Color Decompositions of Gluon Scattering Amplitudes

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New color decompositions for gauge amplitudes at tree and loop level

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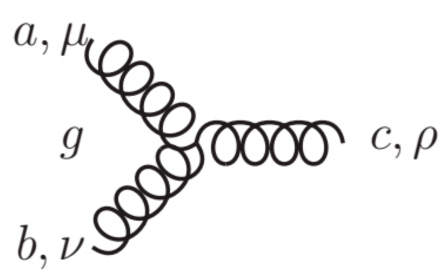
Abstract

Recently, a color decomposition using structure constants was introduced for purely gluonic tree amplitudes, in a compact form involving only the linearly independent subamplitudes. We give two proofs that this decomposition holds for an arbitrary number of gluons. We also present and prove similar decompositions at one loop, both for pure gluon amplitudes and for amplitudes with an external quark–antiquark pair.

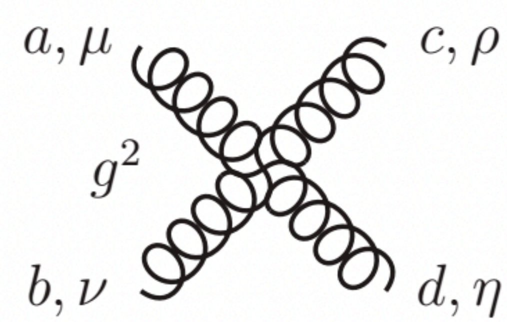
Gluon Scattering

- Massless adjoint spin one particle in adjoint of SU(N)

- Quarks in fundamental of SU(N) have charges in fundamental of SU(N) ,Q
- Gluons must have charges Q+Q*
- fundamental is not its own complex conjugate (true for SU(N) and E_6)



$f^{abc}((k_2 - k_3)_\mu \delta_{\nu\rho} +$



$f^{abe} f^{ecd}((\delta_{\nu\rho}\delta_{\mu\eta} +$

Calculate: "Total Quantum Number Management"

-split amplitude into gauge invariant specific amplitudes

-use helicity of external massless gluon

-"color" of external gluons

-work on-shell

Color Trace Basis

- Replace structure constants by trace of colour matrices

$$f^{abc} = \mathrm{Tr}(T^aT^bT^c) - \mathrm{Tr}(T^aT^cT^b)$$

$$\sum_a \mathrm{Tr}(XT^a)\,\mathrm{Tr}(T^aY) = \mathrm{Tr}(XY),\ \sum_a$$

$$\sum_a \mathrm{Tr}(XT^aYT^a) = \mathrm{Tr}(X)\,\mathrm{Tr}(Y)$$

U(N)

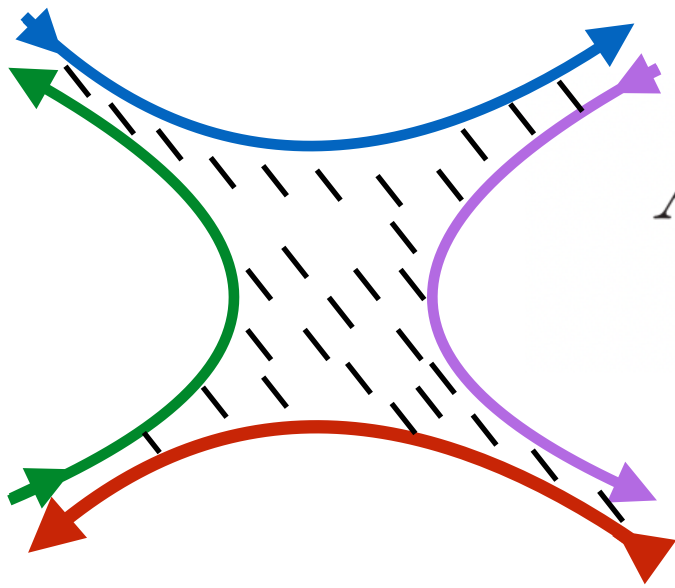
Tree level amplitudes can be expanded

$$\mathcal{A}_n^{(0)}(1,2,3,\cdots,n) = \sum_{S_n/Z_n} \mathrm{Tr}[T^{a_1}T^{a_2}\cdots T^{a_n}]A_n^{(0)}(a_1,a_2,\cdots,a_n)$$

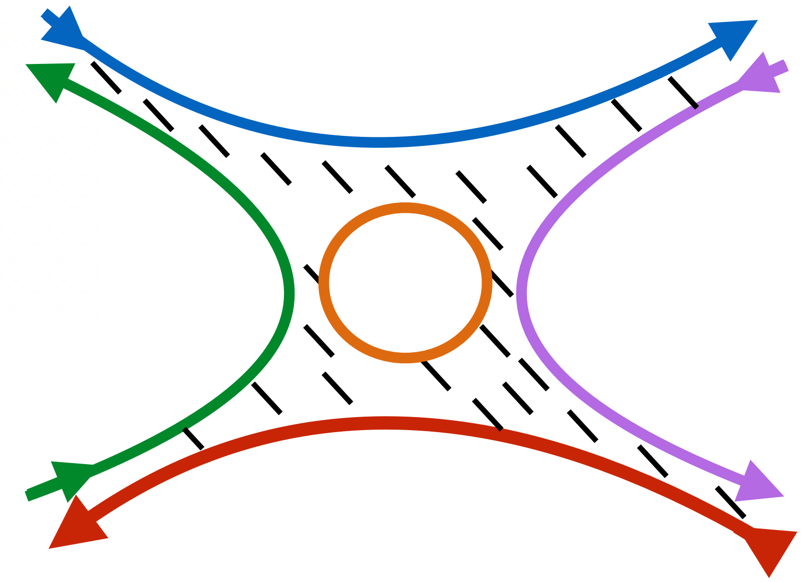
- The partial amplitudes are gauge invariant objects
- Can be computed with colour-ordered formalism

Separation of color and kinematics

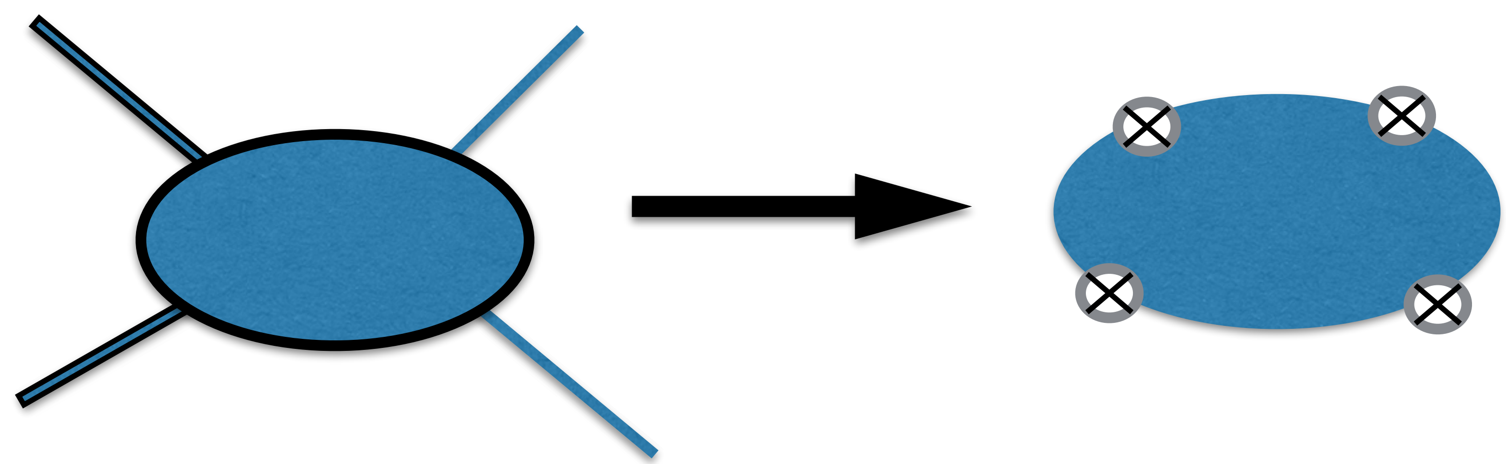
Gluon Scattering in String Theory



$$A \sim \sum_{surface} \sum_{points} e^{-S}$$



- Color charges act at endpoints of an open string (quarks)
- Charges are in fundamental representation
- State is in adjoint



-string theory *starts* with the trace basis expansion

$$\mathcal{A}_n^{(0)}(1,2,3,\cdots,n) = \sum_{S_n/Z_n} \mathrm{Tr}[T^{a_1}T^{a_2}\cdots T^{a_n}] A_n^{(0)}(a_1,a_2,\cdots,a_n)$$

String theory amplitudes depending on *T*: *T -> Infinity recover Yang-Mills*

Kleiss-Kuijf Relations

-color trace terms are independent $\mathrm{Tr}[T^{a_1}T^{a_2}\cdots T^{a_n}]$

-not all partial amplitudes are independent $A_n^{(0)}(a_1, a_2, \cdots, a_n)$

$$A_n^{(0)}(1, \{\alpha\}, n, \{\beta\}) = (-1)^{|\beta|} \sum_{\sigma \in OP(\alpha, \beta^T)} A_n^{(0)}(1, \{\sigma\}, n)$$

Kleiss and Kuijf. 1989

- Reduces number of independent from (n-1)!/2 to (n-2)!
- Quite complex permutation sum

Proof of KK (1):

Del Duca, Dixon and Maltoni

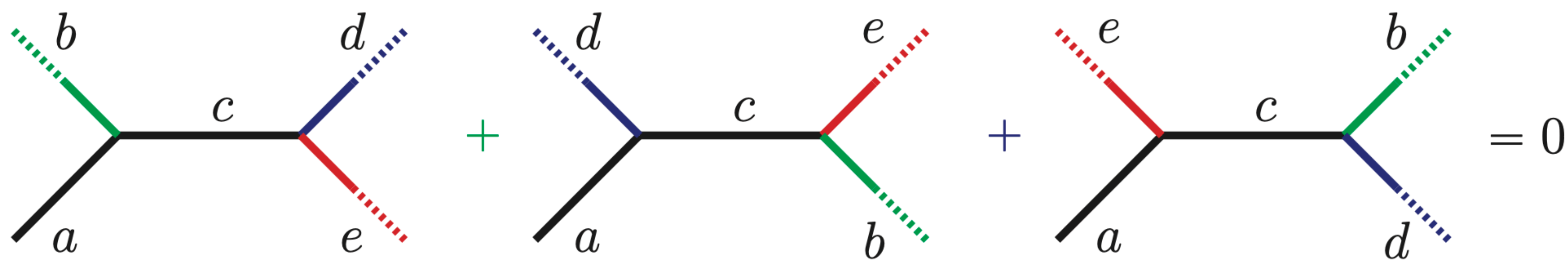
-stay with structure constant expansion, consists of terms which are products of structure constants

$$f^{abc} f^{def} \dots f^{xyz} ,$$

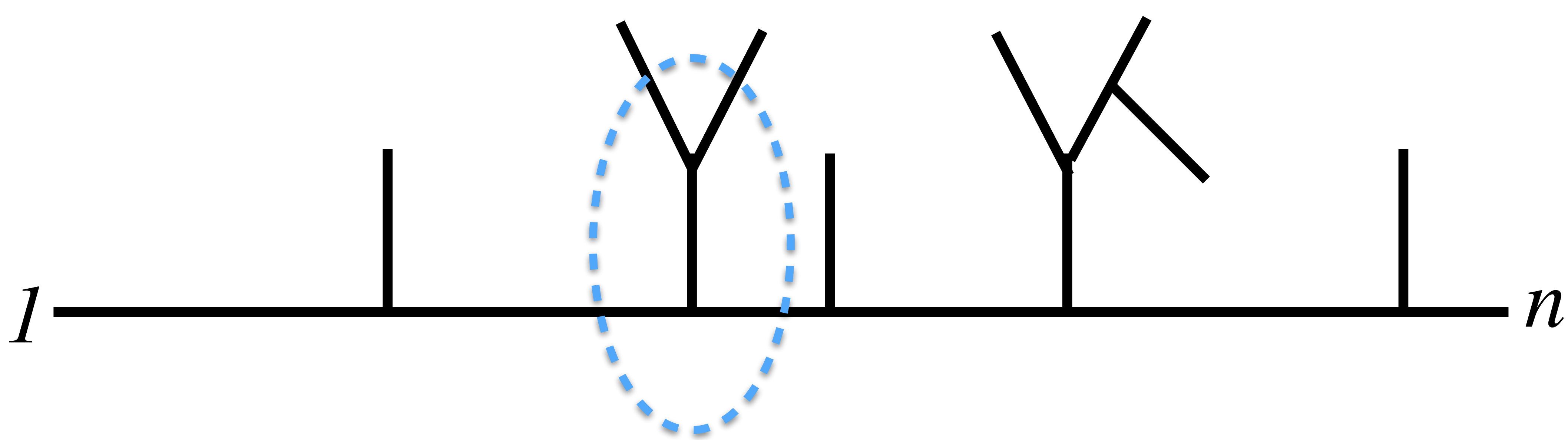
- Turn these into a colour diagram. Cubic vertex for each f^{abc}. Diagrams match cubic Feynman diagrams.
- Not all combinations are independent.
- Jacobi identity links terms

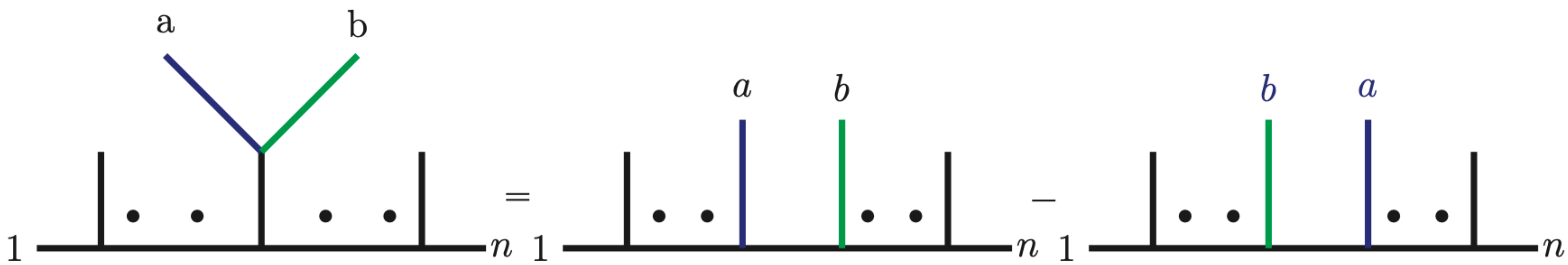
$$f^{abc} f^{cde} + f^{adc} f^{ceb} + f^{aec} f^{cbd} = 0$$

Jacobi:

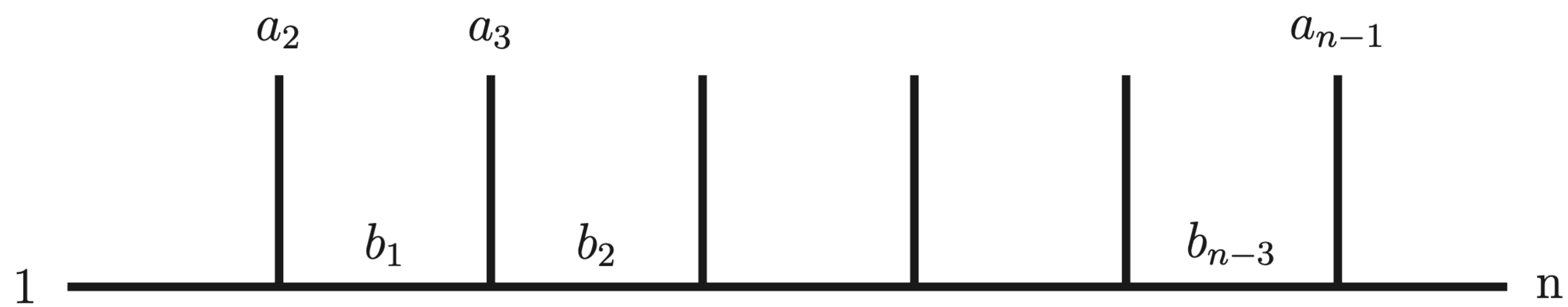


- Choose two legs , *1* and *n* say
- Any term will be trees of legs attached to line joining *1* and *n*





-recursively using Jacobi



$$\mathcal{A}_n^{(0)}(1,2,3,\cdots,n) = \sum_{S_{n-2}} f^{1a_2b_1} f^{b_1a_3b_2} \cdots f^{b_{n-3}a_{n-1}n} \overline{A}_n(1,a_2,\cdots,a_{n-1},n)$$

Del Duca Dixon and Maltoni

In principle these are different but….

(n-2)!

$$f^{1a_2b_1}f^{b_1a_3b_2}\cdots f^{b_{n-1}a_{n-1}n} = \mathrm{Tr}(T^1T^{a_2}\cdots T^{a_{n-1}}T^n) + (-1)^n\,\mathrm{Tr}(T^nT^{a_{n-1}}\cdots T^{a_2}T^1)$$
$$+ \sum \mathrm{Tr}(T^1XT^nY)\ ,$$

Comparing

$$\overline{A}_n(1, a_2\cdots a_{n-1}, n) = A_n^{(0)}(1, a_2\cdots a_{n-1}, n)$$

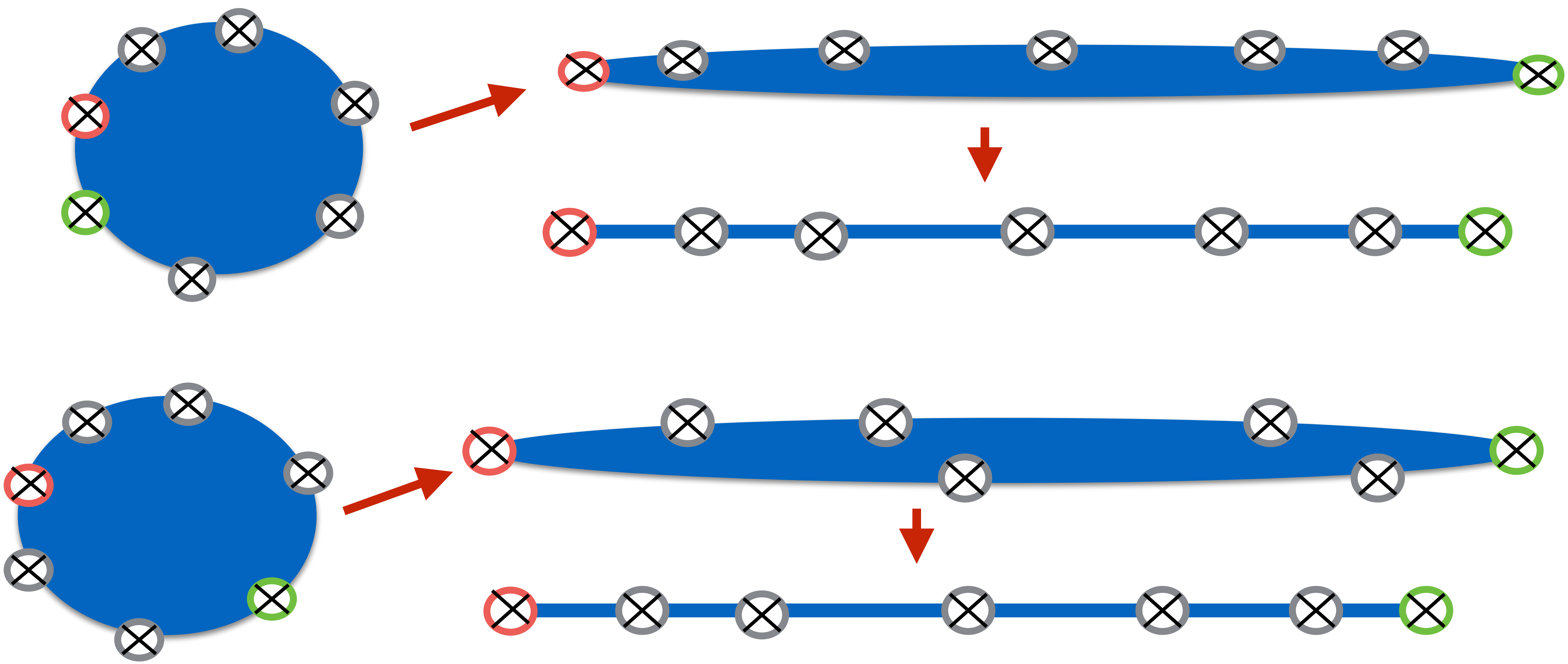
and obtain Kleiss-Kuijf

Why? : connected Feynman diagrams yield a restricted set of terms

String Theory “proof” of Kleiss-Kuijf…

-only contributions to the field theory limit come from boundaries in moduli space

$A_n^{(0)}(1,\{\sigma\},n)$



$A_n^{(0)}(1,\{\alpha\},n,\{\beta\})$

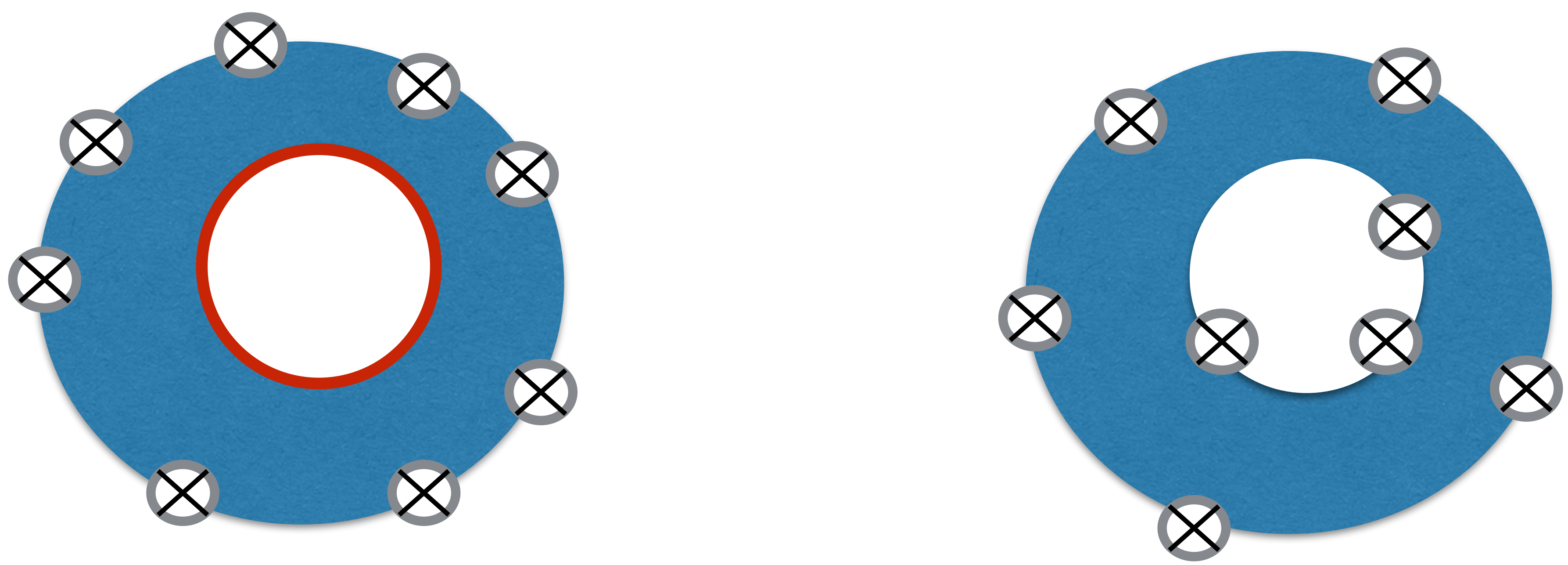
-kinematic argument

One-Loop

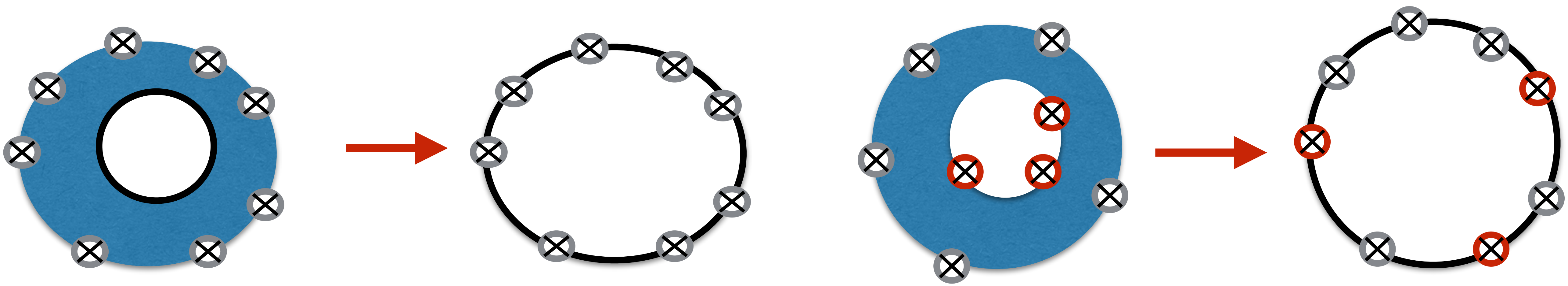
$$\mathcal{A}_n^{(1)}(1,2,3\ ,\cdots\ ,n) = \sum_{S_n/Z_n} N_c \,\mathrm{Tr}[T^{a_1}T^{a_2}\cdots T^{a_n}]A_{n:1}^{(1)}(a_1,a_2,\cdots\ ,a_n)$$

$$+\sum_r \sum_{P_{n:r}} \mathrm{Tr}[T^{a_1}\cdots T^{a_{r-1}}]\,\mathrm{Tr}[T^{a_r}\cdots T^{a_n}]A_{n:r}^{(1)}(a_1,a_2,\cdots\ ,a_{r-1};a_r,\cdots\ ,a_n)$$

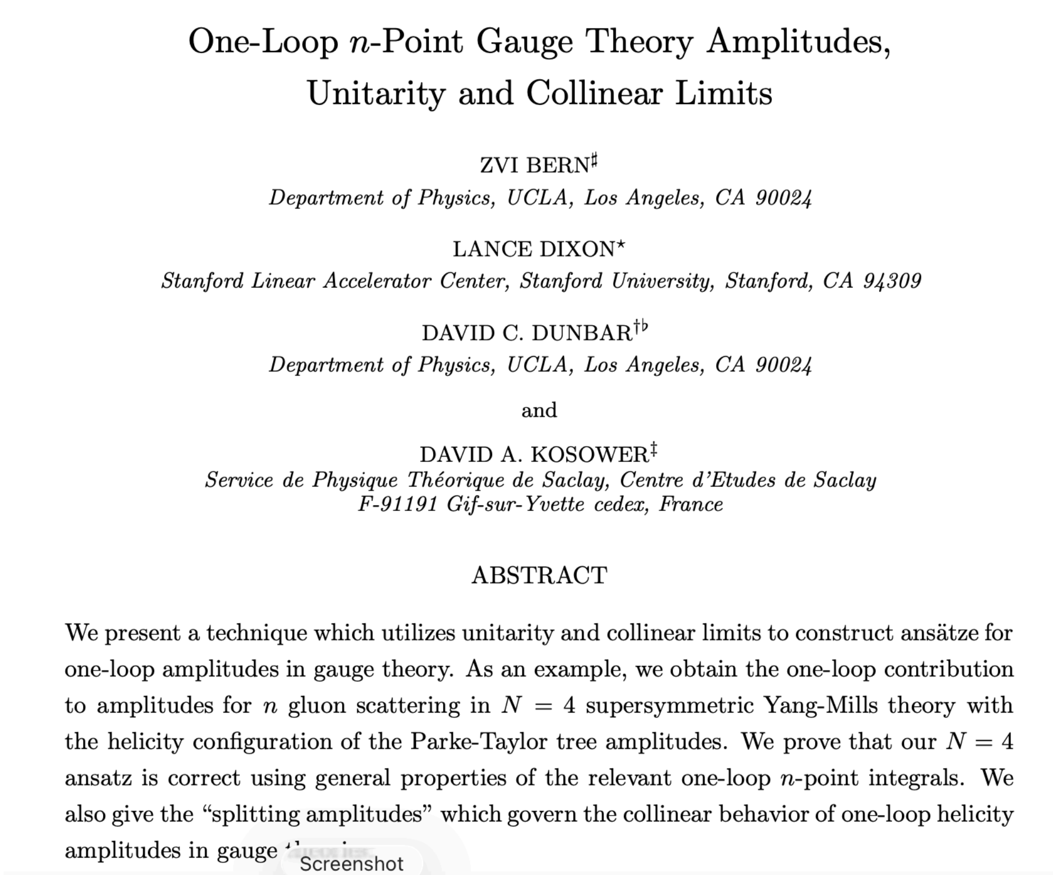
-from string theory: one loop surface is an annulus with two boundaries



Relation between terms



$$A^{(1)}_{n:r}(a_1, a_2, \cdots, a_{r-1}; b_r, \cdots, b_n) = (-1)^r \sum_{\sigma \in COP\{\alpha\}\{\beta^T\}} A^{(1)}_{n:1}(\sigma)$$



One-Loop n-Point Gauge Theory Amplitudes,
Unitarity and Collinear Limits

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ABSTRACT

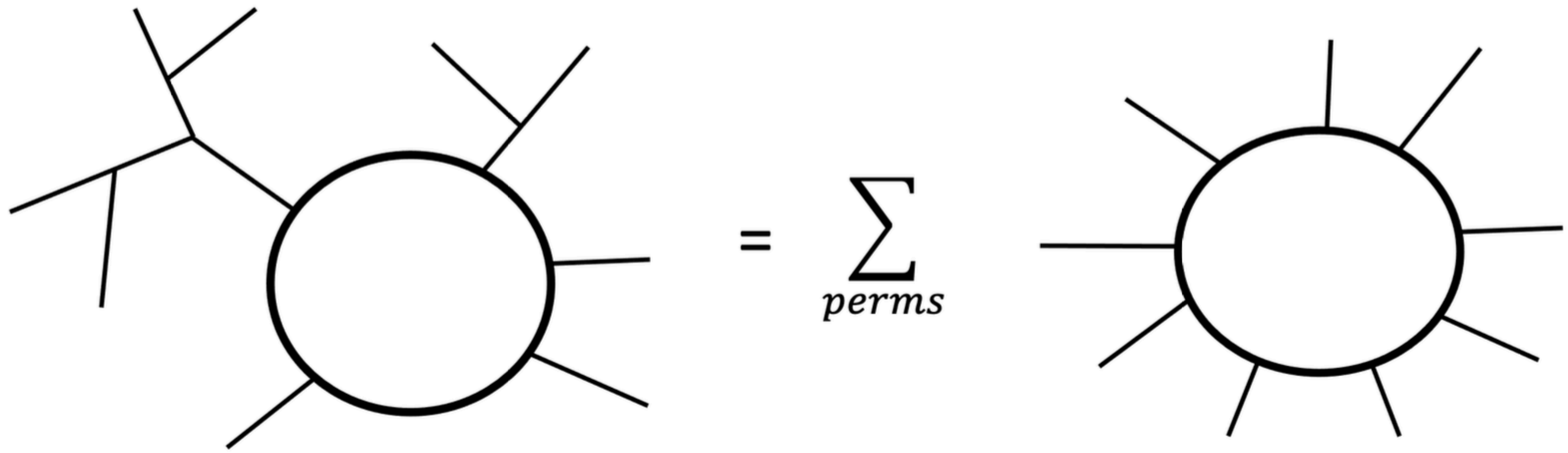
We present a technique which utilizes unitarity and collinear limits to construct ansätze for one-loop amplitudes in gauge theory. As an example, we obtain the one-loop contribution to amplitudes for n gluon scattering in $N = 4$ supersymmetric Yang-Mills theory with the helicity configuration of the Parke-Taylor tree amplitudes. We prove that our $N = 4$ ansatz is correct using general properties of the relevant one-loop n-point integrals. We also give the "splitting amplitudes" which govern the collinear behavior of one-loop helicity amplitudes in gauge

Screenshot

Del Duca, Dixon and Maltoni

Alt proof

- Consider products of colour terms: all contain a



Using Jacobi

$$C^1(a_1, a_2, a_3, \cdots, a_n) \equiv f^{b_n a_1 b_1} f^{b_1 a_2 b_2} \ldots f^{b_{n-1} a_n b_n}$$

$$\mathcal{A}_n = \sum_{S_n/Z_n} C^1(a_1, a_2, a_3, \cdots, a_n) \overline{A}_n^{(1)}(a_1, a_2, a_3, \cdots, a_n)$$

using identity

$$C^1(a_1, a_2, a_3, \cdots, a_n) = \mathrm{Tr}(T^{a_1} T^{a_2} \cdots T^{a_n}) + (-1)^n \,\mathrm{Tr}(T^{a_n} \cdots T^{a_2} T^{a_1}) + \text{double trace terms} .$$

we obtain

$$\overline{A}_n^{(1)}(1, a_2, a_3, \cdots, a_n) = A_{n:1}^{(1)}(1, a_2, a_3, \cdots, a_n)$$

and relation for sub-leading follows

One-Loop all-plus gluon Amplitude

$$A^{(1)}_{n:1}(1^+, 2^+, \cdots, n^+) = \sum_{1 \le k_1 < k_2 < k_3 < k_4 \le n} \frac{\langle k_1\, k_2 \rangle [k_2\, k_3] \langle k_3\, k_4 \rangle [k_4\, k_1]}{\langle 1\, 2 \rangle \langle 2\, 3 \rangle \cdots \langle n\, 1 \rangle}$$

Bern, Chalmers, Dixon and Kosower hep-ph/9312333

~ $n^4/24$ terms

- Tree amplitude vanishes
- Amplitude finite and free of (poly)logarithms (pseudo-tree)
- obtained from collinear limits
- (non manifestly) cyclically symmetric
- Amplitudes of self-dual Yang-Mills Cangemi; Chalmers and Siegal
- cannot be constructed from MHV vertices
- Zero in supersymmetric theory

Least interesting for real experiments!

Note: these relations may not be optimal

Eg. One-Loop all-plus,

~ $n^4/24$ terms

$$A^{(1)}_{n:r}(a_1, a_2, \cdots, a_{r-1}; b_r, \cdots, b_n) = (-1)^r \sum_{\sigma \in COP\{\alpha\}\{\beta^T\}} A^{(1)}_{n:1}(\sigma)$$

$$|COP(\alpha_r, \beta_s)| = \frac{(r+s-1)!}{(r-1)!(s-1)!}$$

but

r=2~ (n-1)(n-2) terms

$$A^{(1)}_{n:r}(1^+, 2^+, \cdots, r-1^+; r^+ \cdots n^+) =$$

$$-2i \frac{(K^2_{1\cdots r-1})^2}{\langle 1\,2\rangle\langle 2\,3\rangle \cdots \langle (r-1)\,1\rangle\langle r\,(r+1)\rangle \cdots \langle n\,r\rangle}$$

DCD, Godwin, Perkins and Strong

Orthogonal Group SO(N)

- Structure constant description identical
- Trace basis structure different
- Eg string theory one-loop expansion

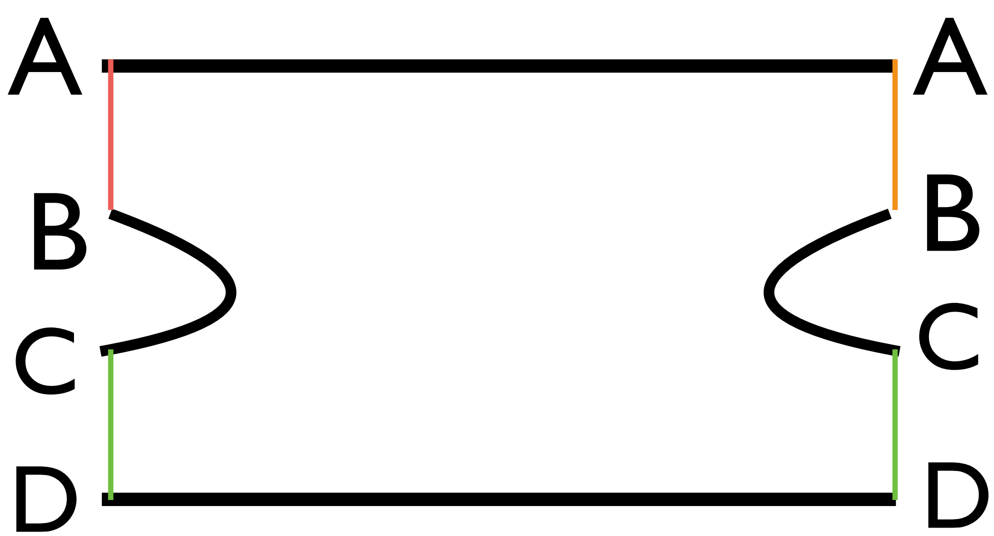
Annulus, two boundaries

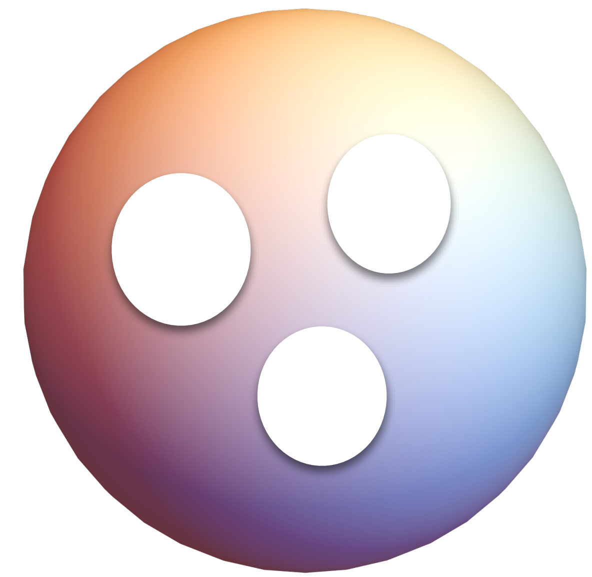
Mobius, one boundary

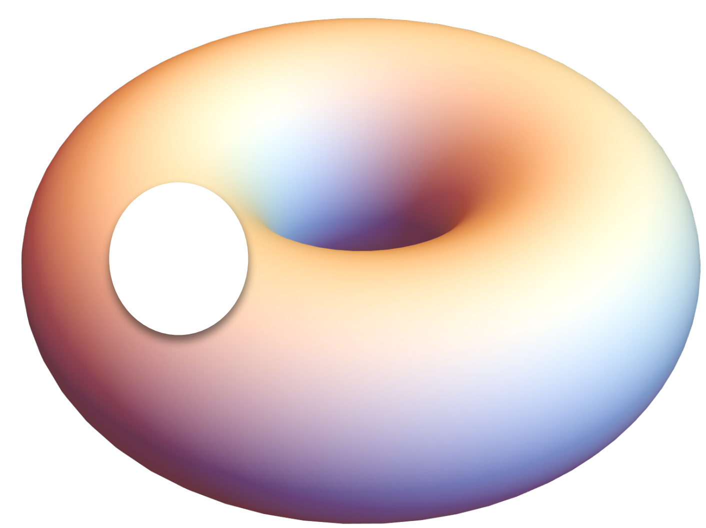
$$\mathcal{A}_n^{(1)}(1,2,3,\cdots,n) = \sum_{S_n/Z_n} N\,\mathrm{Tr}[T^{a_1}T^{a_2}\cdots T^{a_n}]A_{n:1}^{(1)}(a_1,a_2,\cdots,a_n)$$
$$+\sum_r \sum_{P_{n:r}} \mathrm{Tr}[T^{a_1}\cdots T^{a_{r-1}}]\,\mathrm{Tr}[T^{a_r}\cdots T^{a_n}]A_{n:r}^{(1)}(a_1,a_2,\cdots,a_{r-1};a_r,\cdots,a_n)$$
$$+\sum_{S_n/Z_n} \mathrm{Tr}[T^{a_1}T^{a_2}\cdots T^{a_n}]A_{n:1B}^{(1)}(a_1,a_2,\cdots,a_n)$$

Two Loop Amplitudes

Open String Theory Two-Loop Surfaces





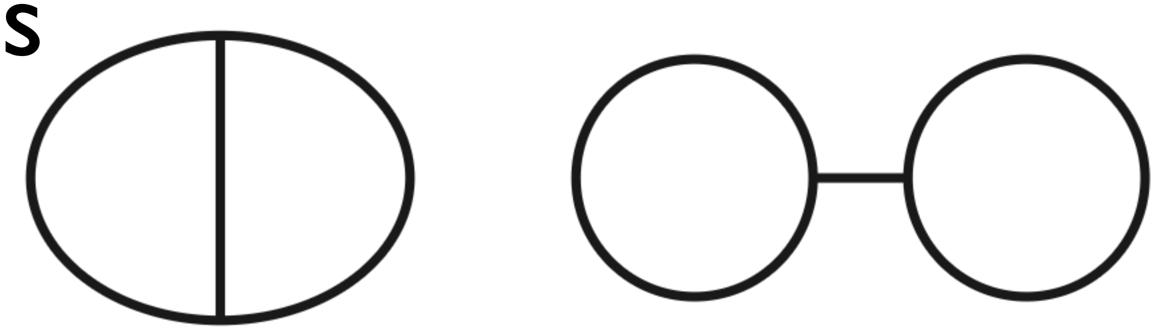


Two-Loop Color Trace Expansion

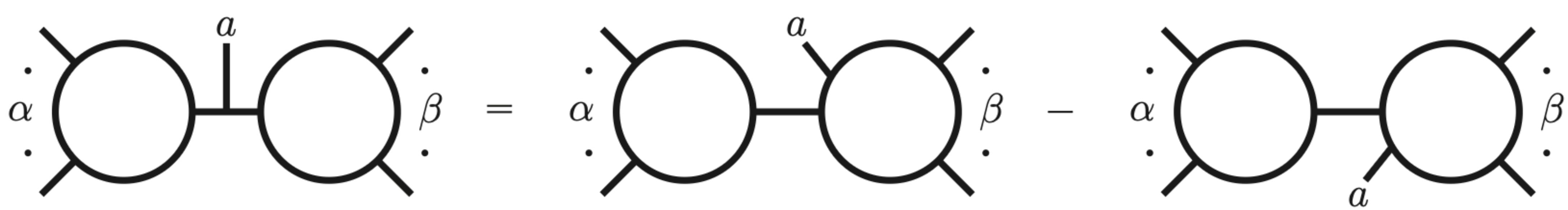
$$
\begin{aligned}
\mathcal{A}_n^{(2)}(1,2,\cdots,n) &= N_c^2 \sum_{S_n/\mathcal{P}_{n:1}} \mathrm{Tr}[T^{a_1}T^{a_2}\cdots T^{a_n}] A_{n:1}^{(2)}(a_1,a_2,\cdots,a_n) \\
&+ N_c \sum_{r=2}^{[n/2]+1} \sum_{S_n/\mathcal{P}_{n:r}} \mathrm{Tr}[T^{a_1}T^{a_2}\cdots T^{a_{r-1}}]\,\mathrm{Tr}[T^{b_r}\cdots T^{b_n}] A_{n:r}^{(2)}(a_1,a_2,\cdots,a_{r-1};b_r,\cdots,b_n) \\
&+ \sum_{s=1}^{[n/3]} \sum_{t=s}^{[(n-s)/2]} \sum_{S_n/\mathcal{P}_{n:s,t}} \mathrm{Tr}[T^{a_1}\cdots T^{a_s}]\,\mathrm{Tr}[T^{b_{s+1}}\cdots T^{b_{s+t}}]\,\mathrm{Tr}[T^{c_{s+t+1}}\cdots T^{c_n}] \\
&\qquad \times A_{n:s,t}^{(2)}(a_1,\cdots,a_s;b_{s+1},\cdots,b_{s+t};c_{s+t+1},\cdots,c_n) \\
&+ \sum_{S_n/\mathcal{P}_{n:1}} \mathrm{Tr}[T^{a_1}T^{a_2}\cdots T^{a_n}] A_{n:1B}^{(2)}(a_1,a_2,\cdots,a_n)\,.
\end{aligned}
$$

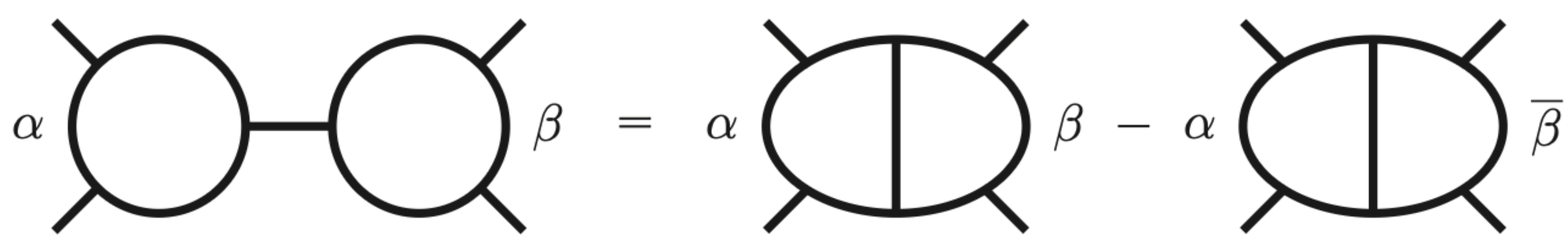
Two-Loop color terms

-two basic topologies

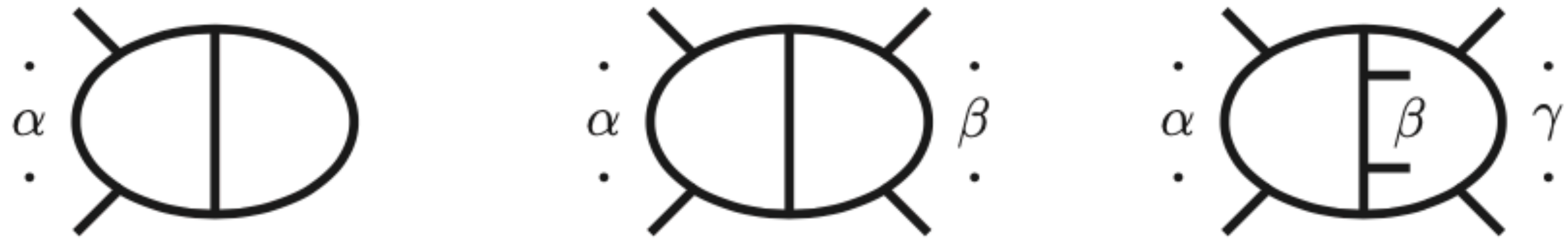


-can eliminate second



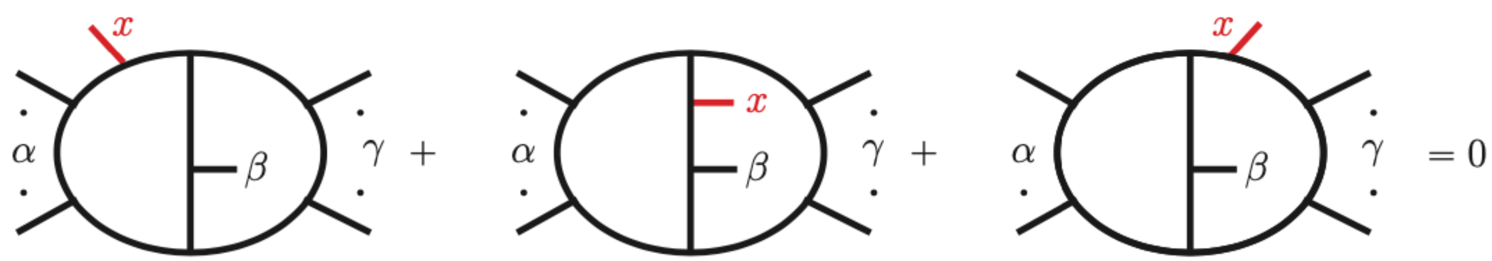


-color terms has spanning set

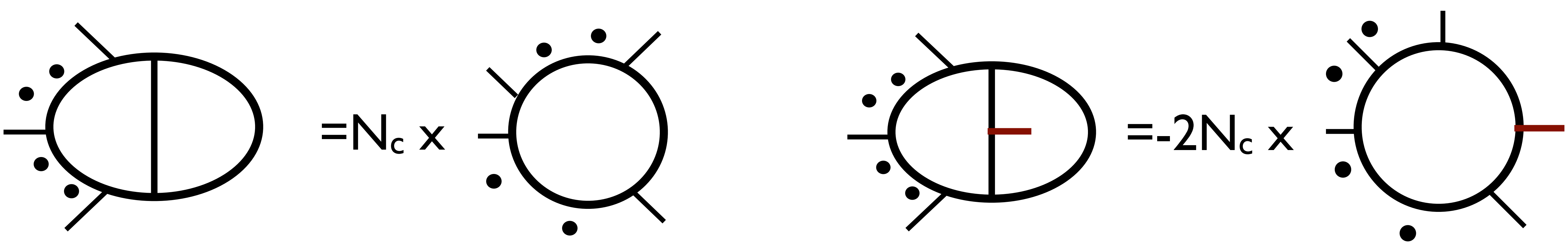


a) b) c)

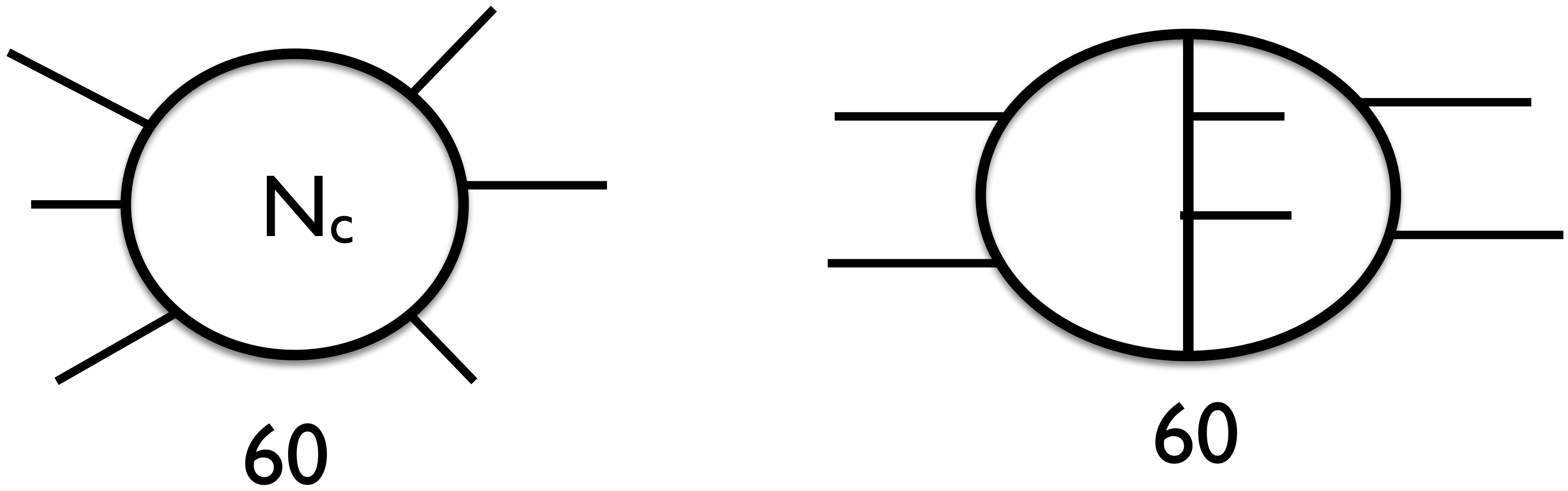
-Jacobi identity still provides relations



-identity



-vary wide choice of basis from spanning set eg. 6-point



-immediately implies 80 relations among partial amplitudes

-all relations apply to all helicities

-nothing beyond these

Difficult to find appealing basis in general

Can work with spanning set

Relations between terms,

- Looking to understand group theory constraints
- Looking for possible linear relations beyond this

-use known amplitudes together with colour expansions

Relations implied purely from colour

Relations true for all helicities

?

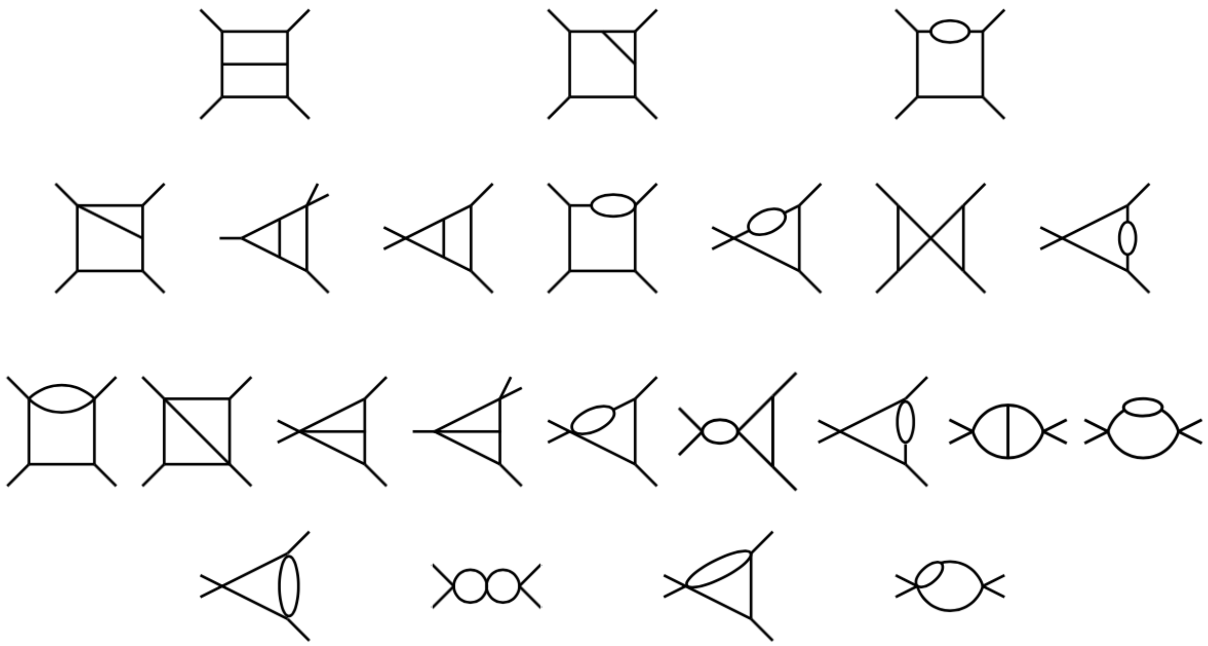
Relations satisfied by explicit amplitudes

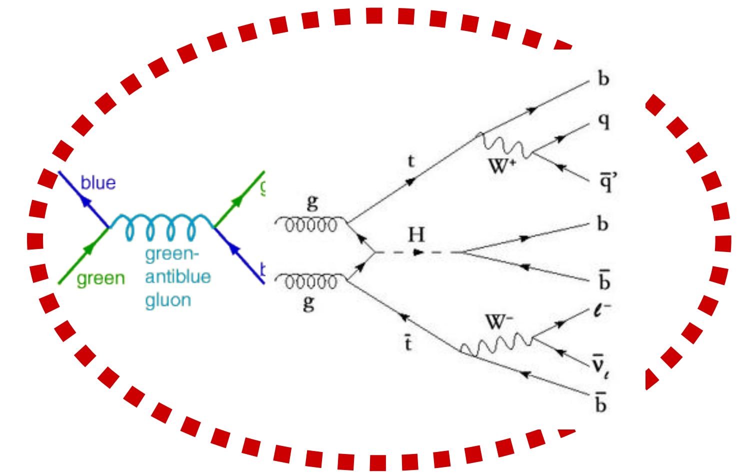
QCD : Two-Loop Amplitudes

-four point obtained in analytic form

Bern, Dixon and Kosower
Glover, Oleari and Tejeda-Yeomans

Bern, De Freitas and Dixon,





-numerical unitarity developed for two-loops for this process

Abreu, Febres Cordero, Ita , Jaquier, Page and Zeng, arXiv:1703.05273

-look beyond four point

-external gluons with identical helicity, leading in color

-either second most interesting amplitude or least interesting

Known Amplitude: Five Gluon Two-Loop All-Plus

looking at leading colour, call plus

$$A_5^{(2)} = A_5^{(1)} \left[- \sum_{i=1}^{5} \frac{1}{\epsilon^2} \left(\frac{\mu^2}{-s_{i,i+1}} \right)^{\epsilon} + \frac{5\pi^2}{12} \right] + F_5^{(2)} + \mathcal{O}(\epsilon)$$

first computed at integrand level

Badger, Frellesvig Y.~Zhang, arXiv:1310.1051.

Badger, Mogull, Ochirov and O'Connell arXiv:1507.08797

subsequently integrated and presented in this form

Gehrman, Henn and Presti, arXiv:1511.05409

- Finite remainder function

$$F_5^{(2)} = F_5^{cc} + R_5^{(2)}$$

  - Singularities match general theorems
  - one-loop amplitude to order epsilon^2

Catani

$$F_5^{cc} = \sum \frac{i}{6} \frac{[a\,b]^2\,[c\,d]\,[d\,e]}{\langle c\,e\rangle} \times$$

$$\left(-\frac{2}{s_{cd}s_{de}}\right)\left[\mathrm{Li}_2\left(1-\frac{s_{ab}}{s_{cd}}\right)+\mathrm{Li}_2\left(1-\frac{s_{ab}}{s_{de}}\right)+\frac{1}{2}\ln^2\left(\frac{s_{cd}}{s_{de}}\right)+\frac{\pi^2}{6}\right]$$

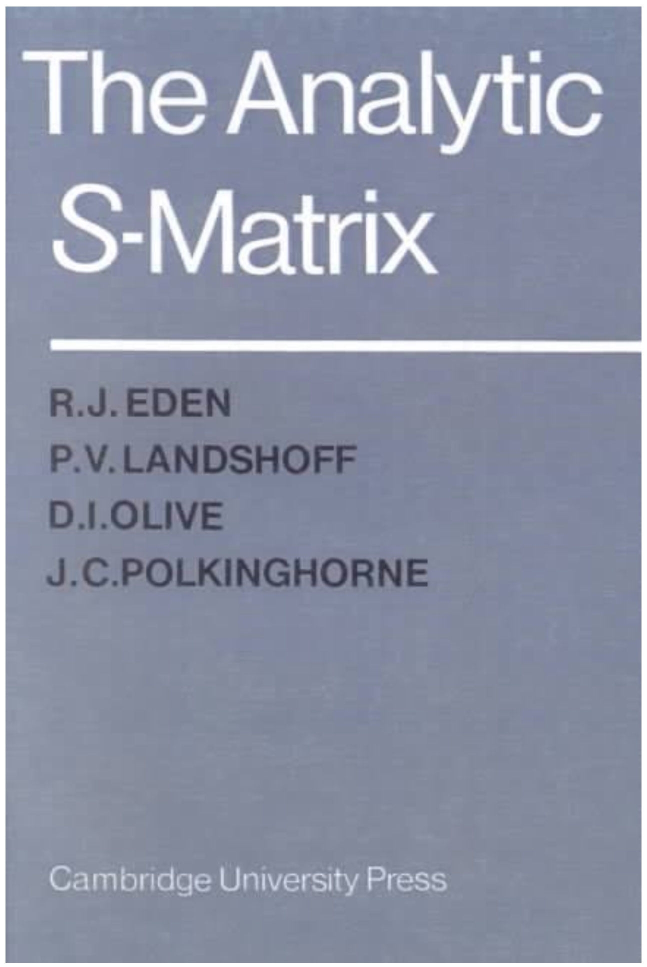
-this combination either truncated or higher dimension box one-loop integral function

$$R_5^{(2)} = \frac{i}{6\,\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,5\rangle\,\langle 5\,1\rangle} \times \left(R_5^a + R_5^b\right)$$

$$R_5^a = \frac{2}{3}\sum \frac{\mathrm{tr}_+^2(4512)}{s_{45}s_{12}}\,,$$

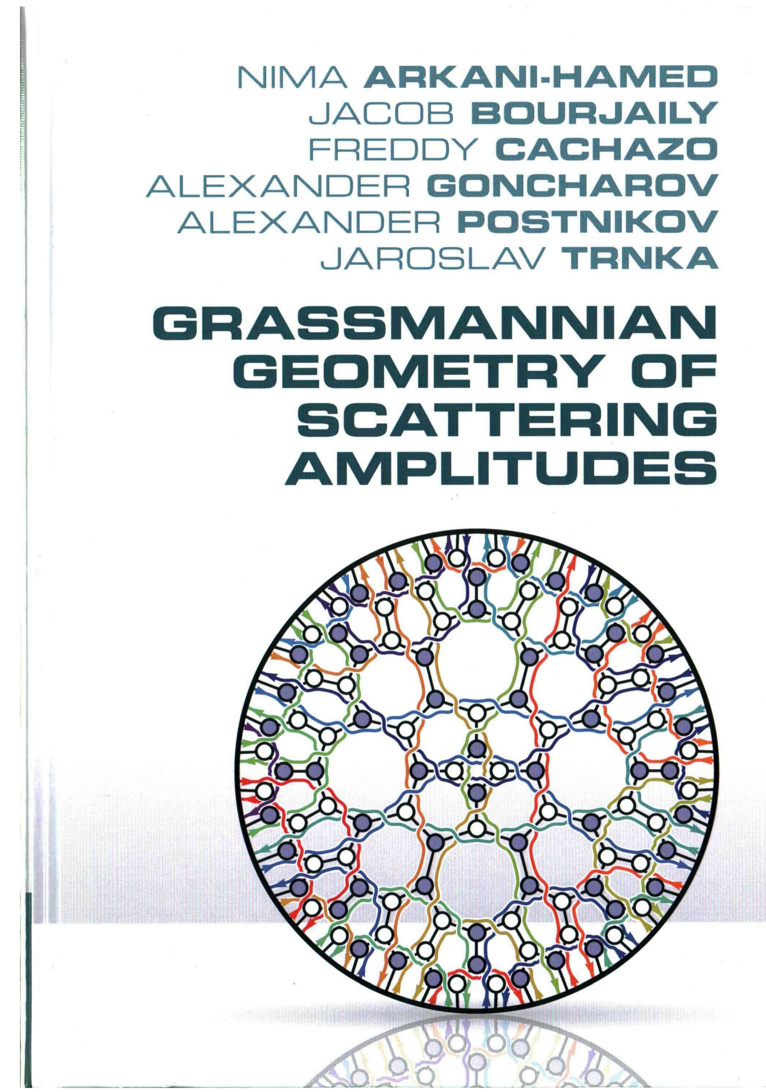
$$R_5^b = \sum\left(\frac{10}{3}s_{12}s_{23} + \frac{2}{3}s_{12}s_{34}\right)$$

Singular Structure is everything?……

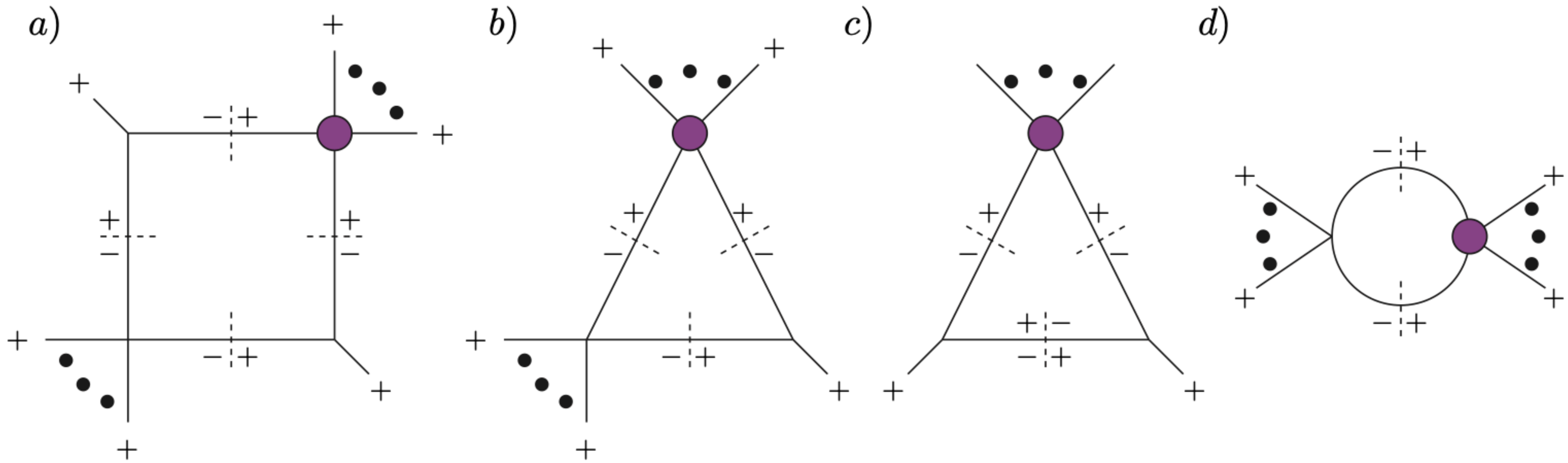


1966

- can amplitudes be constructed from a knowledge of singularities?

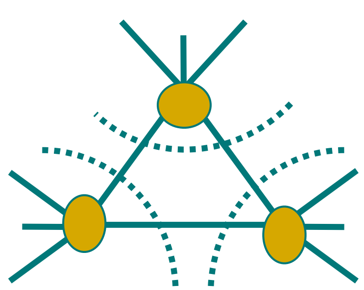


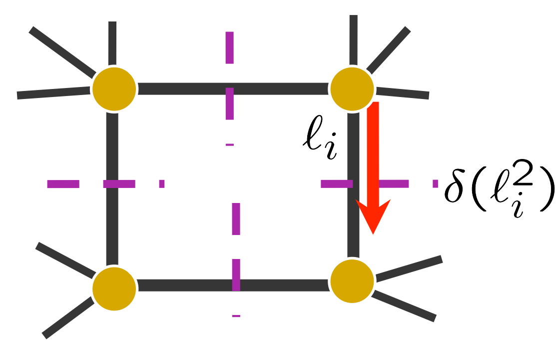
can five point by reconstructed simply and further amplitudes computed?



- Four dimensional unitarity : treat all-plus as a vertex. One-loop unitarity methods are then used

-use information to solve for coefficients



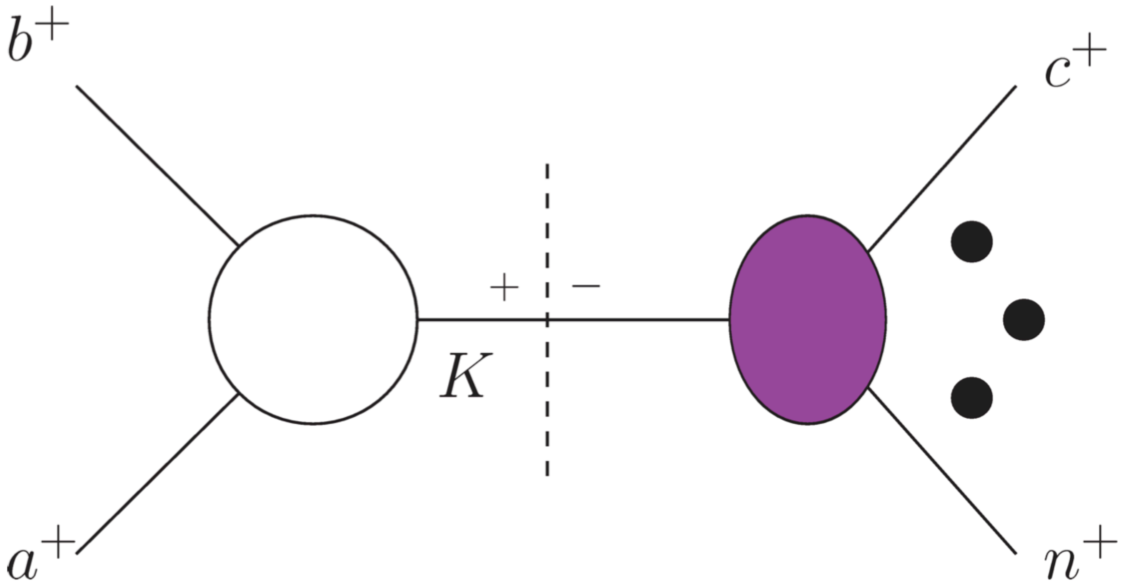


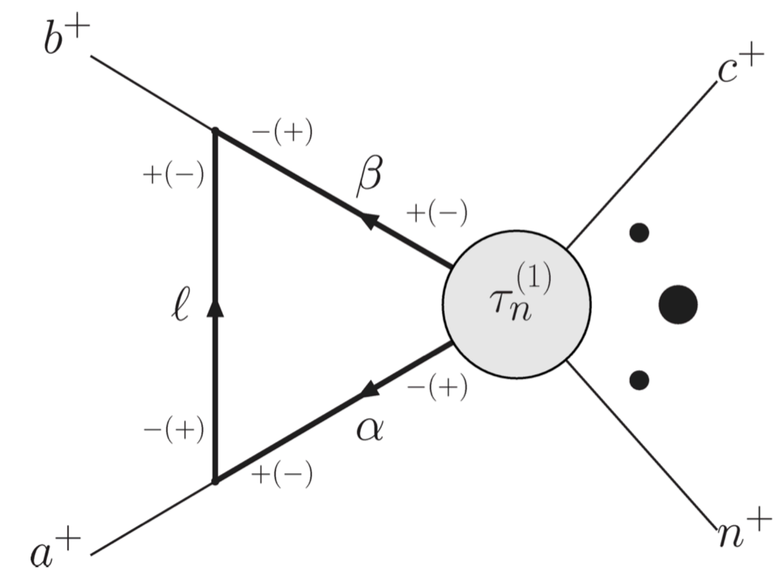
Britto,Cachazo,Feng

-quadruple cuts determine box

coefficients algebraically

Recursion…rational terms contain double poles under BCFW recursion





$$R(z) = \frac{c_{-2}}{(z-z_j)^2} + \frac{c_{-1}}{(z-z_j)} + \mathcal{O}((z-z_j)^0)$$

$$\begin{aligned} \lambda_i \to \lambda_{\hat{i}} &= \lambda_i + z\,[jk]\lambda_\eta\,, \\ \lambda_j \to \lambda_{\hat{j}} &= \lambda_j + z\,[k\,i]\,\lambda_\eta\,, \\ \lambda_k \to \lambda_{\hat{k}} &= \lambda_k + z\,[i\,j]\,\lambda_\eta\,. \end{aligned}$$

Results for all-plus

- Five point leading in color — Badger, Frellesvig Y.~Zhang; Badger, Mogull, Ochirov and O'Connell; Gehrman, Henn and Presti
- Five point full color — Badger, Chicerin, Gehrmann, Heinrich, Henn Peraro et al; DCD Godwin Perkins and Strong
- Six point leading in colour — DCD Jehu, Perkins
- Six point full — Dalgleish, DCD, Perkins and Strong
- Seven point full — Dalgleish, DCD, Perkins and Strong
- n-pt cut constructible
- All-n for sub-sub leading in color — DCD, Perkins and Strong

Least interesting for real experiments!

$$A^{(2)}_{n:1B} = U_{n:1B} + P_{n:1B} + R_{n:1B}$$

$$
\begin{aligned}
P^{(2)}_{n:1B} = -2i \sum_{a<b} \Bigg(& \sum_{(U_1^i:U_2^i)\in Spl_2(U_{ab})} \; \sum_{(V_1^j:V_2^j)\in Spl_2(V_{ab})} c(a,b,U_1^i,V_1^j,U_2^i,V_2^j)\mathrm{F}(a,b;U_1^i\cup V_1^j;U_2^i\cup V_2^j) \\
& + \sum_{(U_1^i:U_2^i)\in Spl_2(U_{ab})} \; \sum_{(V_1^j:V_2^j)\in Spl_2(V_{ab})} c(a,b,U_2^i,V_2^j,V_1^j,U_1^i)\mathrm{F}(a,b;V_2^j\cup U_2^i;U_1^i\cup V_1^j) \\
& - \sum_{(V_1^i:V_2^i:V_3^i)\in Spl_3(V_{ab})} c(a,b,U_{ab},V_2^i,V_1^i,V_3^i)\mathrm{F}(a,b;U_{ab}\cup V_2^i;V_1^i\cup V_3^i) \qquad) \\
& - \sum_{(U_1^i:U_2^i:U_3^i)\in Spl_3(U_{ab})} c(a,b,U_2^i,V_{ab},U_3^i,U_1^i)\mathrm{F}(a,b;U_2^i\cup V_{ab};U_3^i\cup U_1^i) \Bigg)
\end{aligned}
$$

$$R^{(2)}_{n:1B_1}(1^+,2^+,\cdots,n^+) = -2i\, \mathrm{C}_{\mathrm{PT}}(1,2,\cdots,n-1,n) \times \sum_{1\le i<j<k<l\le n} \epsilon(i,j,k,l)$$

$$R^{(2)}_{n:1B_2}(1^+,2^+,\cdots,n^+) =$$

$$4i \sum_{r=1}^{n-4} \sum_{s=r+4}^{n} \sum_{i=r+1}^{s-2} \sum_{j=i+1}^{s-1} \epsilon(\{1,\cdots,r\},j,i,\{s,\cdots,n\})(-1)^{i-j+1} \times \sum_{\alpha\in S_{r,s,i,j}} \mathrm{C}_{\mathrm{PT}}(\{\alpha_{S_{r,s,i,j}}\})$$

Fest

Kleiss-Kuijf like relation for sub-sub leading

$$A^{(2)}_{n:1B}(1,\{\alpha\},n,\{\beta\}) = (-1)^{|\beta|} \sum_{\sigma\in OP(\alpha,\beta^T)} A^{(2)}_{n:1B}(1,\{\sigma\},n)$$

- Satisfied by all-plus amplitude
- Implied by decoupling identities for n <= 6
- Implied by colour decomposition for n=7
- Outside colour relations for n=8

$$A^{(2)}_{8:1B}\left(\ydiagram{5,1,1,1}\right)_1 + 2A^{(2)}_{8:1B}\left(\ydiagram{5,1,1,1}\right)_2 \neq 0$$

For all plus

Not implied by decomposition

Decompose relations according representations of S_n

eg: S_6

		$\vert R_i\vert$	$A^{(2)}_{6:1}$	$A^{(2)}_{6:2}$	$A^{(2)}_{6:3}$	$A^{(2)}_{6:4}$	$A^{(2)}_{6:1,1}$	$A^{(2)}_{6:1,2}$	$A^{(2)}_{6:2,2}$	$A^{(2)}_{6:1B}$
$\vert A^{(2)}_{6:x}\vert$			60	72	45	30	45	60	15	60
R_1	6	1	I	I	I	I	I	I	I	I=0
R_2	5, 1	5	.	I=0	I=0	.	I=0	I=I=0	.	.
R_3	4, 2	9	I,I	I	I,I	I	I,I	I=I	I	I=I
R_4	$4, 1^2$	10	.	.	.	.	.	I=0	.	.
R_5	3^2	5	.	I=0	.	.	.	I=0	.	.
R_6	3, 2, 1	16	I	I=0,I	I	.	I	I=I	.	I=I
R_7	$3, 1^3$	9	I	.	.	.	.		.	I
R_8	2^3	5	I,I	I	I	I	I	.	I=0	I=0, I
R_9	$2^2, 1$	10	.	I=0	.	.	.	.	.	.
R_{10}	$2, 1^4$	5	I	I	.	I	.	.	.	I=0
R_{11}	1^6	1	.	I=0	.	.	.	.	.	.

Young Tableaux

Edison and Naculich

-can find all possible linear relations

Conclusions

- Example of how long-lived nice formula can be
- Example of Lance's versatility

-all plus amplitude places upper limit on relations

-most of these are shown to extend to all helicities as consequence of expansion in structure functions

-small number of relations beyond these

-starting at seven and eight points

Conclusions

- Analytic computations of Amplitudes
- Use all-plus as probe : very special case which allows us to look at large number of legs
- Relations for small number of legs can mislead

-all plus amplitude places upper limit on relations

-most of these are shown to extend to all helicities as consequence of expansion in structure functions

-small number of relations beyond these

-starting at seven and eight points

in-conclusions

-look for null vectors

$$\mathcal{A}_n^{(2)} = \sum_i C_S^i \bar{A}_{n:i} = \sum_a C_T^\lambda A_{n:\lambda}^{(2)}$$

Structure constants

Color trace

-color terms can be decomposed into trace terms

$$C_S^i = \sum_a M_{i\lambda} C_T^\lambda$$

Edison and Naculich

-a null vector , $\sum_a M_{i\lambda} V^\lambda = 0 \quad \forall i$

-will generate a relation

$$\sum_\lambda A_{n:\lambda}^{(2)} V^\lambda = \sum_i \sum_\lambda \bar{A}_{n:i} M_{i\lambda} V^\lambda = 0$$