TRAVELING WITH LANCE TO THE INFRARED SIDE

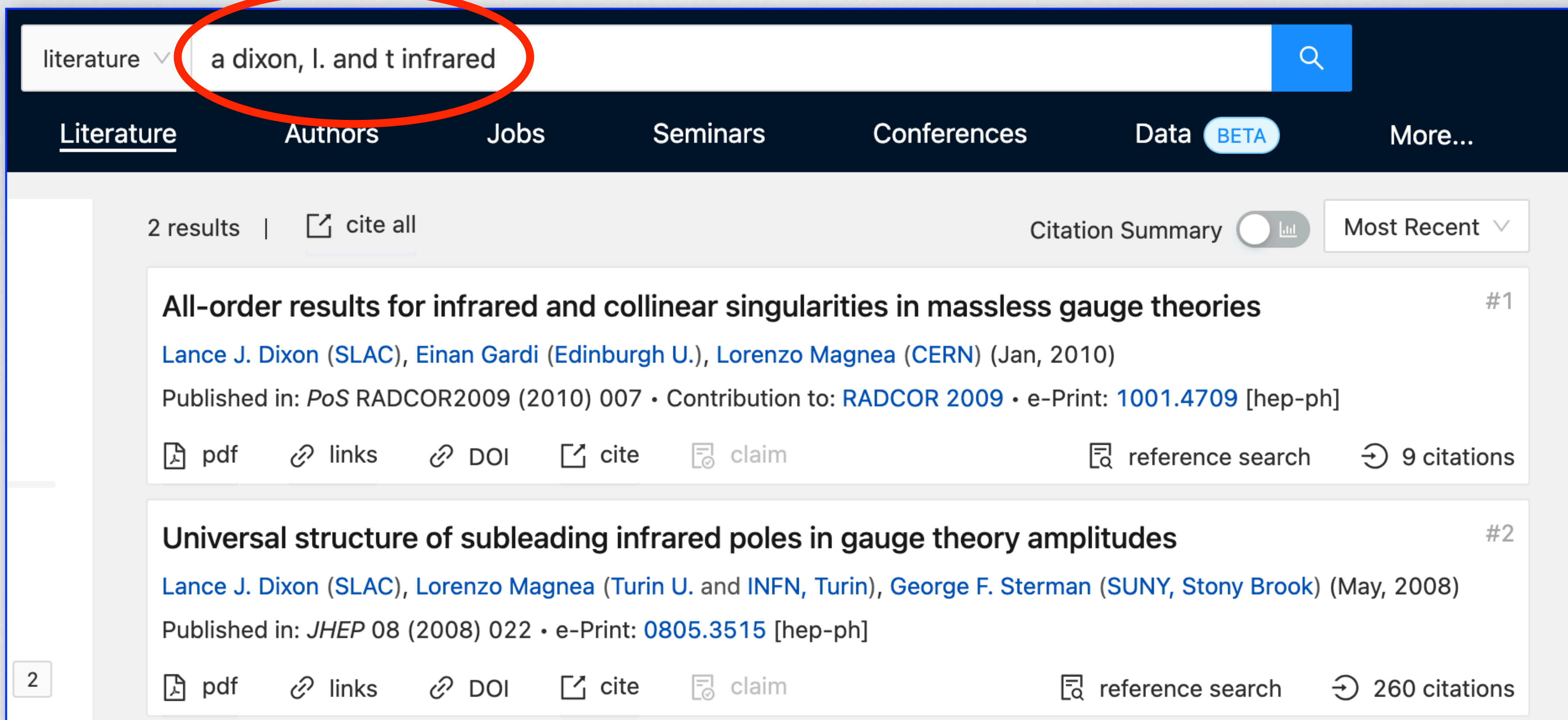
Lorenzo Magnea

Università di Torino - INFN Torino

LanceFest - Edinburgh - 24/06/2026

A LONG-DISTANCE STORY

A LONG-DISTANCE STORY



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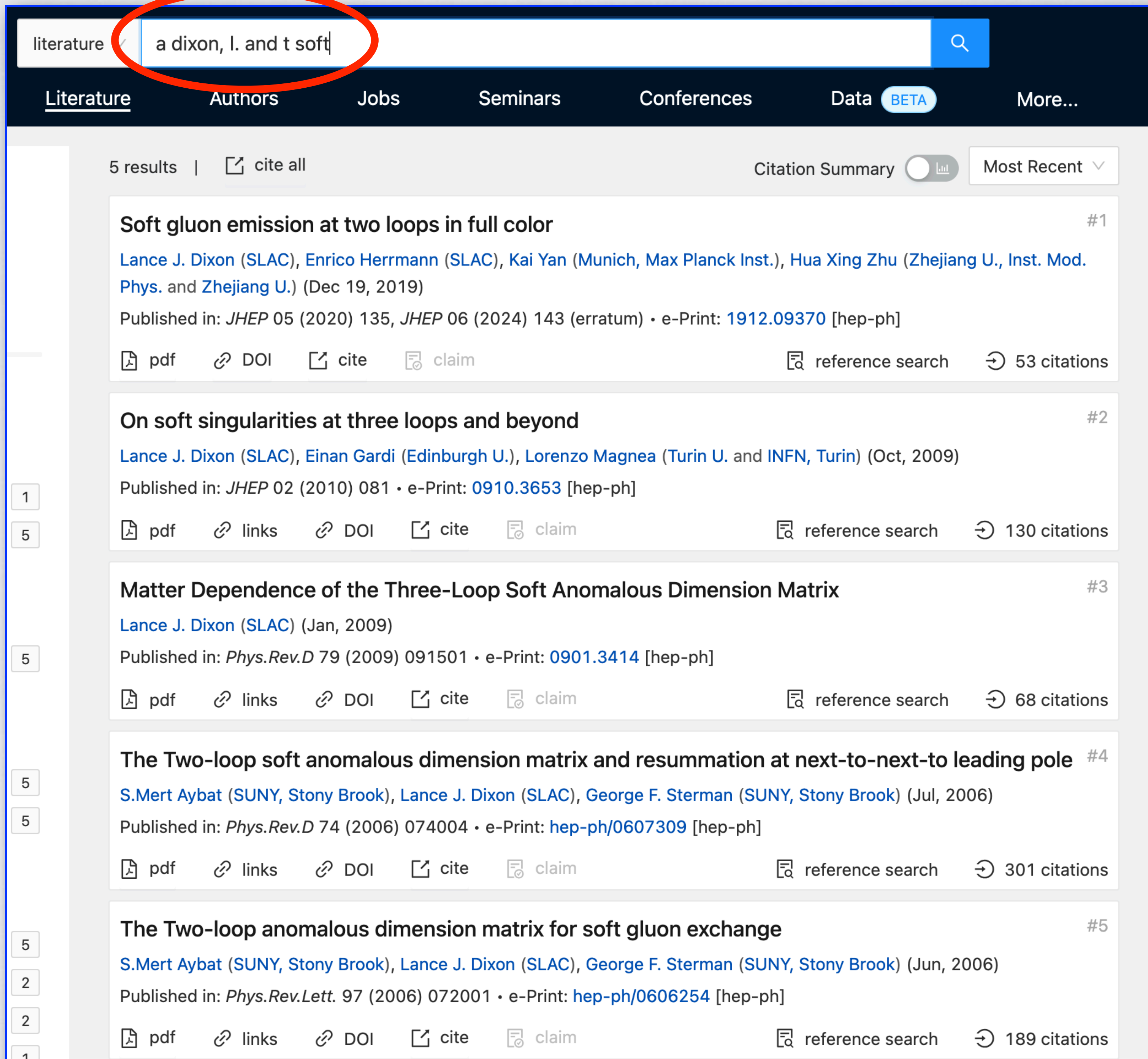

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Lance J. Dixon (SLAC), Enrico Herrmann (SLAC), Kai Yan (Munich, Max Planck Inst.), Hua Xing Zhu (Zhejiang U., Inst. Mod. Phys. and Zhejiang U.) (Dec 19, 2019)
Published in: *JHEP* 05 (2020) 135, *JHEP* 06 (2024) 143 (erratum) • e-Print: 1912.09370 [hep-ph]
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On soft singularities at three loops and beyond #2
Lance J. Dixon (SLAC), Einan Gardi (Edinburgh U.), Lorenzo Magnea (Turin U. and INFN, Turin) (Oct, 2009)
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Matter Dependence of the Three-Loop Soft Anomalous Dimension Matrix #3
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The Two-loop soft anomalous dimension matrix and resummation at next-to-next-to leading pole #4
S.Mert Aybat (SUNY, Stony Brook), Lance J. Dixon (SLAC), George F. Sterman (SUNY, Stony Brook) (Jul, 2006)
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The Two-loop anomalous dimension matrix for soft gluon exchange #5
S.Mert Aybat (SUNY, Stony Brook), Lance J. Dixon (SLAC), George F. Sterman (SUNY, Stony Brook) (Jun, 2006)
Published in: *Phys.Rev.Lett.* 97 (2006) 072001 • e-Print: hep-ph/0606254 [hep-ph]
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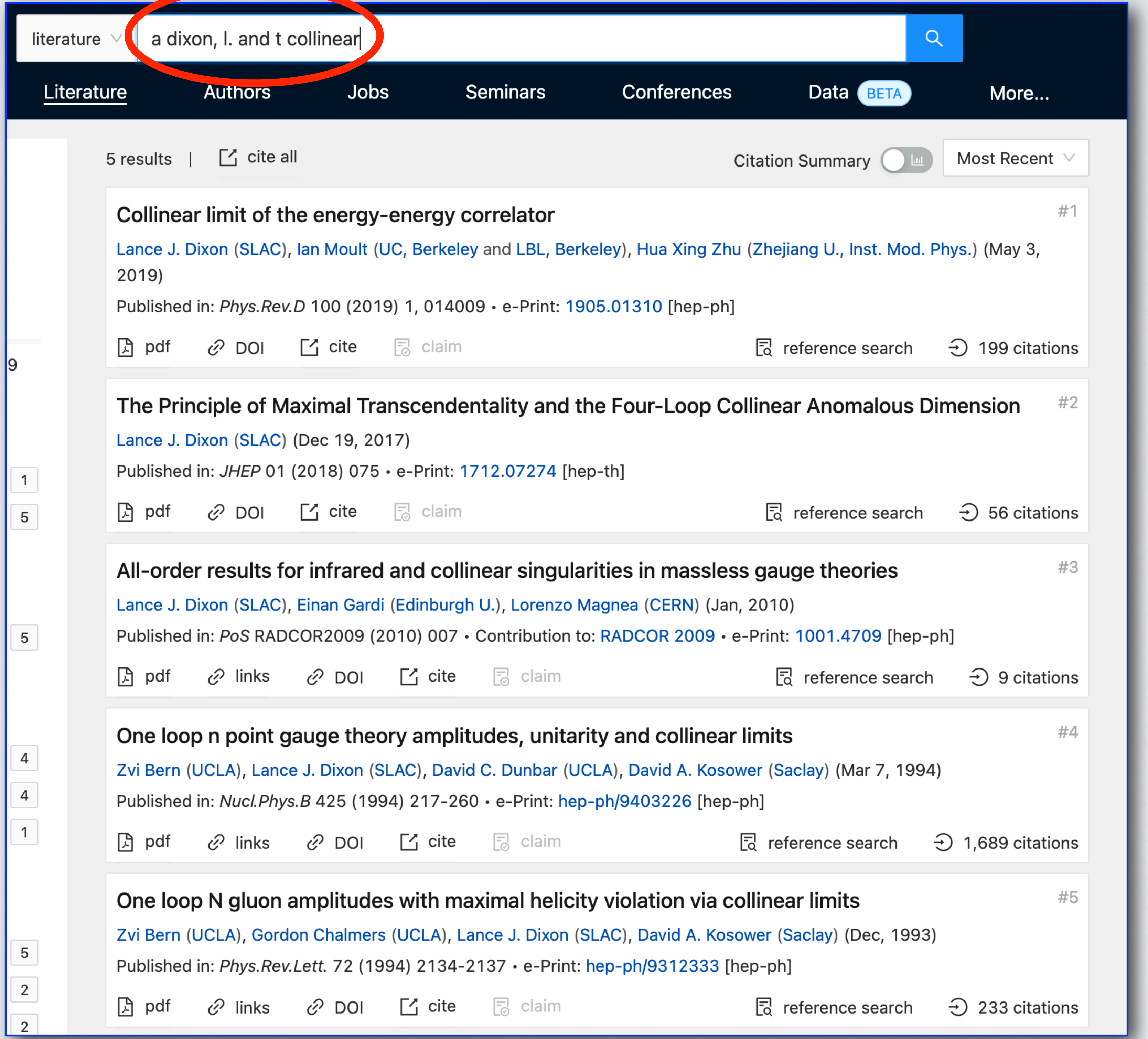

Outline

- Infrared factorisation of non-abelian amplitudes …
- …on the celestial sphere
- Celestial soft radiation
- Outlook

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In collaboration with

Enrico Zunino

[2512.22104]

INFRARED FACTORISATION …

INFRARED FACTORISATION …

The factorised amplitude

A. Sen, A.H. Mueller, J. Collins, G. Sterman, J. Botts, G. Korchemsky, A. Radyushkin, LM, S. Catani, **L. Dixon**, E. Gardi, T. Becher, M. Neubert, I. Feige, M. Schwartz, O. Erdogan, Y. Ma, …

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The factorised amplitude

Infrared divergences in fixed-angle *massless* multi-particle scattering amplitudes factorise

$$\mathcal{A}_n\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \mathcal{Z}_n\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) \mathcal{F}_n\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) ,$$

The infrared factor is a colour operator determined by a finite anomalous dimension matrix

$$\mathcal{Z}_n\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \mathcal{P} \exp\left[\frac{1}{2}\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2}\, \Gamma_n\left(\frac{p_i}{\lambda}, \alpha_s(\lambda^2, \epsilon)\right)\right] ,$$

A. Sen, A.H. Mueller, J. Collins, G. Sterman, J. Botts, G. Korchemsky, A. Radyushkin, LM, S. Catani, **L. Dixon**, E. Gardi, T. Becher, M. Neubert, I. Feige, M. Schwartz, O. Erdogan, Y. Ma, …

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All infrared poles arise from the scale integration, through the d-dimensional running coupling

$$\lambda \frac{\partial \alpha_s}{\partial \lambda} \equiv \beta(\alpha_s, \epsilon) = -2\epsilon\alpha_s - \frac{\alpha_s^2}{2\pi} \sum_{k=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^k b_k \, .$$

For massless theories, the all-order structure of the anomalous dimension in known, up to corrections due to higher-order Casimir operators of the gauge algebra

$$\Gamma_n\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right) = \Gamma_n^{\rm dip}\left(\frac{s_{ij}}{\mu^2}, \alpha_s(\mu^2)\right) + \Delta_n\left(\rho_{ijkl}, \alpha_s(\mu^2)\right) ,$$

$$\rho_{ijkl} = \frac{p_i \cdot p_j \, p_k \cdot p_l}{p_i \cdot p_l \, p_j \cdot p_k} = \frac{s_{ij} s_{kl}}{s_{il} s_{jk}} \, .$$

A. Sen, A.H. Mueller, J. Collins, G. Sterman, J. Botts, G. Korchemsky, A. Radyushkin, LM, S. Catani, **L. Dixon**, E. Gardi, T. Becher, M. Neubert, I. Feige, M. Schwartz, O. Erdogan, Y. Ma, …

The factorised amplitude

Infrared divergences in fixed-angle multi-particle scattering amplitudes factorise

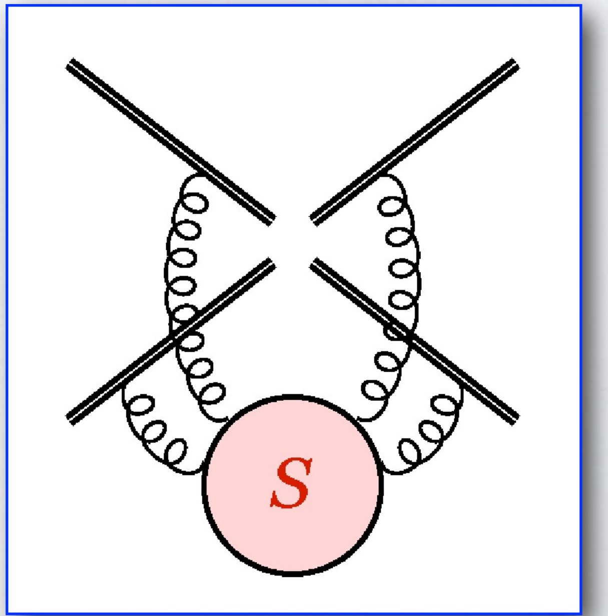
$$\mathcal{A}_n\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \mathcal{Z}_n\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) \mathcal{F}_n\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right),$$

The infrared factor is a colour operator determined by a finite anomalous dimension matrix

$$\mathcal{Z}_n\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \mathcal{P}\exp\left[\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\,\Gamma_n\left(\frac{p_i}{\lambda}, \alpha_s(\lambda^2, \epsilon)\right)\right],$$

Color correlations in the infrared factor are purely soft, and are computed by a correlator of semi-infinite light-like Wilson lines, following the classical particle trajectories.

$$\mathcal{S}_n\left(\beta_i \cdot \beta_j\right) = \langle 0|\, T\left[\prod_{k=1}^{n}\Phi_{\beta_k}(\infty, 0)\right]|0\rangle,$$



The soft factor for fixed-angle scattering amplitudes is naturally defined on the celestial sphere.

E. Gardi, LM; T. Becher, M. Neubert.

The dipole formula

Let's take a closer look at the structure of the infrared anomalous dimension matrix.

The dipole term :

$$\Gamma_n^{\rm dip}\left(\frac{s_{ij}}{\mu^2},\alpha_s(\mu^2)\right) = \frac{1}{2}\widehat{\gamma}_K\left(\alpha_s(\mu^2)\right)\sum_{i=1}^{n}\sum_{j=i+1}^{n}\log\left(\frac{s_{ij}\,{\rm e}^{{\rm i}\pi\lambda_{ij}}}{\mu^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j+\sum_{i=1}^{n}\gamma_i\left(\alpha_s(\mu^2)\right)\,,$$

The cusp anomalous dimension in the `Casimir scaling' limit:

$$\gamma_{K,r}(\alpha_s) = C_r^{(2)}\,\widehat{\gamma}_K(\alpha_s)\,,$$

Corrections start at three loops, with quadrupoles:

$$F_{ijkl}(\{\rho\})\,f_{abe}f_{cd}{}^{e}\,\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\mathbf{T}_l^d\,,$$

Ø. Almelid, C. Duhr, E. Gardi; J. Henn, B. Mistlberger.

- The colour dipole is the natural structure arising at one loop from gluon exchange.
- The fact that it survives at two loops and beyond is a non-trivial consequence of symmetries.
- Field anomalous dimensions in color-uncorrelated terms govern collinear singularities.
- Unitarity phases contain crucial analytic information. For final-state pairs: $\lambda_{ij} = 1$.
- The cusp anomalous dimension plays a very special role: a universal infrared coupling.
- The structure emerges from the constraints of scale invariance in the soft limit.

A Lance story

A Lance story

PRL **97,** 072001 (2006) PHYSICAL REVIEW LETTERS week ending 18 AUGUST 2006

Two-Loop Anomalous-Dimension Matrix for Soft-Gluon Exchange

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[2]*Stanford Linear Accelerator Center, Stanford University, Stanford, California 94309, USA*

(Received 26 June 2006; published 17 August 2006)

The resummation of soft-gluon exchange for QCD hard scattering requires a matrix of anomalous dimensions. We compute this matrix directly for arbitrary $2 \to n$ massless processes for the first time at two loops. Using color-generator notation, we show that it is proportional to the one-loop matrix. This result reproduces all pole terms in dimensional regularization of the explicit calculations of massless $2 \to 2$ amplitudes in the literature, and it predicts all poles at next-to-next-to-leading order in any $2 \to n$ process that has been computed at next-to-leading order. The proportionality of the one- and two-loop matrices makes possible the resummation in closed form of the next-to-next-to-leading logarithms and poles in dimensional regularization for the $2 \to n$ processes.

DOI: 10.1103/PhysRevLett.97.072001 PACS numbers: 12.38.Cy, 11.15.Bt, 12.38.Bx, 12.39.St

A Lance story

PRL **97,** 072001 (2006) PHYSICAL REVIEW LETTERS week ending 18 AUGUST 2006

Two-Loop Anomalous-Dimension Matrix for Soft-Gluon Exchange

PHYSICAL REVIEW D **74,** 074004 (2006)

Two-loop soft anomalous dimension matrix and resummation at next-to-next-to-leading poles

S. Mert Aybat,[1] Lance J. Dixon,[2] and George Sterman[1]

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[2]*Stanford Linear Accelerator Center, Stanford University, Stanford, California 94309, USA*
(Received 28 July 2006; published 10 October 2006)

We extend the resummation of dimensionally regulated amplitudes to next-to-next-to-leading poles. This requires the calculation of two-loop anomalous dimension matrices for color mixing through soft gluon exchange. Remarkably, we find that they are proportional to the corresponding one-loop matrices. Using the color-generator notation, we reproduce the two-loop single-pole quantities $\mathbf{H}^{(2)}$ introduced by Catani for quark and gluon elastic scattering. Our results also make possible threshold and a variety of other resummations at next-to-next-to-leading logarithm. All of these considerations apply to $2 \to n$ processes with massless external lines.

DOI: 10.1103/PhysRevD.74.074004 PACS numbers: 12.38.Cy, 11.15.Bt, 12.38.Bx, 12.39.St

A Lance story

PRL **97,** 072001 (2006) PHYSICAL REVIEW LETTERS week ending 18 AUGUST 2006

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Factorization constraints for soft anomalous dimensions in QCD scattering amplitudes

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On soft singularities at three loops and beyond

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A Lance story

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Factorization c… dimensions in …

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PRL **117,** 172002 (2016) PHYSICAL REVIEW LETTERS week ending 21 OCTOBER 2016

Three-Loop Corrections to the Soft Anomalous Dimension in Multileg Scattering

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[2] *CERN Theory Division, 1211 Geneva 23, Switzerland*

[3] *Center for Cosmology, Particle Physics and Phenomenology (CP3), Université Catholique de Louvain, 1348 Louvain-La-Neuve, Belgium*

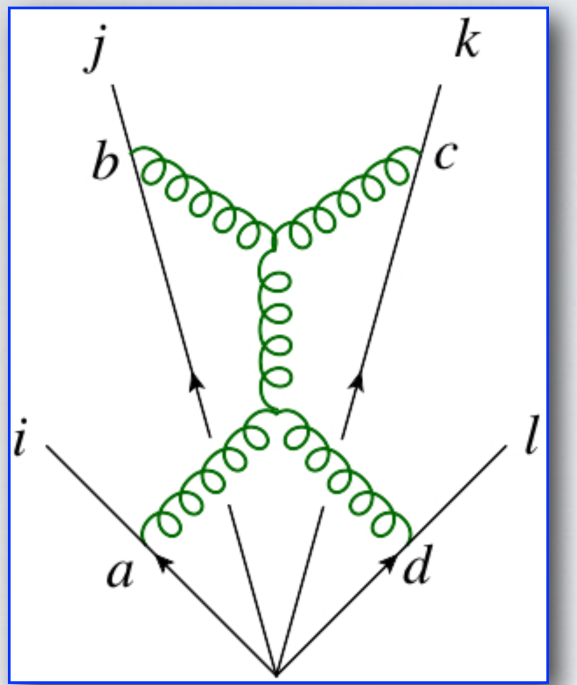
(Received 26 May 2016; published 20 October 2016)

We present the three-loop result for the soft anomalous dimension governing long-distance singularities of multileg gauge-theory scattering amplitudes of massless partons. We compute all contributing webs involving semi-infinite Wilson lines at three loops and obtain the complete three-loop correction to the dipole formula. We find that nondipole corrections appear already for three colored partons, where the correction is a constant without kinematic dependence. Kinematic dependence appears only through conformally invariant cross ratios for four colored partons or more, and the result can be expressed in terms of single-valued harmonic polylogarithms of weight five. While the nondipole three-loop term does not vanish in two-particle collinear limits, its contribution to the splitting amplitude anomalous dimension reduces to a constant, and it depends only on the color charges of the collinear pair, thereby preserving strict collinear factorization properties. Finally, we verify that our result is consistent with expectations from the Regge limit.

DOI: 10.1103/PhysRevLett.117.172002

Life beyond dipoles

- Quadrupole corrections to the correlator at three loops and beyond



Ø. Almelid, C. Duhr, E. Gardi; with A. McLeod and C. White; J. Henn, B. Mistlberger.

$$\Delta_n^{(3)}(\rho_{ijkl}) = 16\, f_{abe} f_{cde} \Big\{ -C \sum_{i=1}^{n} \sum_{\substack{1\leq j<k\leq n \\ j,k\neq i}} \big\{\mathbf{T}_i^a, \mathbf{T}_i^d\big\}\, \mathbf{T}_j^b \mathbf{T}_k^c + \sum_{1\leq i<j<k<l\leq n} \Big[\mathbf{T}_i^a \mathbf{T}_j^b \mathbf{T}_k^c \mathbf{T}_l^d\, \mathcal{F}(\rho_{ikjl}, \rho_{iljk}) + \mathbf{T}_i^a \mathbf{T}_k^b \mathbf{T}_j^c \mathbf{T}_l^d\, \mathcal{F}(\rho_{ijkl}, \rho_{ilkj}) + \mathbf{T}_i^a \mathbf{T}_l^b \mathbf{T}_j^c \mathbf{T}_k^d\, \mathcal{F}(\rho_{ijlk}, \rho_{iklj}) \Big] \Big\},$$

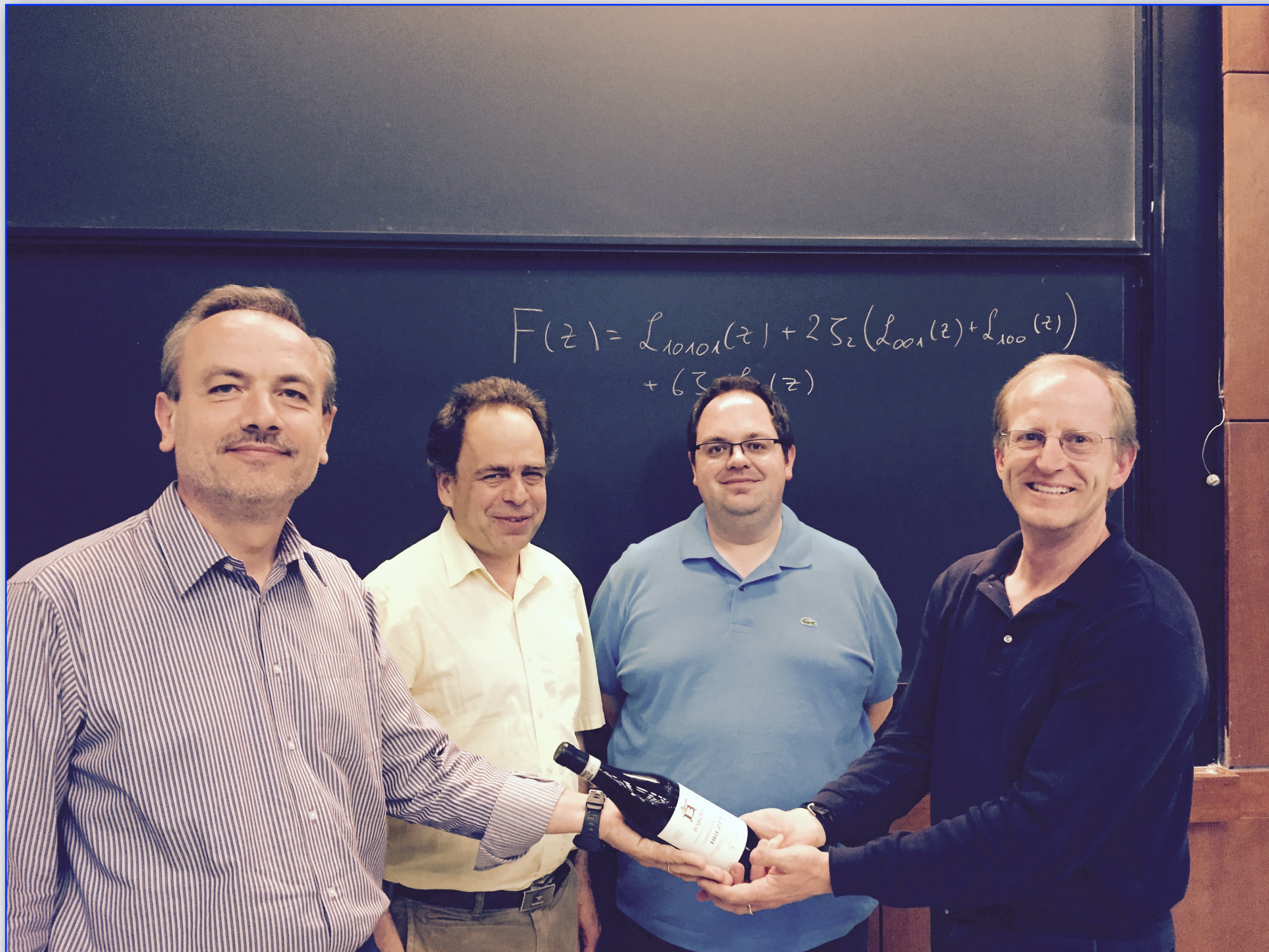
$$\mathcal{F}(\rho_{ijkl}, \rho_{ilkj}) = F(1 - z_{ijkl}) - F(z_{ijkl}),$$

$$z_{ijkl}\bar{z}_{ijkl} = \rho_{ijkl}, \quad (1 - z_{ijkl})(1 - \bar{z}_{ijkl}) = \rho_{ilkj}$$

$$F(z) = \mathcal{L}_{10101}(z) + 2\zeta_2[\mathcal{L}_{001}(z) + \mathcal{L}_{100}(z)].$$

- Multipole corrections to any order must be functions of scale-invariant cross-ratios.
- They are naturally defined on the celestial sphere (punctured by Wilson lines).
- They are further constrained by Bose symmetry and — non-trivially — by gauge invariance.
- At three loops they are entirely given by a simple combination of uniform-weight single-valued harmonic polylogarithms, functions that naturally appear in conformal fields theories.

Barolo beyond dipoles



… ON THE CELESTIAL SPHERE

Credit: NASA/JWST

… ON THE CELESTIAL SPHERE

… ON THE CELESTIAL SPHERE

Celestial coordinates

Crucially, we now parametrise the light-cone momenta in celestial coordinates

$$p_i^\mu = \frac{\omega_i}{\sqrt{2}} \left\{ 1 + z_i \bar{z}_i, z_i + \bar{z}_i, -\mathrm{i}(z_i - \bar{z}_i), 1 - z_i \bar{z}_i \right\} \equiv \omega_i \beta_i^\mu$$

$$\beta_i \cdot \beta_j = |z_{ij}|^2$$

where the energy ω_i and the sphere coordinates z_i have simple transformation properties under the Lorentz group acting as SL(2, C). We also need polarisation vectors

$$\varepsilon_+^\mu(q) = \frac{1}{\omega_q} \partial_z q^\mu = \frac{1}{\sqrt{2}} \{ \bar{z}, 1, -\mathrm{i}, -\bar{z} \}$$

$$\varepsilon_+(q) \cdot \beta_i = \bar{z}_{qi}$$

Mandelstam invariants are distances on the sphere

$$s_{ij} = 2 p_i \cdot p_j = \omega_i \omega_j \, |z_{ij}|^2$$

which unpacks the logarithms

$$\log(-s_{ij} + \mathrm{i}\eta) = \log\left(|z_{ij}|^2\right) + \log \omega_i + \log \omega_j + \mathrm{i}\pi$$

Energies give new singlet terms

$$\Gamma_n^{\text{dipole}} \left(\frac{s_{ij}}{\lambda^2}, \alpha_s(\lambda, \epsilon) \right) \equiv \widehat{\Gamma}_n^{\text{corr.}} \Big(z_{ij}, \alpha_s(\lambda, \epsilon) \Big) + \widehat{\Gamma}_n^{\text{singl.}} \left(\frac{\omega_i}{\lambda}, \alpha_s(\lambda, \epsilon) \right),$$

which take the form

$$\widehat{\Gamma}_n^{\text{singl.}} \left(\frac{\omega_i}{\lambda}, \alpha_s(\lambda, \epsilon) \right) = - \sum_{i=1}^n \gamma_i \big(\alpha_s(\lambda, \epsilon) \big) - \frac{1}{4} \widehat{\gamma}_K \big(\alpha_s(\lambda, \epsilon) \big) \sum_{i=1}^n \ln \left(\frac{- \, \omega_i^2 + \mathrm{i}\eta}{\lambda^2} \right) C_i^{(2)},$$

Celestial dipoles

The colour-correlated term, responsible for all soft poles, is remarkably simple

$$\widehat{\Gamma}_n^{\text{corr.}}\Big(z_{ij}, \alpha_s(\lambda,\epsilon)\Big) = \frac{1}{2}\,\widehat{\gamma}_K\big(\alpha_s(\lambda,\epsilon)\big) \sum_{i=1}^{n} \sum_{j=i+1}^{n} \ln\Big(|z_{ij}|^2\Big)\, \mathbf{T}_i \cdot \mathbf{T}_j \,.$$

Scale and coupling dependence are completely factored from colour and kinematics, and equal for all dipoles. The scale integral can this be performed in full generality, yielding

$$\begin{aligned}
\mathcal{Z}_n^{\text{corr.}}\Big(z_{ij}, \alpha_s(\mu), \epsilon\Big) &\equiv \exp\left[\int_0^{\mu} \frac{d\lambda}{\lambda}\, \widehat{\Gamma}_n^{\text{corr.}}\Big(z_{ij}, \alpha_s(\lambda,\epsilon)\Big)\right] \\
&= \exp\left[- K\big(\alpha_s(\mu),\epsilon\big) \sum_{i=1}^{n} \sum_{j=i+1}^{n} \ln\Big(|z_{ij}|^2\Big)\, \mathbf{T}_i \cdot \mathbf{T}_j\right],
\end{aligned}$$

The scale factor K is well-known in QCD from form-factor calculations, and gives the perturbative Regge trajectory in the high-energy limit of four-point amplitudes. It is

J. Collins, D. Soper; G. Korchemsky, I.A. Korchemskaya;
V. Del Duca, C. Duhr, E. Gardi, LM, C. White;
G. Falcioni, L. Vernazza, …

$$K\big(\alpha_s(\mu),\epsilon\big) = -\frac{1}{2} \int_0^{\mu} \frac{d\lambda}{\lambda}\, \widehat{\gamma}_K\big(\alpha_s(\lambda,\epsilon)\big)\,.$$

The function K can be computed order by order in terms of the cusp and the β function

$$\begin{aligned}
K(\alpha_s,\epsilon) = \;& \frac{\alpha_s}{\pi}\frac{\widehat{\gamma}_K^{(1)}}{4\epsilon} + \left(\frac{\alpha_s}{\pi}\right)^2 \left(\frac{\widehat{\gamma}_K^{(2)}}{8\epsilon} + \frac{b_0\widehat{\gamma}_K^{(1)}}{32\epsilon^2}\right) \\
&+ \left(\frac{\alpha_s}{\pi}\right)^3 \left(\frac{\widehat{\gamma}_K^{(3)}}{12\epsilon} + \frac{b_0\widehat{\gamma}_K^{(2)} + b_1\widehat{\gamma}_K^{(1)}}{48\epsilon^2} + \frac{b_0^2\,\widehat{\gamma}_K^{(1)}}{192\epsilon^3}\right) + \mathcal{O}\big(\alpha_s^4\big),
\end{aligned}$$

$\xrightarrow{\;\beta \to 0\;}$

$$K(\alpha_s,\epsilon) = \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \frac{\widehat{\gamma}_K^{(n)}}{4n\epsilon}\,,$$

A celestial conformal theory

It is natural to mimic the bosonic string, considering free bosons spanning the gauge algebra.

$$S(\phi) \,=\, \frac{1}{2\pi} \int d^2 z\, \partial_z \phi^a(z,\bar{z})\, \partial_{\bar{z}} \phi_a(z,\bar{z})\,,$$

The free bosons could be organised in a matrix field :
gauge generators at different points must then be taken to commute

$$\Phi_r(z,\bar{z}) \,\equiv\, \phi_a(z,\bar{z})\, T^a_{r,\,z}\,,$$

The well-known results for free bosons in d=2 can be directly transcribed.

The equations of motions are:

$$\partial_z\, \partial_{\bar{z}}\, \phi^a(z,\bar{z}) \,=\, 0\,,$$

implying that the derivatives of the fields are (anti)holomorphic

A normal-ordered product can be defined, obeying the classical equation of motion

$$:\phi^a(z,\bar{z})\, \phi^b(w,\overline{w}): \,=\, \phi^a(z,\bar{z})\, \phi^b(w,\overline{w}) + \frac{1}{2}\, \delta^{ab} \log|z-w|^2\,,$$

There is a traceless conserved energy-momentum tensor, and a conserved Noether current

$$T(z) \,=\, -\, :\partial_z \phi^a(z,\bar{z})\, \partial_z \phi_a(z,\bar{z}):\,,$$

$$j^a(z) \,=\, \partial_z \phi^a(z,\bar{z})\,,$$

A. Nande, M. Pate, A. Strominger; N. Kalyanapuram (QED). LM.

Matrix vertex operators

Guided by the QED example, we can tentatively define a matrix-valued vertex operator

$$V(z,\bar{z}) \equiv :\, \mathrm{e}^{\mathrm{i}\kappa\, \mathbf{T}_z \cdot \phi(z,\bar{z})} : = :\, \mathrm{e}^{\mathrm{i}\kappa \Phi(z,\bar{z})} : ,$$

Colour-kinematic dual of the string vertex operator !

In colour space, this is a matrix in the representation of $\mathbf{T}_z$, defined on the boundary sphere and acting on the bulk colour degrees of freedom. But is it a conformal primary field?

For conventional vertex operators (as for example for bosonic strings)

$$V_{\mathrm{c.s.}}(z,\bar{z}) \equiv :\, \mathrm{e}^{\mathrm{i}k^\mu X_\mu(z,\bar{z})} : \quad \longrightarrow \quad h = \frac{1}{4} k^\mu k^\nu \eta_{\mu\nu} = \frac{k^2}{4} ,$$

The same calculation yields

$$V(z,\bar{z}) \equiv :\, \mathrm{e}^{\mathrm{i}\kappa\, \mathbf{T}_z \cdot \phi(z,\bar{z})} : \quad \longrightarrow \quad h = \frac{\kappa^2}{4} \mathbf{T}_z \cdot \mathbf{T}_z = \frac{\kappa^2}{4} C_r^{(2)} ,$$

Crucially, this is a positive real number and not a matrix. For consistency, two-point functions must evaluate to a power of the distance given by the conformal weight $\Delta = h + \bar{h}$. Indeed

$$\langle V(z_1,\bar{z}_1) V(z_2,\bar{z}_2) \rangle \sim |z_{12}|^{-2\Delta} ,$$

by colour conservation $\mathbf{T}_1 + \mathbf{T}_2 = 0$

Note analogies with other constructions.

Vertex operator construction of Kac-Moody algebras: $U^\alpha(z) = z^{\alpha^2/2} :e^{i\alpha\cdot Q(z)}: .$

Reggeon fields for high-energy scattering: $U(z) = e^{i g_s T^a W^a(z)} .$

(Caron-Huot 2013)

A conformal correlator

Our construction from the beginning targeted the n-point correlator

$$\mathcal{C}_n\Big(\{z_i\},\kappa\Big) \equiv \Big\langle \prod_{i=1}^{n} V(z_i,\bar{z}_i) \Big\rangle .$$

The calculation is a textbook exercise: it can be done with oscillators, after expanding the free fields in modes on the sphere, or computing the path integral (Polchinski). The result is

$$\mathcal{C}_n\Big(\{z_i\},\kappa\Big) = C(N_c)\,\exp\left[\frac{\kappa^2}{2}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\ln\Big(|z_{ij}|^2\Big)\,\mathbf{T}_i\cdot\mathbf{T}_j\right],$$

reproducing the structure of the gauge theory infrared operator. Note that

$$\sum_{i=1}^{n}\mathbf{T}_i = 0,$$

- The correlator has support only on colour conserving configurations
- The field normalisation κ maps to the integral K, carrying scale and regulator dependence.
- In a path integral evaluation on a curved surface (say, a finite sphere with radius R) the correlator acquires a scale-dependent `Weyl' factor, which in this setting maps to an (undetermined) colour-singlet collinear contribution.

$$\mathcal{W}_n\Big(\{z_i\},\kappa\Big) = \exp\left[-\frac{1}{2}\sum_{i=1}^{n}C_i^{(2)}\,g(z_i,\bar{z}_i)\right],$$

Preliminary - With Enrico Zunino and Gabriel Macedo

Towards an interacting theory?

- Interactions in the d=2 theory are constrained by gauge and euclidean invariance, whether the theory is conformal or not. With up to four fields one finds

$$\mathcal{L}\left(\phi^a\right) = \frac{1}{2}\,\partial_\mu \phi^a \partial^\mu \phi_a + \mathrm{i}\,\frac{\lambda_1}{6}\,\varepsilon^{\mu\nu}\, f^{abc}\,\phi_a\,\partial_\mu \phi_b\,\partial_\nu \phi_c - \frac{\lambda_2}{24}\, f^{abe} f_e{}^{cd}\,\phi_b\,\phi_d\,\partial_\mu \phi_a\,\partial^\mu \phi_c + \ldots$$

In fact, $\lambda_1 = \lambda_2 = 1$ yields the leading terms of the WZNW action, while $\lambda_1 = 0$, $\lambda_2 = 1$ yields the principal chiral model, which is not conformally invariant.

- Correlators in the WZNW model must obey the Knizhnik-Zamolodchikov equation, but this fails for the gauge-theory correlator, and cannot be compensated by quadrupoles.

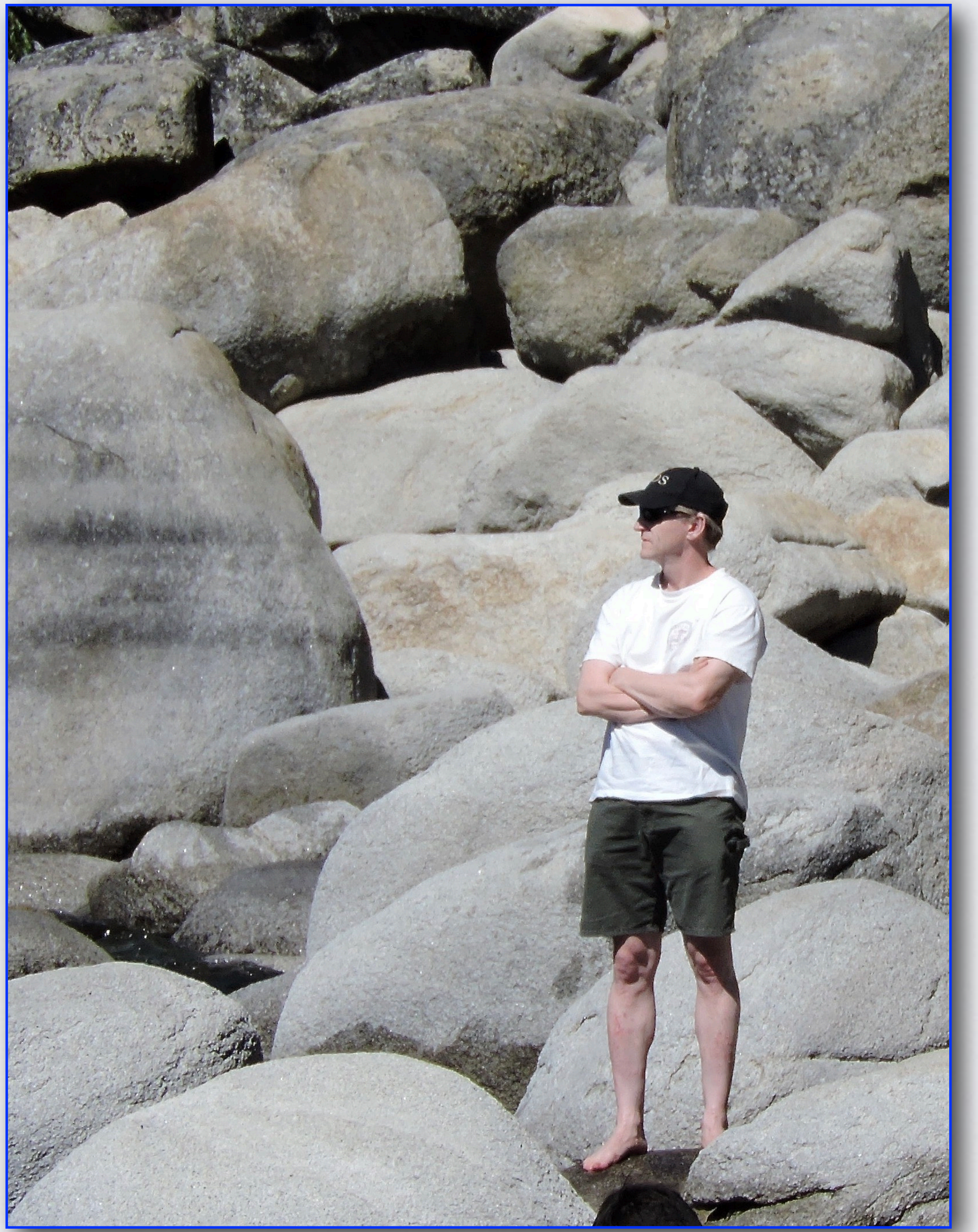
$$\mathcal{Z}_n\left(z_i\right) \equiv \exp\left[\mathcal{E}_n\left(z_i\right)\right] \quad \rightarrow \quad \frac{\partial \mathcal{Z}_n}{\partial z_i} = K\,\mathbf{T}_i \cdot \sum_{k \neq i} \frac{\mathbf{T_k}}{z_k - z_i} + \frac{1}{2}\left[\mathcal{E}_n\left(z_i\right), \frac{\partial \mathcal{E}_n}{\partial z_i}\right] + \ldots$$

H. Nastase, F. Rojas, C. Rubio (2021)

Giving up full d=2 conformal invariance? A new colourful CFT? A deformed CFT? **not** a CFT?

SOFT COLOURED RADIATION

SOFT COLOURED RADIATION



A. Strominger, T. He, P. Mitra, A. Nande, M. Pate,
W. Fan, A. Fotopoulos, T.R. Taylor, …

A tree-level soft theorem

Real emission of a soft massless gauge boson from a fixed angle hard amplitude factorises in any non-abelian theory in the form

$$\langle c| \otimes \langle \lambda| \mathcal{A}_{g,f_1 \ldots f_n}(k, p_1, \ldots, p_n)\rangle_{\rm soft} = \epsilon_\lambda(k) \cdot J^c(k) \, |\mathcal{A}_{f_1 \ldots f_n}(p_1, \ldots, p_n)\rangle \, ,$$

The tree-level soft-gluon current has the classic eikonal form and is gauge-invariant

$$\mathbf{J}^\mu(k) = g \sum_{i=1}^n \mathbf{T}_i \, \frac{\beta_i^\mu}{\beta_i \cdot k} \, ,$$

$$k \cdot \mathbf{J}^\mu(k) = g \sum_{i=1}^n \mathbf{T}_i = 0 \, ,$$

The tree-level soft theorem is reproduced by the Ward identity for the Noether current associated with invariance under field translations in the Lie algebra. Using the conformal operator product expansion one finds

$$\left\langle \partial_z \phi^a(z, \bar{z}) \prod_{i=1}^n V(z_i, \bar{z}_i) \right\rangle \simeq -\frac{\mathrm{i}}{2} \sum_{i=1}^n \frac{\mathbf{T}_i^a}{z - z_i} \, \mathcal{C}_n\Big(\{z_i\}, \kappa\Big) \, .$$

where the poles as $z \to z_i$ are collinear poles, since the celestial theory is energy-independent.

Z. Bern, V. del Duca, C. Schmidt; S. Catani, M. Grazzini

A one-loop soft theorem

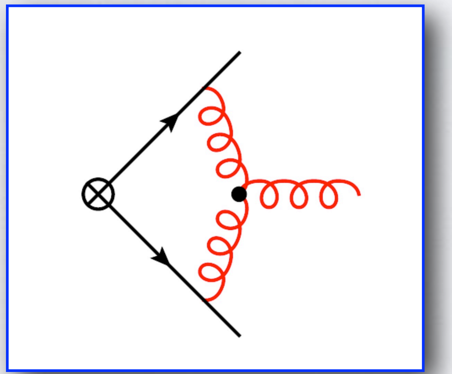
For single emission, the general formalism defines the single soft current to all orders by

$$\mathcal{A}^{\lambda}_{n+1}\left(q;\frac{p_i}{\mu}\right) = \mathbf{S}^{\lambda}(q;\{\beta_i\})\,\mathcal{A}_n\left(\frac{p_i}{\mu}\right) = \varepsilon^{\lambda}(q)\cdot\mathbf{J}(q,\{\beta_i\})\,\mathcal{A}_n\left(\frac{p_i}{\mu}\right)$$

At tree level, one recovers the well-known result: it is remarkably simple in celestial coordinates.

$$\omega_q\,\mathbf{S}^{+,(0)}(q) = \omega_q\sum_{i=1}^{n}\frac{\varepsilon_+\cdot\beta_i}{q\cdot\beta_i}\,\mathbf{T}_i = \sum_{i=1}^{n}\frac{1}{z_{qi}}\,\mathbf{T}_i$$

In massive QED the tree-level current is exact. In massless QED it has corrections starting at three loops (Y. Ma, G. Sterman, A. Venkata). In a non-abelian theory, at one loop,



$$\mathbf{S}^{\lambda,(1)}_a(q) = \mathrm{i}C_1(\epsilon)f_{abc}\sum_{i\neq j}^{n}\mathbf{T}^b_i\mathbf{T}^c_j\left(\frac{\varepsilon^{\lambda}\cdot\beta_i}{\beta_i\cdot q}-\frac{\varepsilon^{\lambda}\cdot\beta_j}{\beta_j\cdot q}\right)\left(\frac{\mu^2 s_{ij}}{s_{qi}s_{qj}}\right)^{\epsilon}$$

$$C_1(\epsilon) = -\frac{e^{\epsilon\gamma_E}}{\epsilon^2}\frac{\Gamma^3(1-\epsilon)\Gamma^2(1+\epsilon)}{\Gamma(1-2\epsilon)} = -\frac{1}{\epsilon^2}-\frac{\zeta(2)}{2}+\frac{7}{2}\zeta(3)\epsilon+\mathcal{O}(\epsilon^2)$$

- The structure is determined by power counting, scale and gauge invariance, and Bose symmetry.
- The expression is for the bare current. It renormalises with the non-abelian coupling.
- The double pole in the pre-factor forces the presence of logarithms in the ε expansion.

With Enrico Zunino: 2512.22104

Choosing scales

A remarkable fact about the soft current helps organise the logarithms, to all orders. We use the d-dimensional running coupling

$$\alpha_s(\mu) = \left(\frac{\mu_0}{\mu}\right)^{2\epsilon} \alpha_s(\mu_0) + \frac{b_0}{4\pi\epsilon}\left(\frac{\mu_0}{\mu}\right)^{2\epsilon}\left[\left(\frac{\mu_0}{\mu}\right)^{2\epsilon} - 1\right]\alpha_s^2(\mu_0) + \mathcal{O}\big(\alpha_s^3(\mu_0)\big)\,.$$

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and we identify the scale factor as the transverse momentum relative to each color dipole

$$\beta_i^\mu = \{1, 0, 0, 1\} \, , \qquad \beta_j^\mu = \{1, 0, 0, -1\} \, , \qquad q^\mu = \{q^0, \mathbf{q}_\perp, q^3\} \, .$$

$$\frac{s_{ij}}{s_{iq} s_{jq}} = \frac{\beta_i \cdot \beta_j}{2(q \cdot \beta_i)(q \cdot \beta_j)} = \frac{1}{q_{ij}^2} \, ,$$

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We come to a remarkable expression for the renormalised one-loop single soft current

$$\frac{\alpha_s(\mu^2)}{4\pi}\, \omega_q\, \mathbf{S}_a^{+,(1)} = \mathrm{i} C_1(\epsilon) f_{abc} \sum_{i\neq j}^{n} \mathbf{T}_i^b \mathbf{T}_j^c \, \frac{z_{ij}}{z_{qi} z_{qj}} \, \frac{\alpha_s(q_{ij}^2, \epsilon)}{4\pi} ,$$

With Enrico Zunino: 2512.22104

Choosing scales

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- The expression is completely dictated by symmetry, power counting and scale naturalness.
- All logarithms in the expansion of the dipole term in the single soft-gluon current are RG logarithms.
- The scale choice has a long history in QCD, and is remarkably captured by dimensional regularisation.
- Does the celestial CFT need to be "logarithmic"? Or is the scale of the coupling external information?

L. Dixon, E. Herrman, K. Yan, H.X. Zhu;
F. Herzog, Y. Ma, B. Mistlberger, A. Suresh

Beyond one loop

The scale structure is preserved at two loops, where new color structures appear

$$\omega_q\, \mathbf{S}^{+,(2)}_{a,\mathrm{dip}}(q) = \mathrm{i}C_2(\epsilon) f_{abc} \sum_{i\neq j} \mathbf{T}^b_i \mathbf{T}^c_j\, \mathcal{Z}^q_{ij} \left(\Omega_q \left|\mathcal{Z}^q_{ij}\right|^2\right)^{2\epsilon}$$

$$\omega_q\, \mathbf{S}^{+,(2)}_{a,\mathrm{trip}}(q) = f^{aeb} f^{bcd} \sum_{i\neq k\neq j} \mathbf{T}^c_i \mathbf{T}^d_j \mathbf{T}^e_k \left(\Omega_q \left|\mathcal{Z}^q_{ij}\right|^2\right)^{2\epsilon} \times \left(1+\epsilon^2\zeta_2\right)\left[\mathcal{Z}^q_{ik} F(z_{iqjk},\epsilon) - \mathcal{Z}^q_{jk} F(z_{ikjq},\epsilon)\right]$$

- The scale factors are naturally raised to power **2ε**. The dipole pre-factor starts at **O(ε^{-4})**.
- The tripole is needed at two loops, to cancel (upon integration) three-loop virtual quadrupoles.
- The function **F** is a weight-4 combination of SVHPLs, depending on conformal cross ratios.
- Even for the tripole term, the scale factor is chosen dipole by dipole. In celestial coordinates

$$\left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^2 \omega_q\, \mathbf{S}^{+,(2)}_{a,\mathrm{dip}}(q) \longrightarrow \mathrm{i}\widetilde{C}_2(\epsilon) f_{abc} \sum_{i\neq j} \mathbf{T}^b_i \mathbf{T}^c_j \frac{z_{ij}}{z_{qi} z_{qj}} \left(\frac{\alpha_s(q^2_{ij})}{4\pi}\right)^2$$

$$\left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^2 \omega_q\, \mathbf{S}^{+,(2)}_{a,\mathrm{trip}}(q) \longrightarrow \left(1+\epsilon^2\zeta_2\right) f^{aeb} f^{bcd} \sum_{i\neq k\neq j} \mathbf{T}^c_i \mathbf{T}^d_j \mathbf{T}^e_k \times \left[\frac{z_{ik}}{z_{qi} z_{qk}} F(z_{iqjk},\epsilon) - \frac{z_{jk}}{z_{qj} z_{qk}} F(z_{ikjq},\epsilon)\right] \left(\frac{\alpha_s(q^2_{ij})}{4\pi}\right)^2$$

Conjecture: ALL logarithms in the single current are of UV origin, resummed by the scale choice.

OUTLOOK

Great achievements …

… call for great accolades

but … rumours of “retirement”?

but … rumours of “retirement”?

but … rumours of "retirement"?

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Perhaps a new career?

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There is always …

There is always …

	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$	$L=6$	$L=7$	$L=8$
Adjacency-allowed	6	102	1830	32,838	589,254	$1.1 \cdot 10^7$	$1.9 \cdot 10^8$	$3.4 \cdot 10^9$
Suffix-forbidden	0	48	864	15,504	278,208	$5.0 \cdot 10^6$	$9.0 \cdot 10^7$	$1.6 \cdot 10^9$
Prefix-forbidden	0	6	78	1,206	21,438	$3.8 \cdot 10^5$	$6.9 \cdot 10^6$	$1.2 \cdot 10^8$
Remaining allowed	6	48	888	16,128	289,608	$5.2 \cdot 10^6$	$9.3 \cdot 10^7$	$1.67 \cdot 10^9$
Zeroes	0	36	252	4,920	25,728	$2.8 \cdot 10^5$	$3.0 \cdot 10^5$	$1.77 \cdot 10^6$
Nonzeroes	6	12	636	11,208	263,880	$4.9 \cdot 10^6$	$9.3 \cdot 10^7$	$1.67 \cdot 10^9$
Nonzero fraction	1	0.25	0.7162	0.6949	0.9112	0.9460	0.9968	0.9989

Table 2. Number of adjacency-allowed elements at loops 1 to 8 of table 1, followed by those forbidden by the suffix rule (3.1). Of those remaining, the number forbidden by the prefix rule (3.2) is given next. The remaining allowed elements are further broken down to zero elements and nonzero elements in the actual L-loop symbol. The fraction of remaining allowed elements that are actually nonzero is over 99.6% by 7 loops!

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There is always … the joy of finding things out!

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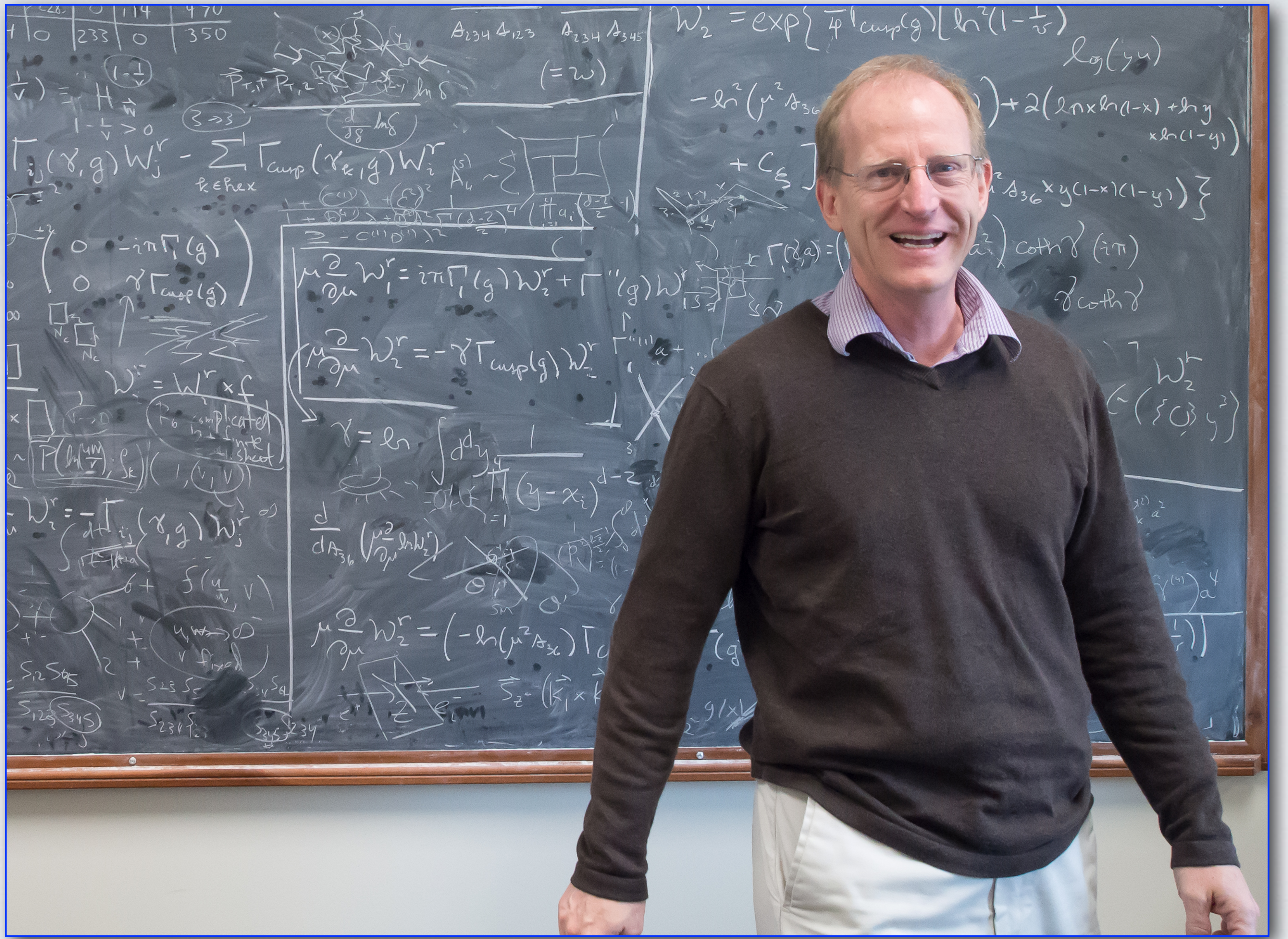

Credit: Jeffrey Richman

THANK YOU LANCE!

THE PRIZE IS THE PLEASURE

OF FINDING THE THING OUT,

THE KICK IN THE DISCOVERY,

THE OBSERVATION THAT OTHER PEOPLE USE IT.

THOSE ARE THE REAL THINGS