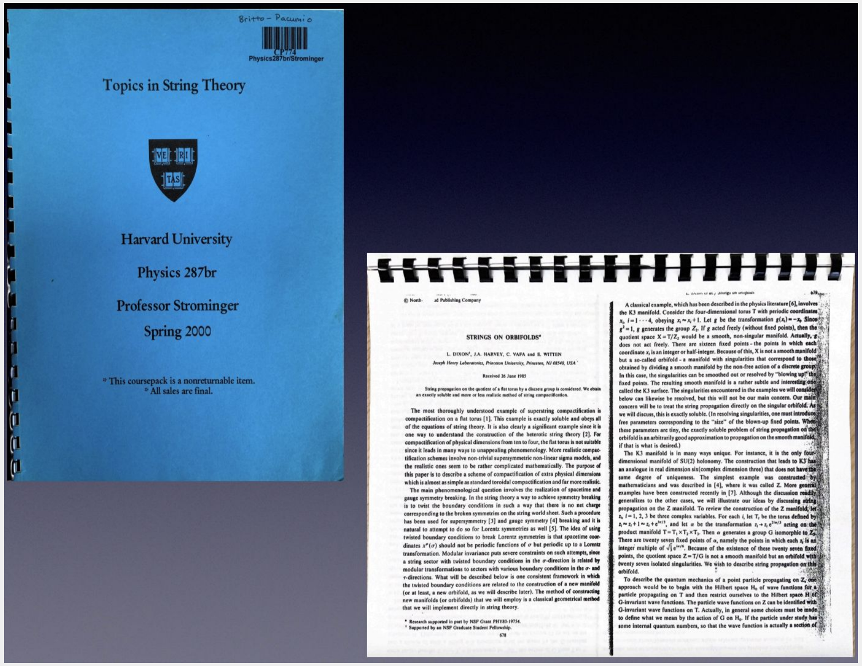
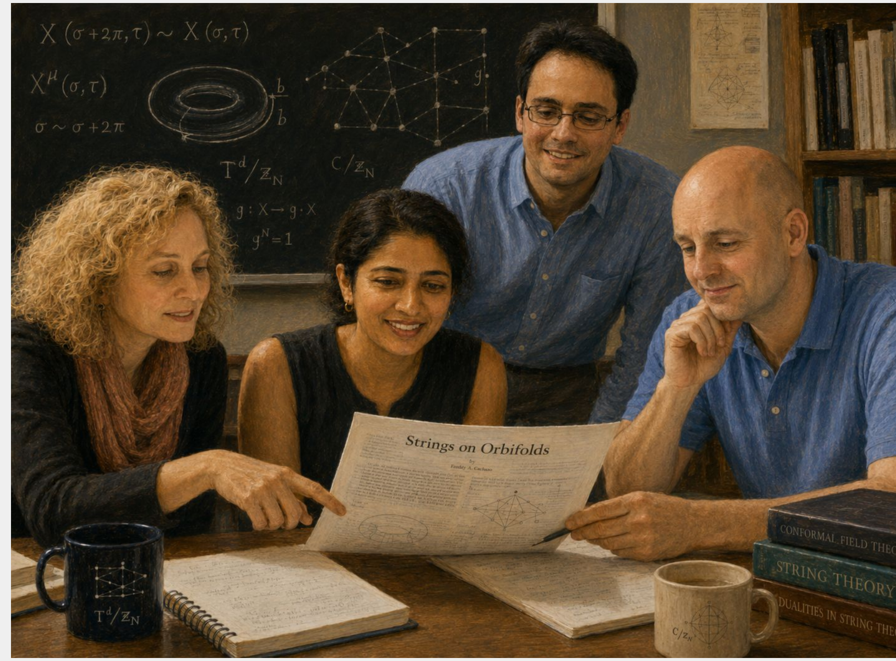
de Sitter Wavefunction from Quadrangular Polylogarithms

Anastasia Volovich
Brown University

Lancefest Workshop
Edinburgh, June 2026





As Ruth already described, in 2000, Ruth, Freddy, Mark and I all were taking a String Theory Class together and Lance’s “Strings on Orbifolds” paper (with Harvey, Vafa and Witten) was one of the assigned papers to read and to present. This is how I first heard about Lance.

It wasn't until
the 2004 KITP Collider Physics program
when I first met Lance in person, and
realized it is the same person!

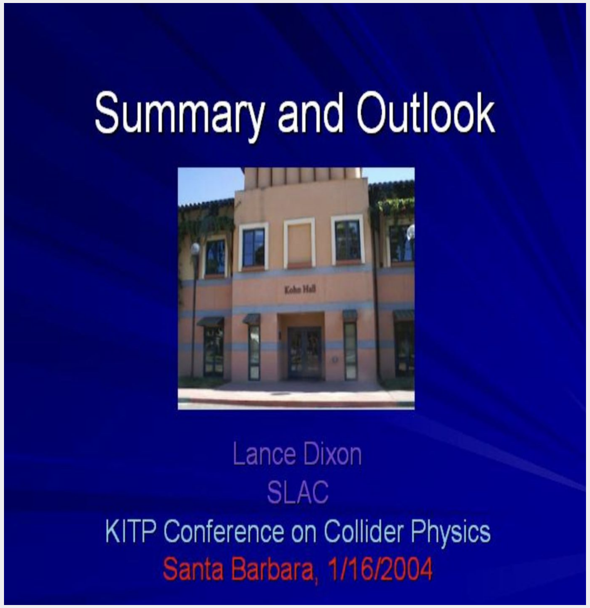
UC SANTA BARBARA
Kavli Institute for
Theoretical Physics

OME DIRECTORY ACTIVITIES PROPOSE ACTIVITY APPLY FOR VISITORS ONLINE T

KITP Program: Collider Physics (Jan 12 - Apr 2, 2004)
Coordinators: Z. Bern, J. Huston, Z. Kunszt, K. Melnikov

Speakers: Please lend your transparencies to the KITP staff members in the front office (Rm 1102) for scanning, or see

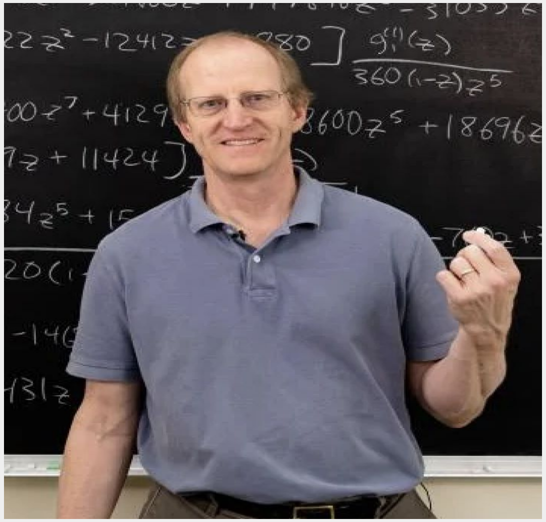
Time	Speaker	Title
1/12-16	**Conference:**	**Collider Physics**
1/20, 2:15 p.m.	Dr. Sergey Alekhin (Moscow) Dr. James Stirling (Durham) Dr. Wu-Ki Tung (MSU)	PDFs Shootout: Views from Three Approaches[Slides][Aud][Cam]
1/22, 12:00 p.m.	Dr. Sergey Alekhin (Moscow) Dr. James Stirling (Durham) Dr. Wu-Ki Tung (MSU)	PDFs Rematch[Slides][Aud][Cam]
1/27, 2:00 p.m.	Dr. Edward Witten IAS	Perturbative Yang-Mills Theory as a String Theory in Twistor Space[
1/28, 3:30 p.m.	Dr. Edward Witten IAS	Discussion
1/29, 10:30 a.m.	Dr. Adrian Signer Durham	Infrared Finite Amplitudes in Gauge Theories[Slides][Aud][Cam]
1/29, 10:30 a.m.	Dr. Zvi Bern UCLA	Supergravity from QCD Amplitudes[Slides][Aud][Cam]



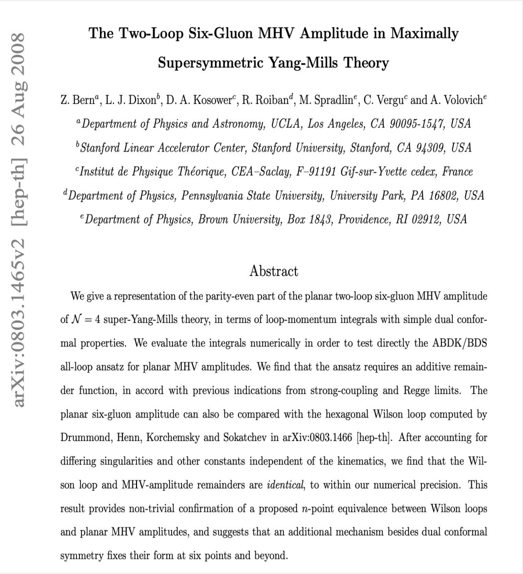
The work that I will describe is motivated by the beautiful mathematical structures and incredible explicit analytic results that have been uncovered in

N=4 Yang-Mills

by Lance and collaborators.



N=4 Yang-Mills


arXiv:0803.1465v2 [hep-th] 26 Aug 2008

The Two-Loop Six-Gluon MHV Amplitude in Maximally Supersymmetric Yang-Mills Theory

Z. Bern[a], L. J. Dixon[b], D. A. Kosower[c], R. Roiban[d], M. Spradlin[e], C. Vergu[e] and A. Volovich[e]

[a]Department of Physics and Astronomy, UCLA, Los Angeles, CA 90095-1547, USA

[b]Stanford Linear Accelerator Center, Stanford University, Stanford, CA 94309, USA

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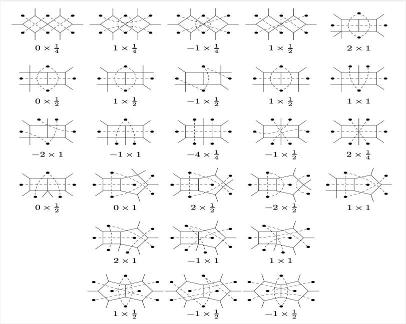
[e]Department of Physics, Brown University, Box 1843, Providence, RI 02912, USA

Abstract

We give a representation of the parity-even part of the planar two-loop six-gluon MHV amplitude of $\mathcal{N}=4$ super-Yang-Mills theory, in terms of loop-momentum integrals with simple dual conformal properties. We evaluate the integrals numerically in order to test directly the ABDK/BDS all-loop ansatz for planar MHV amplitudes. We find that the ansatz requires an additive remainder function, in accord with previous indications from strong-coupling and Regge limits. The planar six-gluon amplitude can also be compared with the hexagonal Wilson loop computed by Drummond, Henn, Korchemsky and Sokatchev in arXiv:0803.1466 [hep-th]. After accounting for differing singularities and other constants independent of the kinematics, we find that the Wilson loop and MHV-amplitude remainders are *identical*, to within our numerical precision. This result provides non-trivial confirmation of a proposed n-point equivalence between Wilson loops and planar MHV amplitudes, and suggests that an additional mechanism besides dual conformal symmetry fixes their form at six points and beyond.

I only wrote one paper with Lance (also with Bern, Kosower, Roiban, Spradlin, Vergu, all are here) in 2008.

We computed (numerically!) the 2-loop 6-point MHV amplitude in N=4 Yang-Mills.



kinematic point	(u_1, u_2, u_3)	R_A
$K^{(0)}$	$(1/4, 1/4, 1/4)$	1.0937 ± 0.0057
$K^{(1)}$	$(1/4, 1/4, 1/4)$	1.076 ± 0.022
$K^{(2)}$	$(0.547253,\ 0.203822,\ 0.881270)$	-1.659 ± 0.014
$K^{(3)}$	$(28/17, 16/5, 112/85)$	-3.6508 ± 0.0032
$K^{(4)}$	$(1/9, 1/9, 1/9)$	5.21 ± 0.10
$K^{(5)}$	$(4/81, 4/81, 4/81)$	11.09 ± 0.50

N=4 Yang-Mills

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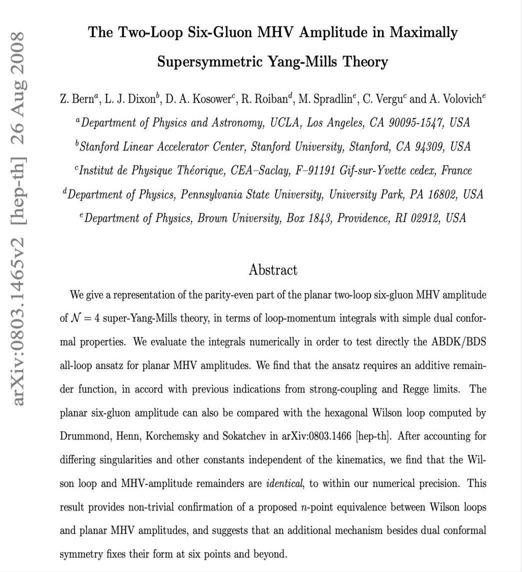

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We computed (numerically!) the 2-loop 6-point MHV amplitude in N=4 Yang-Mills.

Now we consider this amplitude "the simplest non-trivial" amplitude. It exhibits mathematical structures (cluster algebras, cluster adjacency, symbols, multiple polylogarithms), that appear at higher (likely all) loop order.

N=4 Yang-Mills

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We computed (numerically!) the 2-loop 6-point MHV amplitude in N=4 Yang-Mills.

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arXiv > hep-th > arXiv:2606.22380 | Search | Help |

High Energy Physics – Theory

[Submitted on 21 Jun 2026]

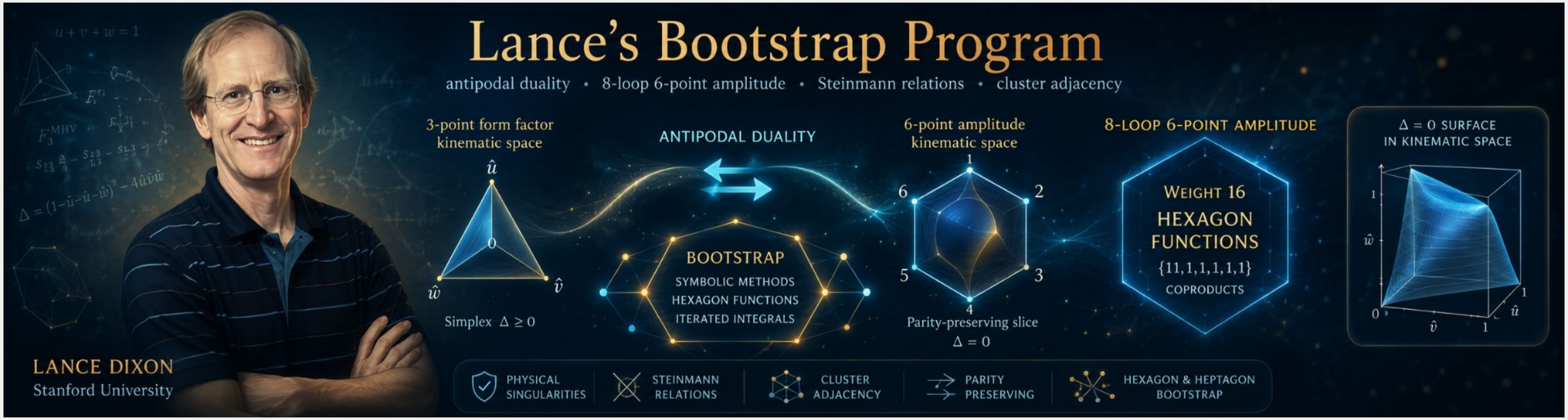
Eight loop form factors, amplitudes and patterns in planar $\mathcal{N} = 4$ super-Yang-Mills theory

Lance J. Dixon, Zhenjie Li

The simplest nontrivial amplitude in planar $\mathcal{N} = 4$ super-Yang-Mills theory is six-gluon scattering in the maximally-helicity-violating configuration. It has been computed to 8 loops

By now, this amplitude has been computed by Lance and collaborators analytically to 8-loops

Lance’s 8-loop results stand at the forefront of perturbative calculations in N=4 Yang-Mills (and quantum field theory in general) and, in my opinion, serve as a benchmark - or a “challenge” - against which any “AI takeover” should be measured.



de Sitter Wavefunction from Quadrangular Polylogarithms

2603.08670, 2605.06542

Paranjape, Skowronek, Spradlin, AV, Weng

Ferro, Lukowski, Ren, Spradlin, AV, Weng, Zhang

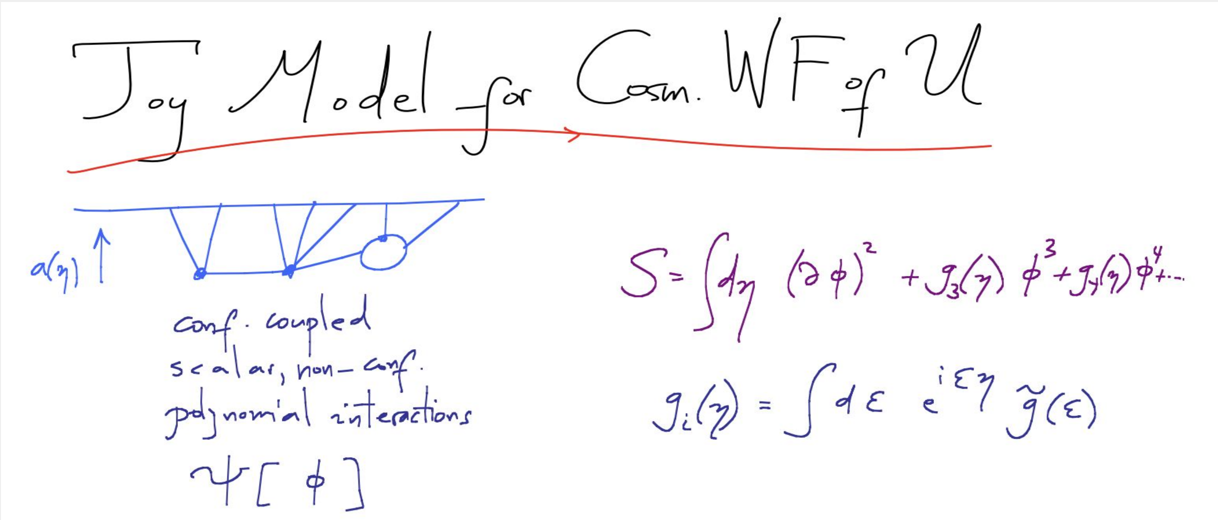
We presented an explicit formula for the n-site chain graph contribution to the cosmological wavefunction for conformally coupled φ^3 theory in de Sitter. Our result relies on the recent finding that symbol of this function satisfies total compatibility with respect to A_{2n-2} cluster algebra, and that Rudenko QLi-polylogs provide a complete basis for such functions.

Outline

- de Sitter wavefunction
- two- and three-site chain functions
- (total!) cluster compatibility
- quadrangular polylogarithms
- recursion relations
- solution of the recursion

The de Sitter Wavefunction

Wavefunction coefficients for a conformally coupled scalar in de Sitter space have been studied for many years.



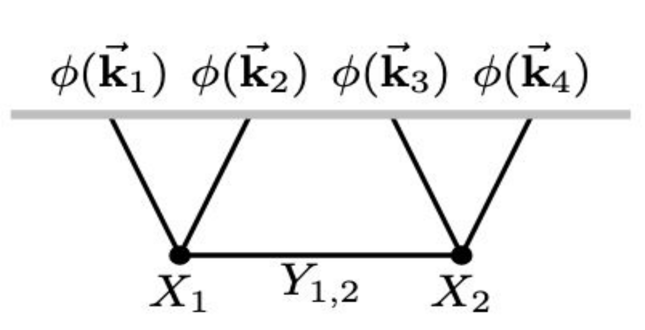
Amplitudes 2017, Edinburgh, Arkani-Hamed

Two-site chain

The simplest example is the integral

$$\psi_2(X_1, X_2; Y_{1,2}) = \int_0^\infty dx_1\, dx_2 \frac{Y_{1,2}}{(x_1 + X_1 + x_2 + X_2)(x_1 + X_1 + Y_{1,2})(x_2 + X_2 + Y_{1,2})}$$

which computes the contribution to the de Sitter wavefunction coefficient from "two-site chain" graph



Arkani-Hamed, Benincasa, Postnikov 1709.02813

$X_1 = |\vec{\mathbf{k}}_1| + |\vec{\mathbf{k}}_2| \qquad X_2 = |\vec{\mathbf{k}}_3| + |\vec{\mathbf{k}}_4| \qquad Y_{1,2} = |\vec{\mathbf{k}}_1 + \vec{\mathbf{k}}_2|$

Two-site chain

Let us embed X and Y variables into the Grassmannian Gr(2,5) via the 2x5 matrix representative

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & X_1 + Y_{1,2} & X_1 - Y_{1,2} & X_1 + X_2 & 1 \end{pmatrix}$$

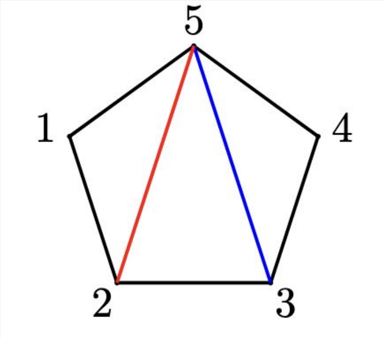
and use Plucker coordinates

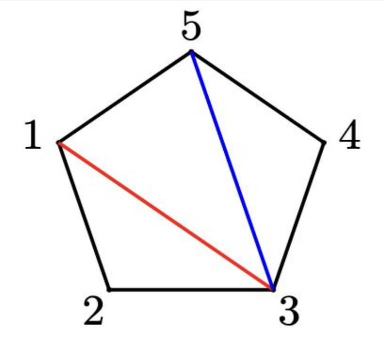
$$\Delta_{ij}$$

Two-site chain

$$\mathcal{S}(\psi_2) = \frac{\Delta_{23}\Delta_{15}}{\Delta_{12}\Delta_{35}} \otimes \frac{\Delta_{13}\Delta_{25}}{\Delta_{12}\Delta_{35}} - \frac{\Delta_{14}\Delta_{23}}{\Delta_{34}\Delta_{12}} \otimes \frac{\Delta_{13}\Delta_{24}}{\Delta_{34}\Delta_{12}} + \frac{\Delta_{23}\Delta_{45}}{\Delta_{25}\Delta_{34}} \otimes \frac{\Delta_{24}\Delta_{35}}{\Delta_{25}\Delta_{34}}$$

The symbol entries are Plucker coordinates on Gr(2,5), which are cluster variables of the A_2 cluster algebra. The adjacent symbol entries correspond to non-crossing chords of the pentagon which manifests cluster compatibility.





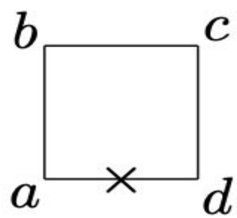
FLRSVWZ 2605.06542

Two-site chain

Unlocking this structure allowed us to write down the function

$$\psi_2 = F_2(5,1,2,3) - F_2(4,1,2,3) + F_2(4,5,2,3)$$

$$F_2(a,b,c,d) =$$



$$= \mathrm{Li}_2(q_{abcd}) - \mathrm{Li}_1(q_{abcd})\log(q_{abcd})$$

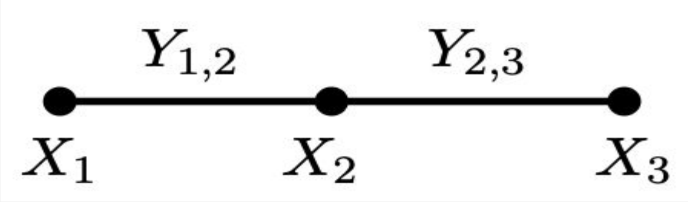
where F2 is associated to a rooted square.

$$q_{ijkl} = \frac{\Delta_{ij}\Delta_{kl}}{\Delta_{jk}\Delta_{li}}$$

FLRSVWZ 2605.06542

Three-site chain

The integral which computes the contribution to the de Sitter wavefunction coefficient from the “three-site chain” graph



is

$$\psi_3(X_1, X_2, X_3; Y_{1,2}, Y_{2,3}) = \int_0^\infty dx_1\, dx_2\, dx_3\, \frac{4Y_{1,2}Y_{2,3}}{B_1 B_2 B_3 B_4}\left(\frac{1}{B_5} + \frac{1}{B_6}\right)$$

$$B_1 = X_1 + x_1 + Y_{1,2} \qquad B_4 = X_1 + X_2 + X_3 + x_1 + x_2 + x_3$$

$$B_2 = X_2 + x_2 + Y_{1,2} + Y_{2,3} \qquad B_5 = X_1 + X_2 + x_1 + x_2 + Y_{2,3}$$

$$B_3 = X_3 + x_3 + Y_{2,3} \qquad B_6 = X_2 + X_3 + x_2 + x_3 + Y_{1,2}$$

Three-site chain

Let us embed X and Y variables into the Grassmannian Gr(2,7) via the 2x7 matrix representative

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & X_1+Y_{1,2} & X_1-Y_{1,2} & X_1+X_2+Y_{2,3} & X_1+X_2-Y_{2,3} & X_1+X_2+X_3 & 1 \end{pmatrix}$$

The symbol entries are Plucker coordinates on Gr(2,7), which are cluster variables of the A_4 cluster algebra.

The symbol also satisfies total compatibility!

Capuano, Ferro, Lukowski, Palazio, Zhang 2603.09965

Three-site chain

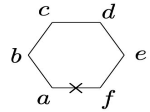
Unlocking this structure allowed us to write down the function

$$\psi_3 = F_3(7,1,2,3,4,5) - F_3(6,1,2,3,4,5) + F_3(6,7,2,3,4,5)$$

where F3 is associated to a rooted hexagon and its quadrangulations.

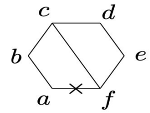
FLRSVWZ 2605.06542

$F_3(a,b,c,d,e,f) =$



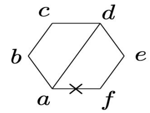
$= -\left(\mathrm{QLi}_3^{+}(a,b,c,d,e,f) + \mathrm{QLi}_2^{+}(a,b,c,d,e,f)\log(q_{abcdef})\right)$

$+$



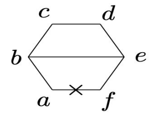
$= -F_2(a,b,c,f)\,\mathrm{Li}_1(q_{cdef})$

$+$



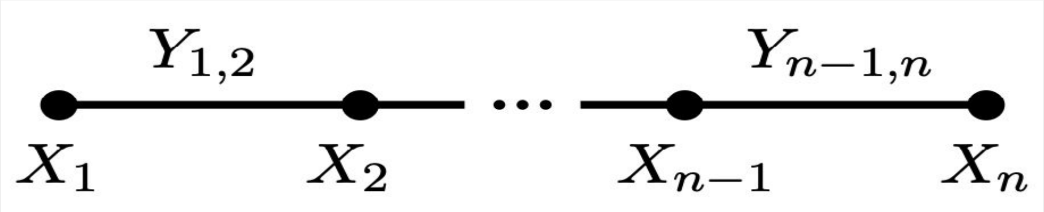
$= -F_2(a,d,e,f)\,\mathrm{Li}_1(q_{abcd})$

$+$



$= +F_2(a,b,e,f)\,\mathrm{Li}_1(1/q_{bcde})$

n-site chain



- The integrand is encoded geometrically in cosmological polytope Arkani-Hamed, Benincasa, Postnikov 1709.02813
- Differential equations for the integrals have been worked out by graphical tubing rules Arkani-Hamed, Baumann, Hillman, Joyce, Lee, Pimentel 2312.05303 De Pokraka 2308.03753 Capuano, Ferro, Lukowski, Palazio 2505.14609
- Explicit results for the integrals are known only up to 3-site Hillman 1912.09450
- The symbol can be recursively computed for all n Hillman 1912.09450 He, Jiang, Liu, Yang, Zhang 2407.17715

n-site chain

- Mazloumi, Xi 2512.14854 and Capuano, Ferro, Lukowski, Palazio 2512.14869 noticed that symbol alphabets for n-site chain graphs are cluster variables for A_{2n-2}-type cluster algebra.
- Paranjape, Skowronek, Spradlin, AV, Weng 2603.08670 showed that they satisfy cluster compatibility.
- Capuano, Ferro, Lukowski, Palazio, Zhang 2603.09965 pointed out that they satisfy much stronger condition of total compatibility.

n-site chain

- Capuano, Ferro, Lukowski, Palazio, Zhang 2603.09965 pointed out that they satisfy much stronger condition of total compatibility.

Remarkably, there is a class of functions satisfying exactly this property that has been constructed by mathematicians and played a critical role in Rudenko's proof of Goncharov's depth conjecture.

Quadrangular Polylogarithms

- These functions are called $QLi_m(0,..2n+1)$: weight-m multiple polylog functions of cross-rations of $z_0,z_1,..z_{2n+1} \in P^1$ whose symbols satisfy total compatibility with respect to A_{2n-2} cluster algebra.

- They were constructed by Rudenko 2012.05599 and 2208.01563 (with Matveiakin).

- We studied these functions when they first appeared, hoping to find some use for them in N=4 SYM (which is still elusive), but they are exactly the functions we need for n-chain graphs!

Quadrangular Polylogarithms

QLis could be constructed recursively in terms of quadrangulations of n-gons and one can express them in terms of ordinary MPLs

$$\mathrm{QLi}_k^+(a,b,c,d) = (-1)^k\,\mathrm{Li}_k(q_{abcd})$$

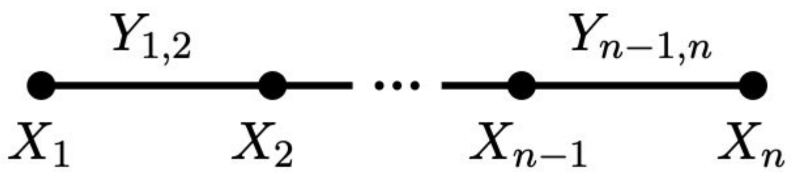
$$\mathrm{QLi}_2^+(a,b,c,d,e,f) = \mathrm{Li}_{1,1}(q_{abcf}, q_{cdef}) - \mathrm{Li}_{1,1}(q_{abef}, 1/q_{bcde}) + \mathrm{Li}_{1,1}(q_{adef}, q_{abcd})\,,$$

$$\begin{aligned}\mathrm{QLi}_3^+(a,b,c,d,e,f) = &- \mathrm{Li}_{2,1}(q_{abcf}, q_{cdef}) - \mathrm{Li}_{1,2}(q_{abcf}, q_{cdef}) \\ &+ \mathrm{Li}_{2,1}(q_{abef}, 1/q_{bcde}) + \mathrm{Li}_{1,2}(q_{abef}, 1/q_{bcde}) \\ &- \mathrm{Li}_{2,1}(q_{adef}, q_{abcd}) - \mathrm{Li}_{1,2}(q_{adef}, q_{abcd})\,,\end{aligned}$$

…

n-site chain

$$\psi_n(X_1, X_2, \ldots, X_n; Y_{1,2}, Y_{2,3}, \ldots, Y_{n-1,n}) =$$



The symbol for n-site chain can be expressed recursively in terms of (n-1)-site with shifted arguments

Hillman 1912.09450

He, Jiang, Liu, Yang, Zhang 2407.17715

Solution of the Recursion

$$\psi_n = F_n(2n{+}1, 1, \ldots, 2n{-}1) - F_n(2n, 1, \ldots, 2n{-}1) + F_n(2n, 2n{+}1, 2, \ldots, 2n{-}1)$$

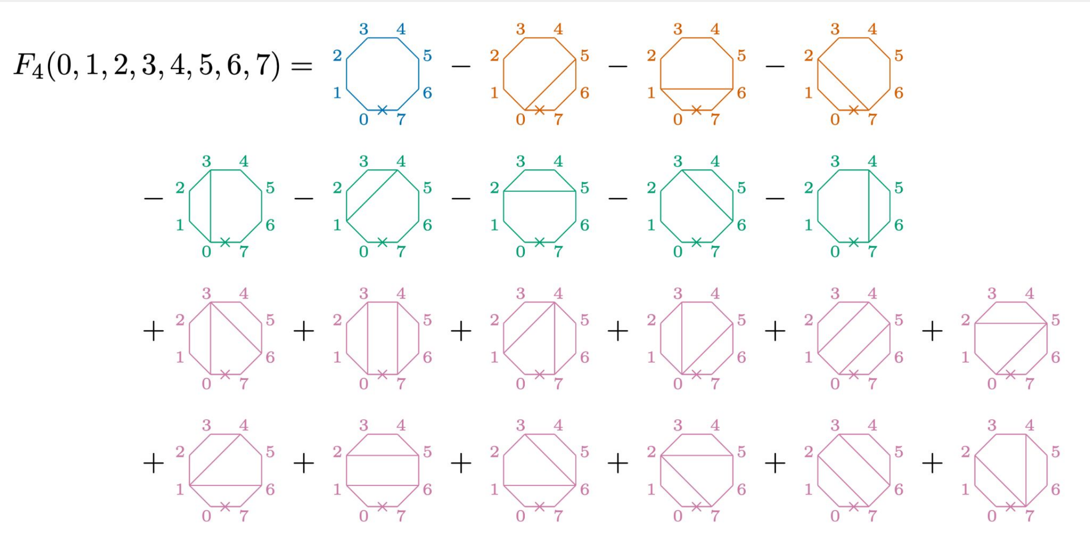
$$F_n(0, \ldots, 2n{-}1) =$$
$$\sum_{k=0}^{n-2} \sum_{\substack{S_0 \sqcup S_1 \sqcup \cdots \sqcup S_k \\ = \{0, \ldots, 2n-1\}}} (-1)^{k+n} \left(\mathrm{QLi}^{+}_{\frac{|S_0|}{2}}(S_0) + \mathrm{QLi}^{+}_{\frac{|S_0|}{2}-1}(S_0) \log(q_{S_0}) \right) \prod_{i=1}^{k} \mathrm{QLi}^{(-)^{S_i(1)}}_{\frac{|S_i|}{2}-1}(S_i)$$

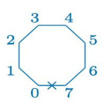
where $S_i(j)$ is the j-th element of S_i. Here the second sum runs over all dissections of the polygon $(0, \ldots, 2n{-}1)$ into even sub-polygons $\{S_0, \ldots, S_k\}$, with S_0 containing the root edge $(0, 2n{-}1)$, to which we associate the cross-ratio given by

$$q_{S_0} = \prod_{i=1}^{\frac{|S_0|}{2}-1} q_{0, S_0(2i), S_0(2i+1), S_0(2i+2)}$$

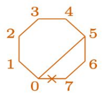
FLRSVWZ 2605.06542

Four-site chain

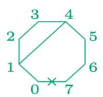




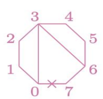
$= \mathrm{QLi}_4^+(0,1,2,3,4,5,6,7) + \mathrm{QLi}_3^+(0,1,2,3,4,5,6,7)\log(q_{01234567})\,,$



$= \big(\mathrm{QLi}_2^+(0,5,6,7) + \mathrm{QLi}_1^+(0,5,6,7)\log(q_{0567})\big)\,\mathrm{QLi}_2^+(0,1,2,3,4,5)\,,$ (5.11)



$= \big(\mathrm{QLi}_3^+(0,1,4,5,6,7) + \mathrm{QLi}_2^+(0,1,4,5,6,7)\log(q_{014567})\big)\,\mathrm{QLi}_1^-(1,2,3,4)\,,$



$= \big(\mathrm{QLi}_2^+(0,3,6,7) + \mathrm{QLi}_1^+(0,3,6,7)\log(q_{0367})\big)\,\mathrm{QLi}_1^+(0,1,2,3)\,\mathrm{QLi}_1^-(3,4,5,6)\,.$

F_4 is associated to a rooted octagon and its quadrangulations.

FLRSVWZ 2605.06542

Summary

- We presented an explicit formula for the n-site chain graph contribution to the cosmological wavefunction for conformally coupled φ^3 theory in de Sitter.
- Our result relies on the recent finding that symbol of this function satisfies total compatibility with respect to A_{2n-2} cluster algebra, and that Rudenko QLi-polylogs provide a complete basis for such functions.
- We proved our formula by showing it satisfies the recursion relations.

Outlook

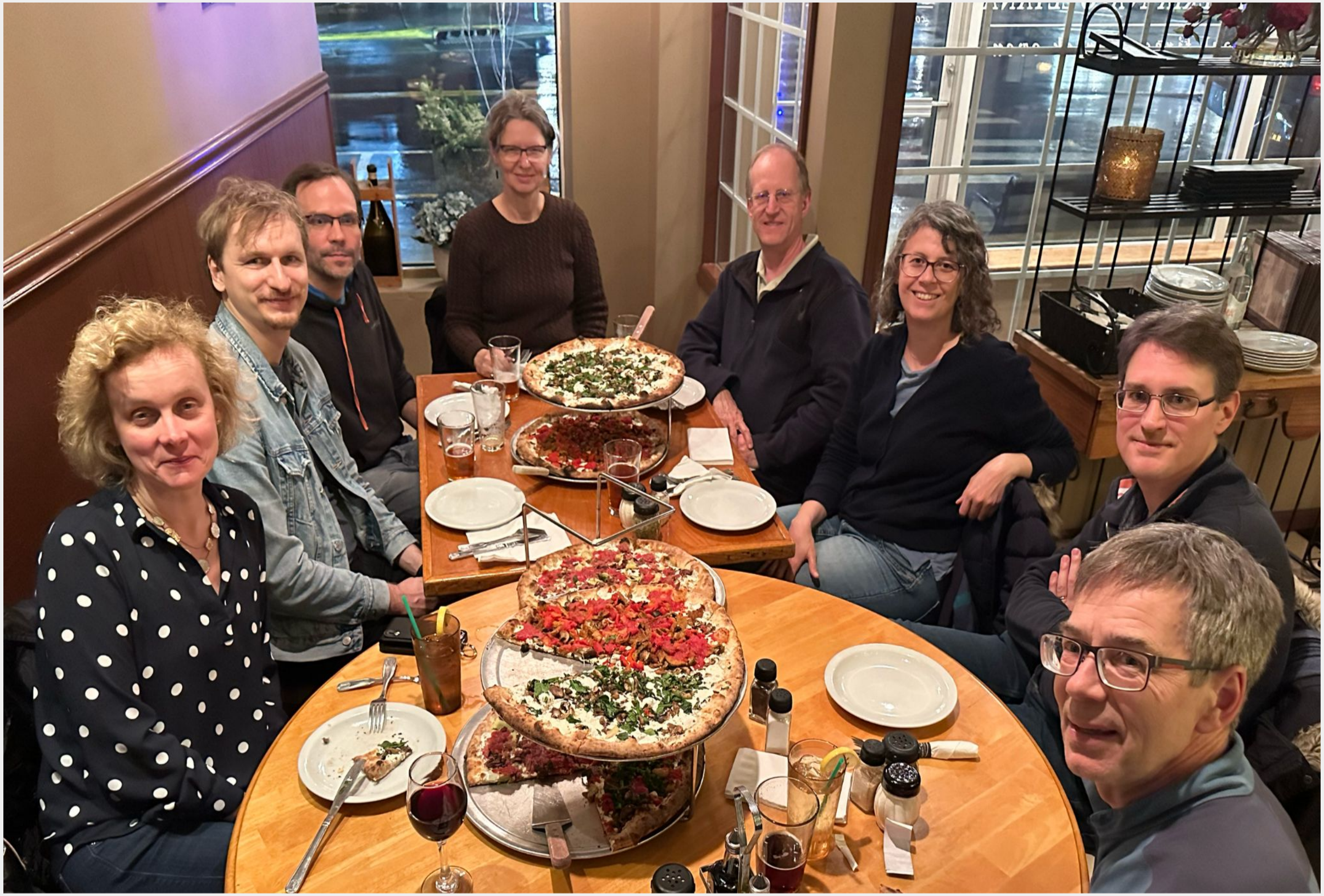
We are currently working on generalizing these results to

- **Loops (B-type cluster algebra)**
- **Correlators**
- **FRW cosmology**

Capuano, Ferro, Lukowski, Palazio, Zhang, Paranjape, Ren, Spradlin, AV, Weng

Thank you,
Lance,
for all of your support, encouragement, and inspiration over the years!

Happy Birthday, Lance!



Happy Birthday, Lance!