Towards All-Loop Amplitudes From Negative Geometries

Jaroslav Trnka

Center for Quantum Mathematics and Physics (QMAP),
University of California, Davis, USA

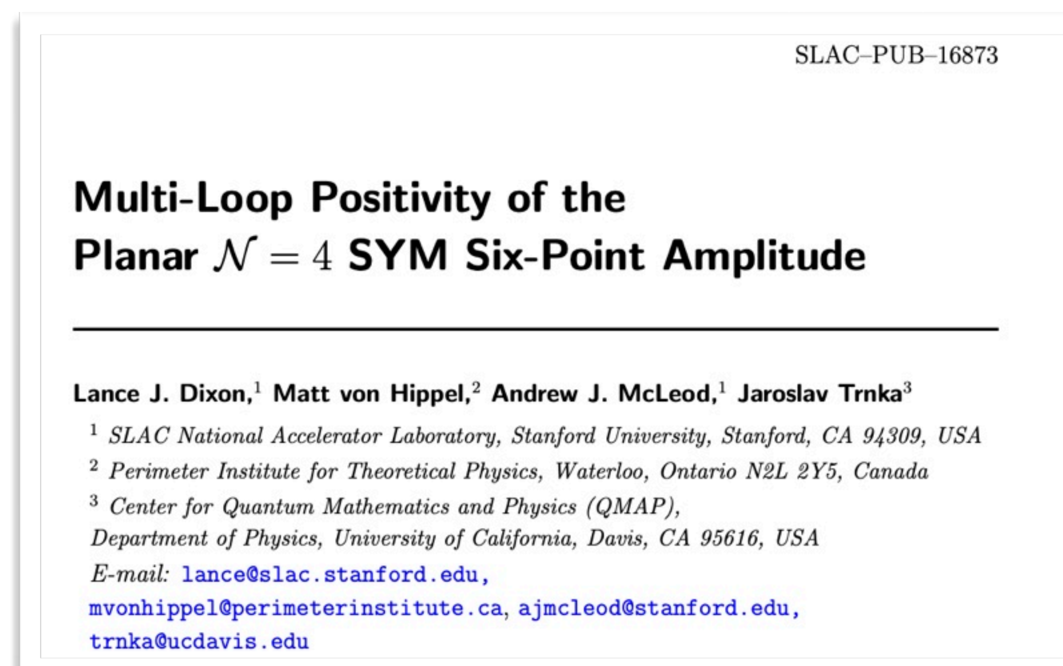
based on **2605.28926**
with Lance Dixon, Umut Oktem, Shruti Paranjape, Yongqun Xu, Shun-Qing Zhang

Lancefest, University of Edinburgh, June 25, 2026

MEETING LANCE

I first met Lance at Amplitudes 2010 in London as a young PhD student and knew him as the founder of the field and the expert who can do "integrals" (while we just played with integrands)

I got to know Lance more in 2014/2015 when he (and Zvi) spent his sabbatical at Caltech - this was a great time!



SLAC–PUB–16873

Multi-Loop Positivity of the Planar $\mathcal{N} = 4$ SYM Six-Point Amplitude

Lance J. Dixon,[1] Matt von Hippel,[2] Andrew J. McLeod,[1] Jaroslav Trnka[3]

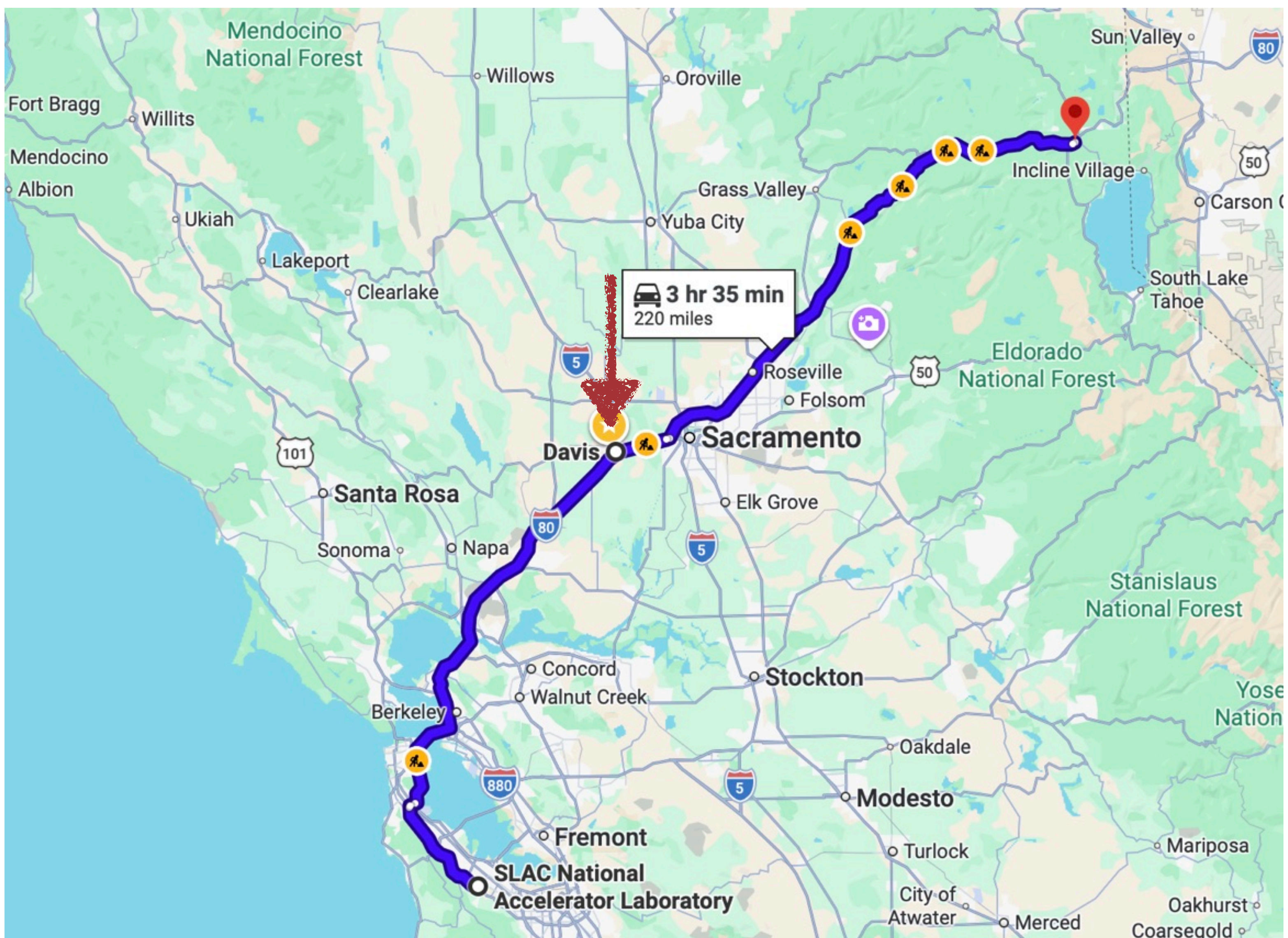
[1] *SLAC National Accelerator Laboratory, Stanford University, Stanford, CA 94309, USA*
[2] *Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada*
[3] *Center for Quantum Mathematics and Physics (QMAP), Department of Physics, University of California, Davis, CA 95616, USA*

E-mail: lance@slac.stanford.edu, mvonhippel@perimeterinstitute.ca, ajmcleod@stanford.edu, trnka@ucdavis.edu

LANCE AS A NEIGHBOR

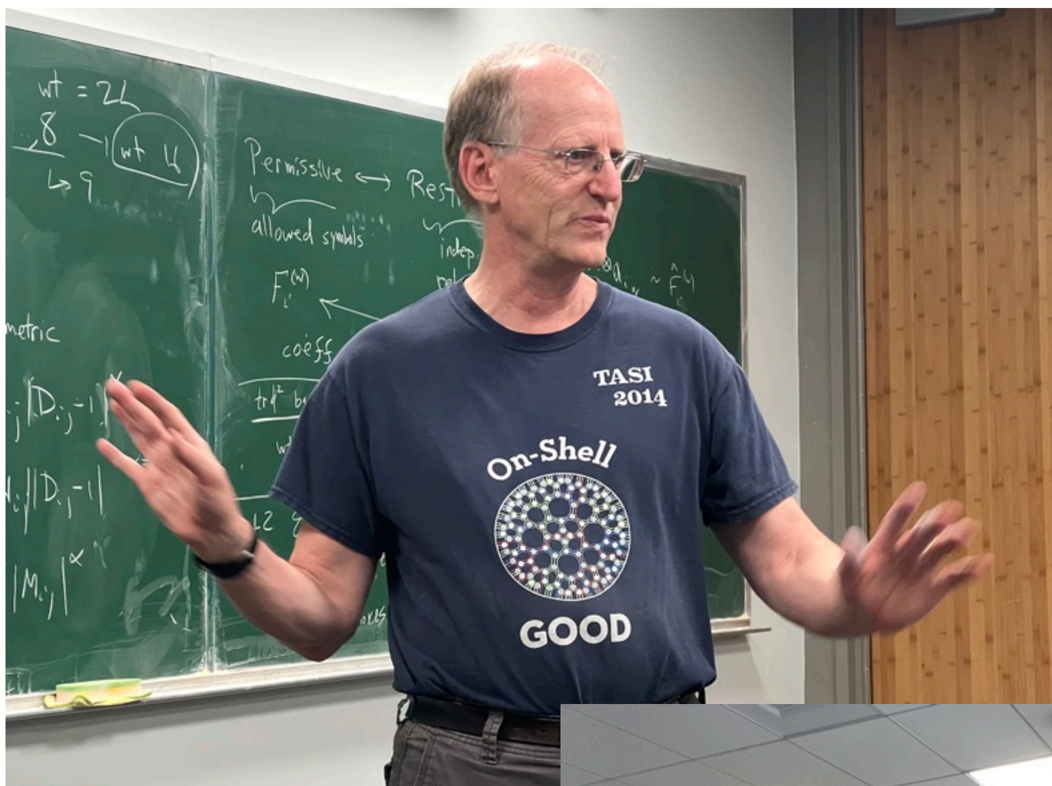
After moving to UC Davis in 2016, I have been very fortunate to be Lance's restroom stop between SLAC and his vacation home

→ Great opportunity for discussions and collaborations!

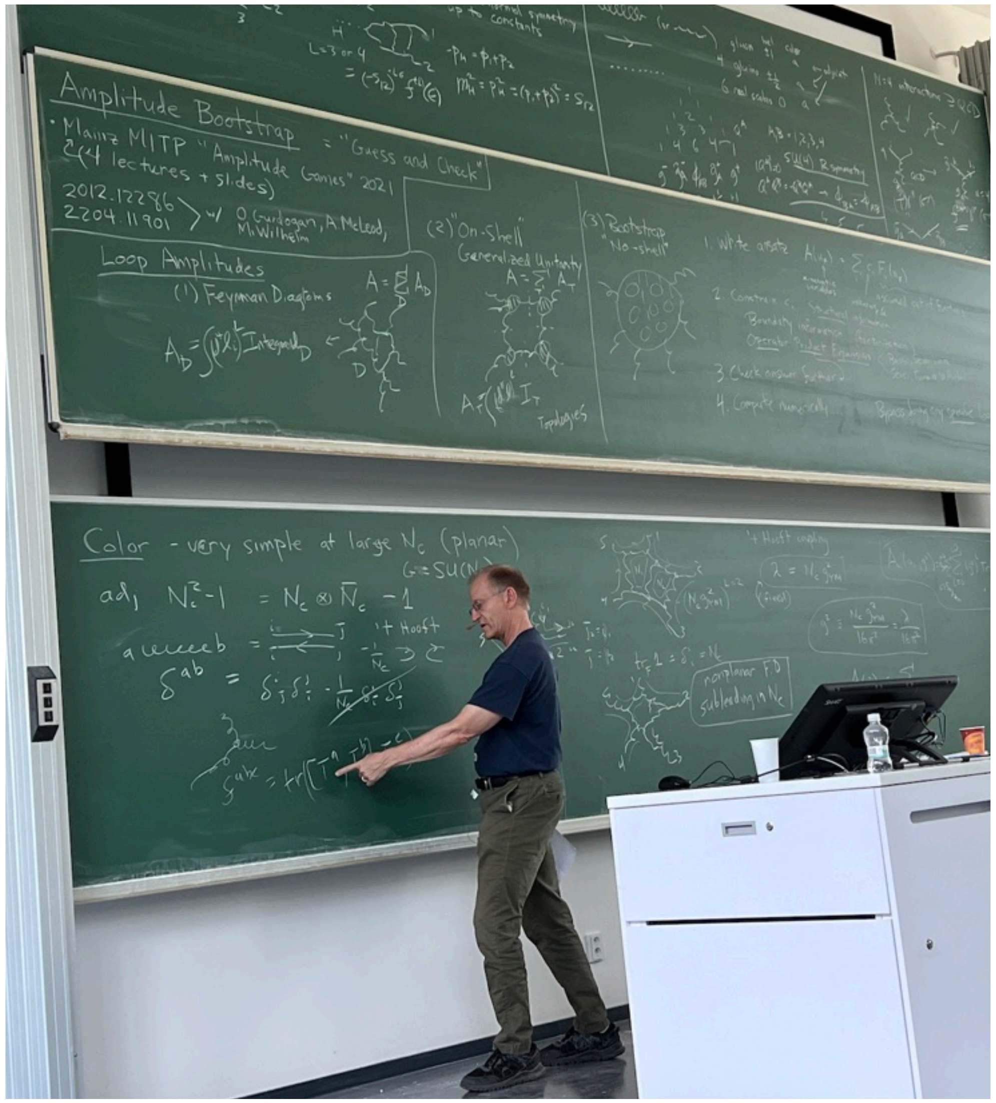


CALIFORNIA AMPLITUDES

Starting 2019, we established annual/bi-annual amplitudes workshops for California-based people

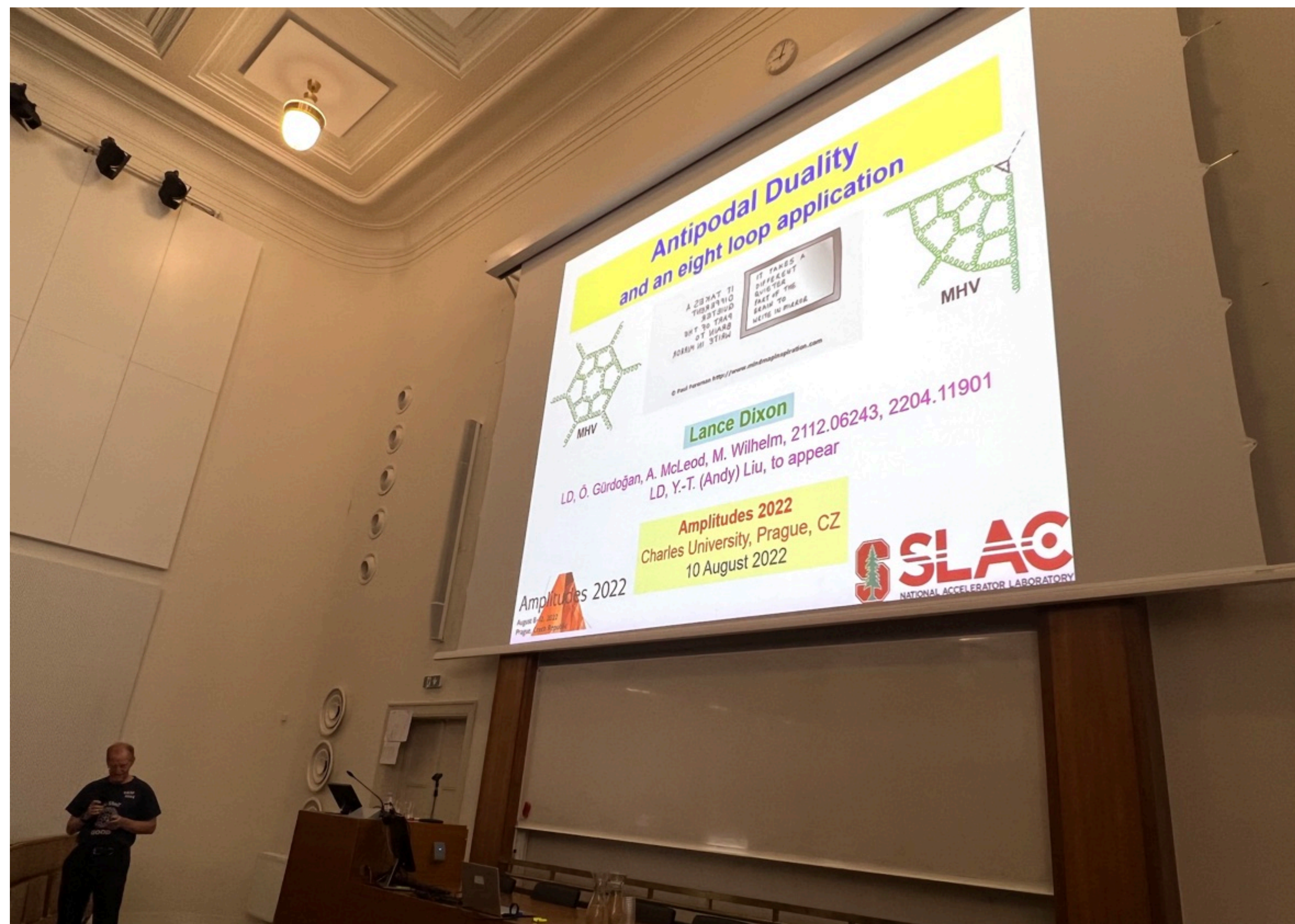


LANCE IN PRAGUE



Amplitudes 2022 conference where we celebrated Lance's induction into the NAS

NATIONAL ACADEMY OF SCIENCES



LANCE IN PRAGUE

Also (unsuccessful) hunt for Lance's ancestors in Klatovy

PROJECT WITH LANCE

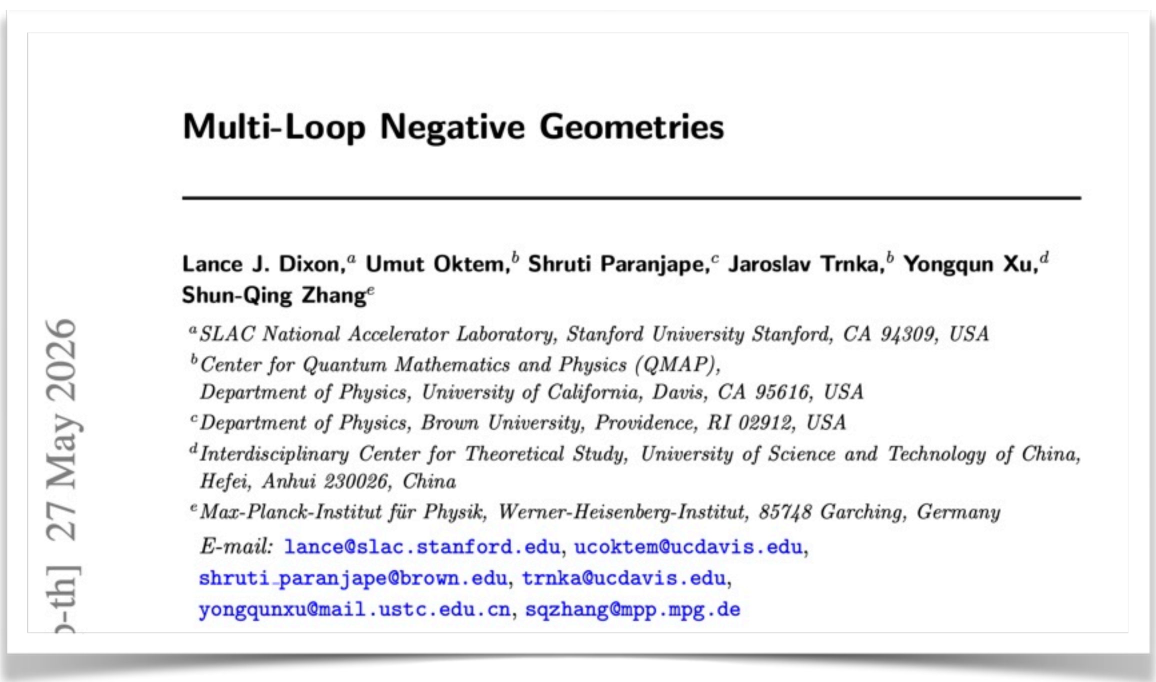
- Our recent paper originated from discussions during Lance's stops in Davis in the last 2 years
- We have been both attacking the question of computing amplitudes in the **planar N=4 SYM theory** to higher loops in perturbation theory:
  - **Integrands**: on-shell diagrams, recursion relations, Amplituhedron (Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Goncharov, Postnikov, JT, ...)
  - **Amplitudes**: hexagon and symbol bootstrap, cluster algebras, anti-podal duality

(Dixon, Drummond, Henn, McLeod, von Hippel, Gurdogan, Papathanasiou, Basso, Caron-Huot, Dulat, Wilhelm, Spradlin, Vergu, Volovich,.....)

PROJECT WITH LANCE

- In the collaboration with Lance, Shruti (Brown), my student Umut (UC Davis -> Arizona), Shun (MPP -> Penn State), Yongqun we evaluated certain higher-loop objects:

negative geometries



Multi-Loop Negative Geometries

Lance J. Dixon,[a] Umut Oktem,[b] Shruti Paranjape,[c] Jaroslav Trnka,[b] Yongqun Xu,[d] Shun-Qing Zhang[e]

[a]SLAC National Accelerator Laboratory, Stanford University Stanford, CA 94309, USA
[b]Center for Quantum Mathematics and Physics (QMAP), Department of Physics, University of California, Davis, CA 95616, USA
[c]Department of Physics, Brown University, Providence, RI 02912, USA
[d]Interdisciplinary Center for Theoretical Study, University of Science and Technology of China, Hefei, Anhui 230026, China
[e]Max-Planck-Institut für Physik, Werner-Heisenberg-Institut, 85748 Garching, Germany

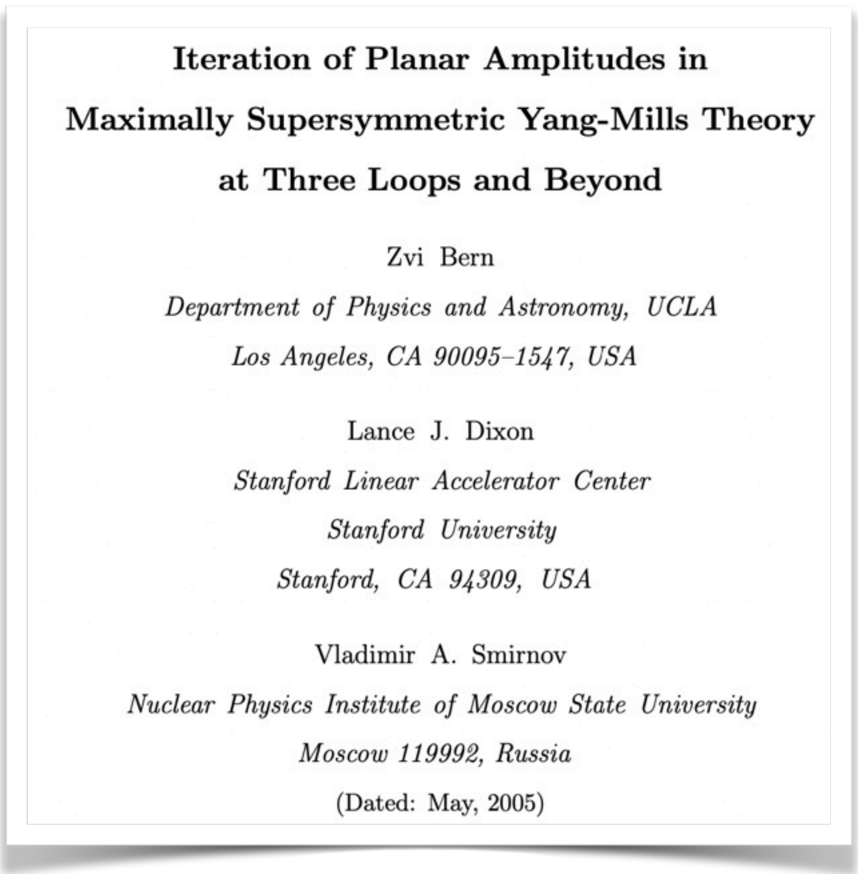
E-mail: lance@slac.stanford.edu, ucoktem@ucdavis.edu, shruti_paranjape@brown.edu, trnka@ucdavis.edu, yongqunxu@mail.ustc.edu.cn, sqzhang@mpp.mpg.de

- Interesting interplay between positive geometries, loop integrands and polylogarithms, SZV and MZVs
- Hopefully a part of a systematic attempt to solve planar N=4 SYM!

FOUR POINT AMPLITUDES

BDS ANSATZ

In 2005, Bern, Dixon and Smirnov calculated 3-loop amplitude



Iteration of Planar Amplitudes in Maximally Supersymmetric Yang-Mills Theory at Three Loops and Beyond

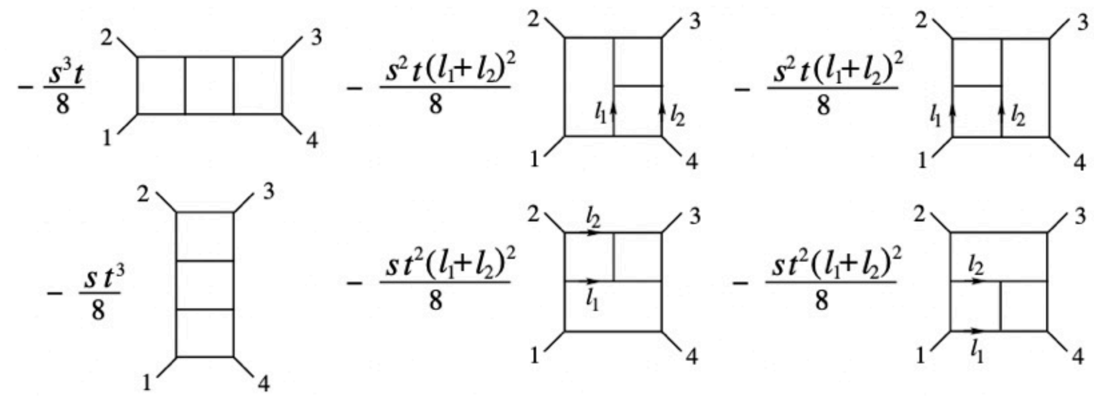
Zvi Bern
Department of Physics and Astronomy, UCLA
Los Angeles, CA 90095–1547, USA

Lance J. Dixon
Stanford Linear Accelerator Center
Stanford University
Stanford, CA 94309, USA

Vladimir A. Smirnov
Nuclear Physics Institute of Moscow State University
Moscow 119992, Russia

(Dated: May, 2005)

$$M_4^{(3)}(\epsilon) = -\frac{1}{8} st \left(s^2 I_4^{(3)a}(s,t) + 2s\, I_4^{(3)b}(t,s) + t^2 I_4^{(3)a}(t,s) + 2t\, I_4^{(3)b}(s,t) \right)$$



The integrand obtained using **unitarity methods**, after integration they found the same iterative structure

$$M_4^{(3)}(\epsilon) \;=\; -\frac{1}{3}\left[M_4^{(1)}(\epsilon)\right]^3 + M_4^{(1)}(\epsilon)\, M_4^{(2)}(\epsilon) + f^{(3)}(\epsilon)\, M_4^{(1)}(3\,\epsilon) + C^{(3)} + \mathcal{O}(\epsilon)$$

Conjecture:

$$\boxed{\mathcal{M}_n \equiv 1 + \sum_{L=1}^{\infty} a^L M_n^{(L)}(\epsilon) = \exp\left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon) \right)\right]}$$

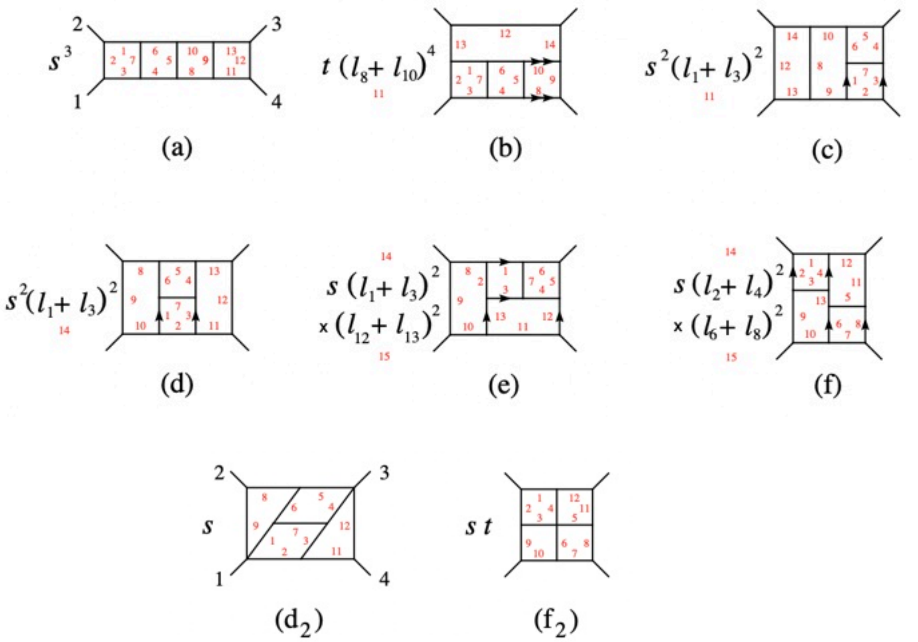
cusp-anomalous dimension $\quad f^{(l)}(\epsilon) = f_0^{(l)} + \epsilon f_1^{(l)} + \epsilon^2 f_2^{(l)}$

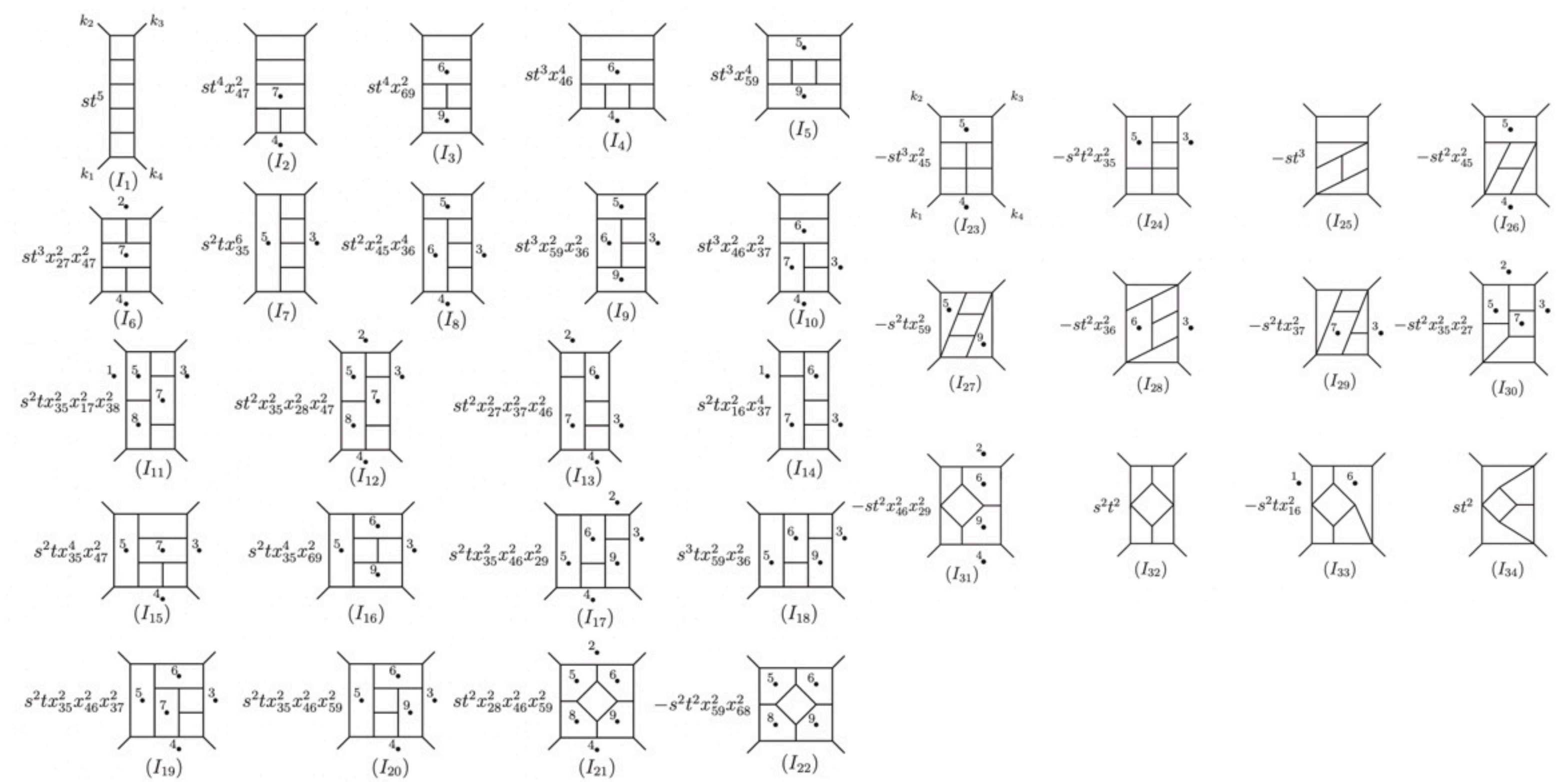
(Beisert, Eden, Staudacher, 2006)

HIGHER LOOPS

In next two years, 4-loop and 5-loop integrands were constructed

(Bern, Czakon, Dixon, Kosower, Smirnov, 2006) (Bern, Carrasco, Johansson, Kosower, 2007)





Numerators can be chosen to be invariant under **dual conformal symmetry**

(Drummond, Henn, Korchemsky, Sokatchev, 2008)

Integrand up to **12 loops** using soft-collinear bootstrap and f-graphs

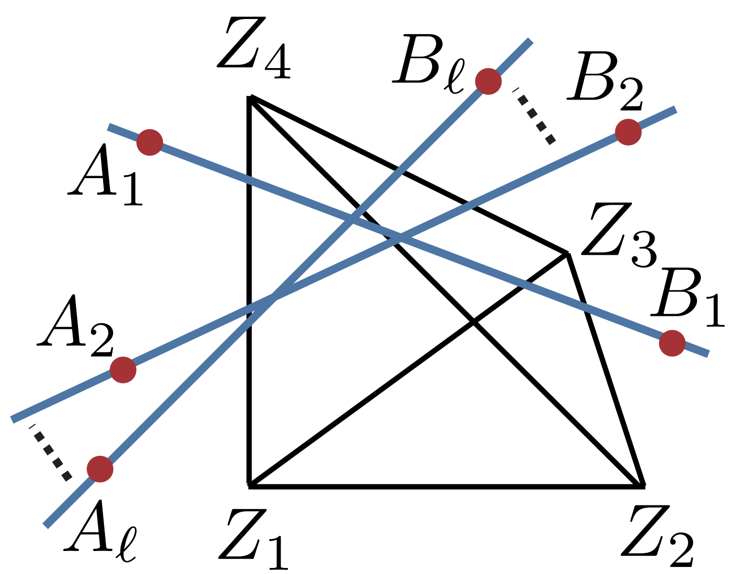
(Bourjaily, DiRe, Skaikh, Spradlin, Volovich, 2011) (Bourjaily, Heslop, Tran, 2015)

(Bourjaily, Heslop, Tran, 2016) (He, Shi, Tang, Zhang 2024) (Bourjaily, He, Shi, Tang, 2025)

LOOP INTEGRAND

In 2010, we took the planar integrand seriously and formulated recursion relations for N=4 SYM in **momentum twistor space**

(Hodges, 2009) (Arkani-Hamed, Cachazo, Cheng, 2009) (Mason, Skinner 2009)

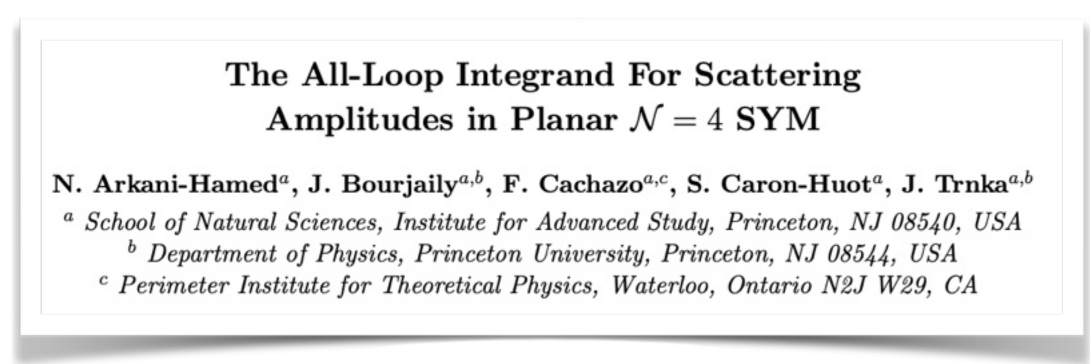


- External momenta parametrized by points in projective space
- Loop momenta correspond to lines
- **Global variables** for planar amplitudes

- the integrand is a **unique rational function**

$$\mathcal{I}_{n,k}^{\ell-\text{loop}}(AB_1, AB_2, \ldots, AB_\ell, Z_1, Z_2, \ldots, Z_n)$$

- connection to on-shell diagrams, positive Grassmannian


The All-Loop Integrand For Scattering Amplitudes in Planar $\mathcal{N}=4$ SYM

N. Arkani-Hamed[a], J. Bourjaily[a,b], F. Cachazo[a,c], S. Caron-Huot[a], J. Trnka[a,b]
[a] School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540, USA
[b] Department of Physics, Princeton University, Princeton, NJ 08544, USA
[c] Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2J W29, CA

(Arkani-Hamed, Bourjaily, Cachazo, Trnka, 2010)
(Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka, 2012)

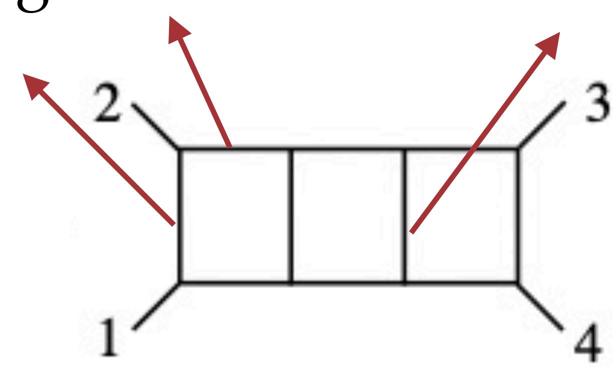
AMPLITUHEDRON

(Arkani-Hamed, JT, 2013) (Arkani-Hamed, Thomas, JT, 2017) (Damgaard, Ferro, Lukowski 2019)

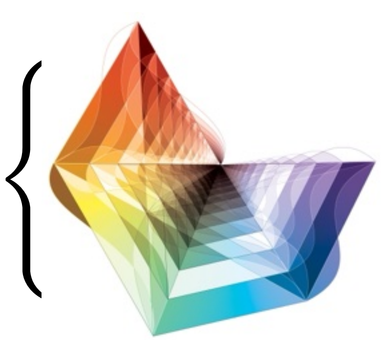
Loop integrand: rational function of 4-brackets

$$\langle XYZW\rangle = \epsilon_{abcd}X^aY^bZ^cW^d$$

$$\mathcal{I}_4^{\ell-\mathrm{loop}} = \frac{N(AB_i, Z_i)\, d\mu}{\underbrace{\prod_i \langle AB_i 12\rangle\langle AB_i 23\rangle\langle AB_i 34\rangle\langle AB_i 41\rangle}_{\text{external propagators}}\ \underbrace{\prod_{ij}\langle AB_i AB_j\rangle}_{\text{internal propagators}}}$$



interpreted as the canonical differential form on the **Amplituhedron**

$$\mathcal{I}_4^{\ell-\mathrm{loop}} \to \Omega_4^{\ell-\mathrm{loop}}\Big\{$$  $$\Big\}$$

triangulations: BCFW recursion (origami), Feynman diagrams,......

IR DIVERGENCIES

Integrate the integrand: amplitudes are IR divergent - regulate

$$M_n^{\mathrm{L-loop}}(\epsilon) = \frac{a}{\epsilon^{2L}} + \frac{b}{\epsilon^{2L-1}} + \ldots$$ at L loops

Exponentiation of IR divergencies:

$$M_n(\epsilon) = \sum_{L=0}^{\infty} g^{2L} M_n^{\mathrm{L-loop}}(\epsilon) = \exp\left(\frac{\gamma(g)}{\epsilon^2} + \ldots\right)$$

$$g^2 = \frac{g_{\mathrm{YM}}^2 N}{16\pi}$$

t'Hooft coupling

Logarithm of the amplitude only mildly divergent:

$$\ln M_n(\epsilon) = \frac{\gamma(g)}{\epsilon^2} + \ldots$$

leading IR divergence

GAMMA CUSP

Leading IR divergence of amplitudes: cusp anomalous dimension

Exact expression known from **integrability** $\gamma^{\text{cusp}}(g)$

(Beisert, Eden, Staudacher 2005)

Reduction of amplitude (function of kinematical variables) to a number at each loop order -- resummed: function of coupling

$$\gamma^{\text{cusp}}(g) = 8g^2 - \frac{8\pi^2}{3}g^4 - \frac{88\pi^4}{45}g^6 - 16\left(\frac{73\pi^6}{630} + 4\zeta_3^2\right)g^8 + \ldots = 2g - \frac{3\ln 2}{2\pi} + \ldots$$

(Basso, Korchemsky, Kotanski 2007)

very special property:
only **single zeta values**
at weak coupling

$$\zeta_k = \sum_{n=1}^{\infty} \frac{1}{n^k}$$

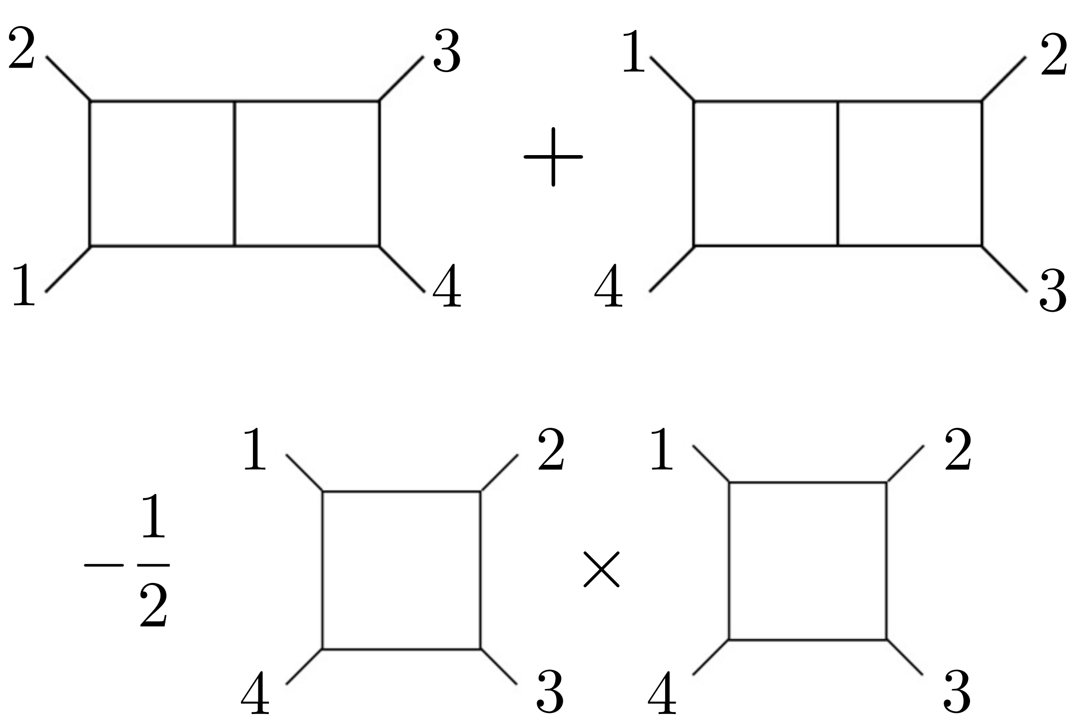
How to derive the all-loop formula from amplitudes?

AMPLITUDE LOGARITHM

Take logarithm seriously: calculate it in the perturbation theory

$$\ln M = \ln\left(1 + g^2 M^{(1)} + g^4 M^{(2)} + g^6 M^{(3)} + \ldots\right)$$

$$= g^2 M^{(1)} + g^4 \Big[\underbrace{M^{(2)} - \frac{1}{2}(M^{(1)})^2}_{\ln M_2}\Big] + g^6 \Big[M^{(3)} - \underbrace{M^{(2)} M^{(1)}}_{\ln M_3} + \frac{1}{3}(M^{(1)})^3\Big] + \ldots$$



each term $1/\epsilon^4$ divergent,
combination only $1/\epsilon^2$ divergent

$\widetilde{\mathcal{I}}_L$: integrand for $\ln M_L$

cut structure encodes
mild IR divergencies

AMPLITUDE LOGARITHM

If we integrate over all loops: IR divergence

Can we extract an **IR finite function** from the integrand?

$\widetilde{\mathcal{I}}_L(\ell_1, \ldots, \ell_L)$ ← rational function

↓ integrate over ℓ_L

⋮ ← weight 2 function

↓ integrate over ℓ_{L-1}

⋮

↓ integrate over ℓ_2

all are IR finite

↓ integrate over ℓ_1

$\ln M_L$ ← IR divergent, BDS ansatz

Do not integrate the last loop!

weight 2(L-1) function

4pt: single cross ratio

$$\mathcal{F}_{L-1}(z)$$

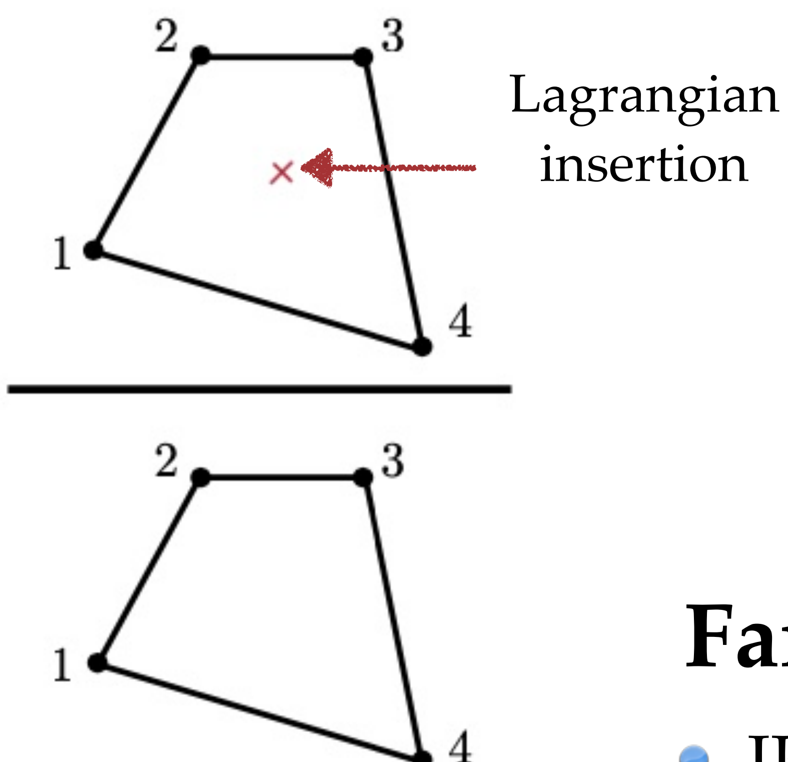
$$z = \frac{\langle AB12\rangle\langle AB34\rangle}{\langle AB23\rangle\langle AB14\rangle}$$

$$\mathcal{F}(g,z) = \sum_{L=0}^{\infty} g^{2L}\mathcal{F}_L(z)$$

WILSON LOOP DUALITY

Same object appeared in the study of Wilson loops

$$\frac{\langle W_F(x_1,x_2,x_3,x_4)\mathcal{L}(x_0)\rangle}{\langle W_F(x_1,x_2,x_3,x_4)\rangle} = \frac{g^2}{\pi^2}\frac{\langle 1234\rangle^2}{\langle AB12\rangle\langle AB23\rangle\langle AB34\rangle\langle AB41\rangle}\mathcal{F}(g,z)$$



- **weak coupling**: expansion in g^2 (known up to 3-loops)

  (Alday, Henn, Sikorowski, 2013) (Henn, Korchemsky, Mistlberger 2019)

- **strong coupling**: expansion in $1/g$

  (Alday, Buchbinder, Tseytlin, 2011) (Engelund, Roiban, 2011, 2012)

$$\mathcal{F}(g,z) = g\frac{z}{(z-1)^3}\left[2(1-z)+(z+1)\log z\right]+\cdots$$

Fantastic object to study!!

- IR finite object, depends only on one cross-ratio, contains 4pt amplitude
- Direct prescription to extract gamma cusp

NEGATIVE GEOMETRIES

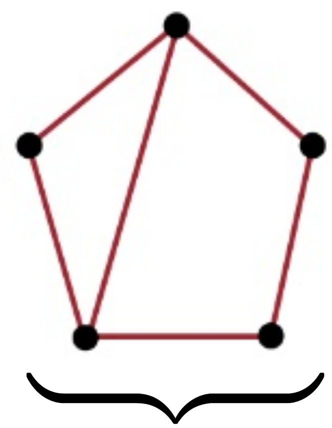
(Arkani-Hamed, Henn, JT, 2021) (Chicherin, Henn, JT, Zhang 2024) (Chicherin, Henn, Mazzucchelli, JT, Yang, Zhang 2025)

(Paranjape, Skowronek, Spradlin, Volovich, Weng 2026) (Dixon, Oktem, Paranjape, JT, Xu, Zhang, 2026)

NEGATIVE GEOMETRIES

Amplituhedron gives us a new type of expansion

$$\widetilde{\Omega}_L = \sum_G (-1)^{E(G)}$$



Each vertex = loop

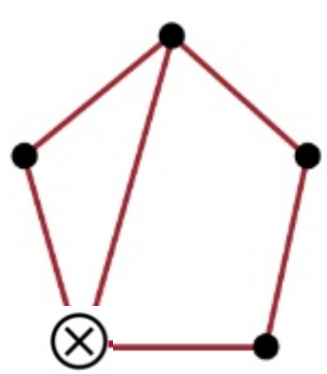
More edges = more complicated

Each diagram is a canonical form on geometry = integrand

From practical point of view, this is an integrand expansion in terms of **new nice objects**

Why special? Finite if we integrate over all loops except one

(this is not manifest in other expansions)

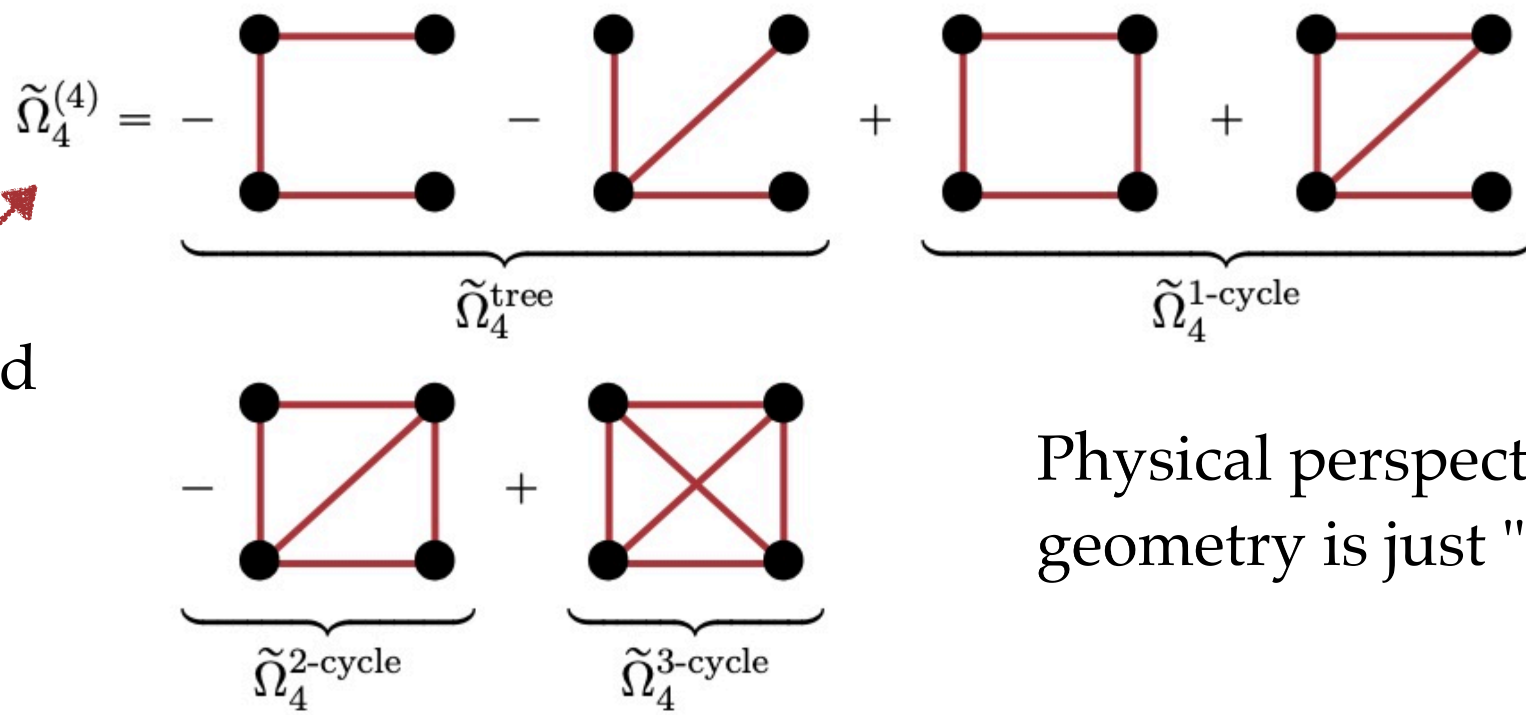


= IR finite function, contribution to the 4-loop object $\mathcal{F}_4(z)$

NEGATIVE GEOMETRIES

Natural hierarchy of objects: number of internal cycles in graphs

Loops of loops expansion



integrand

Physical perspective: each negative geometry is just "some nice integrand"

We can find the **integrand for all 0-cycle (tree) and 1-cycle** negative geometries for any number of loops

That is good but are these sets meaningful in any way?

Can we integrate them? Are low cycles simpler?

EXPLICIT FORMULAS

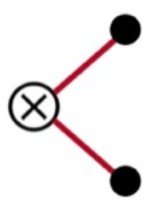
(Arkani-Hamed, Henn, JT, 2021)

One-loop integral: only one contribution = directly $\mathcal{F}_1(z)$

 $= [\pi^2 + \log^2 z]$

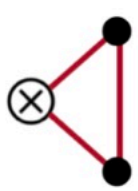
Two-loop integrals: three different negative geometries

Tree graphs:

 $= -\frac{1}{2}\left[\pi^2 + \log^2 z\right]^2$

 $= -\frac{1}{12}\left[\pi^2 + \log^2 z\right] \times \left[5\pi^2 + \log^2 z\right]$

One-cycle graph:



$$= -\frac{1}{6}\log^4 z + \log^2 z\left[-\frac{2}{3}\mathrm{Li}_2\left(\frac{1}{z+1}\right) - \frac{2}{3}\mathrm{Li}_2\left(\frac{z}{z+1}\right) + \frac{\pi^2}{9}\right]$$
$$+ \log z\left[4\mathrm{Li}_3\left(\frac{z}{z+1}\right) - 4\mathrm{Li}_3\left(\frac{1}{z+1}\right)\right] - \frac{2}{3}\left[\mathrm{Li}_2\left(\frac{1}{z+1}\right) + \mathrm{Li}_2\left(\frac{z}{z+1}\right) - \frac{\pi^2}{6}\right]^2$$
$$-\frac{8}{3}\pi^2\left[\mathrm{Li}_2\left(\frac{1}{z+1}\right) + \mathrm{Li}_2\left(\frac{z}{z+1}\right) - \frac{\pi^2}{6}\right] - 8\mathrm{Li}_4\left(\frac{1}{z+1}\right) - 8\mathrm{Li}_4\left(\frac{z}{z+1}\right) - \frac{\pi^4}{18}$$

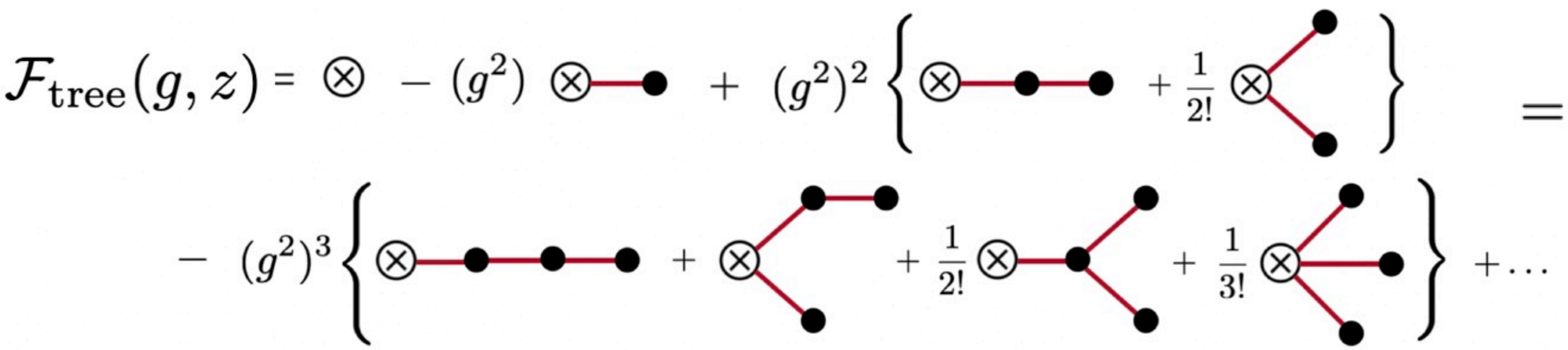
Same transcendentality weight, but more cycles = more complicated functions

TREE-LEVEL RESUMMATION

(Arkani-Hamed, Henn, JT, 2021)

Sum all **tree graphs**

naively do not expect anything sensible



$$= \frac{A^2}{g^2}\frac{z^A}{(z^A+1)^2}$$

where $\dfrac{A}{2g\cos\frac{\pi A}{2}} = 1$

For $\Gamma_{\text{cusp}}(g)$ we get a surprising result:

$$\Gamma_{\text{tree}}(g) \to \begin{cases} 2g - \frac{3\log 2}{2\pi} + \cdots & \longrightarrow \text{ exact} \\ \frac{8}{\pi}g - 1 + \ldots & \longrightarrow \text{ our tree approximation} \end{cases}$$

$\frac{8}{\pi} \approx 2.55$

Very good approximation (maybe a total accident?)

FULL THREE-LOOP RESULT

(Dixon, Oktem, Paranjape, JT, Xu, Zhang, 2026)

We evaluated all 3-loop negative geometries, re-derived the Wilson loop computation (Henn, Korchemsky, Mistlberger 2019)

Negative geometry expansion:

- 0-cycle diagrams = all are logs, understood to any L
- Higher cycles: we can construct the integrands and integrate explicitly -- maximal transcendental IR finite functions

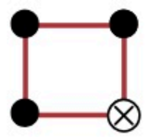
↓

Is there any additional property that distinguishes different cycles?

FULL THREE-LOOP RESULT

(Dixon, Oktem, Paranjape, JT, Xu, Zhang, 2026)

Symbol of the 1-cycle vs 2-cycle diagrams:

Symbol [] =

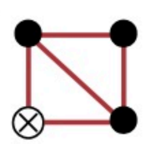
$$16\,\big(3\,\mathrm{SB}(z,z,z,z,z,z) + 2\,\mathrm{SB}(z,z,z,z,z,1{+}z) - \tfrac{1}{2}\,\mathrm{SB}(z,z,z,z,1{+}z,z)$$
$$+ 3\,\mathrm{SB}(z,z,z,z,1{+}z,1{+}z) + 3\,\mathrm{SB}(z,z,z,1{+}z,z,z) - 2\,\mathrm{SB}(z,z,z,1{+}z,z,1{+}z)$$
$$- 4\,\mathrm{SB}(z,z,z,1{+}z,1{+}z,z) + \tfrac{3}{2}\,\mathrm{SB}(z,z,1{+}z,z,z,z) - 5\,\mathrm{SB}(z,z,1{+}z,z,z,1{+}z)$$
$$+ 2\,\mathrm{SB}(z,z,1{+}z,z,1{+}z,z)\big)\,.$$

They both have the same transcendentality weight but differ in the **depth of the function**

$$H_{a_1,a_2,a_3,a_4,a_5,a_6}$$

number of non-zero entries

also easily encoded in the symbol

Symbol [] =

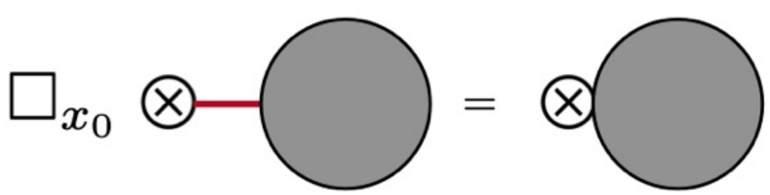
$$8\,\big(4\,\mathrm{SB}(z,z,z,z,b(4),b(4)) + \mathrm{SB}(z,z,z,b(4),z,b(4)) + \mathrm{SB}(z,z,z,b(4),b(4),z)$$
$$+ 4\,\mathrm{SB}(z,z,z,z,z,z) + \mathrm{SB}(z,z,z,z,z,1{+}z) + \mathrm{SB}(z,z,z,z,1{+}z,z)$$
$$+ 6\,\mathrm{SB}(z,z,z,z,1{+}z,1{+}z) + 9\,\mathrm{SB}(z,z,z,1{+}z,z,z) - \mathrm{SB}(z,z,z,1{+}z,z,1{+}z)$$
$$- 5\,\mathrm{SB}(z,z,z,1{+}z,1{+}z,z) - 4\,\mathrm{SB}(z,z,z,1{+}z,1{+}z,1{+}z) + 7\,\mathrm{SB}(z,z,1{+}z,z,z,z)$$
$$- 3\,\mathrm{SB}(z,z,1{+}z,z,z,1{+}z) + \mathrm{SB}(z,z,1{+}z,z,1{+}z,z) - 4\,\mathrm{SB}(z,z,1{+}z,z,1{+}z,1{+}z)$$
$$- 8\,\mathrm{SB}(z,z,1{+}z,1{+}z,z,z) - 4\,\mathrm{SB}(z,z,1{+}z,1{+}z,z,1{+}z) - 4\,\mathrm{SB}(z,z,1{+}z,1{+}z,1{+}z,z)$$
$$+ 8\,\mathrm{SB}(z,z,1{+}z,1{+}z,1{+}z,1{+}z)\big)\,.$$

HIGHER LOOPS

(Dixon, Oktem, Paranjape, JT, Xu, Zhang, 2026)

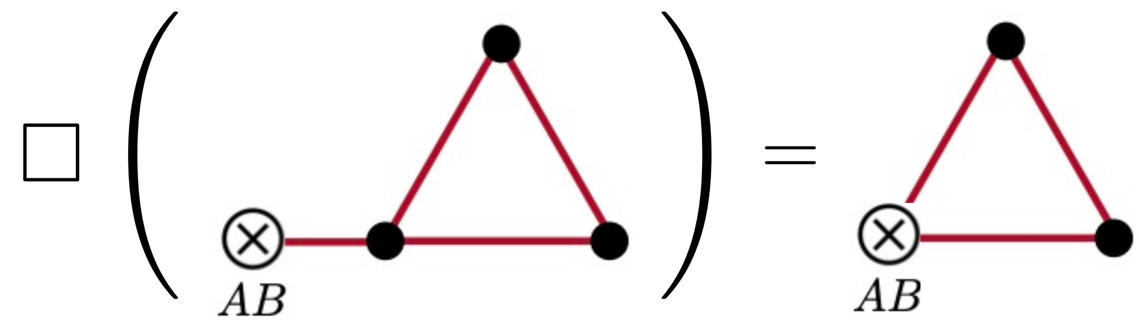
At higher loops, the integration technology does not exist yet

For certain geometries, we can use a differential equation trick:



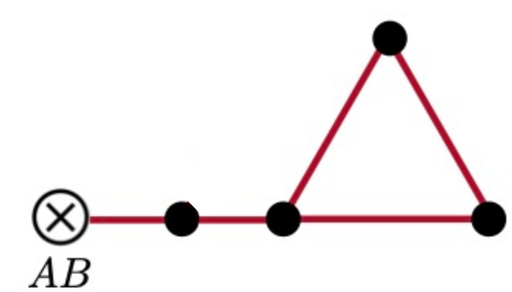
(Arkani-Hamed, Henn, JT, 2021)

for example:



This allows us to solve classes of geometries:

$$
\begin{aligned}
I_{3,2}(z) = & 8H_{0,0,0,0,0,0,0,0,0}(z) + 8H_{0,0,0,0,-1,0,0,0}(z) - 16H_{0,0,0,0,-1,-1,0,0}(z) + 8H_{0,0,0,0,-2,0,0}(z) \\
& - 8\zeta_3(2H_{0,0,0,0,-1}(z) - H_{0,0,0,0,0}(z)) + 4\pi^2(H_{0,0,0,0,-1,0}(z) - 2H_{0,0,0,0,-1,-1}(z) \\
& + H_{0,0,0,0,-2}(z)) + \frac{13\pi^4}{1080}\log^4(z) + \frac{C_{3,1}}{6}\log^3(z) + \frac{D_{3,1}}{2}\log(z)^2 + C_{3,2}\log(z) + D_{3,2}\,,
\end{aligned}
$$



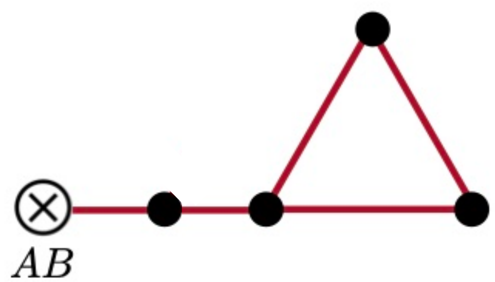
$$C_{3,2} = -\frac{8}{3}\pi^2\zeta_{3,2} - 16\zeta_{5,2} + \frac{127\pi^4\zeta_3}{45} + \frac{8\pi^2\zeta_5}{3} - 168\zeta_7,$$

$$D_{3,2} = 8\pi^2\zeta_{3,3} + 16\zeta_{5,3} + 80\zeta_{6,2} - 8\zeta_3\left(\pi^2\zeta_3 + 32\zeta_5\right) + \frac{6779\pi^8}{75600}.$$

MULTIPLE ZETA VALUES

(Dixon, Oktem, Paranjape, JT, Xu, Zhang, 2026)

We can extract the contribution to gamma cusp from full function

$\Gamma\Big\{$ 

$\Big\}$

$$= g^{10}\left(-\frac{3584}{5}\zeta_{5,3} - \frac{1}{3}256\pi^2\zeta_3^2 - 5120\zeta_5\zeta_3 + \frac{79552\pi^8}{70875}\right)$$

Multiple Zeta Values (MZV) $\quad \zeta_{a,b} = \sum_{n_1>n_2>0} \frac{1}{n_1^a n_2^b}$

Simple analysis suggest that higher cycles: $\zeta_{a_1,a_2,\ldots,a_{\ell+1}}$

They all cancel in the BES formula for gamma cusp!

Mechanism of cancelation? Imprint of integrability?

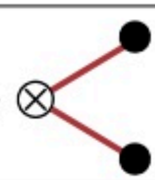
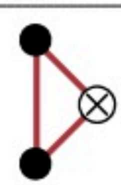
LOW CYCLE DOMINANCE?

(Dixon, Oktem, Paranjape, JT, Xu, Zhang, 2026)

Do low cycles actually dominate at a fixed loop order?

No reason for that, loops of loops is just an ad hoc expansion

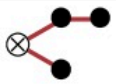
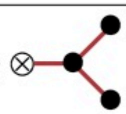
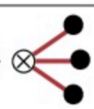
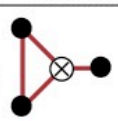
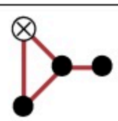
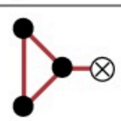
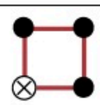
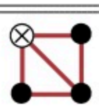
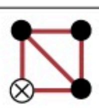
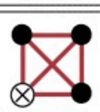
Can not answer for the full amplitude (depends on kinematics), but can answer for the gamma cusp

Graph	Γ_{cusp}	numerical Γ_{cusp}	Fraction $\frac{\Gamma_{\text{diagram}}}{\Gamma_{\text{actual}}}$
	$\frac{64\pi^4}{45}g^6$	$138.54g^6$	0.545
$\frac{1}{2!}$ 	$\frac{32\pi^4}{45}g^6$	$69.27g^6$	0.273
All tree graphs	$\frac{32\pi^4}{15}g^6$	$207.80g^6$	1.09
$-\frac{1}{2!}$  Only 1-cycle graph	$-\frac{8\pi^4}{45}g^6$	$-17.32g^6$	−0.09
Full result	$\frac{88\pi^4}{45}g^6$	$190.48g^6$	1

At 2-loops looks good!

LOW CYCLE DOMINANCE?

(Dixon, Oktem, Paranjape, JT, Xu, Zhang, 2026)

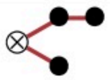
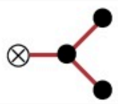
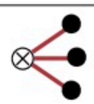
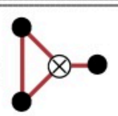
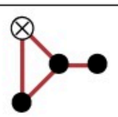
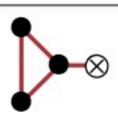
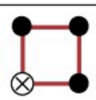
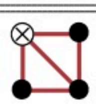
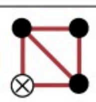
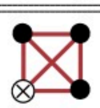
Graph	Γ_{cusp}	numerical Γ_{cusp}	Fraction $\frac{\Gamma_{\text{diagram}}}{\Gamma_{\text{actual}}}$
$-$ 	$-\frac{1}{315}\left(272\pi^6\right) g^8$	$-830.152g^8$	0.443
$-$ 	$-\frac{1}{315}\left(272\pi^6\right) g^8$	$-830.152g^8$	0.443
$-\frac{1}{2!}$ 	$-\frac{1}{35}\left(16\pi^6\right) g^8$	$-439.492g^8$	0.234
$-\frac{1}{3!}$ 	$-\frac{1}{105}\left(16\pi^6\right) g^8$	$-146.497g^8$	0.078
All tree graphs	$-\frac{1}{315}\left(736\pi^6\right) g^8$	$-2246.30g^8$	1.1981
$\frac{1}{2!}$ 	$(\frac{352\pi^6}{2835} - 48\zeta_3^2)g^8$	$50.01g^8$	-0.0267
	$(\frac{704\pi^6}{2835} - 96\zeta_3^2)g^8$	$100.02g^8$	-0.0533
$\frac{1}{2!}$ 	$(\frac{352\pi^6}{2835} - 48\zeta_3^2)g^8$	$50.01g^8$	-0.0267
$\frac{1}{2!}$ 	$-\frac{328\pi^6}{2835}g^8$	$-111.22g^8$	0.0593
All 1-cycle graphs	$\left(\frac{1080\pi^6}{2835} - 192\zeta_3^2\right) g^8$	$88.82g^8$	-0.047
$-\frac{1}{2!}$ 	$(48\zeta_3^2 + \frac{8\pi^6}{189})g^8$	$110.05g^8$	-0.0587
$-\frac{1}{2!}$ 	$(48\zeta_3^2 + \frac{8\pi^6}{189})g^8$	$110.05g^8$	-0.0587
All 2-cycle graphs	$\left(96\zeta_3^2 + \frac{16\pi^6}{189}\right) g^8$	$220.1g^8$	-0.117
$\frac{1}{3!}$  Only 3-cycle graph	$(32\zeta_3^2 + \frac{16\pi^6}{945})g^8$	$62.52g^8$	-0.0333
Full result	$-16\left(4\zeta_3^2 + \frac{73\pi^6}{630}\right) g^8$	$-1874.86g^8$	1

At 3-loops does not quite work

trees dominate, but 2-cycle contribution larger than 1-cycle... not perfect

LOW CYCLE DOMINANCE?

(Dixon, Oktem, Paranjape, JT, Xu, Zhang, 2026)

Graph	π^6 coefficient	Fraction to actual	ζ_3^2 coefficient	Fraction to actual
$-$ 	$-\frac{272}{315}$	0.466	0	0
$-$ 	$-\frac{272}{315}$	0.466	0	0
$-\frac{1}{2}$ 	$-\frac{16}{35}$	0.247	0	0
$-\frac{1}{3!}$ 	$-\frac{16}{105}$	0.082	0	0
All tree graphs	$-\frac{736}{315}$	1.26	0	0
$\frac{1}{2!}$ 	$\frac{352}{2835}$	-0.067	-48	0.75
	$\frac{704}{2835}$	-0.134	-96	1.5
$\frac{1}{2!}$ 	$\frac{352}{2835}$	-0.067	-48	0.75
$\frac{1}{2!}$ 	$-\frac{328}{2835}$	0.062	0	0
All 1-cycle graphs	$\frac{1080}{2835}$	-0.205	-192	3
$-\frac{1}{2!}$ 	$\frac{8}{189}$	-0.023	48	-0.75
$-\frac{1}{2!}$ 	$\frac{8}{189}$	-0.023	48	-0.75
All 2-cycle graphs	$\frac{16}{189}$	-0.046	96	-1.5
$\frac{1}{3!}$  Only 3-cycle graph	$\frac{16}{945}$	-0.009	32	-0.5
Full result	$-\frac{584}{315}$	1	-64	1

At this order we have two structures in gamma cusp

$$a\pi^6 + b\zeta_3^2$$

look at them independently

NICE HIERARCHY

higher-loop data needed but looks promising

OUTLOOK

If we take the loops of loops expansion seriously, we can introduce a new parameter

$$\mathcal{F}(g,z,\xi) = \mathcal{F}^{(0)}(g,z) + \xi\mathcal{F}^{(1)}(g,z) + \xi^2\mathcal{F}^{(2)}(g,z) + \xi^3\mathcal{F}^{(3)}(g,z)\ldots$$

tree graphs | one-cycle graphs | two-cycle graphs

Physical point is $\xi = 1$ but we expand around $\xi = 0$

Future steps: new computational method for higher-cycle graphs (for any L), MZV cancelation in gamma cusp, direct connection to BES equation, …….., **solve planar N=4 SYM**

Happy 65th Birthday, Lance!!!

It has been a great privilege to have you as a mentor, collaborator and a friend. You are a role model for a great human and an excellent scientist. You are the best of us!

Happy 65th Birthday, Lance!!!

It has been a great privilege to have you as a mentor, collaborator and a friend. You are a role model for a great human and an excellent scientist. You are the best of us!

I am making a bet for Lancefest II, August 7-11, 2045 at UC Davis (one year before 85th birthday) with Saturday excursion to Truckee

Happy 65th Birthday, Lance!!!

It has been a great privilege to have you as a mentor, collaborator and a friend. You are a role model for a great human and an excellent scientist. You are the best of us!

I am making a bet for [text hidden behind image] 2045 at UC Davis (one year before 85th [text hidden behind image] cursion to Truckee