Inclusive semileptonic decays using lattice QCD

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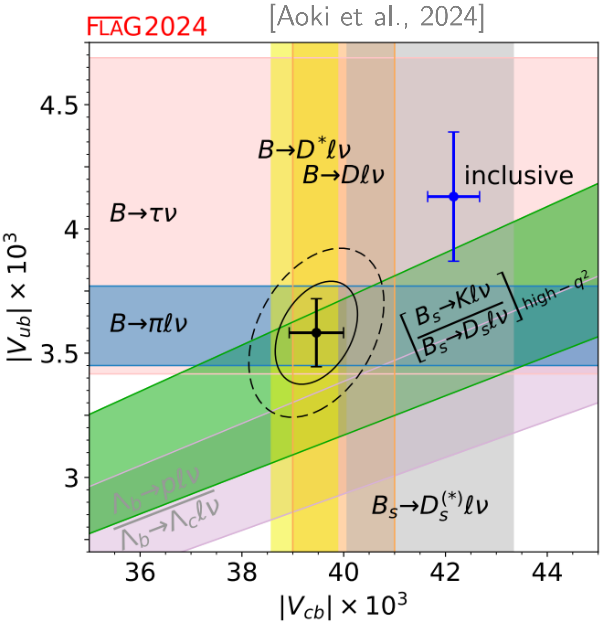
Motivation

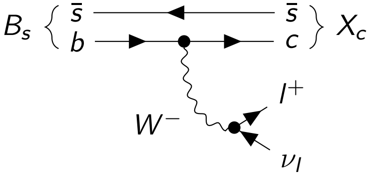
Different method of calculation.

- Exclusive $|V_{cb}|$ = Lattice QCD
- Inclusive $|V_{cb}|$ = Perturbative methods

Our aim is to calculate the **inclusive** $|\boldsymbol{V_{cb}}|$ using Lattice QCD.



Focus on $B_s \to X_c \, \bar{\nu}_l l$



$$H_W = \frac{G_F}{\sqrt{2}} V_{cb} \underbrace{\bar{\nu}_{lL}\gamma^\mu l_L}_{J_L^\mu} \underbrace{\bar{b}_L \gamma_\mu c_L}_{J_\mu}$$

$$\mathcal{M} = \frac{G_F}{\sqrt{2}} V_{cb} \langle l, \bar{\nu}_l | J_L^\mu | 0 \rangle \langle X_c | J_\mu | B_s \rangle$$

$$\frac{d\Gamma}{dq^2 dq_0 dE_l} = \frac{G_F^2 |V_{cb}|^2}{8\pi^3} L_{\mu\nu} W^{\mu\nu}$$

where $q = p_{B_s} - p_{X_c}$

$$\frac{d\Gamma}{dq^2 dq_0 dE_l} = \frac{G_F^2 |V_{cb}|^2}{8\pi^3} L_{\mu\nu} W^{\mu\nu}$$

Leptonic part fully kinematically determined,

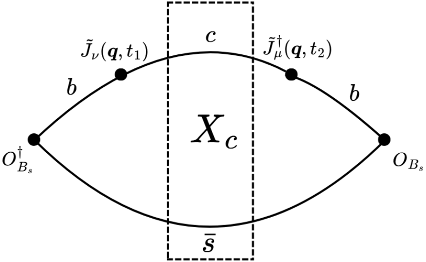
$$L^{\mu\nu} = p_\ell^\mu \, p_{\nu_\ell}^\nu + p_\ell^\nu \, p_{\nu_\ell}^\mu - g^{\mu\nu} \left(p_\ell \cdot p_{\nu_\ell} \right) - i\epsilon^{\mu\alpha\nu\beta} \, p_{\ell,\alpha} \, p_{\nu_\ell,\beta}$$

Hadronic tensor contains non-perturbative info,

$$W^{\mu\nu} \sim \sum_{X_c} \delta^{(4)}(p_{B_s} - q - p_{X_c}) \langle B_s | \tilde{J}^{\mu\dagger}(0) | X_c \rangle \langle X_c | \tilde{J}^{\nu}(0) | B_s \rangle$$

$$C_{\mu\nu}(\boldsymbol{q}, t) = \int_0^\infty d\omega \, W_{\mu\nu}(\boldsymbol{q}, \omega) e^{-\omega t}$$

$$\sim \langle B_s | \tilde{J}_\mu^\dagger(\boldsymbol{q}, 0) \, e^{-\hat{H}t} \tilde{J}_\nu(\boldsymbol{q}, 0) | B_s \rangle$$



where $t = t_2 - t_1$, and $\omega = E_{X_c}$.

[Hansen et al., 2017, Hashimoto, 2017, Hansen et al., 2019, Gambino and Hashimoto, 2020]
[Bailas et al., 2020, Gambino et al., 2022, Barone et al., 2023, De Santis et al., 2025b]

Integrate differential decay width

$$\frac{d\Gamma}{dq^2} \propto \int_{\omega_0}^{\omega_{\max}} d\omega\, k^{\mu\nu}(\boldsymbol{q},\omega) W_{\mu\nu}(\boldsymbol{q},\omega) = \bar{X}(\boldsymbol{q})$$

where ω is the energy of the final state hadron.

$$\bar{X}(\boldsymbol{q}) = \int_{\omega_0}^{\infty} d\omega\, W_{\mu\nu}(\boldsymbol{q},\omega) k^{\mu\nu}(\boldsymbol{q},\omega)\theta(\omega_{\max}-\omega)$$

$$\bar{X}(\boldsymbol{q},\sigma) = \int_{\omega_0}^{\infty} d\omega\, W_{\mu\nu}(\boldsymbol{q},\omega)\, k^{\mu\nu}(\boldsymbol{q},\omega)\theta_{\boldsymbol{\sigma}}(\omega_{\max}-\omega)$$

$$\bar{X}(\boldsymbol{q},\sigma,N) \approx \sum_{t}^{N} c_{\mu\nu,\,t}(\boldsymbol{q},\sigma) \underbrace{\int_{\omega_0}^{\infty} d\omega\, W^{\mu\nu}(\boldsymbol{q},\omega)\, e^{-\omega t}}_{C^{\mu\nu}(\boldsymbol{q},t)}$$

*Sum over $t = t_2 - t_1$ per **channel**

Need to then take appropriate limits

$$\bar{X}(\boldsymbol{q}) = \lim_{\sigma\to 0}\lim_{N\to\infty} \bar{X}(\boldsymbol{q},\sigma,N)$$

[Hansen et al., 2019], [Bailas et al., 2020], [Barone et al., 2023], [De Santis et al., 2025b].

Integrate differential decay width

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$$\bar{X}(\boldsymbol{q}) = \int_{\omega_0}^{\infty} d\omega\, W_{\mu\nu}(\boldsymbol{q},\omega) k^{\mu\nu}(\boldsymbol{q},\omega)\theta(\omega_{\max} - \omega)$$

$$\bar{X}(\boldsymbol{q},\sigma) = \int_{\omega_0}^{\infty} d\omega\, W_{\mu\nu}(\boldsymbol{q},\omega)\, k^{\mu\nu}(\boldsymbol{q},\omega)\theta_{\boldsymbol{\sigma}}(\omega_{\max} - \omega)$$

$$\bar{X}(\boldsymbol{q},\sigma,N) \approx \sum_{t}^{N} \tilde{c}_{\mu\nu,\,t}(\boldsymbol{q},\sigma) \underbrace{\int_0^{\infty} d\omega\, W^{\mu\nu}(\boldsymbol{q},\omega) \boxed{T_t(e^{-\omega})}}_{\sim \boxed{\langle T_t(\boldsymbol{q})\rangle_{\mu\nu}} = \sum_j^t b_t^{(j)} C_{\mu\nu}(j)/C_{\mu\nu}(0)}$$

(1)

Spectral reconstruction using Chebyshevs

Need to then take appropriate limits

$$\bar{X}(\boldsymbol{q}) = \lim_{\sigma\to 0} \lim_{N\to\infty} \left(\lim_{V\to\infty} \right) \bar{X}(\boldsymbol{q},\sigma,N)$$

(2)

Technical simulation details

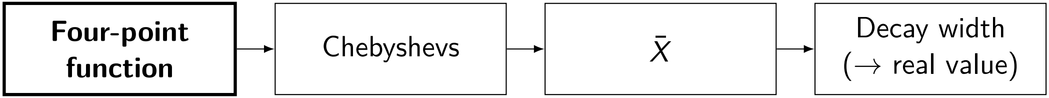
Simulations were carried out on the **DiRAC** Extreme Scaling service at the University of Edinburgh using the Grid [Boyle et al., https://github.com/paboyle/Grid] and Hadrons [Portelli et al., https://github.com/aportelli/Hadrons] software packages. We use RBC/UKQCD ensembles [Allton et al., 2008].

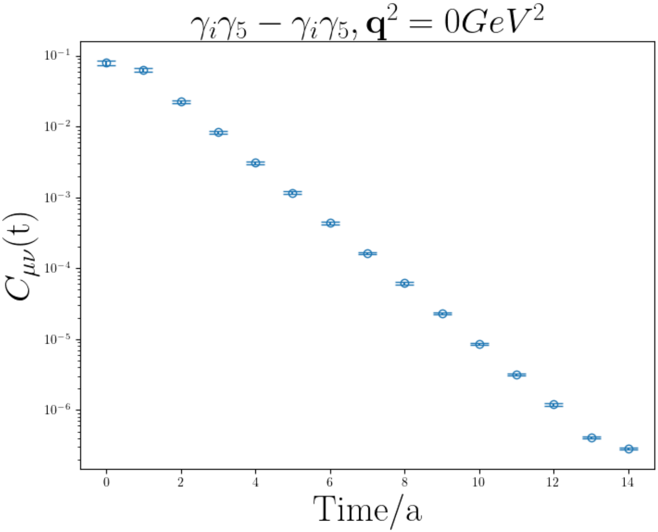
Parameters:

name	a/fm	$L^3 \times T$	M_π/MeV
C1M	0.11	$24^3 \times 64$	276
M1M	0.08	$32^3 \times 64$	286
F1M	0.07	$48^3 \times 96$	232

Actions:

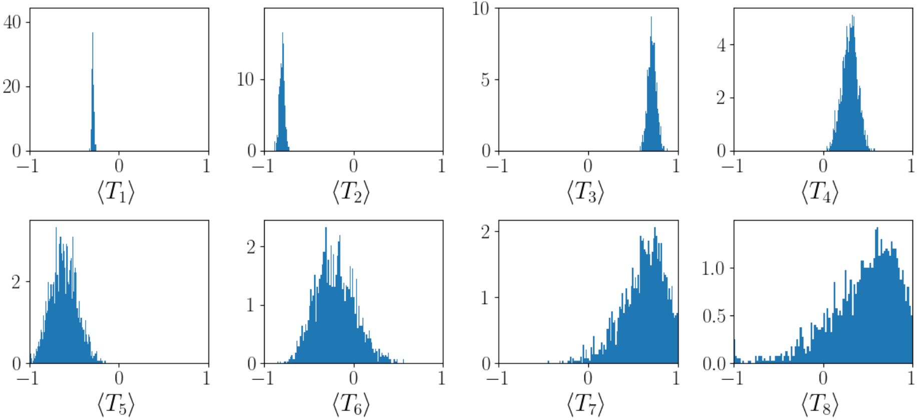
- b quark simulated at its physical mass (**RHQ action**)
  [El-Khadra et al., 1997],[Christ et al., 2007] ,[Lin and Christ, 2007]
- s, c quarks simulated at near-to-physical mass (**DWF action**)
  [Brower et al., 2017]


Four-point function
Chebyshevs
$\bar{X}$
Decay width
($\rightarrow$ real value)


$\gamma_i\gamma_5 - \gamma_i\gamma_5, \mathbf{q}^2 = 0GeV^2$
$C_{\mu\nu}(t)$
Time/a
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
0
2
4
6
8
10
12
14

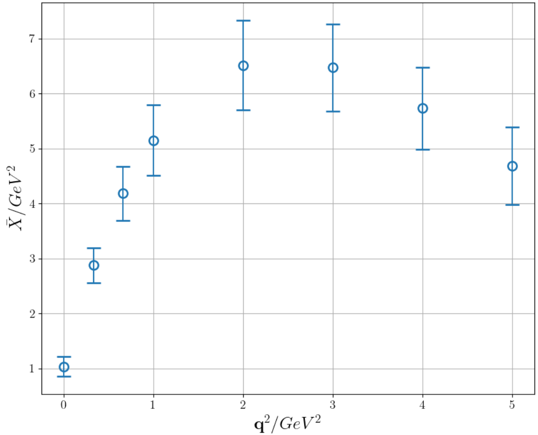
Four-point function → **Chebyshevs** → $\bar{X}$ → Decay width ($\to$ real value)

$$\langle T_t(\boldsymbol{q})\rangle_{\mu\nu} = \sum_j^t b_t^{(j)} C_{\mu\nu}(j)/C_{\mu\nu}(0)$$

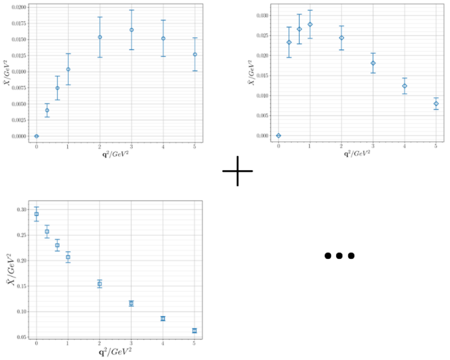


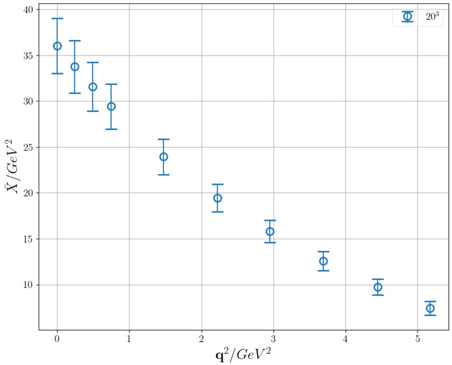
Four-point function → Chebyshevs → **$\bar{X}$** → Decay width ($\rightarrow$ real value)

$$\bar{X}(\boldsymbol{q}^2, \sigma, N) = \sum_{t}^{N} \tilde{c}_{\mu\nu,\, t}(\boldsymbol{q}, \sigma) \left\langle T_t(\boldsymbol{q}) \right\rangle_{\mu\nu}$$

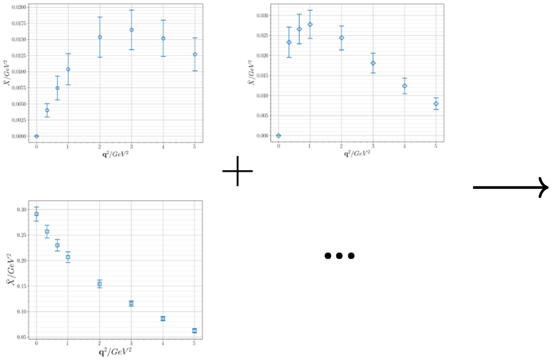


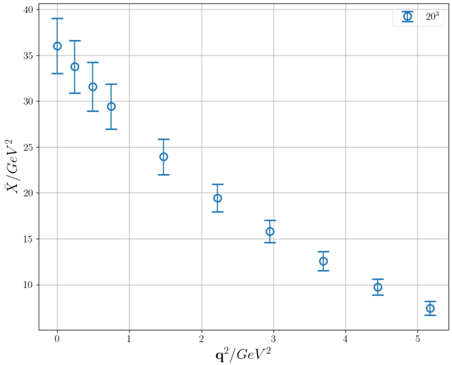
Four-point function → Chebyshevs → $\bar{X}$ → **Decay width ($\rightarrow$ real value)**





Four-point function → Chebyshevs → $\bar{X}$ → **Decay width ($\rightarrow$ real value)**





Need to extrapolate to continuum

Extrapolations

1. Infinite volume extrapolation (study)
2. Continuum extrapolation (+ chiral/heavy quark etc...)
3. Smearing limit*

Smearing of $W_{\mu\nu}$

Introducing the Sigmoid is effectively convolving with smearing kernel.

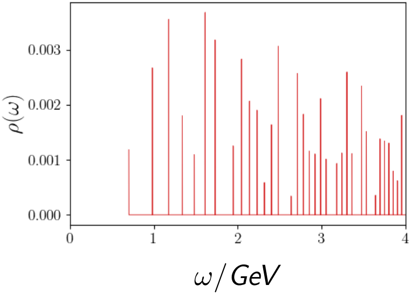
$$\int_0^{\infty} d\omega\, W_{\mu\nu}(\boldsymbol{q},\omega) k^{\mu\nu}(\boldsymbol{q},\omega)\theta_\sigma(\omega_{\text{max}}-\omega) \to \int_{-\infty}^{\infty} dy\, \theta(\omega_{\text{max}}-y)\, \left(W^{\mu\nu}k_{\mu\nu} * \delta'_\sigma\right)(y)$$

Model $W_{\mu\nu}(\omega)$ as two-body non-interacting spectral density, $\rho(\omega)$.

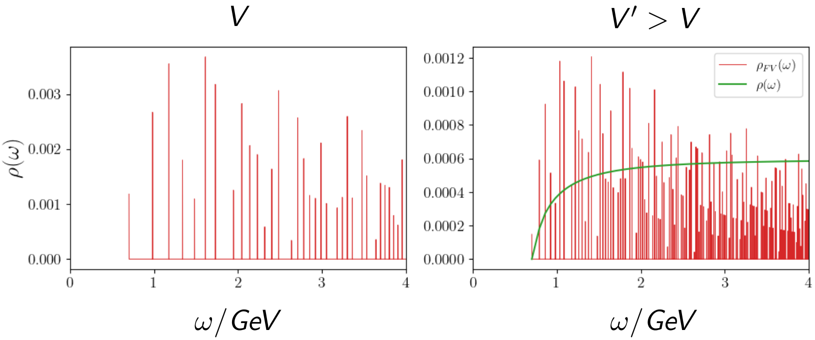
$$\rho(\omega) = \pi \int \frac{d^3q}{(2\pi)^3}\frac{1}{4\epsilon_D\epsilon_K}\delta(\omega-\epsilon_K+\epsilon_D)$$

$$\rho_V(\omega) = \frac{\pi}{V}\sum_{\boldsymbol{q}} \frac{1}{4\sqrt{\boldsymbol{q}^2+m_K^2}\sqrt{\boldsymbol{q}^2+m_D^2}}\delta(\omega-\sqrt{\boldsymbol{q}^2+m_K^2}-\sqrt{\boldsymbol{q}^2+m_D^2})$$

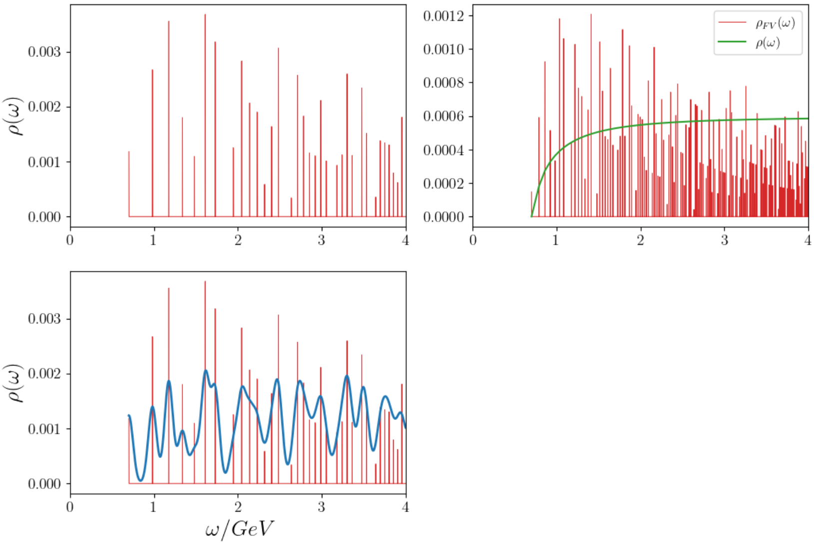
Smearing of $W_{\mu\nu}$



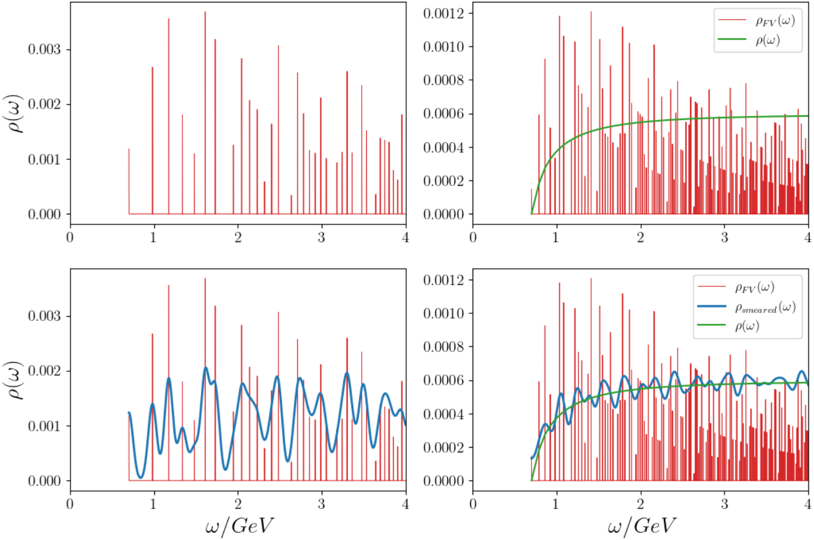
Smearing of $W_{\mu\nu}$



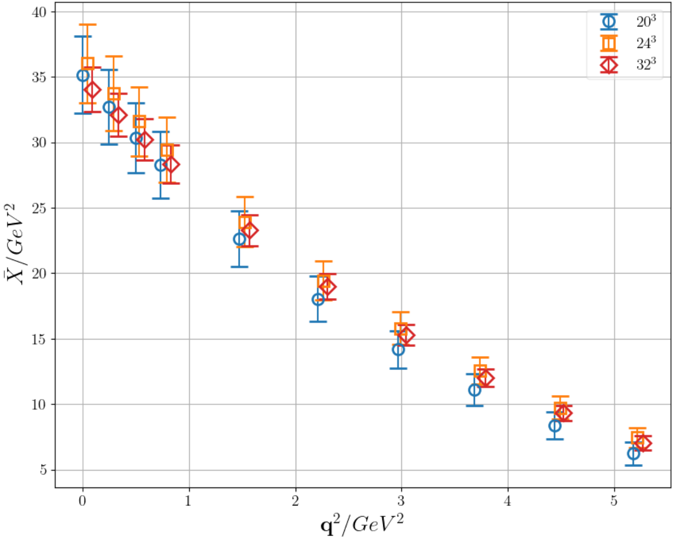
Smearing of $W_{\mu\nu}$



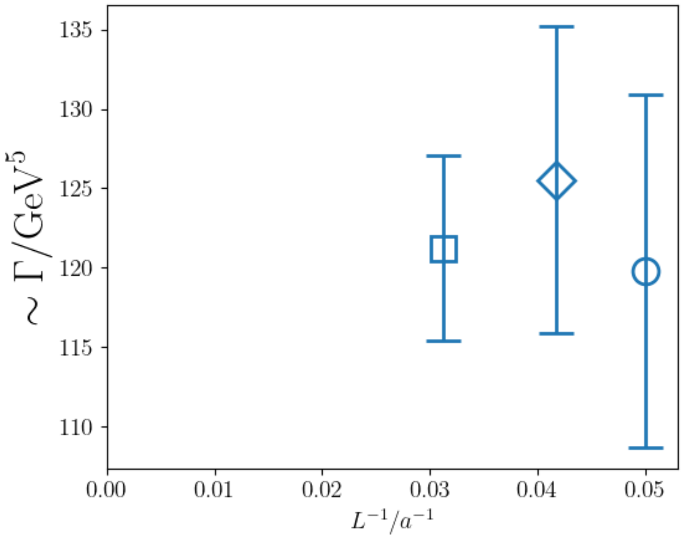
Smearing of $W_{\mu\nu}$



(1) Finite volume effects

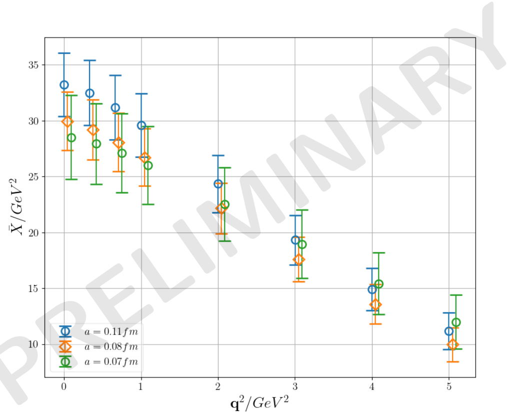


(1) Finite volume effects



(2) Continuum extrapolation - global fits

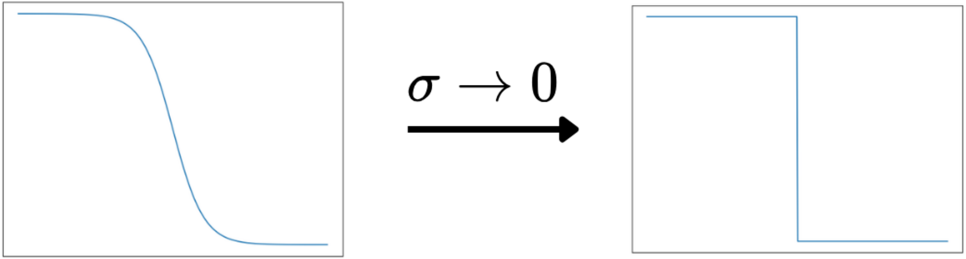
$$\text{fit} = \left(\sum_{i=0}^{N_p} c_i(\boldsymbol{q}^2)^i\right)\left(1 + c_{M_c}\left(\frac{M_{D_s}^{\text{phy}} - M_{D_s}}{\Lambda_c}\right)\right)\left(1 + c_a\,(a\Lambda_c)^2\right)\left(1 + c_{a\boldsymbol{q}}(a\boldsymbol{q})^2\right)$$



- Finest ensemble data still in production.
- Symanzik improvement not yet applied:

$$\mathcal{O}(a) \to \mathcal{O}(a^2)$$

Smearing limit



(3) Error of $\bar{X}$ with σ/N

[Kellermann et al., 2025, Barone et al., 2026]

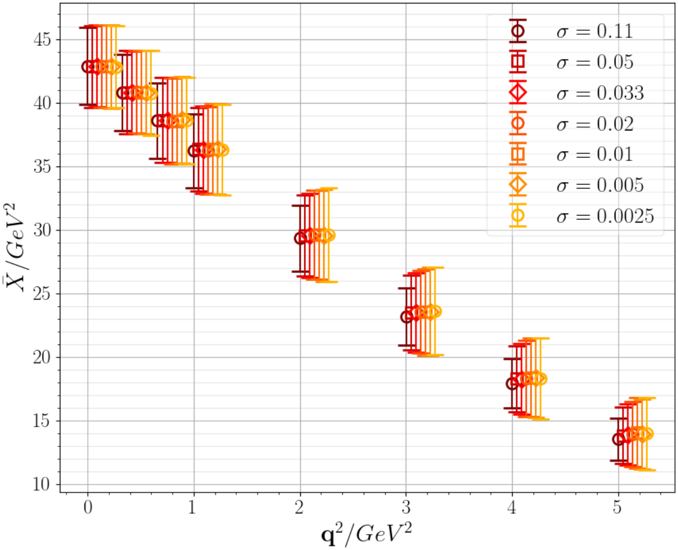
$$k^{\mu\nu}(\boldsymbol{q},\omega)\theta(\omega_{\text{max}}-\omega) \;\rightarrow\; k^{\mu\nu}(\boldsymbol{q},\omega)\theta_{\boldsymbol{\sigma}}(\omega_{\text{max}}-\omega) \;\rightarrow\; \sum_{t=0}^{N}\tilde{c}_{\mu\nu,\,t}(\boldsymbol{q},\sigma)\,T_t(e^{-\omega})$$

- Order of expansion = time separation of currents
- Maximum order is limited by the data.

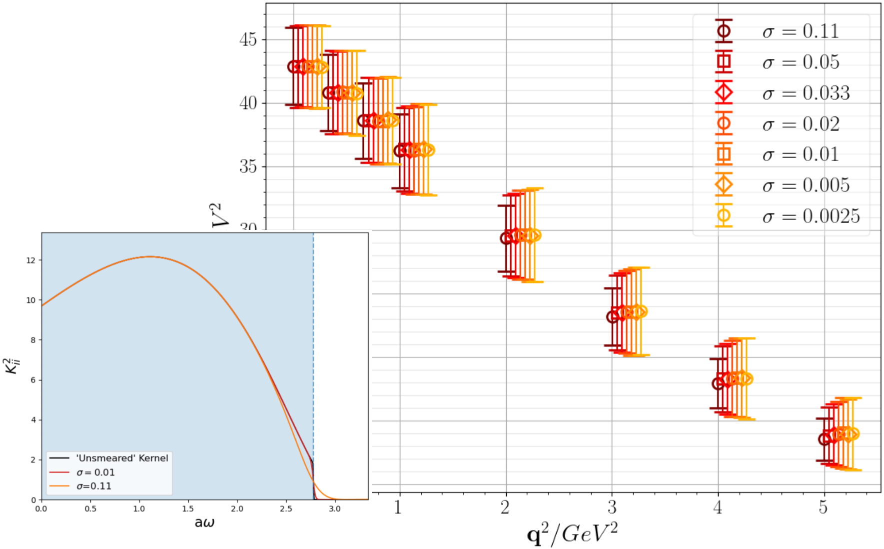
$$\bar{X} = \underbrace{\sum_{t}^{t_{\text{max}}}\tilde{c}_{\mu\nu,\,t}(\sigma)\langle T_t\rangle_{\mu\nu}}_{\text{Signal}} + \sum_{t_{\text{max}}+1}^{N}\tilde{c}_{\mu\nu,\,t}(\sigma)\cdot\text{Noise}$$

- Noise is bounded as $|\langle T_t\rangle_{\mu\nu}| \leq 1$.
- Random sample values for $\langle T_{k>t_{\text{max}}}\rangle$ from $\{-1,1\}$.
- Vary σ to see dependence of $\delta\bar{X}$.

(3) Smearing limit



(3) Smearing limit



Summary

- Explored this alternate calculation of inclusive decays (results for $D_{(s)}$ - [De Santis et al., 2025a, De Santis et al., 2025b, Kellermann et al., 2025, Kellermann et al., 2026])

Future Work

- Full continuum limit with systematics - Symanzik improvement, data production.
- Full phenomenological prediction with quantified errors (very close).

THANK YOU!

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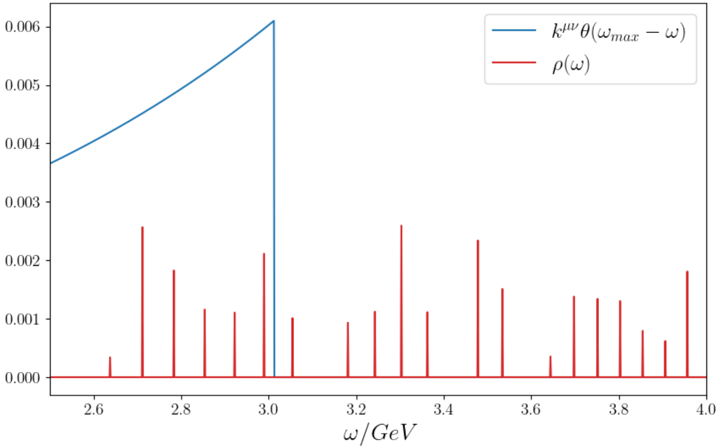
How does $\bar{X}$ behave?

$$\bar{X}(\boldsymbol{q}) = \int_{\omega_0}^{\infty} d\omega\, W_{\mu\nu}(\boldsymbol{q}, \omega) k^{\mu\nu}(\boldsymbol{q}, \omega)\theta(\omega_{\text{max}} - \omega)$$



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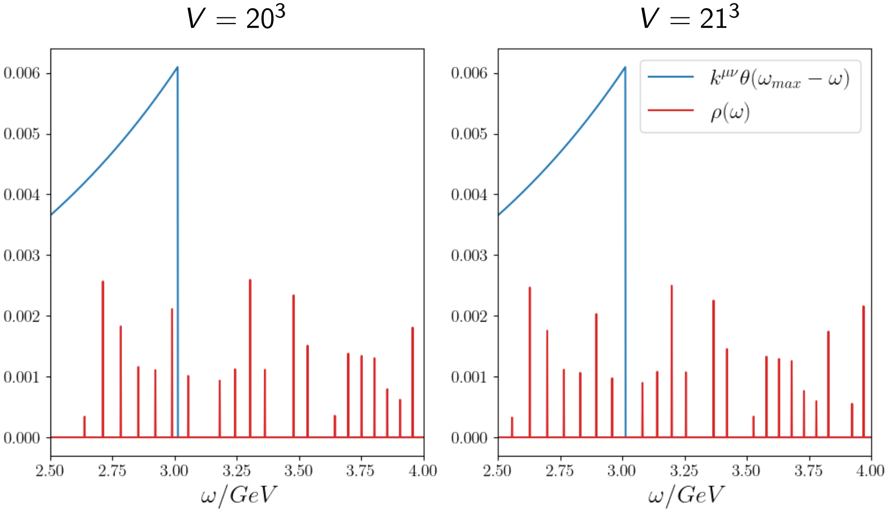
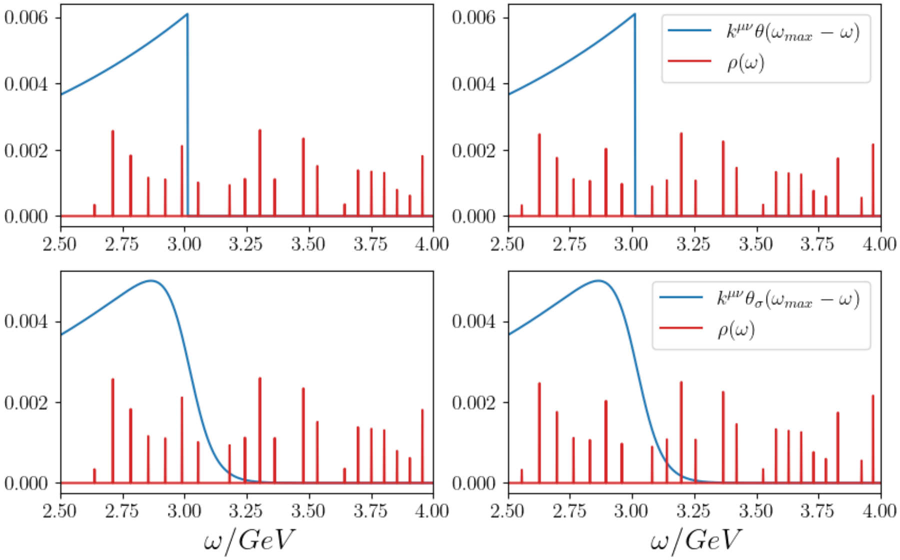


(2) $V \to \infty$ and smearing



Relation between $C_{\mu\nu}$ and $W_{\mu\nu}$?

$$\frac{\left\langle O^S_{B_s}(t_{\text{snk}})\, \tilde{J}^\dagger_\mu(\boldsymbol{q}, t_2)\, \tilde{J}_\nu(\boldsymbol{q}, t_1)\, O^{S\dagger}_{B_s}(t_{\text{src}}) \right\rangle}{\left\langle O^S_{B_s}(t_{\text{snk}}) O^{S\dagger}_{B_s}(t_2) \right\rangle \left\langle O^S_{B_s}(t_2) O^{S\dagger}_{B_s}(t_{\text{src}}) \right\rangle} = \frac{A_{LL}}{2M_{B_s}} \langle B_s \mid \tilde{J}^\dagger_\mu(\boldsymbol{q}, t_2)\, \tilde{J}_\nu(\boldsymbol{q}, t_1) \mid B_s \rangle$$

where we have

$$C_{\mu\nu}(\boldsymbol{q}, t) = \frac{1}{2M_{B_s}} \langle B_s \mid \tilde{J}^\dagger_\mu(\boldsymbol{q}, 0)\, e^{-\hat{H}t}\, \tilde{J}_\nu(\boldsymbol{q}, 0) \mid B_s \rangle$$

This object then relates to $W_{\mu\nu}$ via the Laplace transform.

$$\begin{aligned} C_{\mu\nu}(\boldsymbol{q}, t) &= \int_0^\infty d\omega\, \frac{1}{2M_{B_s}} \langle B_s \mid \tilde{J}^\dagger_\mu(\boldsymbol{q}, 0)\, \delta(\hat{H} - \omega)\, \tilde{J}_\nu(\boldsymbol{q}, 0) \mid B_s \rangle\, e^{-\omega t} \\ &= \int_0^\infty d\omega\, W_{\mu\nu}(\boldsymbol{q}, \omega)\, e^{-\omega t} \end{aligned}$$