What Specifies a Classical Gravitational Scattering State?

Radu Roiban
Penn State University & ETH-ITS

with Asaad Elkhidir and Donal O'Connell

Two of the many threads running through Lance's work:

- Gauge invariance: physics from on-shell, gauge-invariant data
- Observables: IR-safe observables (e.g. energy correlators, …)

What do we want to measure/can we observe in (classical) gravity?

- Are they gauge invariant? Are they IR-safe?

Related: What are three-point S-matrix elements with real momenta & are they unique?

Story for today: large gauge transformations

- affect things we think we'd like to observe
- change 3-pt S-matrix elements with real momenta

Dream: define sufficiently local observables, manifestly free of large gauge ambiguities

At the moment, gravitational observables are not as well-developed as in QCD

Next best things:

- Asymptotically Minkowski: Differential cross sections; a proxy – scattering matrix

$$S \sim \langle \psi_{\rm fin} | \psi_{\rm in} \rangle$$

- Asymptotically (A)dS: boundary/cosmological correlators

Maldacena;
Gubser, Klebanov, Polyakov; Witten

- Inspired by QCD: final-state expectation values

$$\langle \mathcal{O} \rangle = \langle \psi_{\rm fin} | \mathcal{O} | \psi_{\rm fin} \rangle - \langle \psi_{\rm in} | \mathcal{O} | \psi_{\rm in} \rangle$$

Kosower, Maybee, O'Connell
Building on extensive QCD work:
Basham, Brown, Ellis, Love, Belitsky, Korchemsky,
Sterman, Dixon, Moult, Zhu …

- Relational observables

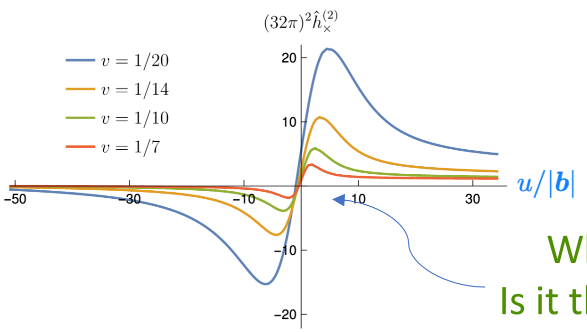
Cheung, Sivaramakrishnan, Wilson-Gerow, Zhou

Some desirable features:

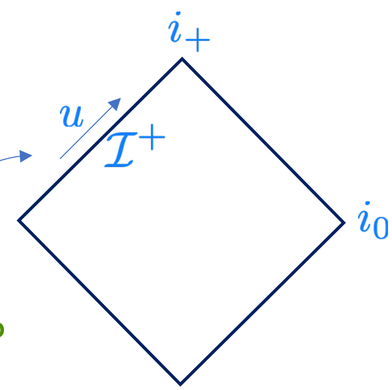
- Invariant under gauge transformations that fall off at infinity
- Inclusive ones – impulse, energy & angular momentum loss – are IR safe

At the moment, gravitational observables are not as well-developed as in QCD

- Not all are IR safe; aspects of all-order factorization remain to be worked out
- Local ones, e.g. waveforms, are not fully gauge invariant: $F(x + \delta x) \neq F(x)$



What defines the origin?
Is it the same in all directions?



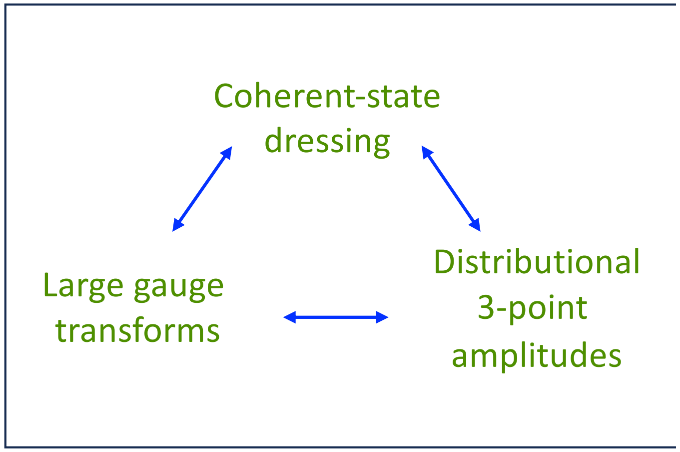
Came up in the comparison of QFT and GR-based computation of $h_{\mu\nu}$ at future null infinity $\mathcal{I}^+$

Bini, Damour, De Angelis, Geralico, Herderschee, RR, Teng; Georgoudis, Heissenberg, Russo

- (almost) time-independent metric fluctuations/zero-frequency gravitons are important
- Technically difficult to keep track of, in general

Kind of a plan

- Charged-matter single-particle states as frame data
- Observables and frames; track gauge data
  - canonical frame, intrinsic frame, XXX-frame, etc
- Three-point amplitudes and large gauge transformations
- Some consequences
  - Conservation laws; BMS supertranslations
  - Gauge dependence of mechanical/field split; conservation of total angular mometum



Charged massive single-particle states are not unique

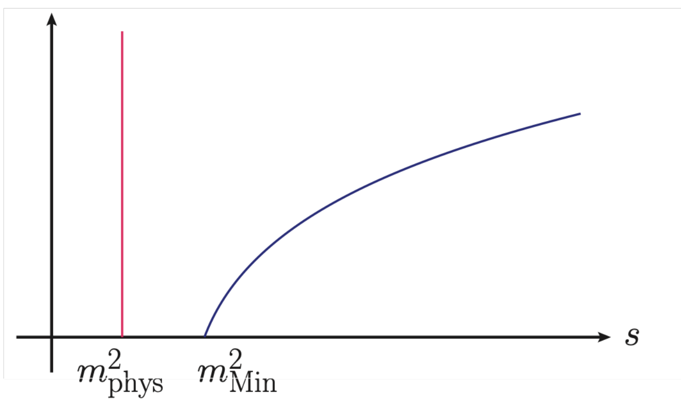
$$i\Delta_F(p) \equiv \int d^4x\, e^{ip\cdot x}\, \langle 0|T\phi(x)\phi(0)|0\rangle \;=\; \int_0^\infty ds\, \frac{\rho(s)}{p^2 - s + i\epsilon}$$

$$\rho(s) = \sum_n |\langle p, n|\phi(0)|0\rangle|^2 \delta(s - m_n^2)$$

Källén
Lehmann

Gapped theories

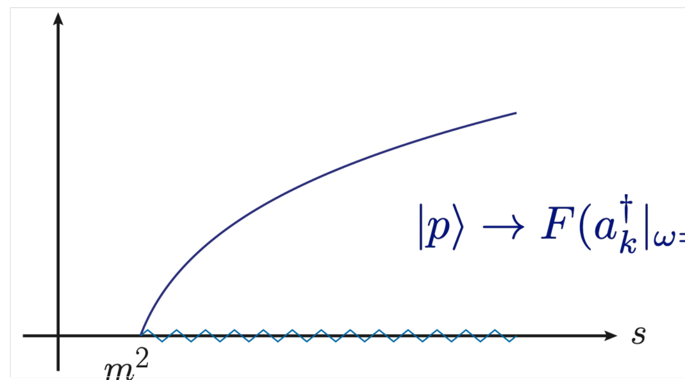
$$\rho(s) = Z\,\delta(s - m^2_{\rm phys.}) + \rho_{\rm cont}(s)\Theta(s - m^2_{\rm Min})$$



isolated pole
defines the on-shell renormalization scheme, etc

Gapless theories

$$\rho(s) \sim (s - m^2)^{\delta - 1}\theta(s - m^2)$$



$$|p\rangle \to F(a_k^\dagger|_{\omega=0})|p\rangle$$

may include arbitrary amounts of
low-energy massless particles

Charged massive particle states are ambiguous to dressing with massless particles

Dressed charged massive single-particle states

- Respect degeneracy – dressing must preserve the quantum numbers

$$ {}^{\text{dressed}}_{\text{in}}\langle p|p\rangle^{\text{dressed}}_{\text{in}} = 1 \qquad {}_{\text{in}}\langle p|\mathbb{P}^\mu|p\rangle_{\text{in}} = p^\mu = {}^{\text{dressed}}_{\text{in}}\langle p|\mathbb{P}^\mu|p\rangle^{\text{dressed}}_{\text{in}} $$

↳ Dressing is: 1. a phase and 2. localized on zero-frequency mediators

$$ |p\rangle^{\text{dressed}}_{\text{in}} = e^{iQ_s[f_p]}\,|p\rangle_{\text{in}} \qquad A^\dagger(p) \to A^\dagger(p)_{\text{dressed}} = e^{iQ_s[f_p]}A^\dagger(p)e^{-iQ_s[f_p]} $$

- Connection to classical physics ⟶ coherent state

$$ Q_s[f_p] = \sum_\eta \int \mathrm{d}\Phi(k)\,\big(\bar{f}^\eta_p(k)\,a^\dagger_\eta(k) + \bar{f}^{\eta*}_p(k)\,a_\eta(k)\big)A^\dagger A(p) \qquad \bar{f}^\eta_p(k) = \delta(\omega)f^\eta_p(n) \qquad k^\mu = \omega n^\mu = \omega(1,\boldsymbol{n})^\mu $$

- Similar w/ Faddeev/Kulish states, *but* with no contributions to IR divergences

- New asymptotic density of states:

$$ \rho_{\text{dressed}}(s) = \sum_n |\langle p,n|\phi_{\text{dressed}}(0)|0\rangle|^2\delta(s-m_n^2) = \sum_n \left|\int \hat{\mathrm{d}}q\Phi(q)\langle p,n|e^{iQ_s[f_p]}A^\dagger(q)|0\rangle\right|^2 \delta(s-m_n^2) $$

Vanishing frequency is an idealization; assumes states at strict infinite distance!

Observables & their values

Kosower, Maybee, O'Connell

- Difference between the corresponding operator in the final and initial state

$$\langle \mathcal{O} \rangle = \langle \psi_{\rm out} | \mathcal{O} | \psi_{\rm out} \rangle - \langle \psi_{\rm in} | \mathcal{O} | \psi_{\rm in} \rangle \qquad |\psi_{\rm out}\rangle = \mathbb{S} |\psi_{\rm in}\rangle \quad \text{where} \quad \mathbb{S} = \mathbb{I} + i\mathbb{T}$$

- Out state as the time evolution of the initial state:

$$|\psi_{\rm out}\rangle = \mathbb{S} |\psi_{\rm in}\rangle \qquad \mathbb{S} = \mathbb{I} + i\mathbb{T} \qquad \mathbb{S} = e^{i\mathbb{N}} \qquad S = e^{i\chi(b,p)}$$

Herrmann, Moult, Kologlu;…

$\mathcal{O}$ = 'measurement function' / detector operators

Dixon, Moult, Zhu; Belitsky, Korchemsky, Sterman;…

e.g. EE, here impulse, field momentum, angular momentum, gravitational field, etc

$$\mathbb{h}_{\mu\nu}(x) = \sum_{h=\pm} \int \hat{\mathrm{d}}\Phi(k) \left[\varepsilon^{(hh)*}_{\mu\nu}(k)\, e^{-ik\cdot x}\, \hat{a}_{hh}(k) + \varepsilon^{(hh)}_{\mu\nu}(k)\, e^{+ik\cdot x}\, \hat{a}^{\dagger}_{hh}(k) \right] \qquad \text{idealization} \quad |\boldsymbol{x}| \to \infty$$

$$\mathbb{h}_{\mu\nu}(x) = -\frac{i}{4\pi|\boldsymbol{x}|} \sum_{\eta} \int_0^{\infty} \hat{\mathrm{d}}\omega_K\, e^{-iu\omega_K}\, \varepsilon^{(\eta)}_{\mu\nu}(K) \hat{a}_{\eta}(K) \Big|_{K=\omega_K(1,\boldsymbol{x}/|\boldsymbol{x}|)} + \mathrm{cc} \qquad \text{idealization} \quad \omega_K |\boldsymbol{x}| \gg 1$$

- Initial state:

$$|\psi_{\rm in}\rangle = \int \mathrm{d}\Phi[p_1] \ldots \phi(p_1) \ldots e^{ip_1\cdot b_1 + \ldots} |p_1 \ldots \rangle$$

Fock state

What else is/should be/could be in the initial state $|\psi_{\rm in}\rangle$?

Asymptotic field (shear) of a massive particle: $h_{\mu\nu}(x) = \langle\psi_{\rm in}|\mathbb{h}_{\mu\nu}(x)|\psi_{\rm in}\rangle = 0$

What else is/should be/could be in the initial state $|\psi_{\rm in}\rangle$?

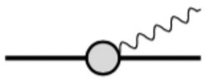
Asymptotic field (shear) of a massive particle: $h_{\mu\nu}(x) = \langle\psi_{\rm in}|\mathbb{h}_{\mu\nu}(x)|\psi_{\rm in}\rangle = 0$

That's odd… - a free point particle has a "radiative" field (boosted lin. Schwarzschild)

$$h^{\rm TT}_{\mu\nu}(\omega,\boldsymbol{x}) \propto \frac{1}{4\pi|\boldsymbol{x}|}\frac{\Pi^{\alpha\beta}_{\mu\nu}p_\alpha p_\beta}{n\cdot p}\delta(\omega) \qquad n=(1,\boldsymbol{x}/|\boldsymbol{x}|) \qquad \sum_\eta \epsilon_{\eta,\mu\nu}\epsilon^{\alpha\beta}_\eta = \Pi^{\alpha\beta}_{\mu\nu}$$

- slight contradiction with $\omega|\boldsymbol{x}|\gg 1$
- depends on the choice of coordinates

- KMOC formalism: give by the on-shell 3-point amplitude w/ real momenta



Both the 'radiative' field of a point particle & on-shell 3-point amplitude w/ real momenta must be a (large) gauge transformation

T(T) field and on-shell 3-point amplitude w/ real momenta are gauge DoFs -- QED illustration

- Asymptotic on-shell vector potential/on-shell 3-point amplitude $k = \omega n$

$$W_{\text{const}}(\omega, n) = \delta(2k \cdot p)\, 2(\varepsilon(k) \cdot p) = \delta(\omega) \frac{\varepsilon(k) \cdot p}{n \cdot p} \longleftrightarrow \begin{aligned} A_\mu dx^\mu &= dn^\mu \partial_\mu \ln n \cdot p \\ &= dF(n \cdot p) \end{aligned}$$

- Obeys the usual Ward identity (anything proportional to $\delta(\omega)$ does: $k^\mu \delta(\omega) \propto \omega\delta(\omega) = 0$)

$$W_{\text{const}}(\omega, n)\big|_{\varepsilon(k) \to k} = 0 \qquad \delta(\omega)\varepsilon(k) \cdot \partial f(n, \ldots)\big|_{\varepsilon(k) \to k} = 0$$

► Gauge invariant but also a (large) gauge transform!

- From classical physics: $A = A_u\, du + A_r\, dr + A_A\, d\theta^A$

$$v_0 = (1, \mathbf{0}) \longrightarrow v = \gamma(1, \boldsymbol{v})$$

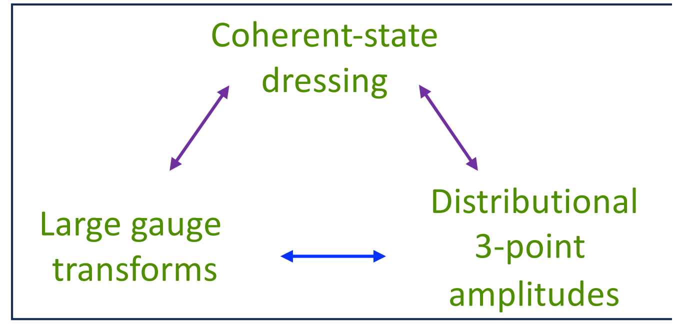
$$A_\mu dx^\mu = \frac{Q}{4\pi} \frac{dt}{|\boldsymbol{x}|} = \frac{Q}{4\pi} \frac{v_0 \cdot dx}{\sqrt{(v_0 \cdot x)^2 - x^2}} \longrightarrow A_\mu dx^\mu = \frac{Q}{4\pi} \frac{v \cdot dx}{\sqrt{(v \cdot x)^2 - x^2}}$$

$$A_A = 0 \qquad\qquad A_A = \partial_A \ln v \cdot n$$

Vocab: ‘frame’ = configuration of 0-energy mediators at $\mathcal{I}^-$

Coherent state dressing ⟷ (supertranslation of the) shear

Elkhidir, O'Connell, RR



- The dressed single-particle wave packet:

$$|\psi\rangle_{\rm in} \to |\psi\rangle_{\rm in}^{\rm dressed} = \int \mathrm{d}\Phi(p)\varphi(p)e^{iQ_s[f]}|p\rangle = e^{iQ_s[f]}|\psi\rangle_{\rm in}$$

- The single-particle shear:

$$h_{\mu\nu} = {}_{\rm in}^{\rm dressed}\langle\psi|\mathbb{h}_{\mu\nu}(x)|\psi\rangle_{\rm in}^{\rm dressed} = \frac{1}{4\pi|\boldsymbol{x}|}\sum_\eta \epsilon_{\eta\mu\nu}\bar{f}_p^\eta(n) + \text{h.c.}$$

$$\bar{f}_p^\eta(n) = \eth^2 T_p(n)$$

Spin-2 function $= \epsilon^\eta_{\alpha\beta}\partial^\alpha\partial^\beta T_p(n)$

$$h_{\mu\nu}dx^\mu dx^\nu = |\boldsymbol{x}|d\theta^A d\theta^B(\Omega_{AB}D^2 - 2D_A D_B)T_p(n)$$

i.e. the metric change of a general supertranslation

- For $T_p(n) = 2GM(n\cdot v)\ln(n\cdot v)$ yields the shear in Schwarzschild coordinates

Coherent-state dressing can generate any shear and any 3-pt on-shell amp. w/ real momenta

⟶ Choose it to vanish, and reinstate it at the end of calculations

The remaining part of the supertranslation: $u \to u + T_p(n)$

-- main source of non-invariance of GR asymptotic local observables

Final state expectation values:

$$ {}^{\text{dressed}}_{\text{in}}\langle p|\$^\dagger \mathcal{O}\$|p\rangle^{\text{dressed}}_{\text{in}} = {}_{\text{in}}\langle p|e^{-iQ_s[T_p]}\$^\dagger \mathcal{O}\$ e^{iQ_s[T_p]}|p\rangle_{\text{in}} $$

Hope:

$$ = {}_{\text{in}}\langle\psi|e^{iQ_h}e^{-i(Q_s[T_p]+Q_h)}\$^\dagger\mathcal{O}\$e^{i(Q_s[T_p]+Q_h)}e^{-iQ_h}|\psi\rangle_{\text{in}} $$

$$ = {}_{\text{in}}\langle\psi|e^{iQ_h}\$^\dagger e^{-i(Q_s[T_p]+Q_h)}\mathcal{O}e^{i(Q_s[T_p]+Q_h)}\$e^{-iQ_h}|\psi\rangle_{\text{in}} $$

Aim to find Q_h from the commutator $[\$, Q_s[T_p]]$

The hope for the shear: $\mathbb{h}_{\mu\nu}(x) = -\frac{i}{4\pi|\boldsymbol{x}|}\sum_\eta \int_0^\infty \hat{\mathrm{d}}\omega_K\, e^{-iu\omega_K}\varepsilon^{(\eta)}_{\mu\nu}(K)\hat{a}_\eta(K)\Big|_{K=\omega_K(1,\boldsymbol{x}/|\boldsymbol{x}|)} + \text{cc}$

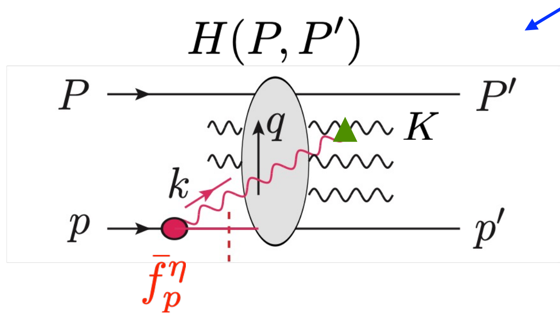
$$ e^{-i(Q_s[T_p]+Q_h)}\mathbb{h}_{\mu\nu}(\omega_K)e^{i(Q_s[T_p]+Q_h)} \overset{?}{=} \hat{\delta}(\omega_K)h^{\text{Schwz., TT}}_{\mu\nu}(x) + e^{-i\omega_K T_p}\mathbb{h}_{\mu\nu}(\omega) $$

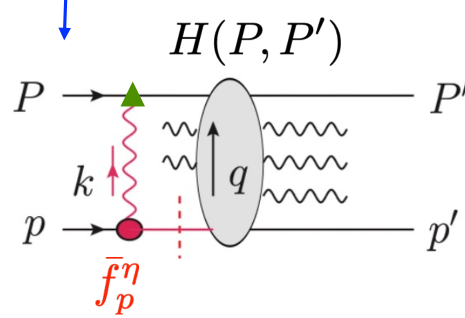
Pushing Q_s past $\mathbb{S}$ ⟶ can Q_s be completed to a symmetry generator?

- $e^{-iQ_s}\mathbb{S}e^{iQ_s}$ governed by $[\mathbb{S}, Q_s]$; focus on one term
- $Q_s \equiv \sum_\eta \int \mathrm{d}\Phi(k)\hat{\delta}(\omega)\bar{f}_p^\eta\, a_\eta^\dagger(k) + \mathrm{h.c.}$ talks *individually* to each external graviton of $\mathbb{S}$
- Q_s localized on zero-energy gravitons; picks out each term of the leading soft limit

$$C \equiv \sum_\eta \int \mathrm{d}\Phi(k)\hat{\delta}(\omega)\bar{f}_p^\eta(k)\,[\mathbb{S}, a_\eta^\dagger(k)] = C_0 + C_M$$

E.g.





Contributions to the commutator:

Elkhidir, O'Connell, RR

$$C \equiv \sum_\eta \int \mathrm{d}\Phi(k)\hat{\delta}(\omega)\bar{f}^\eta_p(k)\,[\mathbb{S}, a^\dagger_\eta(k)] = C_0 + C_M \qquad \bar{f}^\eta_p(k) = e^{AB}_\eta(k)\,\big(\Omega_{AB}D^2 - 2D_AD_B\big)\,T_p(n)$$

C_0: Soft graviton attached to external graviton

C_M: Soft graviton attached to external matter

1. Massless hard charge: $C_0 + [\mathbb{S}, Q_{h,0}] = 0$

proves a recent conjecture of Bini, Damour, Geralico

$$C_0 \sim \sum_\eta \int \mathrm{d}\Phi(l)\hat{\delta}_\beta(\omega_l)\frac{K^\mu K^\nu e(l)^\eta_{\mu\nu}e(l)^{AB}_\eta(D_AD_B - \frac{1}{2}\Omega_{AB}D^2)T_p(n_l)}{-2K\cdot l + i\epsilon} = \frac{1}{2\pi}\,\omega_K\,T_p^{\ell\geq 2}(\hat{K})$$

► $$Q_{h,0} = \sum_\eta \int \mathrm{d}\Phi(k)\,Q^0_h(k)\,a^\dagger_\eta(k)a_\eta(k) \quad , \qquad Q_{h,0}(k) = \frac{1}{2\pi}\omega_k\sum_p T_p^{\ell\geq 2}(\hat{k})$$

2. Massive hard charge: $C_M + [\mathbb{S}, Q_{h,M}] = 0$

► $$Q_{h,M} = \sum_i \int \mathrm{d}\Phi(p)\,Q_{h,M,i}(p)\,a^\dagger_i a_i(p) \qquad Q_{h,M,i}(p,\dots) = \dots$$

Q_s has a completion, $Q = Q_s + Q_{h,0} + Q_{h,M}$, which is conserved $[\mathbb{S}, Q] = 0\ (\forall)\ T_p(n)$

Two consequences of Q_s completion to a conservation law $[\mathbb{S}, Q_s + Q_h] = 0$ Elkhidir, O'Connell, RR

1. The hope for the shear is realized and the second part of supertranslations is as desired

$$e^{-i(Q_s[T_p]+Q_h)}\mathbb{h}_{\mu\nu}(\omega_K)e^{i(Q_s[T_p]+Q_h)} \overset{\checkmark}{=} \hat{\delta}(\omega_K)h^{\text{Schwz., TT}}_{\mu\nu}(x) + e^{-i\omega_K T_p}\mathbb{h}_{\mu\nu}(\omega) \quad \Longrightarrow \quad u \to u + T_p(n)$$

2. Frame independence of total angular momentum; an example

$$\,_{\text{in}}^{\text{dressed}}\langle p|\mathbb{S}^\dagger \mathcal{O}\mathbb{S}|p\rangle_{\text{in}}^{\text{dressed}} = \,_{\text{in}}\langle\psi|e^{iQ_h}\mathbb{S}^\dagger e^{-i(Q_s+Q_h)}\mathcal{O}e^{i(Q_s+Q_h)}\mathbb{S}e^{-iQ_h}|\psi\rangle_{\text{in}}$$

$$\mathcal{O} = \mathbb{J} = \mathbb{J}^{\text{mech.}} + \mathbb{J}^{\text{rad}} \; ; \qquad \mathbb{J}^{\text{rad}}_{\alpha\beta} = -i\sum_\eta \int \hat{\mathrm{d}}\Phi(k)\; a^\dagger_\eta(k)\; k_{[\alpha}\overleftrightarrow{\partial}_{\beta]}\; a_\eta(k)$$

$$a^\dagger_\eta(k) \to e^{-i\omega T(n)}a^\dagger_\eta(k) + f_{\eta,p}(n)$$

a. $\mathbb{J}^{\text{rad}}$ depends on frame: - Linear in f_η: Damour's $\mathcal{O}(G^2)$

- Independent of f_η : canonical + $\mathcal{O}(G^4)$

b. $\mathbb{J}^{\text{mech.}}$ also depends on frame, through the state $\mathbb{S}e^{-iQ_h}|\psi\rangle_{\text{in}}$

Use eikonal representation of the S matrix & saddle-point approximation for integrals

General expressions available in terms of Q_h

An explicit LO illustration

QED: no massless hard charge; $T_p(n) = \frac{e^2}{4\pi} \ln v \cdot n$

$$\delta b^\mu = \frac{e_1 e_2}{2\pi m_1 m_2} \left(\frac{1}{3} \left(\frac{e_1 m_2}{e_2 m_1} + \frac{e_2 m_1}{e_1 m_2} \right) - \frac{1}{\gamma v^2} + \frac{\text{arctanh}(v)}{\gamma^3 v^3} \right) \Delta p^\mu + \mathcal{O}(\Delta p^3),$$

Gravity: $T_p(n) = 2Gmv \cdot n \ln v \cdot n$

$$\delta b^\mu = \frac{\kappa^2}{8\pi} \left(\frac{16}{3} - \frac{2}{v^2} + \frac{2(1-3v^2)\text{arctanh}(v)}{v^3} \right) \Delta p^\mu + \mathcal{O}(\Delta p^3)$$

Change in mechanical angular momentum: $J^{\text{mech.}} = p\, b \longrightarrow \delta J^{\text{mech.}} = p\, \delta b = p\, \frac{b_\mu \delta b^\mu}{b}$

Compensates frame dependence of J^{rad}

QED LO: Saketh, Vines, Steinhoff, Buonanno
GR LO: Bini, Damour

1. Order is restored! ($[\$, Q_s + Q_h] = 0$ is to all orders)
2. Split mechanical-field is ambiguous

Summary: classical gravitational scattering state = hard quantum numbers + a choice of frame

Coulomb field of a particle/
large gauge transformations

0-energy
mediators

asymptotic
charged-particle data

S matrix elements w/
singular kinematic support

- On-shell 3-point amplitudes w/ real momenta are gauge/frame dependent

- Capture the long-distance field of a particle; can be changed by coherent-state dressing

- Any coherent-state dressing Q_s can be completed to a conservation law

- Separation into “mechanical” and “field” is gauge/frame dependent

… and some questions

- Is the supertranslation frame observable?

  $h(u) \longrightarrow h(u+T(n))$ e.g. via a network of distant synchronized detectors?

- Finite-distance/finite energy regularization instead of zero-energy localization?

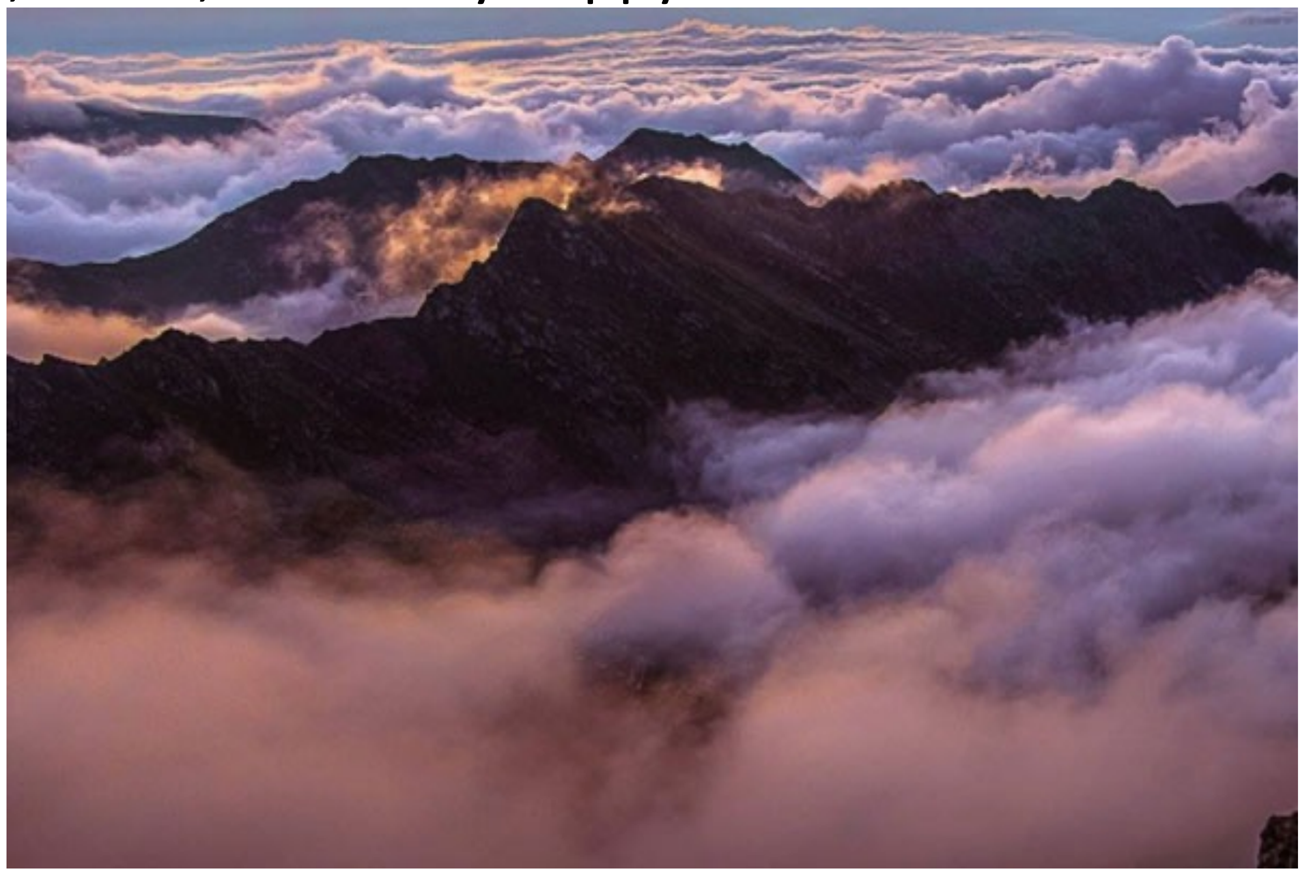
- IR divergences cause breakdown of expected falloff of metric coefficients;
  From GR perspective may interfere with Bondi coordinates. From QFT always $[Q, \mathbb{S}] = 0$
  Lessons for asymptotic symmetries?

- Construction goes through for scalars, so gauge symmetries are not a prerequisite for asymptotic symmetries; what are the consequences?

- Higher dimensions?

More to learn about QFT with inspiration & prompts from classical gravity

Happy 65th, Lance, and many happy returns!



Looking forward to learning more from you, and
to the new routes you'll open in the years ahead.