delta}_{\vec{\lambda}}(z_I, \bar{z}_I; \epsilon) = \sum_{n=2} \left(\frac{\alpha_s}{4\pi}\right)^n \boldsymbol{\Delta}^{(n)}_{\vec{\lambda}}(z_I, \bar{z}_I; \epsilon)$ $h_\lambda = \pm 1$

$$\boldsymbol{\Delta}^{(2)}_{\vec{\lambda}} = 2i\pi\,\bar{\mu}^{2\epsilon} \sum_{I \in \text{out}} \mathcal{T}_{3,1,I} \left[\left(\frac{2}{\epsilon^2} - 2\zeta_2\right)(\ln|z_I|^2 + 2i\pi) + 8\zeta_3 + h_{\vec{\lambda}}\frac{2}{3}\left(\ln^2\frac{z_I}{\bar{z}_I} + 4\pi^2\right)\ln\frac{z_I}{\bar{z}_I}\right],$$

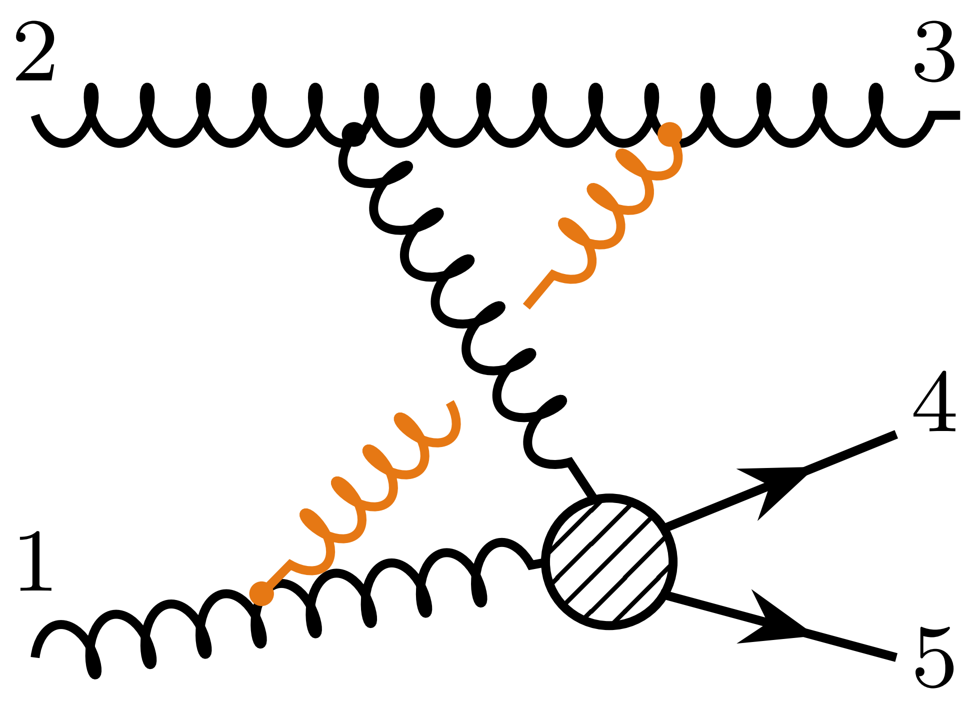
colour tripole operator: $\mathcal{T}_{i,j,k} \equiv [\mathbf{T}_i \cdot \mathbf{T}_j, \mathbf{T}_i \cdot \mathbf{T}_k] = i f^{abc}\, \mathbf{T}^a_i \mathbf{T}^b_j \mathbf{T}^c_k$

properties:

- it appears also in the two-loop corrected single-soft gluon emission [Dixon, Herrmann, Yan, Zhu 1912.09370] (and it's the same)
- it is the same as in $\mathcal{N} = 4$ sYM [Henn, Ma, Xu, Yan, Zhang, Zhu 2406.14604]
- encodes non-trivial dependence on colour charge and kinematics of spectators and on spin of collinear partons
- NB: tripole operator is anti-hermitian (actually anti-symmetric): $\left(\mathcal{T}_{i,j,k}\right)^\dagger = -\mathcal{T}_{i,j,k}$

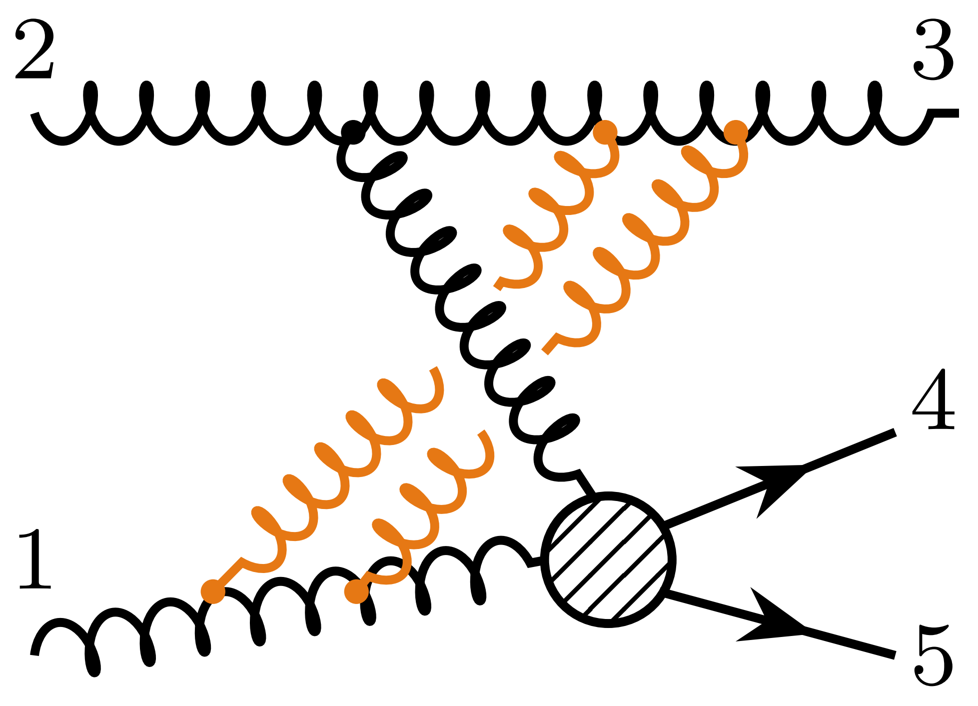
Dipole and tripole diagrams

strict-collinear factorisation violation at 1-loop

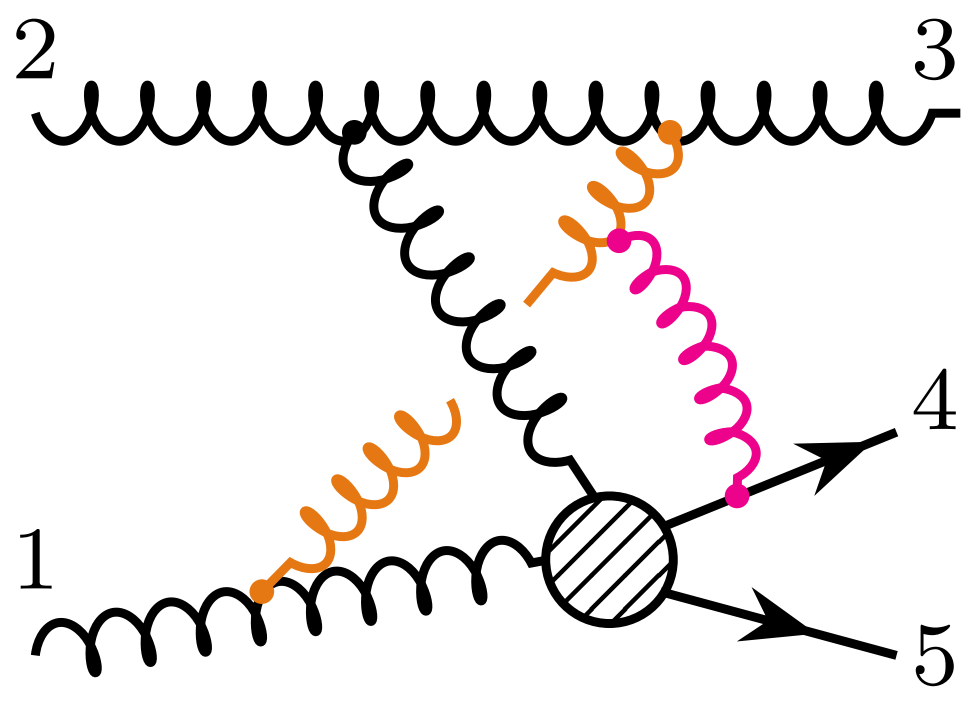


one-loop dipole

strict-collinear factorisation violation at 2-loop



two-loop dipole



two-loop tripole

Beyond dipole/tripole

perturbative expansion: $\mathbf{R}_{ab,\vec{\lambda}}(z) = \sum_{n=0} \left(\frac{\alpha_s}{4\pi}\right)^n \mathbf{R}^{(n)}_{ab,\vec{\lambda}}(z)$ $\quad \mathbf{R}^{(0)}_{ab,\vec{\lambda}}(z) = \mathbf{T}^c_{ab}, \quad \mathbf{R}^{(1)}_{ab,\vec{\lambda}}(z) = 0$

small caveat for $\mathbf{Sp}_{+}(g^{\pm}, g^{\pm})$ splitting (see paper for details)

new contributions at two loops (unknown in the literature before); not present in $\mathcal{N} = 4$ sYM:

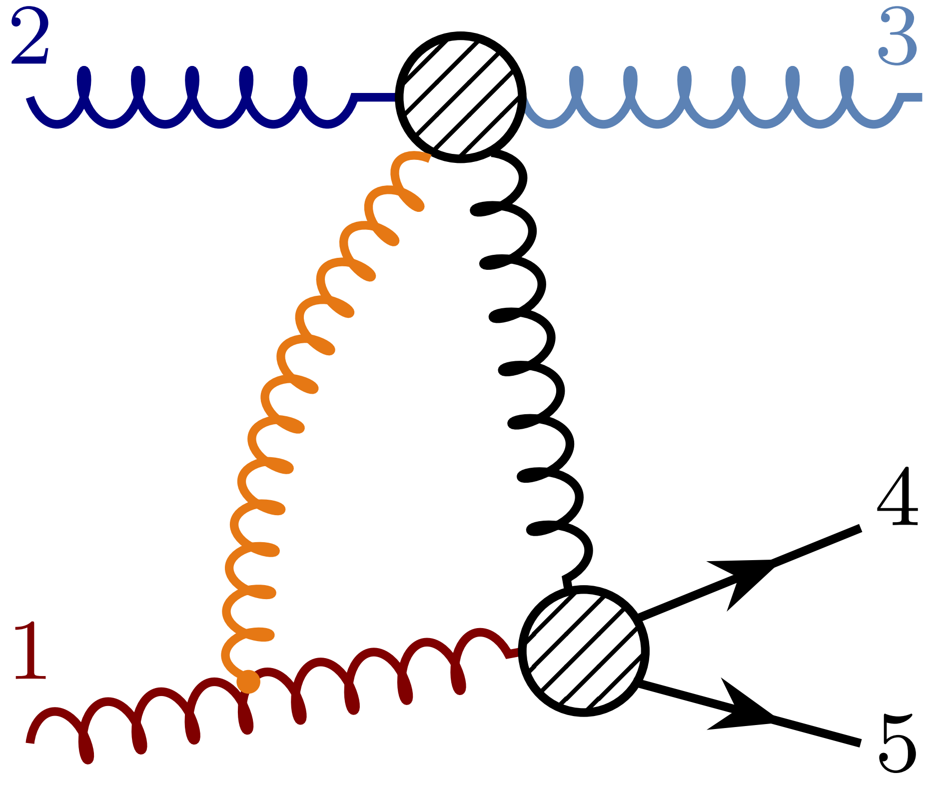
$$\mathbf{R}^{(2)}_{ab} = (2\pi i)\mathscr{R}_{1,\lambda_A}(a^{\lambda_a}, b^{\lambda_b})\mathscr{T}_{b,1,a} \cdot \mathbf{T}^c_{ab} + (2\pi i)\mathscr{R}_{2,\lambda_A}(a^{\lambda_a}, b^{\lambda_b})\,\mathbf{C}^{c,e}_{ab} \cdot \mathbf{T}^e_1$$

colour tripole operator

non-dipole/tripole

violate strict coll. fact.

- depends on quantum numbers of collinear partons and the colour charge of the incoming spectator
- it is flavour dependent
- the operator $\mathbf{C}^{c,e}_{ab}$ vanishes when the outgoing collinear parton b is a quark
- the functions $\mathscr{R}_{1,2}$ are IR finite

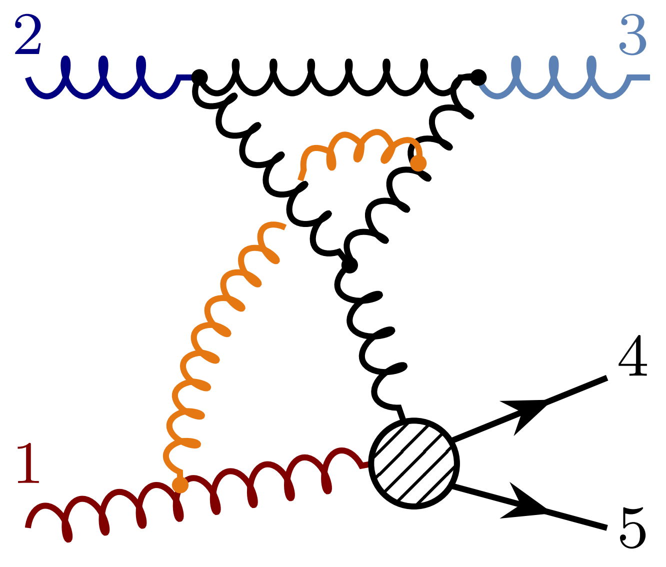


Beyond dipole/tripole: operators

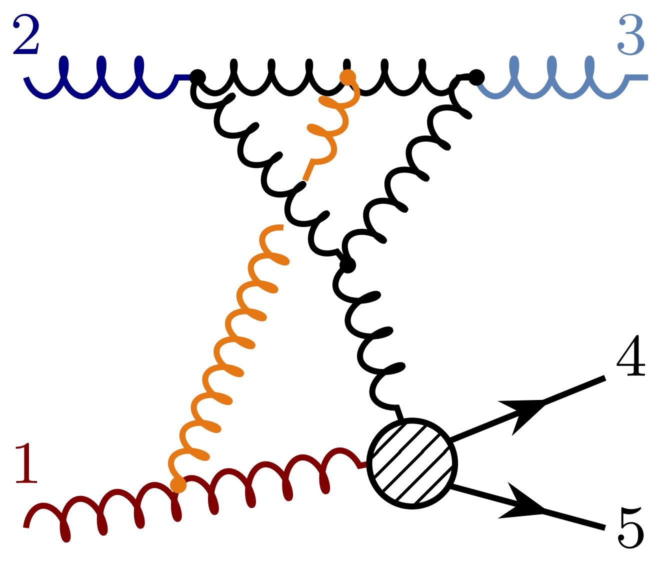
When the outgoing collinear parton is a gluon we find:

$$\mathbf{C}_{gg}^{c,e} = N_f \left[\mathrm{Tr}\left(T^{a_2} T^e T^{a_3} T^c\right) + \mathrm{Tr}\left(T^c T^{a_3} T^e T^{a_2}\right) \right] \qquad \mathbf{C}_{qg}^{c,e} = \left(T^d T^e T^{a_3} T^d\right)_{ci_2} + \frac{1}{N_c^2}\left(T^d T^{[e} T^{a_3]} T^d\right)_{ci_2}$$

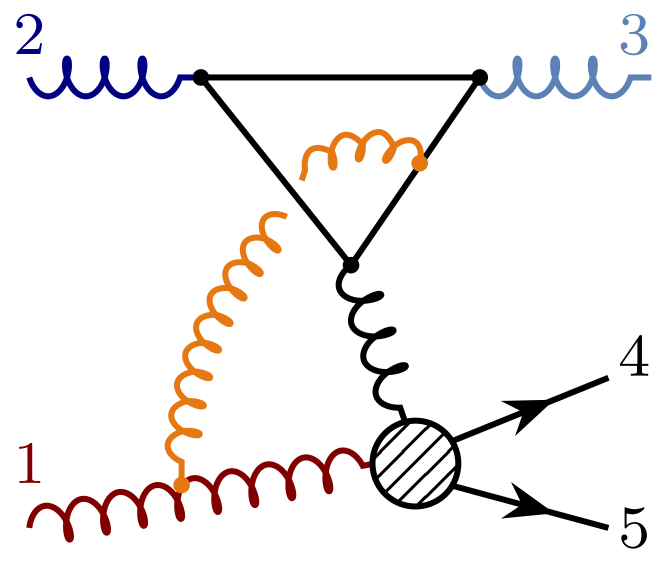
Example: $g \to gg$ splitting



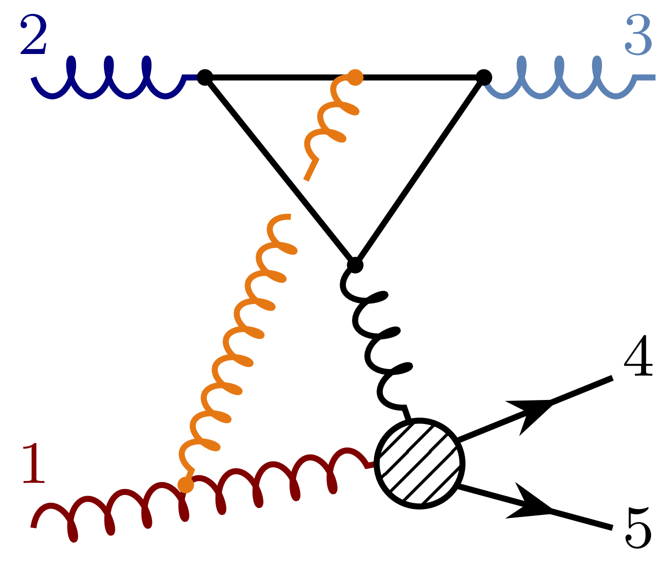
(a1) planar



(a2) non-planar



(b1) planar



(b2) non-planar

- non-planar: (a2) ~ tripole operator $\mathscr{T}_{b,1,a}$; (b2) contributes to $\mathbf{C}_{gg}^{c,e}$; planar ones can be decomposed onto those + dipole (1,3)
- (a2) and (b2) vanish if the system (4,5) is a colour singlet (Higgs or Z boson) → cannot be extracted from [Badger and Glover hep-ph/0405236]
- (b2) also present in an abelian theory (QED): real emission to Compton scattering $e(p_1)\gamma(p_2) \to \gamma(p_3)\gamma(p_4)e(p_5)$

Beyond dipole/tripole: functions

The **functions** $\mathscr{R}_{1,2}$ are **remarkably simple**: consider the splitting $g^{\pm} \to g^{\pm} g^{\pm}$

$$\mathscr{R}_{1,\pm}(g^{\pm}, g^{\pm}) = \frac{4z\left(11z^2 + 21z + 12\right)}{3(1+z)^3}\left(\mathrm{Li}_2(-z) - \zeta_2\right) + \frac{22}{3}\ln^2(1+z) - \frac{4z(3+2z)}{3(1+z)^2} - \frac{8z\ln(1+z)}{3(1+z)^2}$$

$$\mathscr{R}_{2,\pm}(g^{\pm}, g^{\pm}) = \mathscr{R}_{1,\pm}(g^{\pm}, g^{\pm}) - \frac{12z}{1+z}\left(\zeta_2 - \frac{(1+z)}{2z}\ln^2(1+z) - \mathrm{Li}_2(-z)\right)$$

- they **vanish in the soft limit** $z \to 0$ (couldn't be predicted from 2-loop soft function [Dixon, Herrmann, Yan, Zhu 1912.09370])
- in the **high-energy** limit (***small x***) defined by $z \to \infty$ they **reduce to constants**

$$\mathscr{R}_1 = \frac{8}{3} + \frac{44\pi^2}{9}, \qquad \mathscr{R}_2 = \frac{8}{3} + \frac{8\pi^2}{9}$$

- interesting observations:

  I. if both $\mathscr{R}_1$ and are $\mathscr{R}_2$ non vanishing, then $\mathscr{R}_1 - \mathscr{R}_2 \sim$ uniform transcendental weight 2 $\in \{\zeta_2, \ln^2(1+z), \mathrm{Li}_2(-z)\}$

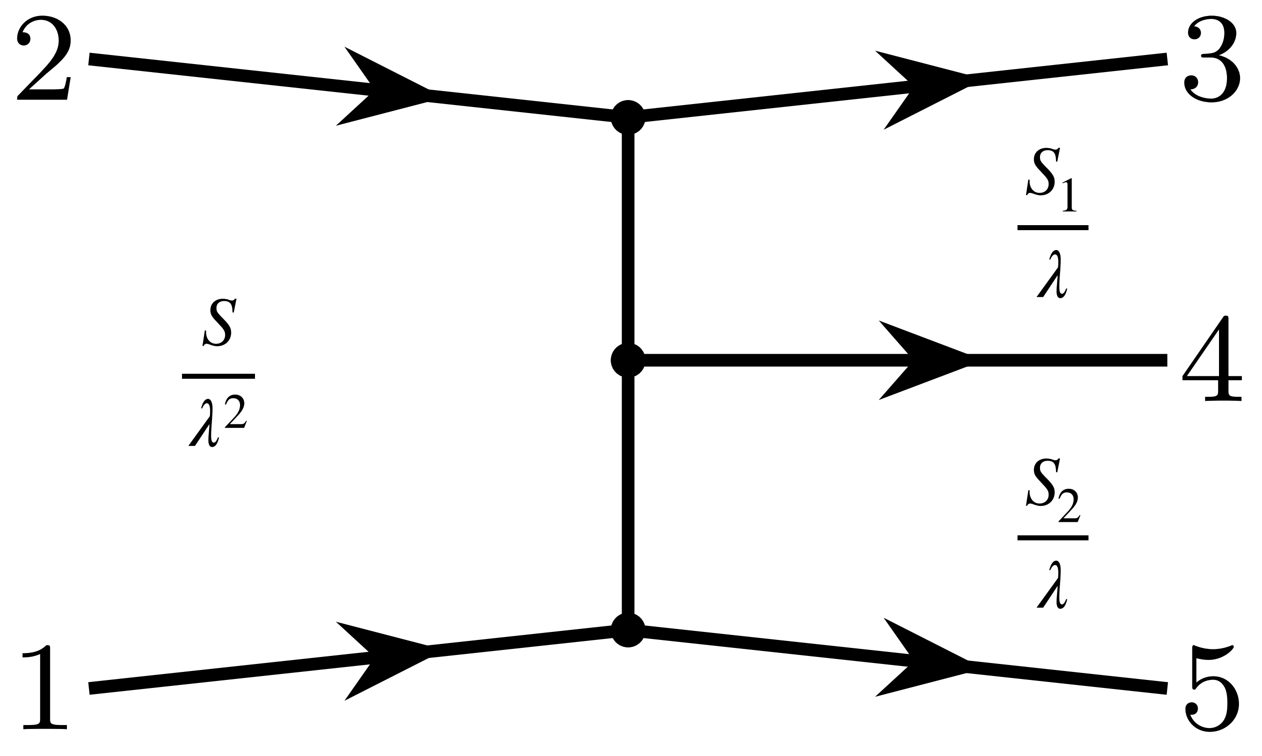
  II. if $\mathscr{R}_2$ vanishes, then $\mathscr{R}_1$ uniform transcendental weight 2

Connection with Multi-Regge Kinematics (MRK)

Interesting to study the space like collinear limit from a high-energy perspective (and viceversa)

MRK: a) final-state particles strongly ordered in rapidity and b) their transverse components $p_{T,i}$ commensurate and $p_{T,i} \ll \sqrt{s_{12}}$

$$s_{12} = \frac{S}{\lambda^2}, \quad s_{23} = -\frac{S_1 S_2}{S} u\bar{u}, \quad s_{34} = \frac{S_1}{\lambda}, \quad s_{45} = \frac{S_2}{\lambda}, \quad s_{51} = -\frac{S_1 S_2}{S}(1-u)(1-\bar{u}), \qquad \lambda \ll 1$$



physically: we are investigating the *small-x* limit of the splitting amplitudes

technically:

1. $\delta \to 0$ controls collinear limit, $\lambda \to 0$ controls MRK limit
2. rescale invariants accordingly and take limits in opposite orders
3. check vs our previous MRK at 2-loop QCD [FB, Caola, Devoto and Gambuti 2411.14050]

see talks by Giulios tomorrow!

collinear kinematics → MRK

$$\left\{ x \to 1 - x\lambda,\ z \to \frac{1+z}{\lambda} - 1,\ \delta \to \lambda\delta\sqrt{\frac{z}{1+z}} \right\}$$

take $\lambda \to 0$ limit

MRK → collinear kinematics

$$\left\{ s \to S(1+z),\ S_1 \to xs(1+z),\ S_2 \to s,\ u\bar{u} \to \frac{4}{x} z\delta^2 \right\}$$

take $\delta \to 0$ limit

full agreement

Connection with Multi-Regge Kinematics (MRK)

observation from kinematics: $\left|\frac{z_I}{\bar{z}_I}\right| = \left|\frac{u}{\bar{u}}\right|$ describe only the kinematics in the transverse plane

$$\mathbf{\Delta}^{(2)}_{\vec{\lambda}} = 2i\pi\,\bar{\mu}^{2\epsilon} \sum_{I\in\text{out}} \mathcal{T}_{3,1,I}\left[\left(\frac{2}{\epsilon^2} - 2\zeta_2\right)(\ln|z_I|^2 + 2i\pi) + 8\zeta_3 + h_{\vec{\lambda}}\,\frac{2}{3}\left(\ln^2\frac{z_I}{\bar{z}_I} + 4\pi^2\right)\ln\frac{z_I}{\bar{z}_I}\right],$$

MRK in $\mathcal{N} = 4$ (equivalent to QCD in this matter) [Caron-Huot et at 2003.03120]

As we have seen in Section 4.4, the $\mathcal{N} = 4$ super Yang-Mills hard function in the multi-Regge limit involves several functions which are not manifestly real analytic in the complex z plane. It is worthwhile to highlight that they appear only in the order-log^0 x components $h^{(2)}_{a,0}$ with $a \in \{1, 2, 5, 6, 11, 12, 14, 15\}$. Among these, $h^{(2)}_{12,0}$ and $h^{(2)}_{15,0}$ are particularly simple, as they only involve – of the functions which are not manifestly real analytic – the logarithms (4.51). For instance,

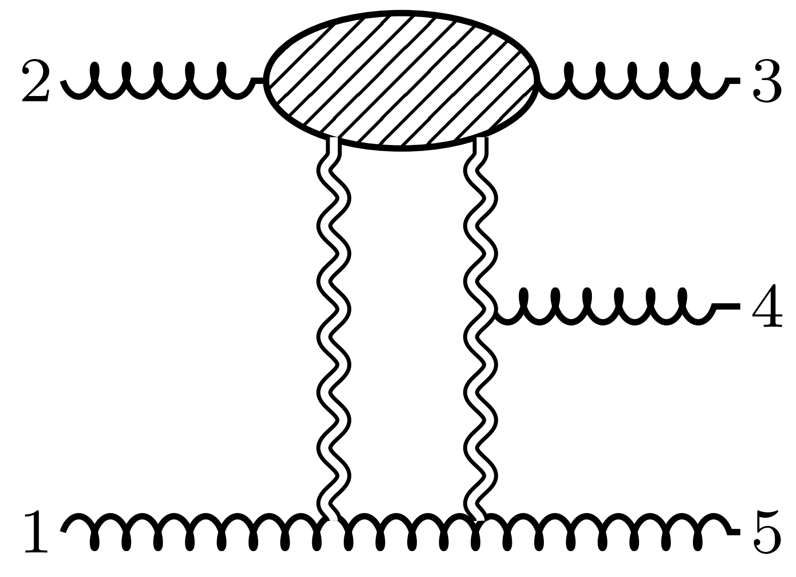
$$h^{(2)}_{12,0} = \frac{i\pi}{6}\left(\left(g_7^{(1)}\right)^3 - \left(g_6^{(1)}\right)^3\right) + \frac{2i\pi^3}{3}\left(g_7^{(1)} - g_6^{(1)}\right) + (\text{analytic})\,. \qquad (5.28)$$

$$g_6^{(1)} = \ln(u) - \ln(\bar{u})$$

$$g_7^{(1)} = \ln(1-u) - \ln(1-\bar{u})$$

$$I^{(1)\prime}_{\text{tri}} = \int \frac{d^{2-2\epsilon}\ell}{\pi^{1-\epsilon}e^{-\epsilon\gamma}}\,\frac{\bar{\mathbf{p}}_3(\mathbf{p}_3-\ell)(\bar{\mathbf{p}}_5+\bar{\ell})\mathbf{p}_5}{|(\mathbf{p}_3-\ell)^2(\mathbf{p}_5+\ell)^2\ell^2|} \times \frac{1}{4}\left[\log\left(\frac{\ell}{\ell+\mathbf{p}_5}\right) - \log\left(\frac{\bar{\ell}}{\bar{\ell}+\bar{\mathbf{p}}_5}\right)\right]^2$$

$$= \frac{1}{12}(\log(z)-\log(\bar{z}))^3 + \frac{\pi^2}{3}(\log(z)-\log(\bar{z})) \;+ (\text{analytic}).$$



one-loop corrected impact factor with 2-reggeon emission

[Fadin, Fiore, Kotsky, Papa hep-ph/9908264]

Connection with Multi-Regge Kinematics (MRK)

observation from kinematics: $\left|\frac{z_I}{\bar{z}_I}\right| = \left|\frac{u}{\bar{u}}\right|$ describe only the kinematics in the transverse plane

signature amplitudes and $\mathbf{C}_{ab}$ operator:

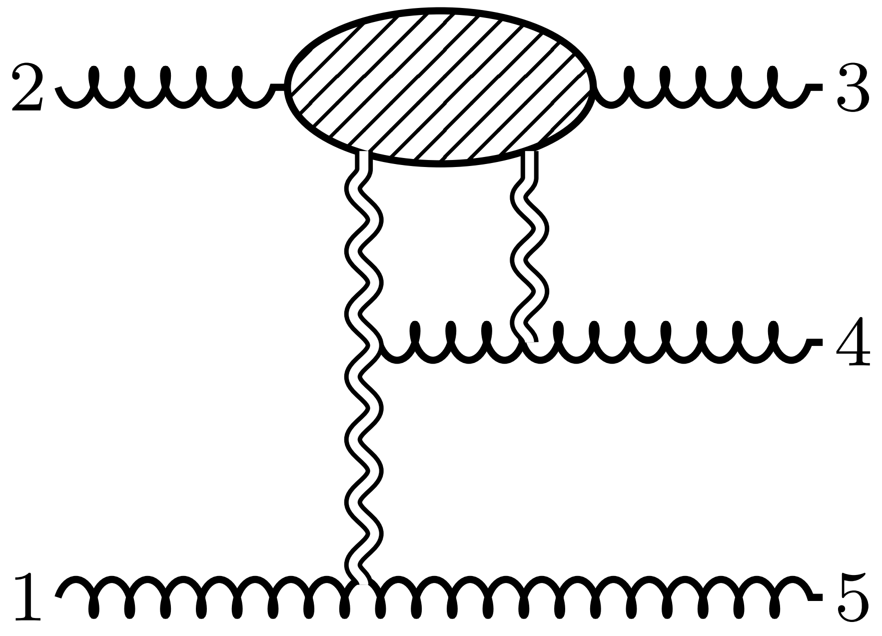
in MRK signature amplitudes very clear meaning: exchange or odd/even # of reggeons → irreducible representations of SU(N)

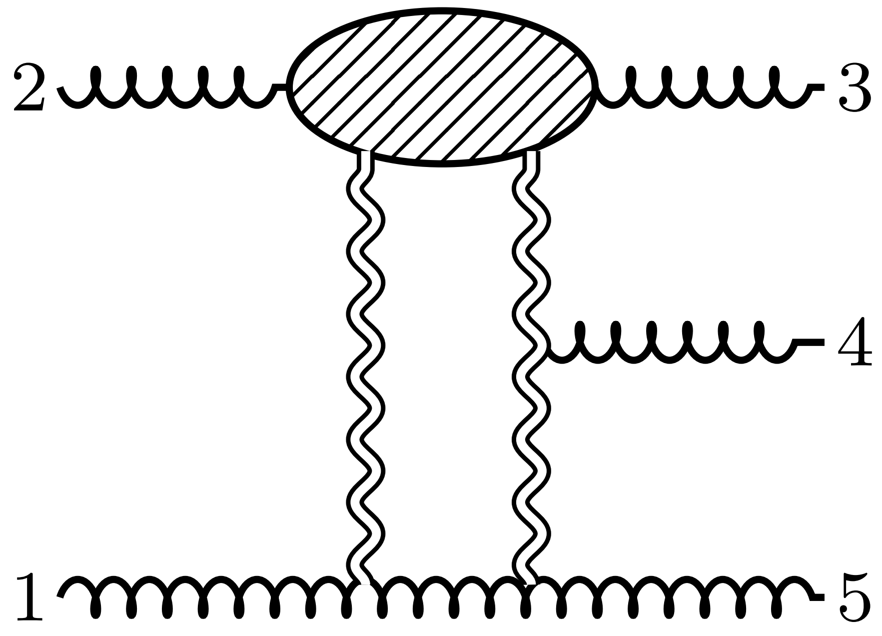
Example: gg → ggg scattering

(r_1, r_2)	$\mathcal{C}_i^{[gg]}$
$(8_a, 8_a)$	$i f^{a_5 b a_1}\, i f^{a_3 c a_2}\, i f^{b c a_4}$
★ $(8_a, 8_s)$	$\frac{N_c^2}{N_c^2-4}\, i f^{a_5 b a_1}\, d^{a_3 c a_2}\, d^{b c a_4}$
$(8_s, 8_a)$	$\frac{N_c^2}{N_c^2-4}\, d^{a_5 b a_1}\, i f^{a_3 c a_2}\, d^{b c a_4}$
★ $(8_s, 8_s)$	$\frac{N_c^2}{N_c^2-4}\, d^{a_5 b a_1}\, d^{a_3 c a_2}\, i f^{b c a_4}$
$(1, 8_a)$	$\frac{N_c^2}{N_c^2-1}\, \delta^{a_5 a_1} i f^{a_3 a_4 a_2}$
★ $(8_a, 1)$	$\frac{N_c^2}{N_c^2-1}\, i f^{a_5 a_4 a_1} \delta^{a_3 a_2}$
★ $(8_a, 0)$	$i f^{a_5 b a_1} \mathcal{T}_0^{a_4 b a_2 a_3}$
$(0, 8_a)$	$\mathcal{T}_0^{a_4 b a_1 a_5} i f^{a_3 b a_2}$
★ $(8_a, 27)$	$i\, f^{a_5 b a_1} \mathcal{T}_{27}^{a_4 b a_2 a_3}$
$(27, 8_a)$	$\mathcal{T}_{27}^{a_4 b a_1 a_5} i f^{a_3 b a_2}$
$(8_a, 10)$	$i f^{a_5 b a_1} \mathcal{T}_{10+\overline{10}}^{a_4 b a_2 a_3}$

(r_1, r_2)	$\mathcal{C}_i^{[gg]}$
$(10, 8_a)$	$\mathcal{T}_{10+\overline{10}}^{a_4 b a_1 a_5} i f^{a_3 b a_2}$
$(8_s, 10)$	$\frac{N_c^2}{N_c^2-4}\, d^{a_5 b a_1} \mathcal{T}_{10-\overline{10}}^{a_4 b a_2 a_3}$
$(10, 8_s)$	$\frac{N_c^2}{N_c^2-4}\, \mathcal{T}_{10-\overline{10}}^{a_4 b a_1 a_5} d^{a_3 b a_2}$
$(27, 27)$ ★	$\mathcal{T}_{27}^{b c a_2 a_3} \mathcal{T}_{27}^{b e a_1 a_5}\, i f^{c a_4 e}$
$(0, 0)$ ★	$\mathcal{T}_0^{b c a_2 a_3} \mathcal{T}_0^{b e a_1 a_5}\, i f^{c a_4 e}$
$(10, 0)$	$\mathcal{T}_{10+\overline{10}}^{b c a_1 a_5} \mathcal{T}_0^{b e a_2 a_3}\, i f^{e a_4 c}$
$(0, 10)$	$\mathcal{T}_0^{b e a_1 a_5} \mathcal{T}_{10+\overline{10}}^{b c a_2 a_3}\, i f^{c a_4 e}$
$(10, 27)$	$\mathcal{T}_{10+\overline{10}}^{b e a_1 a_5} \mathcal{T}_{27}^{b c a_2 a_3}\, i f^{c a_4 e}$
$(27, 10)$	$\mathcal{T}_{27}^{b e a_1 a_5} \mathcal{T}_{10+\overline{10}}^{b c a_2 a_3}\, i f^{c a_4 e}$
$(10, 10)_1$	$\mathcal{T}_{10+\overline{10}}^{b e a_1 a_5} \mathcal{T}_{10+\overline{10}}^{b c a_2 a_3}\, i f^{c a_4 e}$
$(10, 10)_2$	$\mathcal{T}_{10+\overline{10}}^{b e a_1 a_5} \left(\mathcal{T}_{10-\overline{10}}^{b c a_2 a_3}\, d^{c a_4 e} - \mathcal{T}_{10+\overline{10}}^{b c a_2 a_3}\, i f^{c a_4 e}/N_c\right)$

two possible contributions:





one-loop corrected impact factor with two-reggeon emission

[Fadin, Fiore, Kotsky, Papa hep-ph/9908264]

Cross-section level results

Jet cross sections at N3LO QCD

Fiducial m-jets cross section at N3LO in QCD

$$\mathrm{d}\sigma_m^{\delta\mathrm{N^3LO}} = \int_\Lambda \mathrm{d}\Phi_m |\mathscr{M}^{(3l)}_{2\to m}|^2 + \int_\Lambda \mathrm{d}\Phi_{m+1} |\mathscr{M}^{(2l)}_{2\to m+1}|^2 + \int_\Lambda \mathrm{d}\Phi_{m+2} |\mathscr{M}^{(1l)}_{2\to m+2}|^2 + \int_\Lambda \mathrm{d}\Phi_{m+3} |\mathscr{M}^{(0l)}_{2\to m+3}|^2$$

VVV — no collinear emissions, IR structure known

RVV — potentially CFV

RRV — potentially CFV

RRR — no Glauber phases

this talk →

- two-loop single collinear emission

- one-loop triple-collinear emission
- one-loop double-collinear emission

Q1: do CFV terms survive at cross-section level? (in the collinear limit, i.e. DGLAP factorisation)

Q2: if they cancel, what is the mechanism?

NB: for cross-section predictions, we need to sum over colour

Cancellation of CFV contributions. I

Let's start with the IR-finite operator $\mathbf{R}^{(2)}_{ab}$: we have a contraction of the type $\mathbf{R}^{(0),\dagger} \cdot \mathbf{R}^{(2)}$, with $\mathbf{R}^{(0),\dagger} = (\mathbf{T}^{\bar{c}}_{ab})^\dagger$

A. tripole contribution $\mathcal{T}_{b,1,a}$, easy to check that

$$(\mathbf{T}^{\bar{c}}_{ab})^\dagger \cdot \left(i f^{e_1 e_2 e_3} \mathbf{T}^{e_1}_b \mathbf{T}^{e_2}_1 \mathbf{T}^{e_1}_a \right) \cdot \mathbf{T}^{c}_{ab} = 0$$

B. non-tripole contribution ($\mathbf{C}_{ab}$ operator):

$$\mathbf{R}^{(0),\dagger} \cdot \mathbf{R}^{(2)} \sim (\mathbf{T}^{\bar{c}}_{ab})^\dagger \cdot \mathbf{C}^{c,e}_{ab} \cdot \mathbf{T}^{e}_{1}$$

1. $g \to gg$ splitting:

   for $(a, b) = (g, g)$ $\mathbf{T}^{\bar{c}}_{ab}$ is anti-symmetric in a_2, a_3 whereas $\mathbf{C}^{c,e}_{ab}$ is symmetric → contraction is zero upon summing over colour

2. $q \to qg$ splitting:

   explicitly: $(\mathbf{T}^{\bar{c}}_{ab})^\dagger \cdot \mathbf{C}^{c,e}_{ab} = \left(T^d T^e T^{a_3} T^d T^{a_3} \right)_{c\bar{c}} + \frac{1}{N_c^2} \left(T^d T^{[e} T^{a_3]} T^d T^{a_3} \right)_{c\bar{c}} = 0$

Cancellation of CFV contributions. II

operator: $\mathbf{\Delta}^{(2)} = c_1 \mathbf{O}_{\mathscr{T}} + i c_2 \mathbf{O}_{\mathscr{T}}$, with $\mathbf{O}_{\mathscr{T}} \equiv$ tripole, thus $\mathbf{O}^T_{\mathscr{T}} = - \mathbf{O}_{\mathscr{T}}$

at cross-section level we are interested in:

$$\sum_{\text{col}} \left(\langle \mathscr{A}^{(2)} | \mathscr{A}^{(0)} \rangle + \langle \mathscr{A}^{(0)} | \mathscr{A}^{(2)} \rangle \right) = \sum_{\text{col}} \left(\langle \mathscr{A}^{(0)} | \mathbf{\Delta}^{(2),\dagger} + \mathbf{\Delta}^{(2)} | \mathscr{A}^{(0)} \rangle \right) = 2 i c_2 \sum_{\text{col}} \langle \mathscr{A}^{(0)} | \mathbf{O}_{\mathscr{T}} | \mathscr{A}^{(0)} \rangle$$

cancellation of tripole operator based on reality of tree-level amplitudes (or common trivial phase) [Forshaw, Seymour, Siodmok 1206.6363]

$$\langle \mathscr{A}^{(0)} | \mathbf{O}_{\mathscr{T}} | \mathscr{A}^{(0)} \rangle = \mathrm{Tr} \left[\left(| \mathscr{A}^{(0)} \rangle \langle | \mathscr{A}^{(0)} \right) \mathbf{O}_{\mathscr{T}} \right] = 0$$

symmetric — anti-symmetric

this is strictly valid only if $\mathscr{A}_0$ are real: in general, **even in QCD, tree-level amplitudes are not real!**

starting from $2 \to n$ with $n > 2$, non-trivial phases: phases cancel pairwise in parity conjugated helicity configurations

for cancellation at XS level (in a parity invariant theory): **colour + spin sum**

Q1: argument breaks down in the EW theory (CP violation, phases do not cancel upon spin sum). Factorisation breaking?

Q2: possible to devise spin-correlated/sensitive observables? see eg. very recently [CMS 2603.03689]

Cancellation of CFV contributions. III

both $\mathbf{\Delta}^{(2)}$ and $\mathbf{R}^{(2)}$ cancel upon summing over colour (and spin)

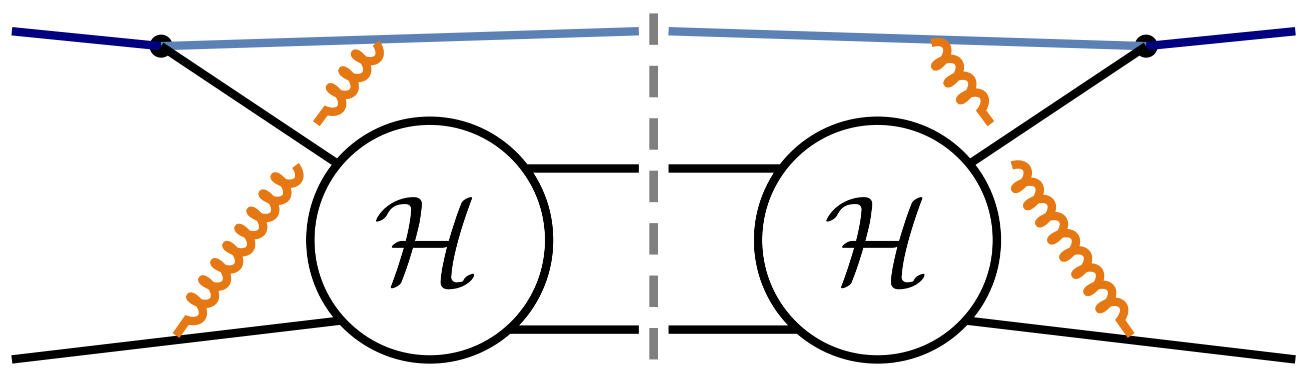
dipole operator (up to this perturbative order) can be written in exponential form

$\exp\left[\mathscr{G}_{\vec{\lambda}}(z;\epsilon)\right]$ ⟶ with $\mathscr{G}_{\vec{\lambda}}(z;\epsilon) \sim$ singlet + dipole ⟶ [singlet, dipole] = 0 + dipole ~ anti-hermitian

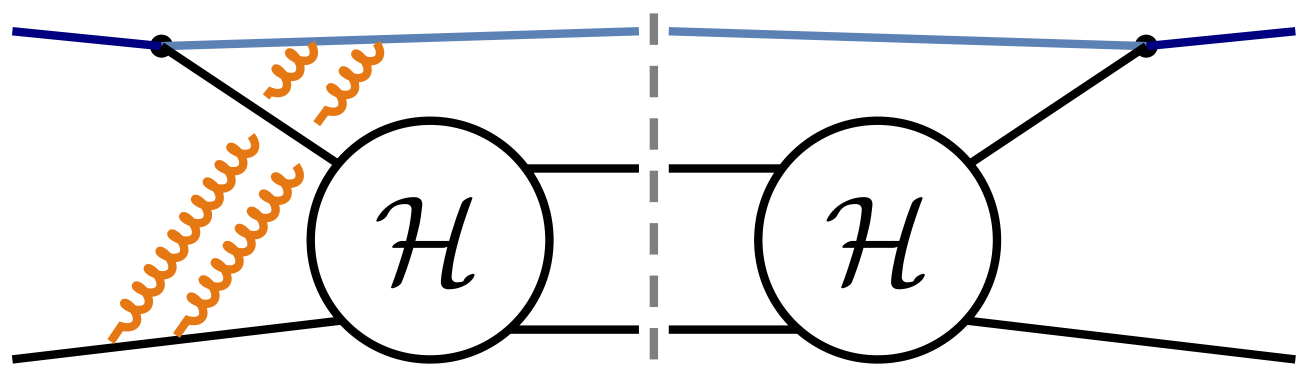
dipole operator → pure phase: trivially vanishes in a squared matrix element

perturbatively, up to two-loops:

$$|\mathscr{A}^{(1)}|^2 \qquad\qquad \mathscr{A}^{(2)*}\mathscr{A}^{(0)} \qquad\qquad \mathscr{A}^{(0)*}\mathscr{A}^{(2)}$$



$+$



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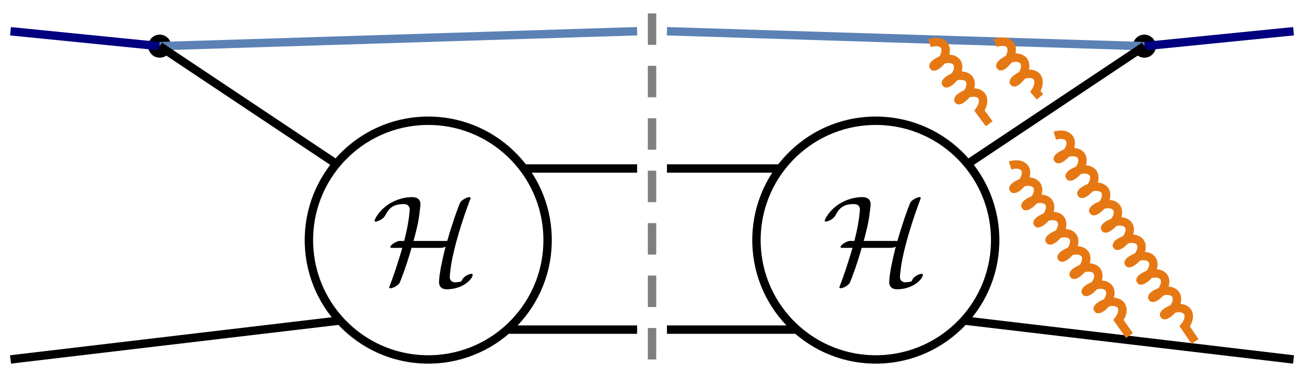


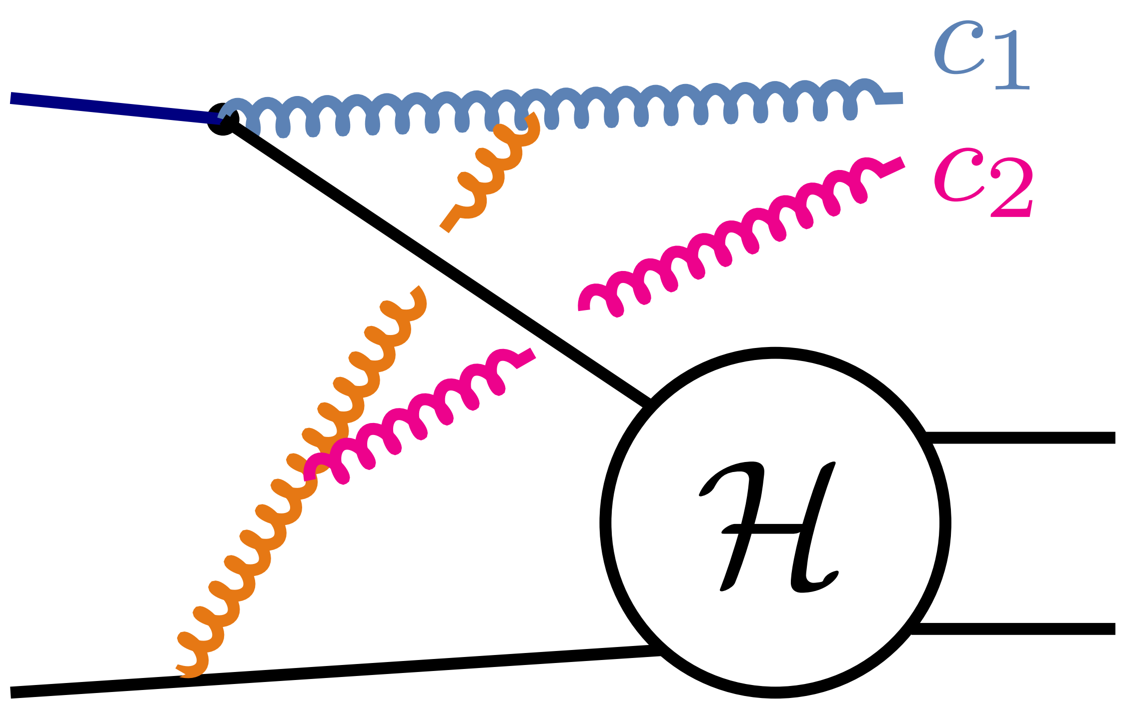
$$\sim +\pi^2 \qquad\qquad \sim -\pi^2/2 \qquad\qquad \sim -\pi^2/2$$

Beyond RVV: CFV at XS level in RRV?

Last place with potential CFV terms remaining at XS level: RRV (double-unresolved emissions)

Relevant ingredients available in the literature:

- one-loop double soft current [Zhu 2009.08919; Czakon, Eschment, Schellenberger 2211.06465]
- one-loop triple collinear splitting [Czakon, Sapeta 2204.11801]



very recently investigated in the literature [Cieri, Dhani, Rodrigo 2402.14749]

(spacelike) generalised collinear and soft-collinear factorisation: multi collinear case

does it survive at XS level?

only source of CFV at XS level in dijet at N3LO?

> and higher orders, the SCF breaking effects are partially canceled also for squared amplitudes [39, 66, 70–72]. To the best of our knowledge, we have identified for the first time a factorization breaking term $\propto \pi^2$ at $\mathcal{O}(\epsilon^0)$ in the SL triple-collinear limit of one-loop double soft-gluon current, which contributes at the level of squared amplitude. Further details will be provided in Ref. [69].

Summary and outlook

Summary

- Strict collinear factorisation at amplitude level challenges the factorisation formula for hadronic cross-sections
- Potential trouble in perturbation theory, first appear at N3LO QCD for jet cross sections (Glauber exchanges)
- often argued/proved to cancel at N3LO for colour-summed squared ME
- clear presence of strict collinear factorisation in 2-loop 5-point QCD amplitudes with one collinear emission
- thanks to **advances in scattering amplitude calculation** → **possible to extract quantitive information**

in **this talk**:

- extracted **complete set of two-loop spacelike splitting amplitudes in QCD** for d=4 (all partonic configurations, all helicities)
- identified all known sources of CFV + so far new unaccounted for contributions (absorptive terms involving incoming spectator)
- (in pure QCD) after **colour + spin sum**: all CFV terms cancel at squared amplitude level, hence at XS level (single coll emission)
- hard to imagine beyond d=4 terms induce CFV terms at XS level → all IR counterterms for jet XS in hh collisions from TL results

Outlook

Several line investigations opened up in fixed-order QCD:

- are **all sources of CVF** at amplitude (and XS) level **accounted for at N3LO**?
- role of **parity** and EW theory?
  A. beyond 4-parton hard scattering process, helicity phases non trivial
  B. will CP violation definitely break down collinear factorisation? (eg jets + Z boson)
- is **collinear factorisation definitely violated at N4LO**? Are there cancellation mechanisms?

Formal investigations:

- intimate connection between Glauber and Reggeon dynamics
  A. intriguing connections in **collinear + high-energy** (small-x) regimes; see also $\mathcal{N} = 4$ planar sYM [Duhr, Venkata, Zhang 2507.05355]
  B. extraction of **universal** functions/**structures**/data from either limits
  C. could offer a path forward when full investigation out of reach, eg. $2 \to 4$ scattering (spectator/spectator interactions)