Domain-wall-seeded bubble nucleation on the lattice

Jaakko Hällfors
University of Helsinki

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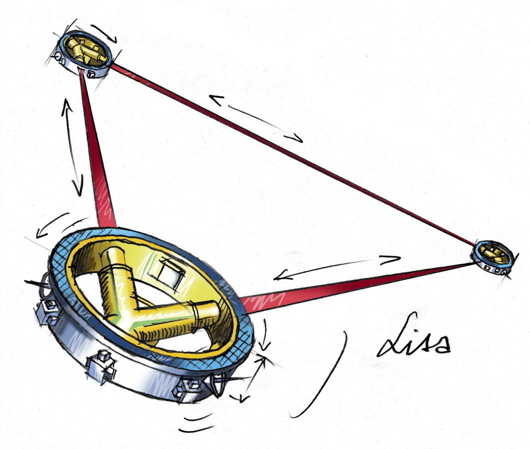
[arXiv:TBA]

With Simone Blasi, Andreas Ekstedt and Kari Rummukainen

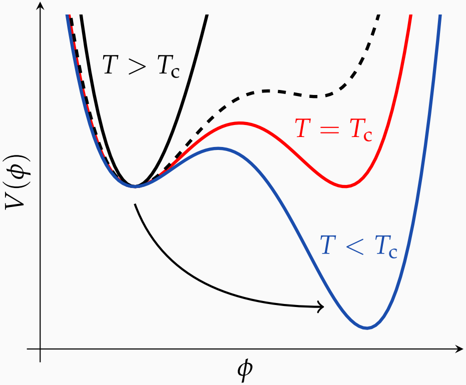
Cosmological phase transitions

- Many extensions of the standard model can have first order phase transitions
- These are expected to produce a model-dependent gravitational wave background
- Future GW observatories might be able to rule out models based on the detected background



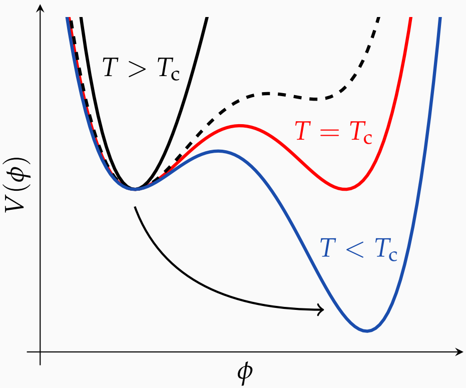
[European Space Agency]

First-order phase transitions



- First-order phase transitions happen through bubble nucleation from the metastable to the stable phase
- The nucleation rate tells us how often bubbles of the stable phase appear

First-order phase transitions



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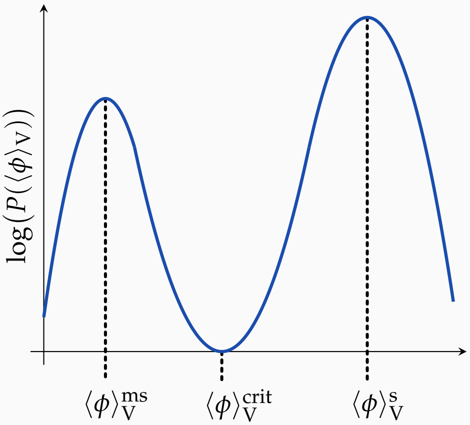


Computing the nucleation rate (perturbatively)

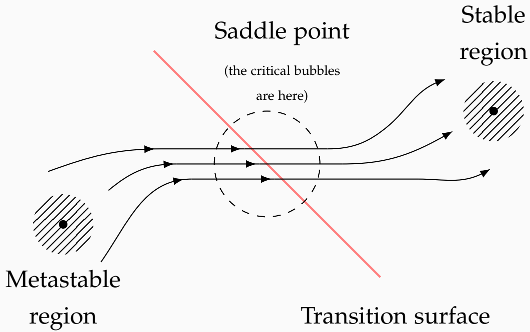
- For supressed transitions the rate Γ is mostly determined by the probability of critical configurations

- A saddle-point approximation gives

$$\Gamma \approx A \exp\{-B\},$$

where B is the "bounce-action" and A contains dynamical information

- Both can be computed semi-analytically for spherical bubbles

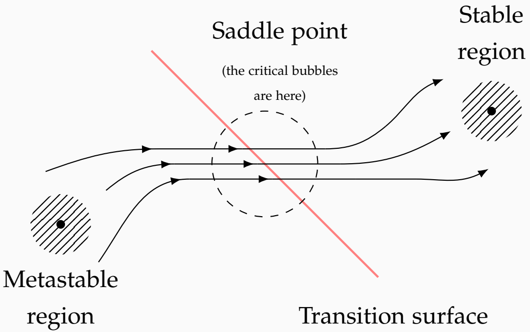


[Langer 1969; Coleman 1977; Callan et al. 1977]

Computing the nucleation rate (non-perturbatively)

- Two scenarios:

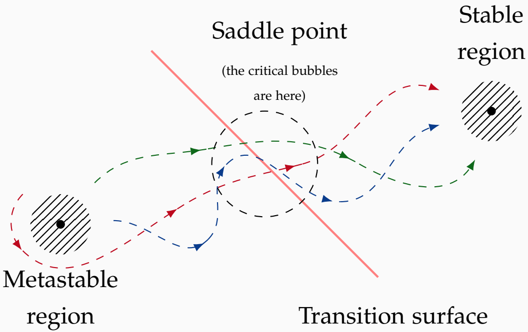
  1. Small nucleation rate
     $\implies$ sample critical configurations using MCMC [Moore et al. 2001]



- Two scenarios:

  1. Small nucleation rate
     $\Longrightarrow$ sample critical configurations using MCMC [Moore et al. 2001]

  2. Large nucleation rate
     $\Longrightarrow$ real-time simulations of the metastable phase



Seeded nucleation

- Almost all observed nucleation is seeded by impurities
- Inhomogeneities such as topological defects could also enhance nucleation rates in the early universe
  - Cosmic strings [Steinhardt 1981]
  - Monopoles [Preskill et al. 1993]
  - Domain walls

  ⋮

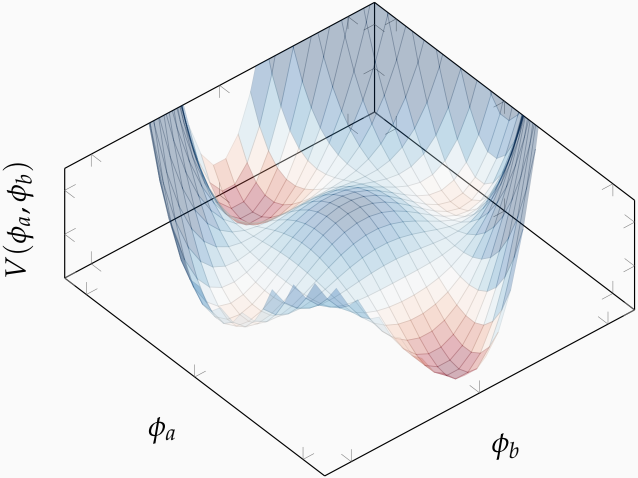
[Wikimedia Commons]

The setup: two scalar fields in 2 + 1 D

$$\mathcal{L} = \sum_{\alpha=\mathrm{a,b}} \left[\frac{1}{2} (\partial_\mu \phi_\alpha)^2 - \frac{1}{2} \mathrm{m}_\alpha^2 \phi_\alpha^2 \right] + \frac{\lambda_\mathrm{a}}{24} \phi_\mathrm{a}^4 + \frac{\lambda_\mathrm{b}}{24} \phi_\mathrm{b}^4 + \frac{\mu}{4} \phi_\mathrm{a}^2 \phi_\mathrm{b}^2$$

$$\mathrm{m}_\mathrm{a}^2 = 0.09, \quad \mathrm{m}_\mathrm{b}^2 = 0.226,$$
$$\lambda_\mathrm{a} = 0.006, \quad \lambda_\mathrm{b} = 0.096,$$
$$\mu = 0.026\,(0.028)$$

(In units of some mass scale M^2)



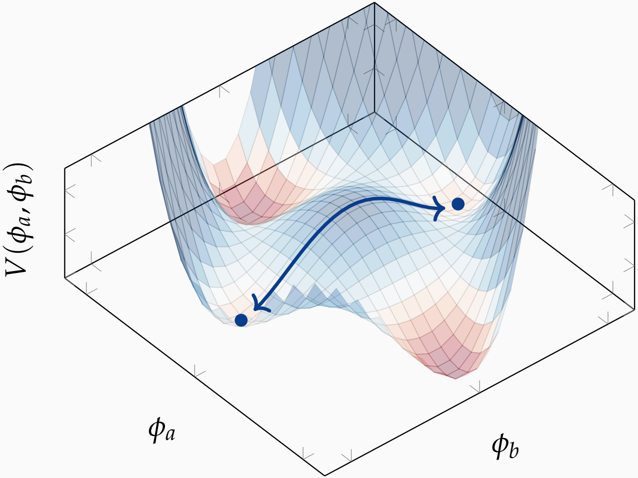
Domain-wall-seeded nucleation

- Two-step transition:

$$(\phi_\mathrm{a}, \phi_\mathrm{b}) : (0, 0)$$

$$\xrightarrow{\text{DWs form}} \quad (0, \pm \mathrm{v_b})$$

$$\xrightarrow{\text{1st order}} \quad (\pm \mathrm{v_a}, 0)$$



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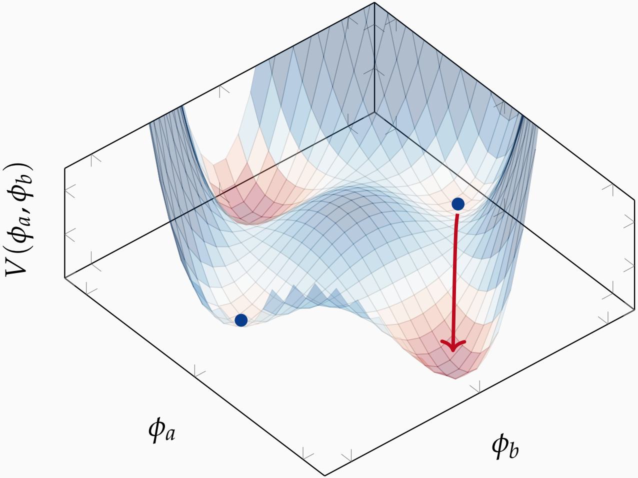
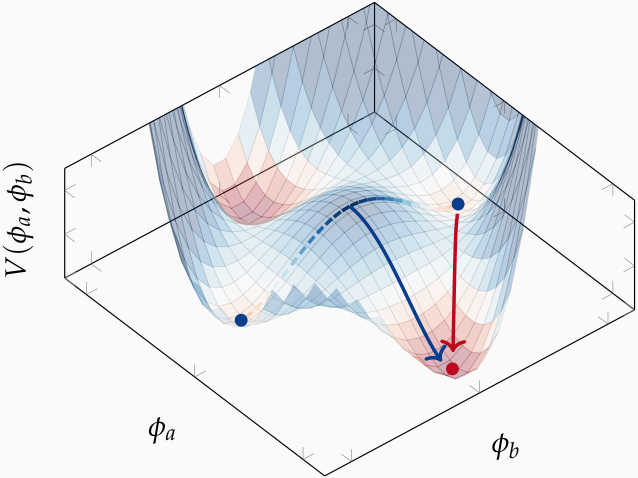


Domain-wall-seeded nucleation

- Two-step transition:

$$(\phi_a, \phi_b): (0,0) \xrightarrow{\text{DWs form}} (0, \pm v_b) \xrightarrow{\text{1st order}} (\pm v_a, 0)$$

- At the domain wall $\phi_b \approx 0$ and only gradient terms prevent the transition



How to estimate the rate analytically?

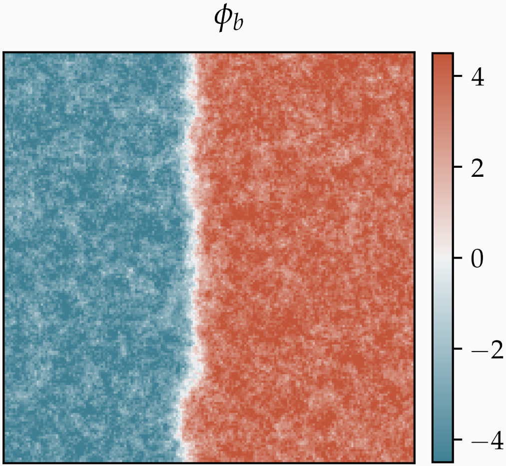
[Rajaraman 1982; Blasi et al. 2022]

- Decompose both fields perpendicular to the domain wall

$$\phi_{\mathrm{b}} = \mathrm{DW}(\mathrm{x}) + \sum_{\mathrm{n}} \phi_{\mathrm{b,n}}(\mathrm{y})\psi_{\mathrm{n}}(\mathrm{x})$$

$$\phi_{\mathrm{a}} = \sum_{\mathrm{n}} \phi_{\mathrm{a,n}}(\mathrm{y})\xi_{\mathrm{n}}(\mathrm{x}),$$

- Perform integrals numerically to obtain a 1D-theory with infinitely many fields



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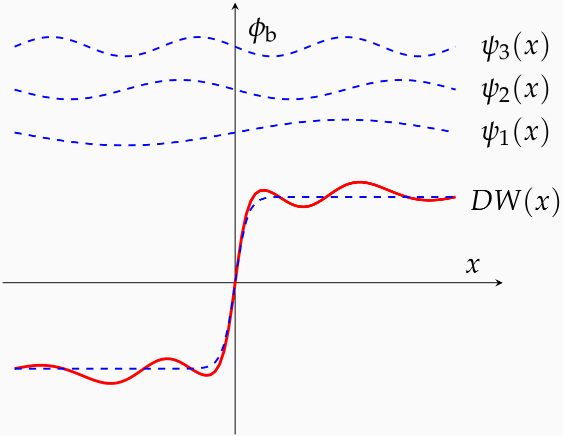


$$\implies \mathcal{L}^{1D} = \sum_{i=0}^{\infty} \sum_{\alpha=a,b} \left[\frac{1}{2} (\partial_y \phi_{\alpha,i})^2 + \frac{1}{2} m_{\alpha,i}^2 \phi_{\alpha,i}^2 \right] + V(\phi_{a,j}, \phi_{b,k}),$$

$$\text{where } m_{\alpha,0}^2 < m_{\alpha,1}^2 < \ldots .$$

- Heavier fields can be integrated out to yield an EFT for $\phi_{a,0}$:

$$\mathcal{L}_{a,0}^{1D} = \frac{1}{2} (\partial_y \phi_{a,0})^2 + \frac{1}{2} m_{a,0}^2 \phi_{a,0}^2 + V(\phi_{a,0})$$

$\implies$ a single field homogeneous nucleation problem in 1D

On the lattice:

- Antiperiodic boundary conditions perpendicular to the wall
- Stochastic Hamiltonian equations

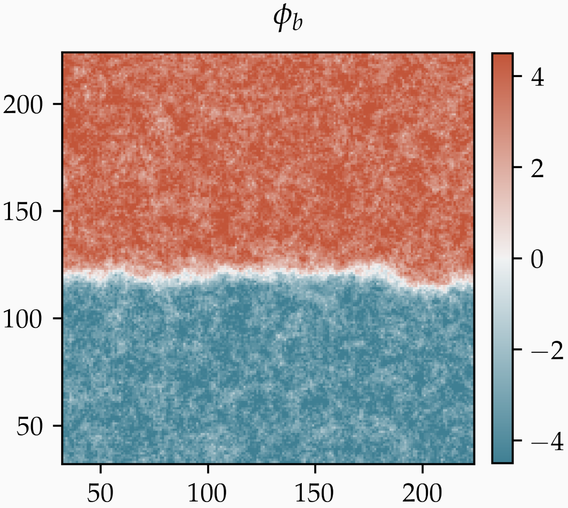
$$\frac{\partial \phi_\alpha}{\partial t} = \frac{\delta H}{\delta \pi_\alpha}, \quad \frac{\partial \pi_\alpha}{\partial t} = -\frac{\delta H}{\delta \phi_\alpha} - \gamma \frac{\delta H}{\delta \pi_\alpha} + \xi$$

- Integrate numerically and take ξ into account by updating every $\Delta t \ll a$

$$\pi_\alpha(x, t) = (1 - \epsilon^2)^{1/2} \pi_\alpha(x, t) + \epsilon \eta_\alpha(x, t),$$

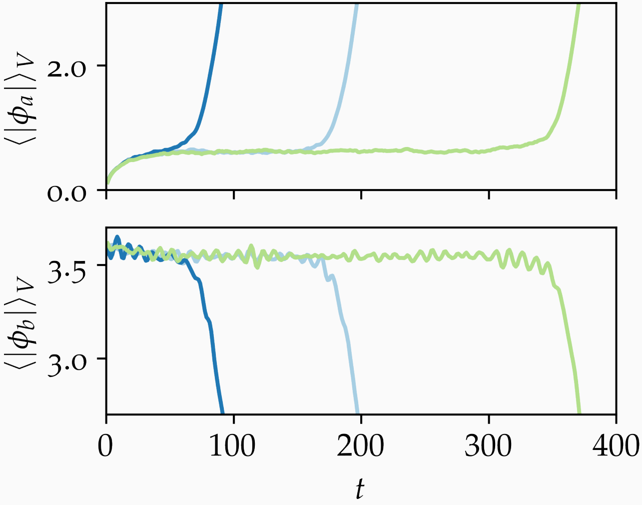
where $\epsilon^2 = 1 - \exp(-2\gamma \Delta t)$.

[Moore et al. 2001]

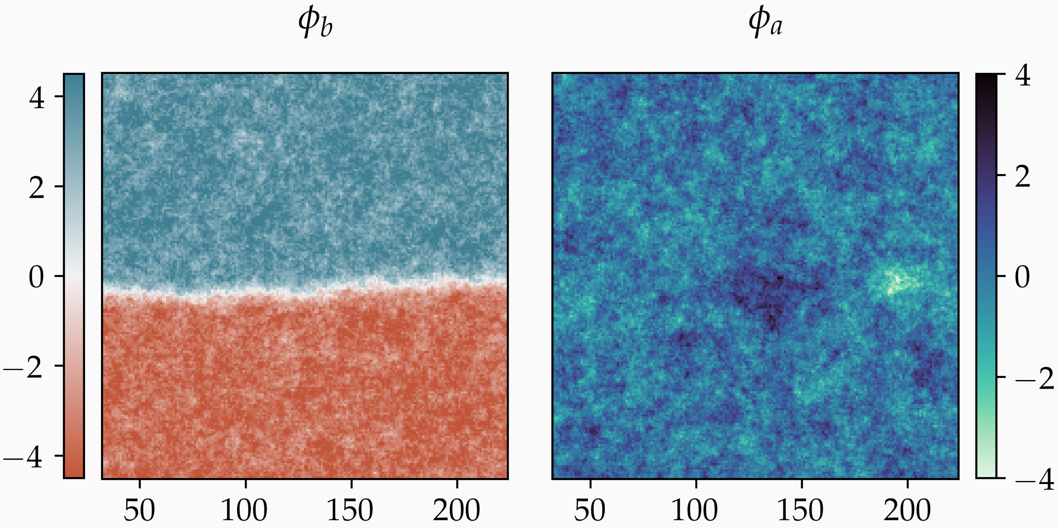


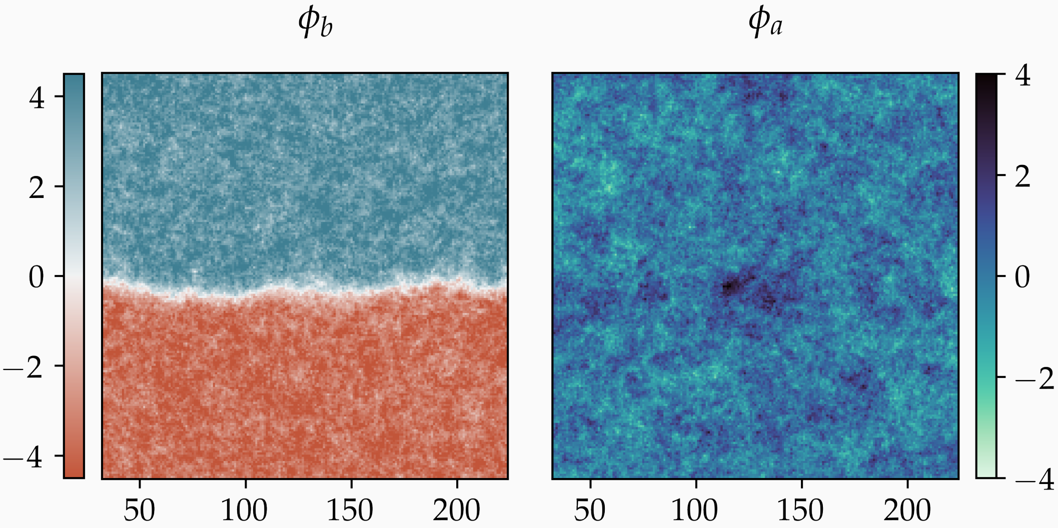
Simulation procedure

1. Initialize fields with anti-periodic boundary conditions to one direction
2. Evolve only ϕ_{b} to form the wall
3. Evolve both fields until $\langle|\phi_{\mathrm{a}}|\rangle_{\mathrm{V}}$ reaches some semi-arbitrary threshold value
4. Record $\mathrm{t_i}$
5. See 1.

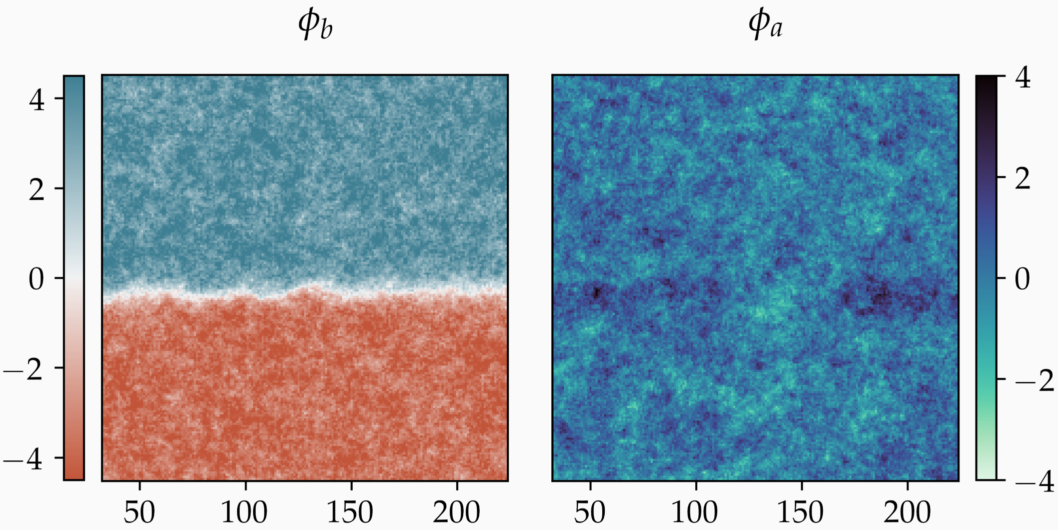


What it looks like




ϕ_b
ϕ_a
4
2
0
−2
−4
50
100
150
200
50
100
150
200
4
2
0
−2
−4

What it looks like



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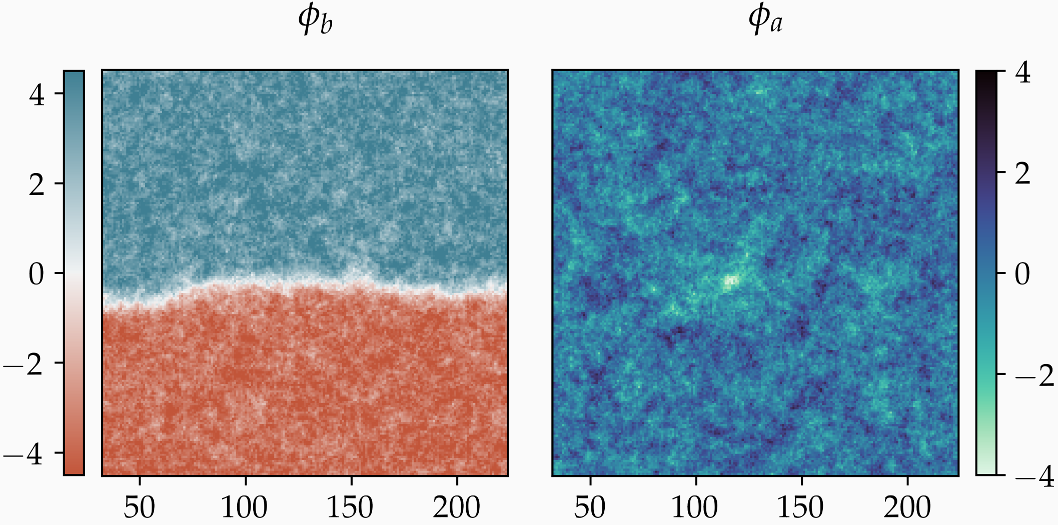

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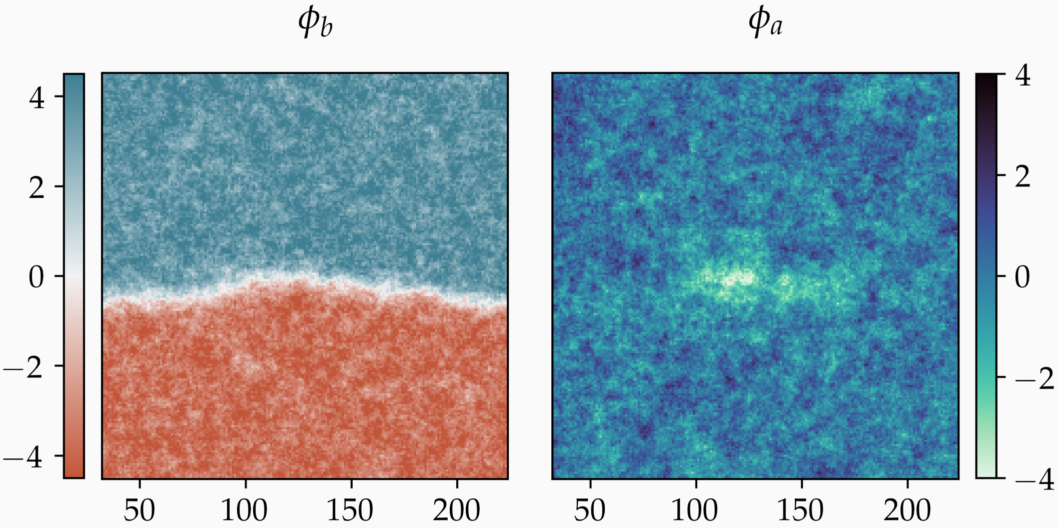

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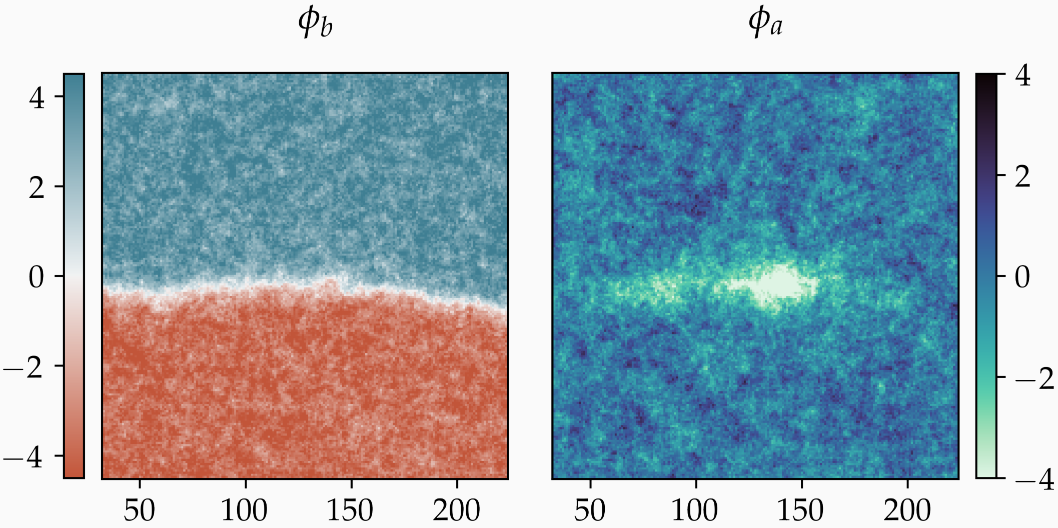

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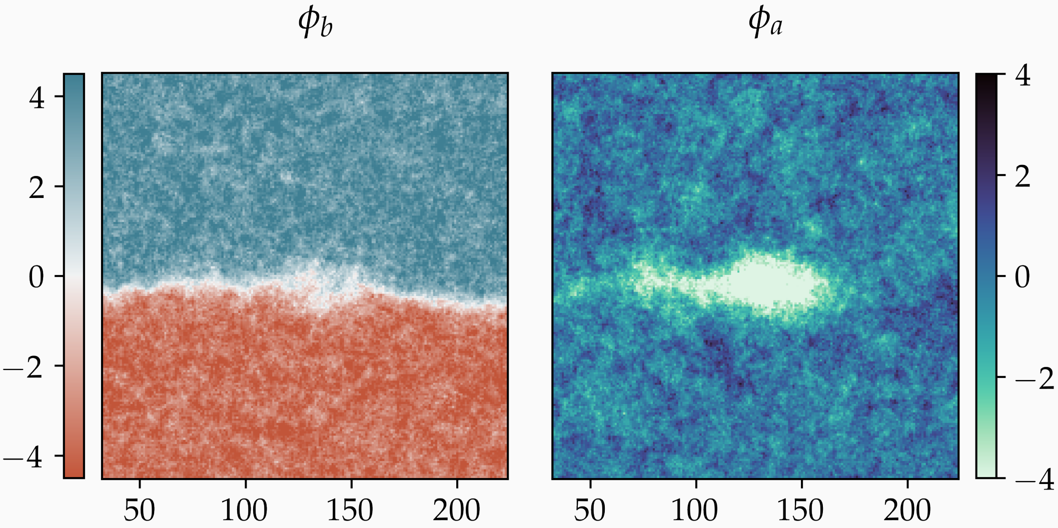

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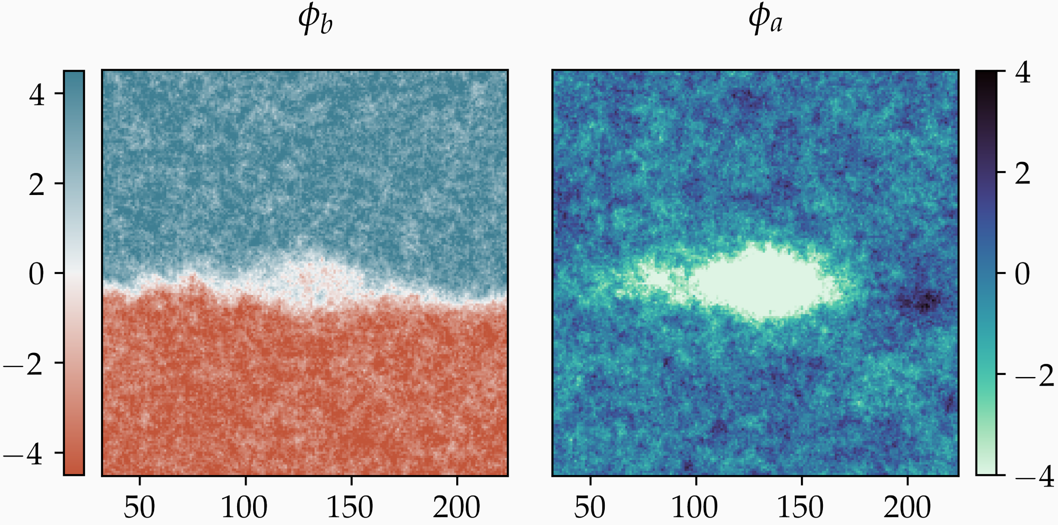

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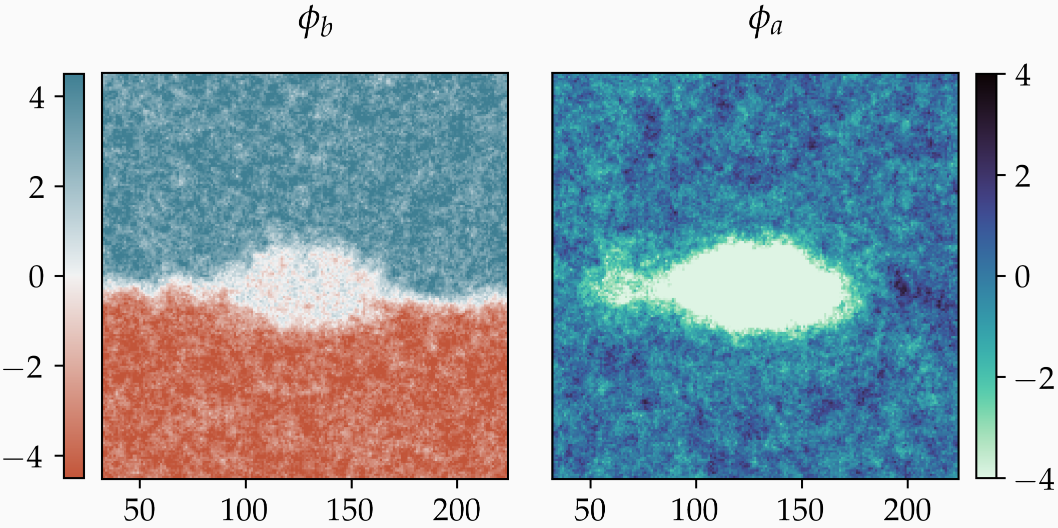
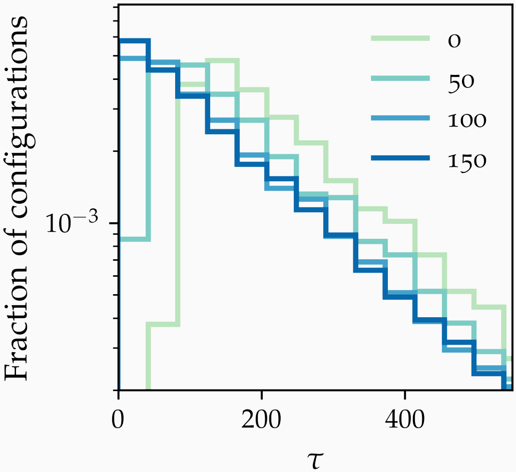


Analysing the nucleation times

- To control dependence on the initial conditions, construct from the full set of nucleation times $\{t_i\}$ the sets

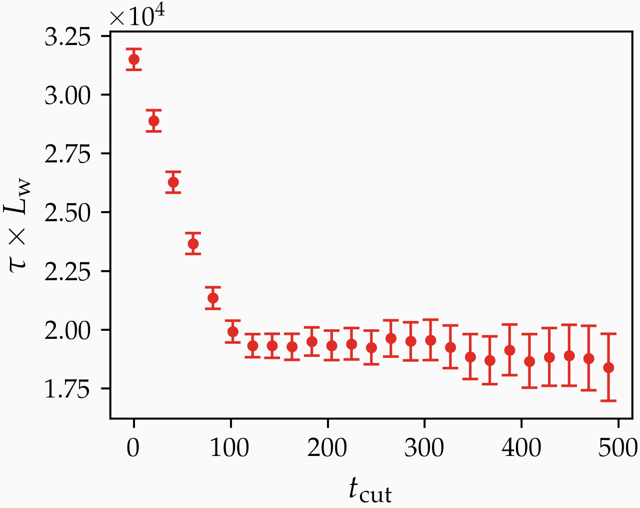
$$\{\, t_i - t_{cut} \mid t_i > t_{cut} \,\}$$

- Compute the mean nucleation time τ for larger and larger values of t_{cut}
- Extract the result from behaviour at large t_{cut}



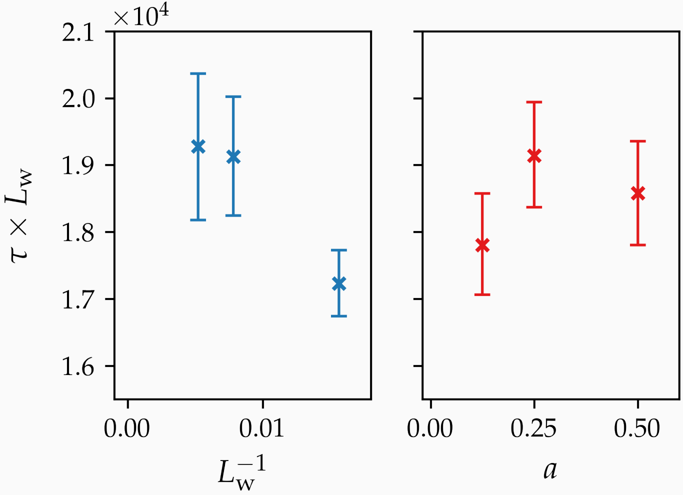
Analysing the nucleation times

- τ stabilises as $t_{cut} \to \infty$
- Works well if the thermalisation time is not much longer than the average nucleation time τ
- Estimate statistical errors by resampling the initial set of nucleation times $\{t_i\}$

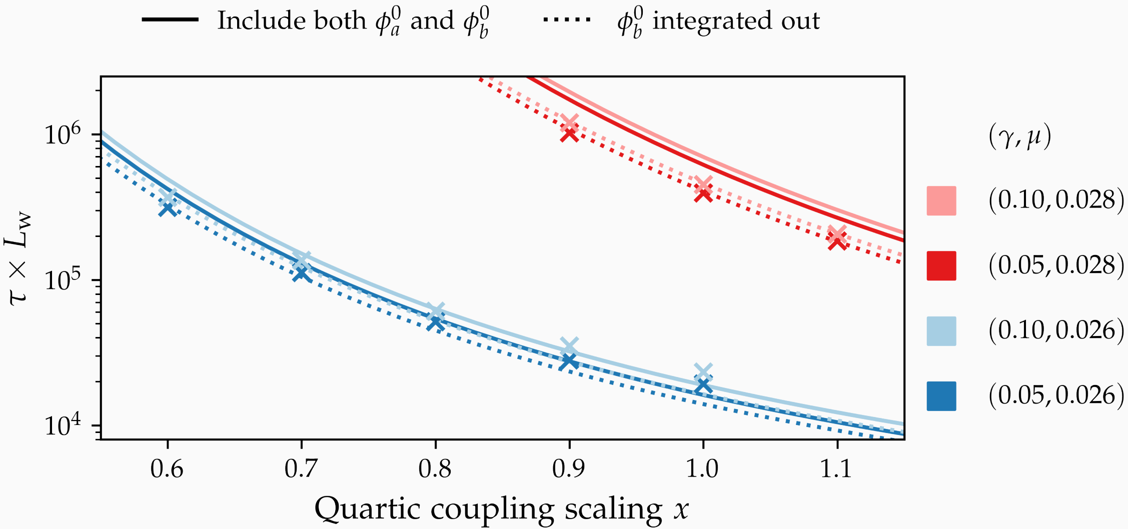


Lattice dependency

- Moderate dependency on the lattice spacing
- Small dependency on the length of the domain wall
- No discernible dependency on the value of $\langle|\phi_\mathrm{a}|\rangle$ used as the nucleation threshold



Results



Include both ϕ_a^0 and ϕ_b^0
ϕ_b^0 integrated out
$\tau \times L_W$
10^6
10^5
10^4
0.6
0.7
0.8
0.9
1.0
1.1
Quartic coupling scaling x
(γ, μ)
$(0.10, 0.028)$
$(0.05, 0.028)$
$(0.10, 0.026)$
$(0.05, 0.026)$

What's next?

DW-seeded nucleation in the Singlet Extended SM:

- Mapping rules to a dimensionally reduced 3D theory exist [Niemi et al. 2024]

  $\implies$ Easy to simulate on a lattice

- Too slow to use pure real-time approach

  $\implies$ Sample bubbles using MCMC