# Bound states in gauge theory

QED : Successful & Well understood

→ Used as model for less well understood QCD

But not for bound states : hadrons

* Intro to QM : Schrödinger eq & H-atom

$\mathcal{L}_{QED} = \bar{\Psi}(i\not{\partial} - e\not{A} - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ gives S-eq. How?

Textbooks / lecture courses typically omit even basics of bound states

- Myself, did not include this in my courses (no excuse!)
- How many of you have derived the S-eq. from $\mathcal{L}_{QED}$?

The good news : Few others did, either.

Valuable contributions need not be technically difficult.

** Motivations from hadron physics (slides)

* Recall canonical quantization (scalar field)

$$[\varphi(t,\vec{x}), \pi(t,\vec{y})] = i\delta^3(\vec{x}-\vec{y})$$ Equal-time comm rel.

$P = (E, \vec{P})$ :

$$|\text{Positronium}, P\rangle = \Phi^{P}_{e^+e^-}(\vec{x}_1; \vec{x}_2)\,|e^+e^-; P\rangle + \Phi^{P}_{e^+e^-\gamma}(\ldots)\,|e^+e^-\gamma, P\rangle + \ldots$$

Expansion in complete Fock state basis

Each Fock state described by its own "ET" wf.

H mixes Fock components (dynamical), $\hat{P}$ does not (kinematical)

Bound states are invariant under time- & space-translations

$$P = (E, \vec{P}) : \quad H|P\rangle = E|P\rangle \; ; \; \hat{\vec{P}}|P\rangle = \vec{P}|P\rangle$$

$\gamma$

* Notion of equal $t$ is frame dependent

$[H, \vec{K}] \neq 0 \qquad \vec{K}$ : Boost generators

$U(\Lambda) = \exp[i \vec{K} \cdot \vec{\xi}]$ operator corresponding to: $x' = \Lambda x$

H: $U(\Lambda) \hat{\varphi}(x) U^{+}(\Lambda) = \hat{\varphi}(\Lambda x)$

H: $U(\Lambda) | P \rangle = | \Lambda P \rangle$ State with momentum $\Lambda P$ (in original frame)

$\boxed{\text{Dirac: Boosting bound states as hard a solving BSE from scratch with } \vec{P} \neq 0}$

Fock state wf's $\Phi_{ete-,\ldots}$ transform dynamically.

* Physical quantities are frame independent

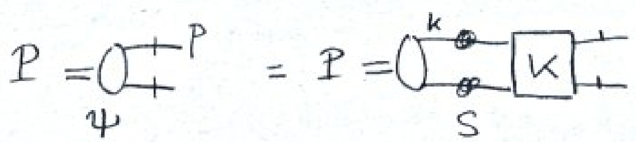
E.g: Scattering matrix $S$:

$$S_{fi} = {}_{out}\langle f | T\exp[-i \int d^4x\, \mathcal{H}_I(x)] | i \rangle_{in}$$

- $|in\rangle$, $|out\rangle$ states treated as free ⇒ Lorentz covariant
  ← Not quite correct → IR problems
- $\mathcal{O}(\alpha^n)$ L-invariant, $\forall n$, since $\alpha$ free parameter

Bethe - Salpeter formalism of bound states

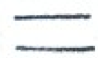
* Schrödinger eq. (1926) only lowest order approx in QED
  Search for a L-covariant, formally exact BSE
  resulted in Bethe Salpeter eq. (1951)




Propagator $S$
2-particle irr kernel $K$

$$\Phi(P, p) = \int \frac{d^4k}{(2\pi)^4} \Phi(P, k) S(k) K(k \to p)$$

- We'll derive this at lowest order

$S =$ 

$K =$ 

* S and K are expanded perturbatively in $\alpha$

$$\Phi(P; x_1 - x_2)\, e^{-iP\cdot(x_1+x_2)/2} \equiv \langle 0 | T\{\varphi(x_2)\varphi(x_1)\} | P\rangle$$

- Translation: $x_1 \to x_1 + a$, $x_2 \to x_2 + a$ gives phase $e^{-iP.a}$ : Plane wave

- Lorentz transf: $x' = \Lambda x$ ; $P' = \Lambda P$ ; $U(\Lambda)\varphi(x)U^\dagger(\Lambda) = \varphi(\Lambda x)$

$$\Phi(P, x_1 - x_2)\, e^{-iP'\cdot(x_1'+x_2')/2} = \langle 0 | U(\Lambda)\, T\{\ \}\, U^\dagger(\Lambda) U(\Lambda) | P\rangle$$

$$= \langle 0 | T\{\varphi(x_1')\varphi(x_2')\} | \Lambda P\rangle$$

$$\Rightarrow \Phi(P, x_1 - x_2) = \Phi(\Lambda P, x_1' - x_2') \quad \text{L-covariant}$$

* But: If $x_1^0 = x_2^0$ then $\Phi(P, x_1 - x_2)$ is the 2-particle Fock state wf.

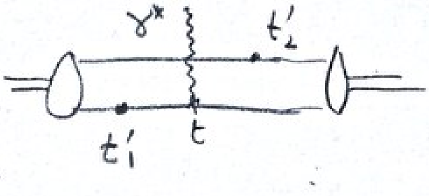
but $x_1^{0\prime} \neq x_2^{0\prime}$ if $\Lambda$ is a boost;

$\Rightarrow \Phi(\Lambda P, x_1', x_2')$ is not simply related to Fock wf's

$$e^{-i(t_2'-t_1')H}\, \varphi(t_1, \vec{x}_1')\, e^{i(t_2'-t_1')H} = \varphi(t_2, \vec{x}_1'),$$

but due to interactions $\varphi(t_2, \vec{x}_1')$ is a multiparticle operator

* Also: If a bound state interacts with external sources, the knowledge of its wf at unequal times is not useful



* No analytic solution $\Phi(\vec{P}, \vec{p})$ known even for single photon kernel $\Rightarrow$ Difficult in practice to calculate higher order corrections using the BS-eq.
  - A solution is found in terms of $\phi_{e^+e^-}$ and $\phi_{e^+e^-\gamma}$ ET wf's at any $\vec{P}$ (below) and lowest order in $\alpha$.

* It was realized (1977) that the BS-eq. remains exact for any choice of propagator $S$, e.g. the NR propagator
  - Choice of propagator determines the kernel $K$
  - $\Rightarrow$ Simplifies higher order calculations

* The optimal method is to use NRQED (1986): Effective theory for which $\mathcal{L}_{QED}$ is expanded in $\frac{1}{m_e}$
  - Only $\sim 10^{-11}$ probability for relative momentum $p \sim m_e$ in positronium $\square$ $\sqcap$

* Similarly, NRQCD has been developed for quarkonia in QCD.

* Requirement of manifest L-covariance has mostly been given up for QED bound states, and calculations confined to $\vec{P} = 0$
  - Results for physical quantities are L-invariant (cf gauge inv.)

* Dyson-Schwinger formalism pursued in QCD
  - Exact relations between Green functions $\sqcap$
  - Eq's do not close $\Rightarrow$ truncate & use as model for NP effects
  - Outside the scope of these lectures.

EdC

# Contents: Bound states in gauge theory (Edinburgh)

## $\underline{\text{V}}$ Dirac eq. from Feynman diagrams

- Leading order in $e^2 Z \gg e^2$
- Crossed Coulomb exchanges - $e^+e^-$ pairs in wf. } Interference
- Loops in external field implicitly included }
- "single electron" wf describes $\infty$ $e^+e^-$ pairs - cf hadrons
- $|\psi|^2 \sim$ inclusive density

## $\underline{\text{VI}}$ Dirac wf. in $D = 1+1$

- $2\times 2$ Dirac matrices, $[e] = [m]$, Linear potential
- Analytic solution $({}_1F_1)$
- $\int dx\, |\psi|^2 = \infty$, continuous spectrum (1932)
- Cf, Klein-Gordon in $D = 3+1$ with linear potential
- NR limit of Dirac eq: Discrete spectrum

## $\underline{\text{VII}}$ Hamiltonian formulation of Dirac state

- Analogous to Positronium, but relativistic, $V_{ext}$
- $e^+e^-$ pairs included implicitly

## $\underline{\text{VIII}}$ Relativistic $f\bar{f}$ in $D = 1+1$, linear potential of QED

- $A^1 = 0$ in any frame (Gauge choice; $A^+ = 0$ in $D = 1+1$)
- Poincaré covariance (provided $V$ is linear)
- Discrete spectrum, unnormalizable wf.
- Gauge invariant EM form factor
- DIS with sea fermions

$\Rightarrow D = 3+1$; QCD