Thermodynamics from Entanglement Entropy

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Based on [2603.07635]

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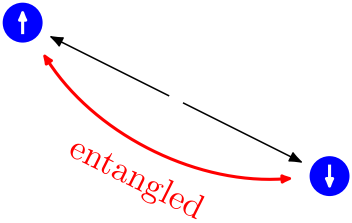
Structure

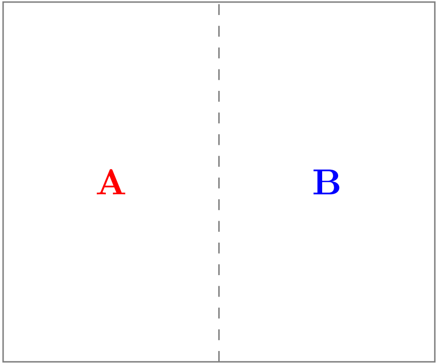
Entanglement

- A fundamental property of quantum systems arrising from conservation laws
- Describes the correlations of quantum states
- For QFTs, applications in
  - Critical phenomena
  - RG flow
  - Confinement
  - Thermodynamics



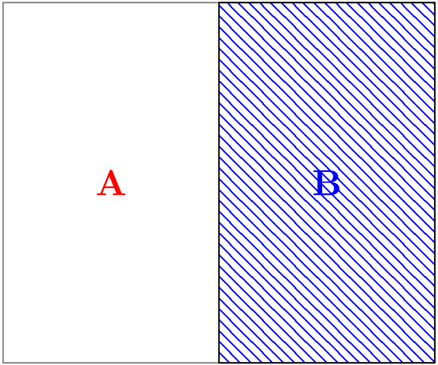
Entanglement Entropy

- An entanglement measure for a bipartite system



Entanglement Entropy

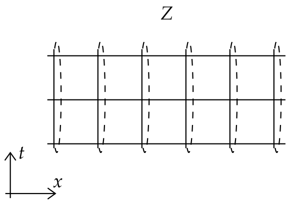
- An entanglement measure for a bipartite system
- Reduced density matrix
  - $\rho_A = \mathrm{tr}_B(\rho_{AB})$
- Entanglement entropy:
  - $S_{EE}(A) = -\mathrm{tr}(\rho_A \log(\rho_A))$

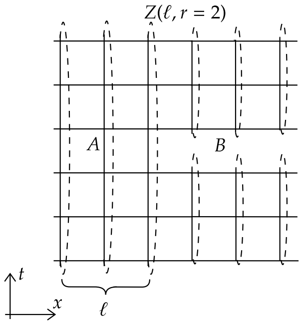


EE in QFTs and the replica trick

- The replica trick [Calabrese, Cardy; hep-th/0405152], $\mathrm{tr}(\rho_A^r) = \frac{\tilde{Z}(\ell,r)}{Z^r}$
- Entanlement entropy:

$$S_{\mathrm{EE}}(\ell) = -\lim_{r\to 1} \frac{\partial \log \mathrm{tr}(\rho_A^r)}{\partial r}$$
$$= -\lim_{r\to 1} \left(\frac{\partial \log \tilde{Z}(\ell, r)}{\partial r} - \log Z \right).$$





EE in QFTs and the replica trick

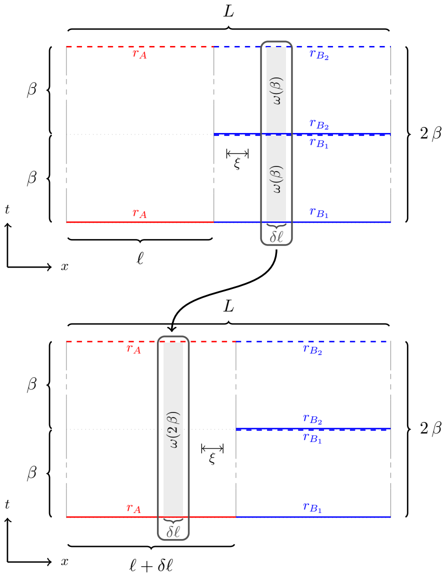
- Problem: UV-divegence
- Measure derivative w.r.t ℓ instead

$$\frac{\partial S_{\text{EE}}}{\partial \ell} = \lim_{r\to 1} \frac{\partial}{\partial \ell}\left(\frac{\partial \log \tilde{Z}(\ell, r)}{\partial r} - \log Z\right)$$

$$= \lim_{r\to 1} \frac{\partial^2 \log \tilde{Z}(\ell, r)}{\partial \ell\, \partial r}$$

Entanglement thermodynamics

- Free energy density $\omega(\beta,\mu) = \frac{1}{V}\log Z(\beta,\mu,V)$
- Thermal entropy density: $s = \beta\,\frac{\partial\omega}{\partial\beta} - \omega$



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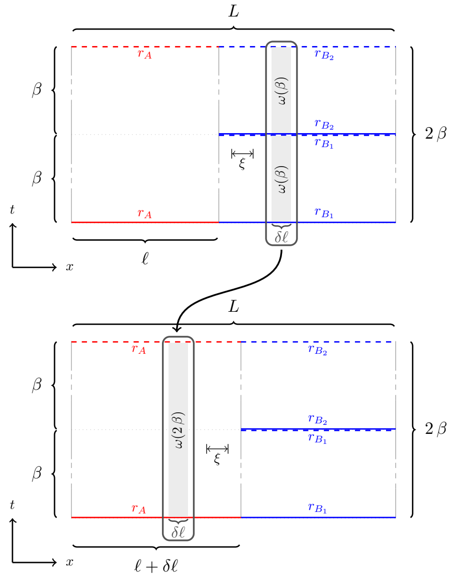
- EE derivative: $\partial_\ell S_{\text{EE}}(\ell) = \lim_{r \to 1} \frac{\partial^2 \log \tilde{Z}(\ell, r)}{\partial r \, \partial \ell}$
- ∂_ℓ can be expressed as

$$\frac{\log \tilde{Z}(\ell + \delta\ell, \beta, V, \mu, r) - \log \tilde{Z}(\ell, \beta, V, \mu, r)}{\delta\ell}$$



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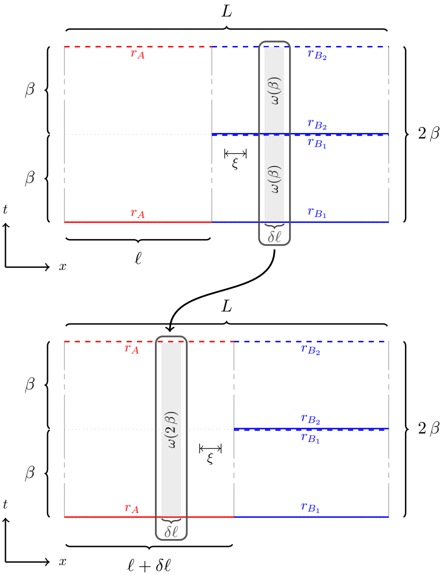
- at $\xi \ll \ell \ll L$

$$\frac{1}{V_\perp}\frac{\log\tilde{Z}(\ell,r)}{\partial\ell} = \omega(r\,\beta,\mu) - r\,\omega(\beta,\mu)$$



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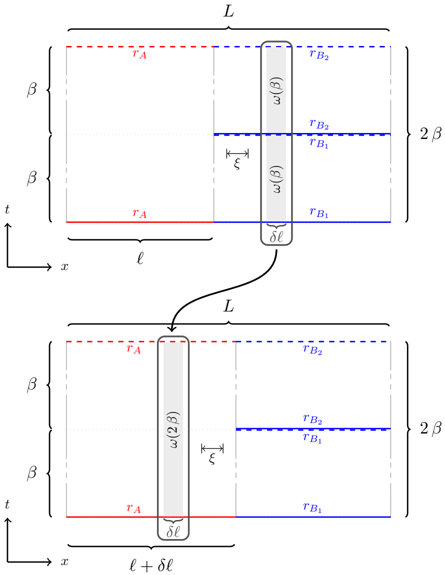
- So,

$$\frac{1}{V_\perp}\frac{\partial S_{\text{EE}}(\ell)}{\partial\ell} = \beta\,\frac{\partial\omega}{\partial\beta} - \omega = s$$



$O(N)$ models

- A QFT with a N-component scalar field
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- Action with a chemical potential and source terms:

$$S[\phi] = \sum_x \left(-\frac{\kappa}{2} \sum_{\nu=1}^{d} e^{2\,\mu\tau_3\delta_{\nu,d}}\, \phi_x\, \phi_{x+\hat{\nu}} + e^{-2\,\mu\tau_3\delta_{\nu,d}}\, \phi_{x-\hat{\nu}}\, \phi_x \right)$$
$$+ \,(\phi_x \cdot \phi_x) + \lambda\,((\phi_x \cdot \phi_x) - 1)^2 - (j \cdot \phi_x)$$

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- With non-zero μ, the action is in general complex
  - Importance sampling can't be done in terms of ϕ
  - Can be solved by going to dual variables

Partition function with dual variables

- With series expansions, redefinitions and integrations, we get [Rindlisbacher et al.;1602.09017,1512.05684]

$$Z = \sum_{\{k,l,\chi,p,q,n\}} \prod_x \Bigg\{ \delta\Big(p_x + \sum_\nu (k_{x,\nu} - k_{x-\hat{\nu},\nu})\Big) \Bigg(\prod_{i=3}^{N} \delta_2 \Bigg(\sum_\nu \Big(\chi^{(i)}_{x,\nu} + \chi^{(i)}_{x-\hat{\nu},\nu} \Big) + n^{(i)}_x \Bigg) \Bigg)$$

$$\times\, \mathrm{e}^{\mu\, k_{x,d}} \Bigg(\prod_{\nu=1}^{d} w_{\mathrm{l}}(L_{x,\nu};\kappa) \Bigg)\, w_{\mathrm{s}}(L_x, M_x; \lambda, j) \Bigg\}$$

- Monomer $p_x, q_x, n^{(i)}_x$, and Flux $k_{x,\nu}, l_{x,\nu}, \chi^{(i)}_{x,\nu}$, numbers
- Link and site weights
- A coupling of μ to k-variables in the time direction
- Delta function and evenness constraints

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- Monomer $p_x, q_x, n_x^{(i)}$, and Flux $k_{x,\nu}, l_{x,\nu}, \chi^{(i)}_{x,\nu}$, numbers
- Link and site weights
- A coupling of μ to k-variables in the time direction
- Delta function and evenness constraints
  - $k, \chi^{(i)}, p$ and $n^{(i)}$ cannot be sampled with regular local Metropolis

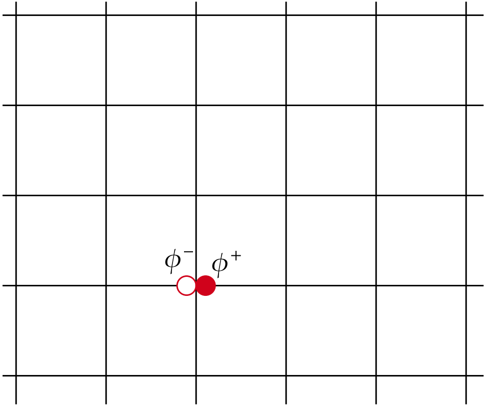
Worm algorithm

- Originally in spin models as an alternative to cluster algorithms [Prokof'ev, Svistunov; cond-mat/0103146]
- Our method based on [Rindlisbacher et al.;1602.09017,1512.05684]
- The basic idea is to introduce a source and sink pair and move one of them around

Worm algorithm

1. Start by proposing to insert an external $\phi^+\phi^-$ pair at some site x

$$\delta(p_x + 1 - 1 + \sum_\mu (k_{x,\mu} - k_{x-\hat{\mu},\mu}))$$

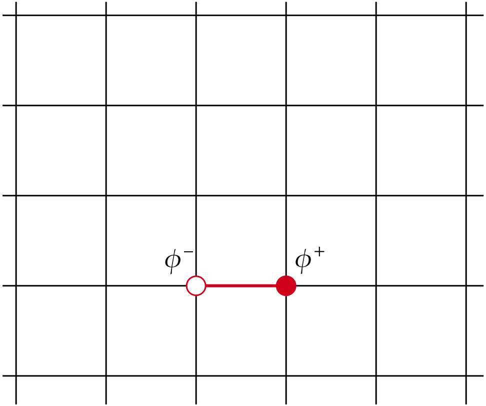


Worm algorithm

2. Propose to move ϕ^+ to a random neighboring site $x' = x + \hat{\nu}$

$$\delta(p_x - 1 + \sum_\mu (k_{x,\mu} - k_{x-\hat{\mu},\mu}))\delta(p_{x'} + 1 + \sum_\mu (k_{x',\mu} - k_{x'-\hat{\mu},\mu}))$$

- Simultaneously shift $k_{x,\nu} \to k_{x,\nu} \pm 1$ to respect the constraints



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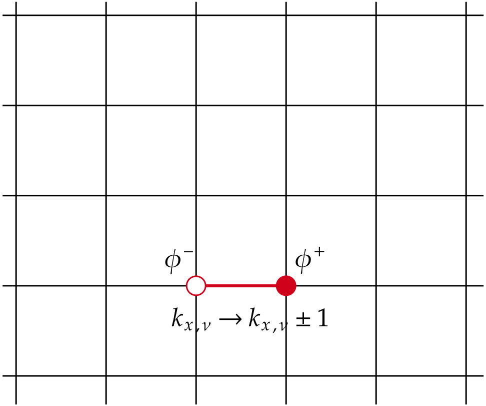

Worm algorithm

3. If $\phi^+\phi^-$ are at the same site, one can propose to remove them

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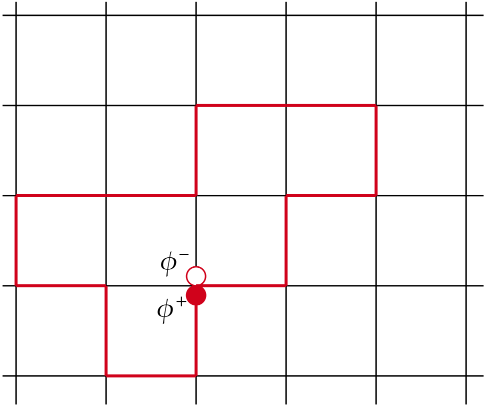

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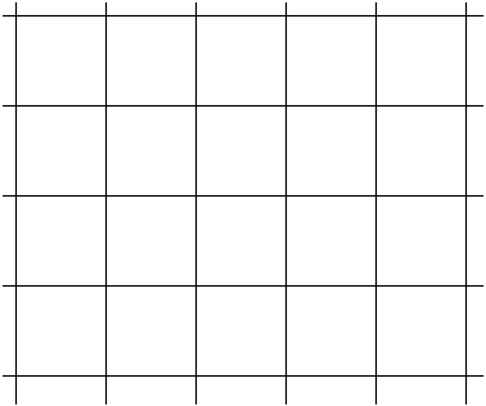

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3. If $\phi^+\phi^-$ are at the same site, one can propose to remove them

$$\delta(p_x + \sum_\mu (k_{x,\mu} - k_{x-\hat{\mu},\mu}))$$

- Works similarly for $\chi^{(i)}$ variables and the evenness constraints
- Disconnected moves for $j_i \neq 0$ to sample p and $n^{(i)}$

Entanglement entropy on the lattice

- $\lim_{r\to 1} \partial_r \partial_\ell$ in $\partial_\ell S_{\text{EE}}$ have to be discretized.

$$\left.\frac{\partial S_{EE}(\ell)}{\partial \ell'}\right|_{\ell'=\ell+1/2} \approx -\log \tilde{Z}(\ell+1,2) + \log \tilde{Z}(\ell,2)$$

- Corresponds to approximating S_{EE} as the 2nd Rényi entropy $H_2(\ell)$

$$\begin{aligned} H_r(\ell,\beta,V,\mu) &= \frac{1}{1-r}\log \operatorname{tr}(\rho_A^r) \\ &= \frac{1}{1-r}\Big(\log \tilde{Z}(\ell,\beta,V,\mu,r) - r\,Z(\beta,V,\mu)\Big) \end{aligned}$$

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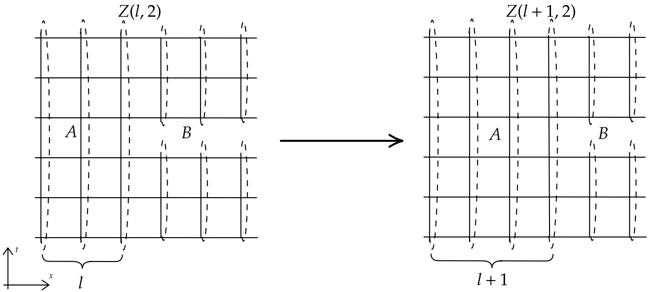

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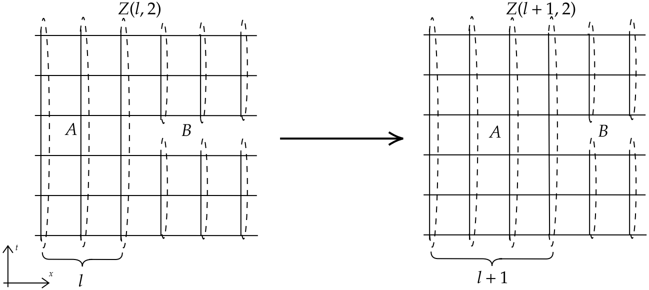
- The change from ℓ to $\ell+1$ is highly non-local $\to$ bad overlap



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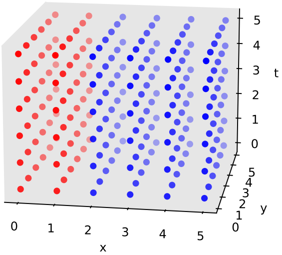

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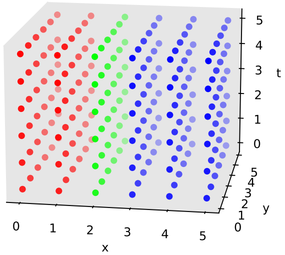

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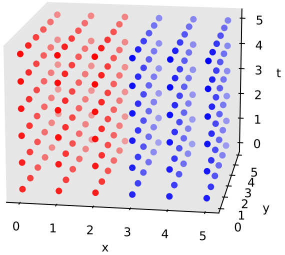
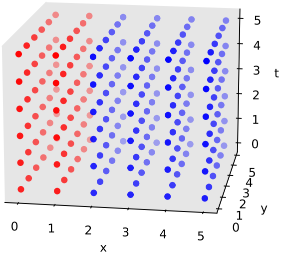


Boundary deformation method

- Deform the entangling surface and interpolate between ℓ and $\ell+1$ [Jokela et al.; 2304.08949]
- histograms h_i for each boundary configuration

$$\left.\frac{\partial S_{\mathrm{EE}}(\ell')}{\partial \ell'}\right|_{\ell'=\ell+1/2} \approx \log(h_N) - \log(h_0)$$



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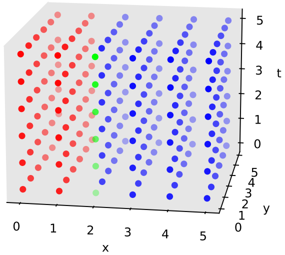

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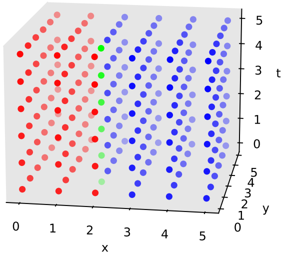

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- Deform the entangling surface and interpolate between ℓ and $\ell+1$ [Jokela et al.; 2304.08949]
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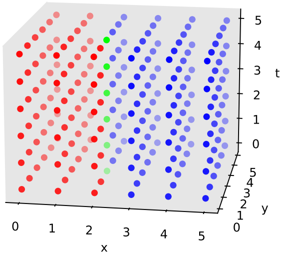

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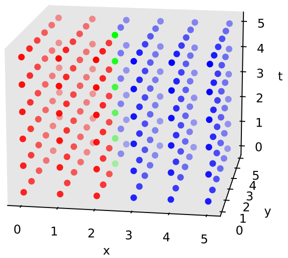

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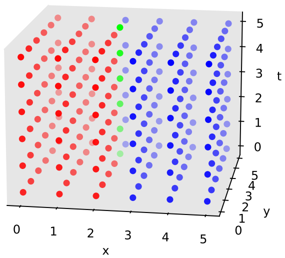

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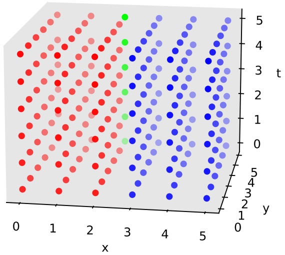

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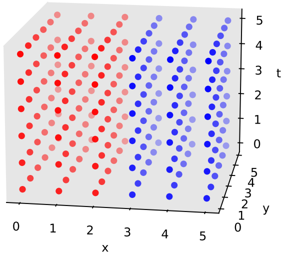
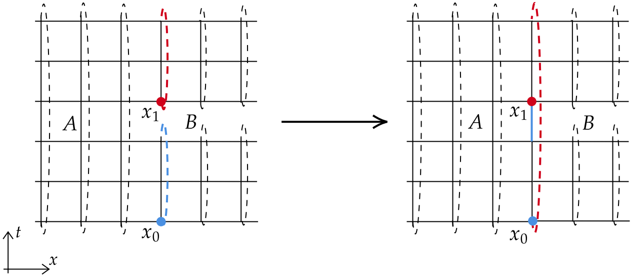


BC flips for lattice $O(N)$ model

- As BC are updated, incoming temporal flux variables $(k,\ l, \chi^{(i)})$ are exchanged between corners.

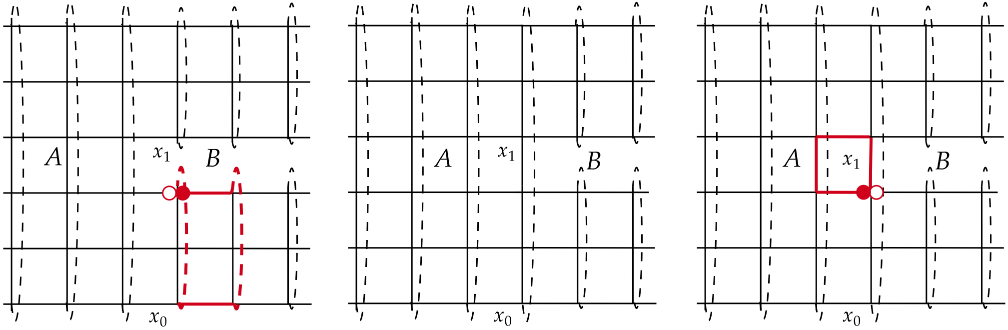


- This can cause violations to the delta function and evenness constraints
- The k and $\chi^{(i)}$ sectors need to be manipulated before changing the BCs

Boundary update with plaquettes

- Change the problematic k and $\chi^{(i)}$ variables with plaquette updates

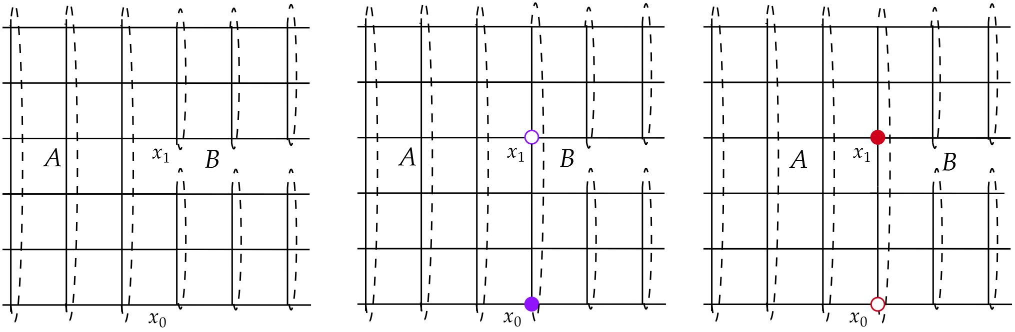
1. Plaquettes are performed s.t. $k_{x_1,t} - k_{x_0,t} = 0$ and $\chi_{x_1,t} - \chi_{x_0,t} = 0 \pmod 2$
2. The boundary conditions are changed
3. Plaquettes are performed to restore $k_{x_1,t} - k_{x_0,t}$ and $\chi_{x_1,t} - \chi_{x_0,t} \pmod 2$ to their original values



Boundary update with worms

- Manipulate expression in constraint with insertion and removal of worms

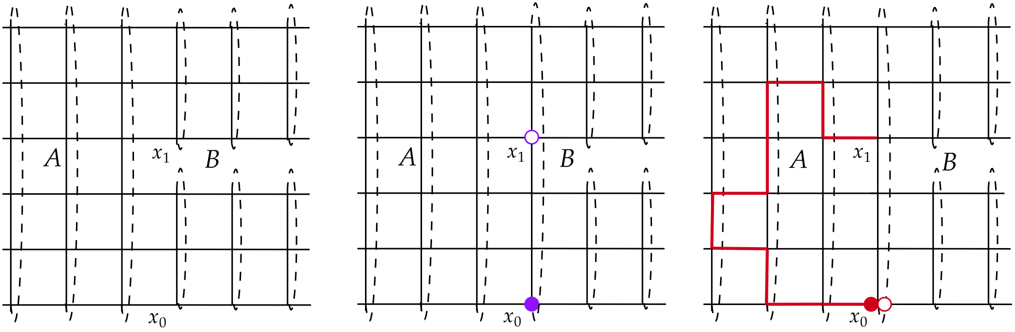
1. Perform BC flip which produces defects
2. Insert head and tail to counter defects



Boundary update with worms

- Manipulate expression in constraint with insertion and removal of worms

1. Perform BC flip which produces defects
2. Insert head and tail to counter defects
3. Move head and tail to same site and terminate

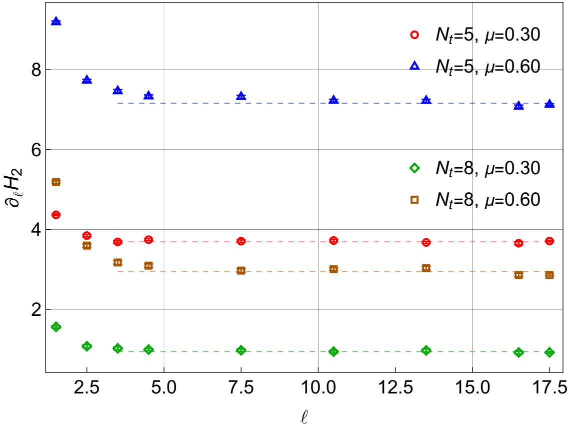


Simulation parameters

- O(4)
- $\kappa = 1.2$ and $j_3 = 0.2$, symmetry broken phase and finite correlation length
- $d = 3$, lattices of size $2N_t \times N_x \times N_s$, $N_x = 36$, $N_s = 12$
- multiple μ and N_t values

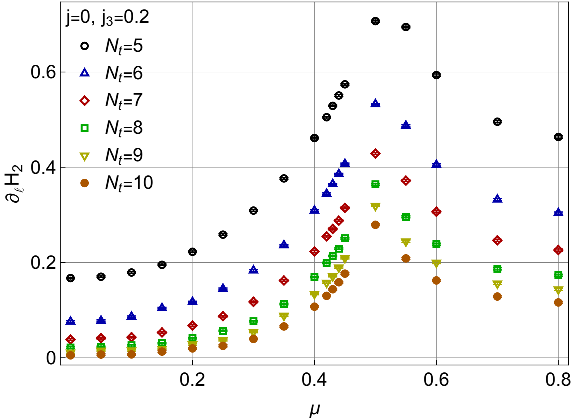
Results

- $\partial_\ell H_2$ plateaus quickly as a function of ℓ:



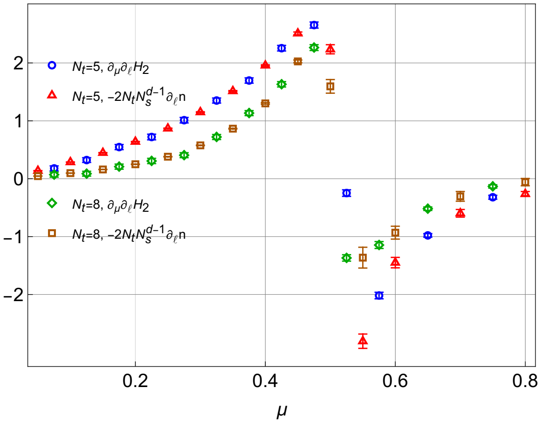
Results

- $\partial_\ell H_2$ as a function of μ
- Finite density phase transition clearly visible with $\mu_c \approx 0.5$



Results

- Derivative of H_2,
  $\partial_\ell H_2 = \partial_\ell(-\log \tilde{Z}(\ell, 2))$
- Charge density
  $\tilde{n}(\ell, 2) = \frac{1}{2N_t V}\partial_\mu \log \tilde{Z}(\ell, 2)$
- Both can be used to estimate
  $\partial_\ell \partial \mu \log \tilde{Z}(\ell, 2)$
- Derivatives should match
  $\partial_\mu \partial_\ell H_2 = -2N_t V \partial_\ell \tilde{n}(\ell, 2)$



Results

- Remember that the connection between S_{EE} and s relied on

$$\frac{1}{V_\perp}\frac{\partial \log \tilde{Z}(\ell,\beta,V,\mu,r)}{\partial \ell} \overset{\xi \ll \ell}{=} \omega(r\,\beta,\mu) - r\,\omega(\beta,\mu)$$

Results

- Set $r = 2$ and take ∂_μ

$$-\frac{1}{V_\perp}\frac{\partial^2 \log \tilde{Z}(\ell, 2)}{\partial\mu\,\partial\ell} \overset{\xi \ll \ell}{=} -2\,N_t\,(n(2\,N_t, \mu) - n(N_t, \mu))$$

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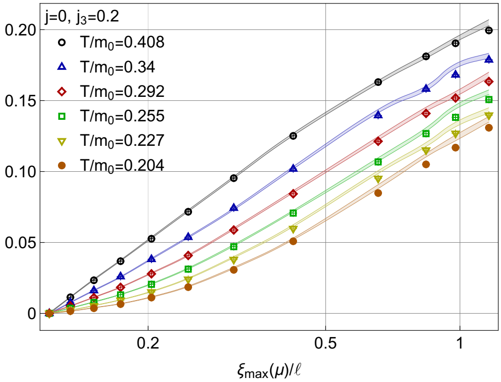
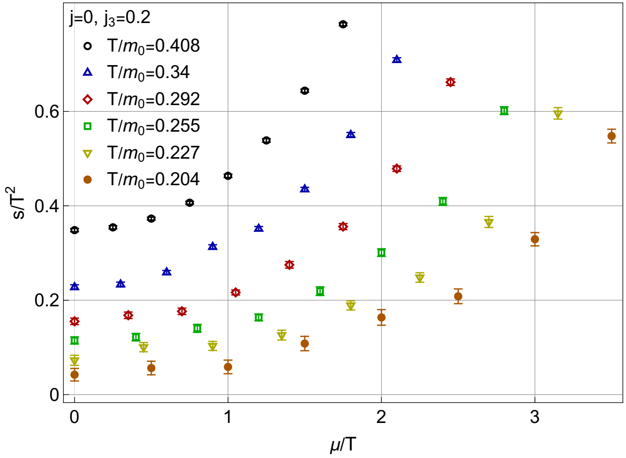
- Both sides can be computed on the lattice
- Relation seems to hold up to $\xi_{\max}(\mu)/\ell \approx 0.5 \to \mu \lesssim 0.4$



Results

- Results for s/T^2 as a function of μ/T



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Thank You!!!