Quantum Computing of Lattice Gauge Theories is simpler with Non-Compact Variable![†]

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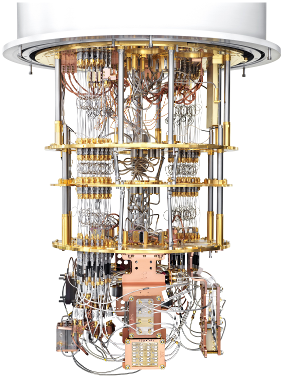
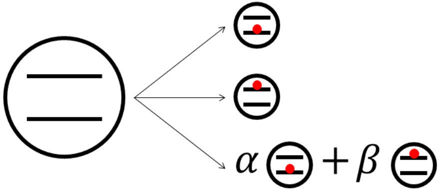
Nordic Lattice Meeting 2026, May 19th, Higgs Centre, University of Edinburgh

[†][arXiv:2604.15132] in collaboration with: Georg Bergner and Masanori Hanada

Quantum Hardware and Qubits



- Quantum superposition
- Entanglement



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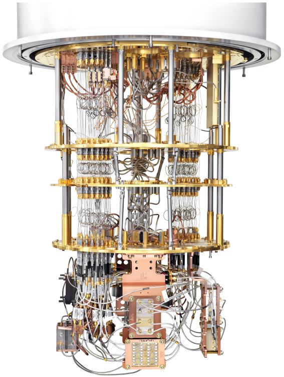

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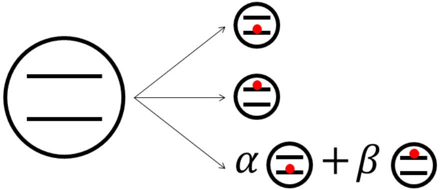
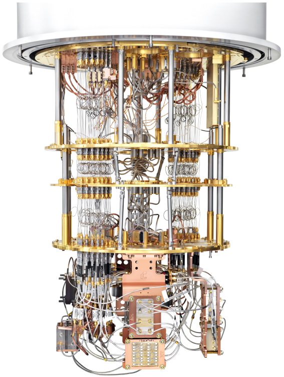


- Thanks to qubits Hamiltonian formalism becomes computationally feasible!

Quantum Hardware and Qubits



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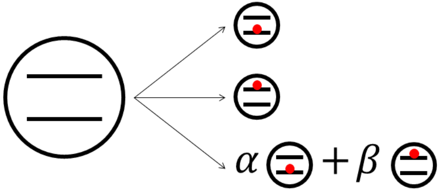
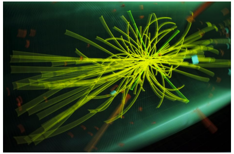


- Thanks to qubits Hamiltonian formalism becomes computationally feasible!
- N spin-1/2 $\rightarrow 2^N$ states $\Rightarrow$ Classical computer 2^N memory slots; **Quantum computer N qubits**

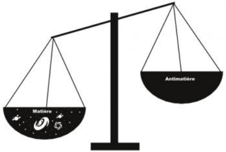
Hamiltonian formulation $\Longrightarrow$ No Sign problem

Real-time evolution

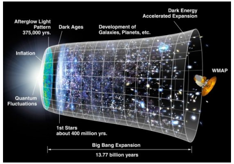
Non-zero chemical potential



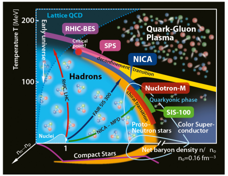
propagations/collisions



Imbalance matter-antimatter

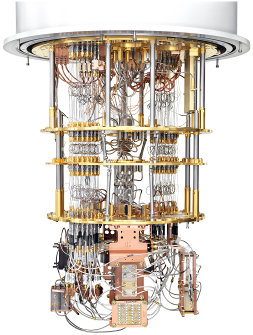


Early Universe



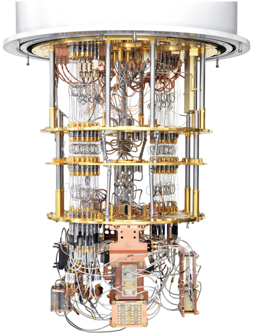
QCD phase diagram

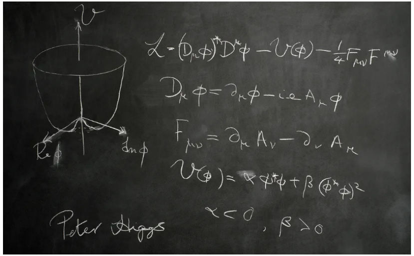
Challenges in using quantum computers: Hardware & Formalism



- Few qubits
- Low qubit connectivity
- Noisy gates
- No error correction $\Rightarrow$ **Error mitigation**!

Challenges in using quantum computers: Hardware & Formalism





- Few qubits
- Low qubit connectivity
- Noisy gates
- No error correction $\Rightarrow$ **Error mitigation**!

- Hamiltonian formulation
- Representation of the Hamiltonian
- Scalability of the continuum limit
- Scalability of the thermodynamic limit

Encoding challenges

The seminal work of Byrnes and Yamamoto

PHYSICAL REVIEW A **73**, 022328 (2006)

Simulating lattice gauge theories on a quantum computer

Tim Byrnes*
National Institute of Informatics, 2-1-2 Hitotsubashi, Chiyoda-ku, Tokyo 101-8430, Japan

Yoshihisa Yamamoto
E. L. Ginzton Laboratory, Stanford University, Stanford, California 94305, USA and National Institute of Informatics, 2-1-2 Hitotsubashi, Chiyoda-ku, Tokyo 101-8430, Japan
(Received 4 October 2005; published 17 February 2006)

We examine the problem of simulating lattice gauge theories on a universal quantum computer. The basic strategy of our approach is to transcribe lattice gauge theories in the Hamiltonian formulation into a Hamiltonian involving only Pauli spin operators such that the simulation can be performed on a quantum computer using only one- and two-qubit manipulations. We examine three models, the U(1), SU(2), and SU(3) lattice gauge theories, which are transcribed into a spin Hamiltonian up to a cutoff in the Hilbert space of the gauge fields on the lattice. The number of qubits required for storing a particular state is found to have a linear dependence on the total number of lattice sites. The number of qubit operations required for performing the time evolution corresponding to the Hamiltonian is found to be between a linear to quadratic function of the number of lattice sites, depending on the arrangement of qubits in the quantum computer. We remark that our results may also be easily generalized to higher SU(N) gauge theories.

[arXiv:0510027]

- "Hamiltonian formulation of Wilson's lattice gauge theories" by Kogut and Susskind [PhysRevD.11.395]
- Building blocks for Hamiltonian construction in U(1), SU(2) and SU(3)
- Guidelines for future quantum simulations on qubits

The nightmare... of using compact variables

$$\Box_1 |\psi_{\text{initial}}\rangle = \sum_{M_E}\sum_{M'_E}\sum_{M_J}\sum_{M'_J}\sum_{M_F}\sum_{M'_F}\sum_{M_I}\sum_{M'_I}\sum_{m_A}\sum_{m'_A}\sum_{m_B}\sum_{m'_B}\sum_{m_C}\sum_{m'_C}\cdots\sum_{m_L}\sum_{m'_L}\sum_{J_F}\sum_{J_E}\sum_{J_I}\sum_{J_J}\sum_{s_1}\sum_{s_2}\sum_{s_6}\sum_{s_5}$$
$$(-1)^{s_1+s_2+s_6+s_5}(-1)^{-2j_E-2j_J-2j_F-2j_I+M_E+M'_E+M_J+M'_J+M_F+M'_F+M_I+M'_I}$$
$$\sqrt{2j_E+1}\sqrt{2J_E+1}\sqrt{2j_J+1}\sqrt{2J_J+1}\sqrt{2j_F+1}\sqrt{2J_F+1}\sqrt{2j_I+1}\sqrt{2J_I+1}$$
$$\begin{pmatrix} j_A & j_E & j_I \\ m'_A & m_E & m_I \end{pmatrix}\begin{pmatrix} j_C & j_E & j_J \\ m_C & m'_E & m_J \end{pmatrix}\begin{pmatrix} j_C & j_G & j_K \\ m'_C & m_G & m'_K \end{pmatrix}\begin{pmatrix} j_A & j_G & j_L \\ m_A & m'_G & m_L \end{pmatrix}$$
$$\begin{pmatrix} j_B & j_F & j_I \\ m'_B & m_F & m'_I \end{pmatrix}\begin{pmatrix} j_D & j_F & j_J \\ m_D & m'_F & m'_J \end{pmatrix}\begin{pmatrix} j_D & j_H & j_K \\ m'_D & m_H & m'_K \end{pmatrix}\begin{pmatrix} j_B & j_H & j_L \\ m_B & m'_H & m'_L \end{pmatrix}$$
$$\begin{pmatrix} j_E & \frac{1}{2} & J_E \\ m_E & -s_1 & -M_E \end{pmatrix}\begin{pmatrix} j_E & \frac{1}{2} & J_E \\ m'_E & s_2 & -M'_E \end{pmatrix}\begin{pmatrix} j_J & \frac{1}{2} & J_J \\ m_J & -s_2 & -M_J \end{pmatrix}\begin{pmatrix} j_J & \frac{1}{2} & J_J \\ m'_J & s_6 & -M'_J \end{pmatrix}$$
$$\begin{pmatrix} j_F & \frac{1}{2} & J_F \\ m_F & s_5 & -M_F \end{pmatrix}\begin{pmatrix} j_F & \frac{1}{2} & J_F \\ m'_F & -s_6 & -M'_F \end{pmatrix}\begin{pmatrix} j_I & \frac{1}{2} & J_I \\ m_I & s_1 & -M_I \end{pmatrix}\begin{pmatrix} j_I & \frac{1}{2} & J_I \\ m'_I & -s_5 & -M'_I \end{pmatrix}$$
$$|j_A, m_A, m'_A\rangle |j_B, m_B, m'_B\rangle |j_C, m_C, m'_C\rangle |j_D, m_D, m'_D\rangle |J_E, M_E, M'_E\rangle |J_F, M_F, M'_F\rangle$$
$$|j_G, m_G, m'_G\rangle |j_H, m_H, m'_H\rangle |J_I, M_I, M'_I\rangle |J_J, M_J, M'_J\rangle |j_K, m_K, m'_K\rangle |j_L, m_L, m'_L\rangle\ . \tag{A15}$$

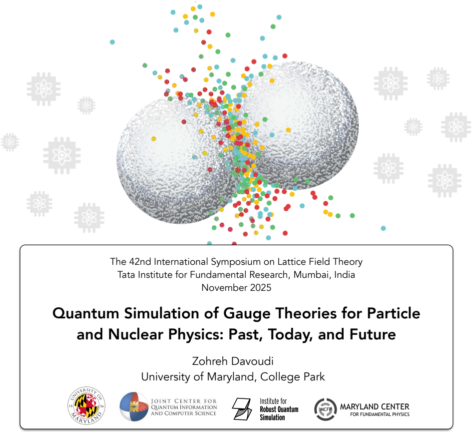
Applying a final state to that result allows all sums to be performed and the answer simplifies to

$$\langle\psi_{\text{final}}|\,\Box_1\,|\psi_{\text{initial}}\rangle = (-1)^{j_A+j_B+j_C+j_D+2J_E+2J_F+2j_I+2j_J}$$
$$\sqrt{2j_E+1}\sqrt{2J_E+1}\sqrt{2j_J+1}\sqrt{2J_J+1}\sqrt{2j_F+1}\sqrt{2J_F+1}\sqrt{2j_I+1}\sqrt{2J_I+1}$$
$$\begin{Bmatrix} j_A & j_E & j_I \\ \frac{1}{2} & J_I & J_E \end{Bmatrix}\begin{Bmatrix} j_B & j_F & j_I \\ \frac{1}{2} & J_I & J_F \end{Bmatrix}\begin{Bmatrix} j_C & j_E & j_J \\ \frac{1}{2} & J_J & J_E \end{Bmatrix}\begin{Bmatrix} j_D & j_F & j_J \\ \frac{1}{2} & J_J & J_F \end{Bmatrix} \tag{A16}$$

1-plaquette SU(2) [arXiv:2205.09247]

Progress of Quantum Computing for LGT

- For a review of the progress in **Quantum Simulation** and **Hamiltonian Formulations** of LGT:



[Lattice 2025 plenary by Zohreh Davoudi]

[Lattice 2025 plenary by Indrakshi Raychowdhury]

The encoding challenge

$$\hat{H} = \frac{g^2}{2}\left(\sum_{i=\text{links}} \hat{E}_i^2 - 2x \sum_{i=\text{plaquettes}} \hat{\square}_i\right)$$

Gate decomposition

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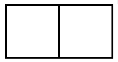
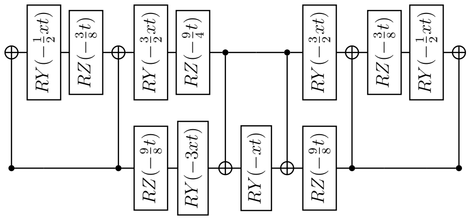


$$H = \frac{3}{8}(7 - 3Z_0 - Z_0 Z_1 - 3Z_1) - \frac{x}{2}(3 + Z_1)X_0 - \frac{x}{2}(3 + Z_0)X_1$$

$$\Downarrow$$

Real-time evolution operator circuit:



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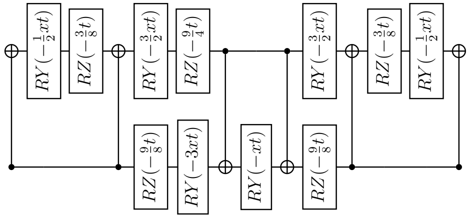

- Spin-like systems $H \longrightarrow$ quantum circuit (Ok)

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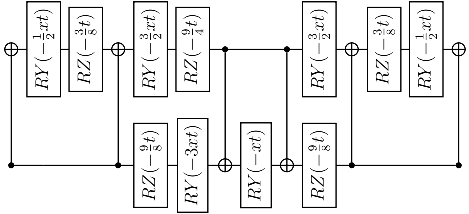

- Spin-like systems $H \longrightarrow$ quantum circuit (Ok)
- No spin-like systems:
  1. Matrix representation (**scales exponentially**)
  2. Convert matrix to quantum gates, Pauli decomposition (worst-case $\mathcal{O}(4^Q)$ [arXiv:2310.13421])

The encoding of Kogut-Susskind Hamiltonian

arXiv > quant-ph > arXiv:2505.02553

Quantum Physics

[Submitted on 5 May 2025 (v1), last revised 9 Oct 2025 (this version, v3)]

Exponential improvement in quantum simulations of bosons

Masanori Hanada, Shunji Matsuura, Emanuele Mendicelli, Enrico Rinaldi

Hamiltonian quantum simulation of bosons on digital quantum computers requires truncating the Hilbert space to finite dimensions. The method of truncation and the choice of basis states can significantly impact the complexity of the quantum circuit required to simulate the system. For example, a truncation in the Fock basis where each boson is encoded with a register of Q qubits, can result in an exponentially large number of Pauli strings required to decompose the truncated Hamiltonian. This, in turn, can lead to an exponential increase in Q in the complexity of the quantum circuit. For lattice quantum field theories such as Yang-Mills theory and QCD, several Hamiltonian formulations and corresponding truncations have been put forward in recent years. There is no exponential increase in Q when resorting to the orbifold lattice Hamiltonian, while we do not know how to remove the exponential complexity in Q in the commonly used Kogut-Susskind Hamiltonian. Specifically, when using the orbifold lattice Hamiltonian, the continuum limit, or, in other words, the removal of the ultraviolet energy cutoff, is obtained with circuits whose resources scale like Q, while they scale like $\mathcal{O}(\exp(Q))$ for the Kogut-Susskind Hamiltonian: this can be seen as an exponential speed up in approaching the physical continuum limit for the orbifold lattice Hamiltonian formulation. We show that the universal framework, advocated by three of the authors (M.~H., S.~M., and E.~R.) and collaborators, provides a natural avenue to solve the exponential scaling of circuit complexity with Q, and it is the reason why using the orbifold lattice Hamiltonian is advantageous.

Whether an efficient encoding is achievable remains an open question!

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Whether an efficient encoding is achievable remains an open question!

- It is difficult to write quantum circuits explicitly
- Exponential classical preprocessing is often required
- No systematic analysis of resource scaling for quantum simulations

A scalable formalism? maybe the Orbifold lattice

Why do I like the Orbifold Lattice for QCD and Yang-Mills ?

- The Hamiltonian is given explicitly for any gauge group, lattice size and truncation level
- Quantum circuit can be written explicitly
- Gate count grows polynomially with qubits
- The Kogut-Susskind Hamiltonian is obtained as Orbifold large-mass limit
- **It requires a substantial number of bosons!**

Orbifold Lattice for Quantum Computing: A Timeline

- **2002** D. B. Kaplan, E. Katz and M. Unsal [arXiv:hep-lat/0206019]
  (Orbifold projection of BFSS matrix model to supersymmetric Yang-Mills for exact Supersymmetry on the lattice)

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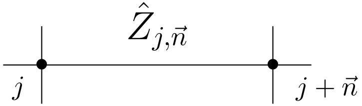
- **2002** D. B. Kaplan, E. Katz and M. Unsal [arXiv:hep-lat/0206019]
  (Orbifold projection of BFSS matrix model to supersymmetric Yang-Mills for exact Supersymmetry on the lattice)
- **2020** A. J. Buser, H. Gharibyan, M. Hanada, M. Honda and J. Liu [arXiv:2011.06576]
  (Orbifold for quantum simulations of U(k))
- **2024** G. Bergner, M. Hanada, E. Rinaldi, A. Schafer [arXiv:2401.12045]
  (Orbifold lattice for quantum simulations of QCD)
- **2024** J. C. Halimeh, M. Hanada, S. Matsuura, F. Nori, E. Rinaldi and A. Schäfer [arXiv:2411.13161]
  (Universal framework for quantum simulations of SU(N) Yang-Mills)
- **2025** G. Bergner, M. Hanada and E. Mendicelli [arXiv:2506.00755]
  (Lattice simulations (2+1)d: Kogut-Susskind as Orbifold limit)
- **2025** J. C. Halimeh, M. Hanada and S. Matsuura [arXiv:2506.18966]
  (Universal framework with fermions)
- **2025** M. Hanada, S. Matsuura, A. Schafer, J. Sun [arXiv:2512.22932]
  (Gauge symmetry by Orbifold)
- **2026** G. Bergner, M. Hanada and E. Mendicelli [arXiv:2604.15132]
  (Kogut-Susskind limit with smaller m^2 for Orbifold and Orbifoldish)

Orbifold lattice approach for SU(N)

Compact Variables $\longrightarrow$ Cartesian Coordinate:

- $\mathrm{SU}(N) \subset \mathbb{C}^{N^2} \cong \mathbb{R}^{2N^2}$



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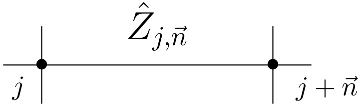

$$Z_{j,\vec{n}} = \sqrt{\frac{a^{d-2}}{2g_d^2}}\, W_{j,\vec{n}}\, U_{j,\vec{n}}$$

- $W_{j,\vec{n}} = \exp\left(a g_d \phi_{j,\vec{n}}\right)$ positive-definite Hermitian $\longrightarrow$ ϕ_j adjoint scalar field
- $U_{j,\vec{n}} = \exp(\mathrm{i} a g_d A_{j,\vec{n}})$, unitary link variable $\longrightarrow$ A_j gauge field
  (Note: $\det(U)$ is not fixed to 1)

Toward a Yang-Mills theory coupled to scalars

The $SU(N)$ Orbifold Hamiltonian

$$\hat{H} = \sum_{\vec{n}} \mathrm{Tr}\left(\sum_{j=1}^{d} \hat{P}_{j,\vec{n}} \hat{\bar{P}}_{j,\vec{n}} + \frac{g_{\mathrm{d}}^2}{2a^d} \left| \sum_{j=1}^{d} \left(\hat{Z}_{j,\vec{n}} \hat{\bar{Z}}_{j,\vec{n}} - \hat{\bar{Z}}_{j,\vec{n}-\hat{j}} \hat{Z}_{j,\vec{n}-\hat{j}} \right) \right|^2 + \frac{2g_{\mathrm{d}}^2}{a^d} \sum_{j<k} \left| \hat{Z}_{j,\vec{n}} \hat{Z}_{k,\vec{n}+\hat{j}} - \hat{Z}_{k,\vec{n}} \hat{Z}_{j,\vec{n}+\hat{k}} \right|^2 \right) + \Delta\hat{H}$$

where

$$\Delta\hat{H} = \frac{m^2 g_d^2}{2a^{d-2}} \sum_{\vec{n}} \sum_{j=1}^{d} \mathrm{Tr} \left| \hat{Z}_{j,\vec{n}} \hat{\bar{Z}}_{j,\vec{n}} - \frac{a^{d-2}}{2g_d^2} \right|^2 + \frac{m_{\mathrm{U}(1)}^2 a^{d-2}}{2g_d^2} \sum_{\vec{n}} \sum_{j=1}^{d} \left| \left(\frac{a^{d-2}}{2g_d^2} \right)^{-N/2} \det(\hat{Z}_{j,\vec{n}}) - 1 \right|^2$$

- first term forces $W \to \mathbf{1}_N$
- second term forces $\det(U) \to 1$

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$$\hat{H} = \sum_{\vec{n}} \mathrm{Tr}\left(\sum_{j=1}^{d} \hat{P}_{j,\vec{n}} \hat{\bar{P}}_{j,\vec{n}} + \frac{g_{\mathrm{d}}^2}{2a^d} \left| \sum_{j=1}^{d} \left(\hat{Z}_{j,\vec{n}} \hat{\bar{Z}}_{j,\vec{n}} - \hat{\bar{Z}}_{j,\vec{n}-\hat{j}} \hat{Z}_{j,\vec{n}-\hat{j}} \right) \right|^2 + \frac{2g_{\mathrm{d}}^2}{a^d} \sum_{j<k} \left| \hat{Z}_{j,\vec{n}} \hat{Z}_{k,\vec{n}+\hat{j}} - \hat{Z}_{k,\vec{n}} \hat{Z}_{j,\vec{n}+\hat{k}} \right|^2 \right) + \Delta \hat{H}$$

where

$$\Delta \hat{H} = \frac{m^2 g_d^2}{2a^{d-2}} \sum_{\vec{n}} \sum_{j=1}^{d} \mathrm{Tr} \left| \hat{Z}_{j,\vec{n}} \hat{\bar{Z}}_{j,\vec{n}} - \frac{a^{d-2}}{2g_d^2} \right|^2 + \frac{m_{\mathrm{U}(1)}^2 a^{d-2}}{2g_d^2} \sum_{\vec{n}} \sum_{j=1}^{d} \left| \left(\frac{a^{d-2}}{2g_d^2} \right)^{-N/2} \det(\hat{Z}_{j,\vec{n}}) - 1 \right|^2$$

- first term forces $W \to \mathbf{1}_N$
- second term forces $\det(U) \to 1$

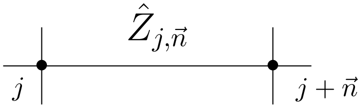
The Kogut-Susskind limit: m^2 and $m_{\mathrm{U}(1)}^2 \to \infty$:

- $(W = \mathbf{1}_N)$, scalar fields ϕ_j with large mass decouple
- $\det(U) = 1$

$$\Longrightarrow \quad Z \equiv U \longrightarrow \in SU(N)$$

Pure Yang-Mills is recovered!

Quantum computing for $SU(N)$ Orbifold lattice



- Z and P $\in \mathbb{C}^{N^2} \to \mathbb{R}^{2N^2} \longrightarrow 2N^2$ components $(x_1, \ldots x_{2N^2}), (p_1, \ldots p_{2N^2}) \longrightarrow 2N^2$ bosons
- Q qubits per each boson, truncation level $\Lambda = 2^Q$

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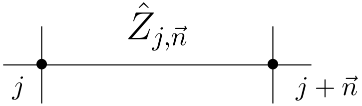

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Λ point on 2R circle $\delta_x = 2R/\Lambda$

Λ point on 2R circle $\delta_p = \pi/R$

$$\hat{x}_a = -\frac{\delta_x}{2} \cdot \left(\hat{\sigma}_{z;a,1} + 2 \cdot \hat{\sigma}_{z;a,2} + \cdots + 2^{Q-1} \cdot \hat{\sigma}_{z;a,Q}\right)$$

$$\hat{p}_a = -\frac{\delta_p}{2} \cdot \left(\hat{\sigma}_{z;a,1} + 2 \cdot \hat{\sigma}_{z;a,2} + \cdots + 2^{Q-1} \cdot \hat{\sigma}_{z;a,Q}\right)$$

- $\hat{x}_a$ basis $\Leftrightarrow$ Quantum Fourier Transform $\Leftrightarrow$ $\hat{p}_a$

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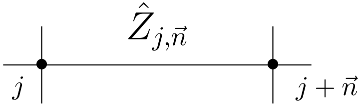

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- $\hat{x}_a$ basis $\Leftrightarrow$ Quantum Fourier Transform $\Leftrightarrow$ $\hat{p}_a$
- $\#Q_{\text{tot}} = d\, N_{\text{site}}\ 2N^2\ Q_{\text{boson}}$
- $\#Gates \propto N_{\text{site}}\ Q^4_{\text{boson}} N^4$

Computationally cheaper SU(N) Hamiltonians

Two cheaper SU(N) Orbifoldish Hamiltonians

$$\hat{H} = \sum_{\vec{n}} \mathrm{Tr}\left(\sum_{j=1}^{d} \hat{P}_{j,\vec{n}} \hat{\bar{P}}_{j,\vec{n}} + \frac{g_{\mathrm{d}}^2}{2a^d} \left| \sum_{j=1}^{d} \left(\hat{Z}_{j,\vec{n}} \hat{\bar{Z}}_{j,\vec{n}} - \hat{\bar{Z}}_{j,\vec{n}-\hat{j}} \hat{Z}_{j,\vec{n}-\hat{j}} \right) \right|^2 + \frac{2g_{\mathrm{d}}^2}{a^d} \sum_{j<k} \left| \hat{Z}_{j,\vec{n}} \hat{Z}_{k,\vec{n}+\hat{j}} - \hat{Z}_{k,\vec{n}} \hat{Z}_{j,\vec{n}+\hat{k}} \right|^2 \right) + \Delta \hat{H}$$

- In the K-S limit $(m^2 \to \infty) \Longrightarrow (\hat{Z}_{j,\vec{n}} \hat{\bar{Z}}_{j,\vec{n}} \to \mathbf{1}_N)$, the second term $\to 0$, so it can be omitted:

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- Additionally, other terms $(\hat{Z}_{j,\vec{n}} \hat{\bar{Z}}_{j,\vec{n}} \to \mathbf{1}_N)$ inside the modulus can be omitted:

$$\boxed{\hat{H}_2 = \sum_{\vec{n}} \mathrm{Tr}\left(\sum_{j=1}^{d} \hat{P}_{j,\vec{n}} \hat{\bar{P}}_{j,\vec{n}} - \frac{2g_{\mathrm{d}}^2}{a^d} \sum_{j<k} \left(\hat{Z}_{j,\vec{n}} \hat{Z}_{k,\vec{n}+\hat{j}} \hat{\bar{Z}}_{j,\vec{n}+\hat{k}} \hat{\bar{Z}}_{k,\vec{n}} + \mathrm{h.c.} \right) \right) + \Delta\hat{H}}$$

Two cheaper SU(N) Orbifoldish Hamiltonians

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- **Is the K-S limit well under control for H_1 and H_2? $\longrightarrow$ Monte Carlo Lattice Simulations**

How to use H, H_1 and H_2 towards some physics?

$$\hat{H}$$

- Choose a system (e.g. 2-plquette ...)
- Fix m^2 and $a \longrightarrow$ remove field truncation
- Extrapolation $a \to 0$ and $L \to \infty$

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$\hat{H}_1$ and $\hat{H}_2$

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Please bare in mind "naturalness issue for scalar fields" $m^2_{\mathrm{phys}} \sim m^2_{\mathrm{bare}} + \frac{\Lambda_0}{a^2}$

Orbifold for $SU(2)$

$SU(2)$ Orbifold-ish Lattice on $\mathbb{R}^4$

- Let's halve the number of boson per link:

$SU(2)$ Orbifold-ish Lattice on $\mathbb{R}^4$

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$$\text{Orbifold} \longrightarrow \text{Orbifold-ish}$$

$$\mathrm{SU}(2) \subset \mathbb{C}^4 \cong \mathbb{R}^8 \longrightarrow \mathrm{SU}(2) \cong \mathbb{S}^3 \subset \mathbb{R}^4$$

8 bosons per link $\longrightarrow$ **4 bosons per link**

$SU(2)$ Orbifold-ish Lattice on $\mathbb{R}^4$

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Orbifold	$\longrightarrow$	Orbifold-ish
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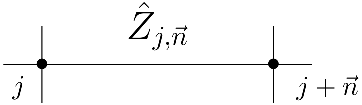
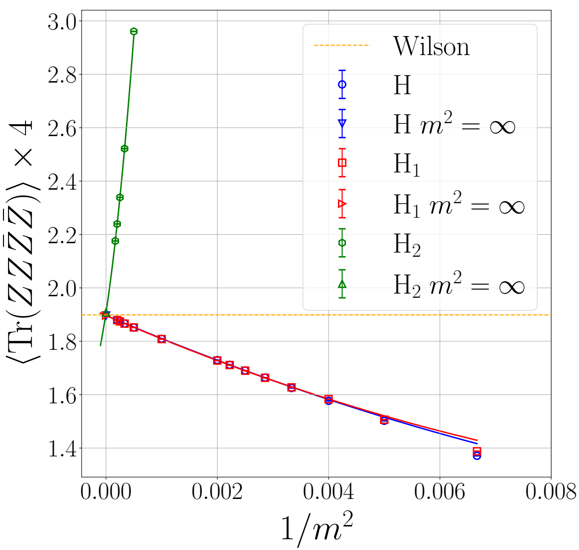


$$Z_{j,\vec{n}} = \sqrt{\frac{a^{d-2}}{2g_d^2}}\, W_{j,\vec{n}}\, U_{j,\vec{n}}$$

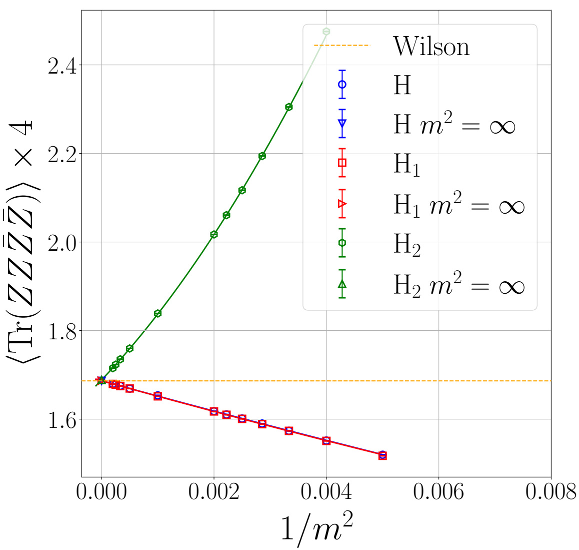
$$m^2 \to \infty \quad \longrightarrow \quad Z \to\in SU(2)$$

The Kogut-Susskind limit for $SU(2)$ Orbifoldish Lattice on $\mathbb{R}^4$ $\left(m^2 \to \infty\right)$

Spatial plaquette $\langle \mathrm{Tr}(ZZ\bar{Z}\bar{Z})\rangle$



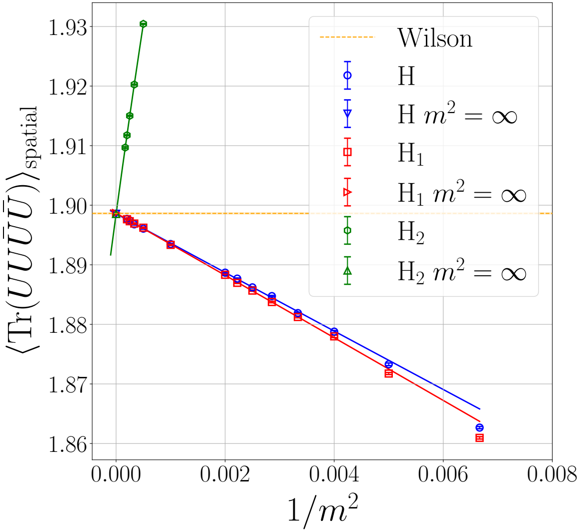
- $8^2 \times 8 \quad a_s = a_t = 0.1$



- $8^2 \times 8 \quad a_s = a_t = 0.3$

- Orbifold lattice agrees with Wilson action

Spatial plaquette $\langle \mathrm{Tr}(UUU^\dagger U^\dagger) \rangle_{\text{spatial}}$



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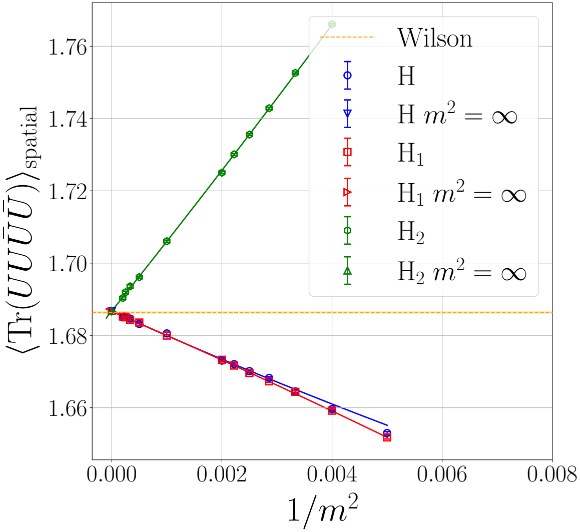

- $8^2 \times 8 \quad a_s = a_t = 0.3$

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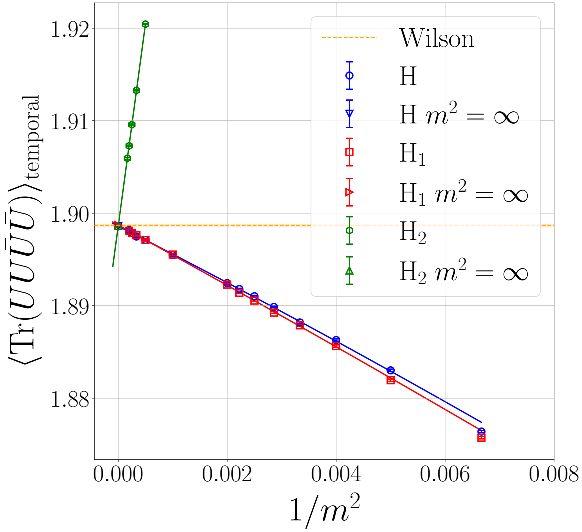
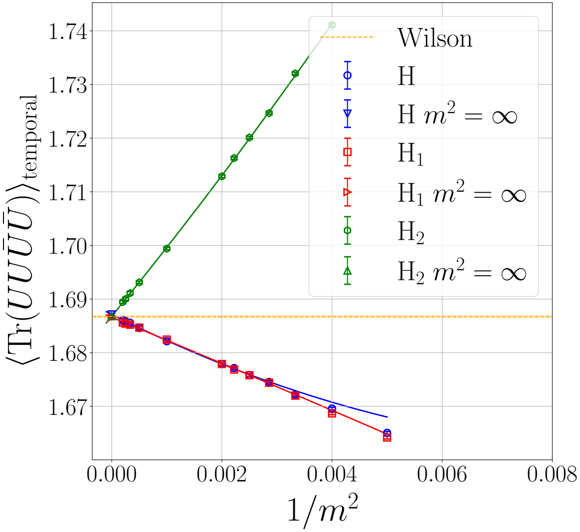
Spatial plaquette $\langle \mathrm{Tr}(UUU^\dagger U^\dagger)\rangle_{\mathrm{temporal}}$



- $8^2 \times 8 \quad a_s = a_t = 0.1$



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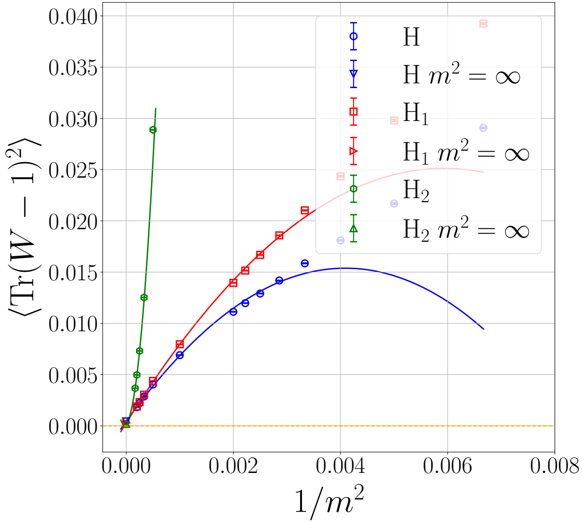
$\langle \mathrm{Tr}(W - \mathbf{1}_N)^2 \rangle$



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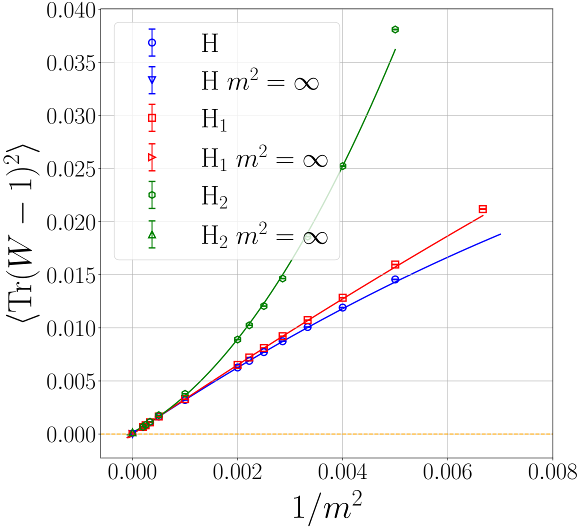


- $8^2 \times 8 \quad a_s = a_t = 0.3$

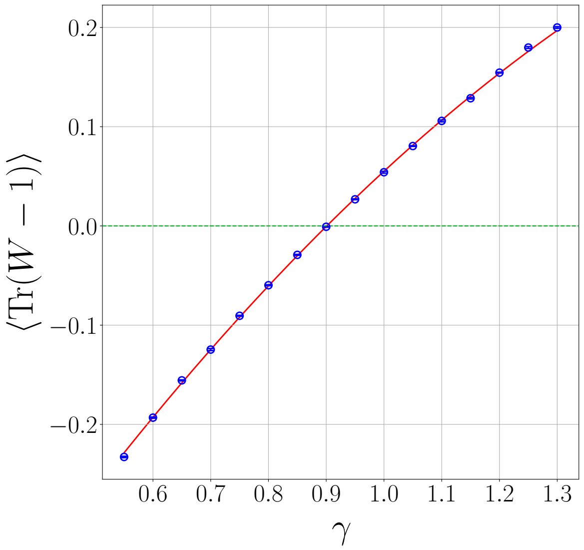
- Numerical confirmation that the large mass scalar ϕ decouples ($W = \exp(a g_d \phi) \to \mathbf{1}_N$)

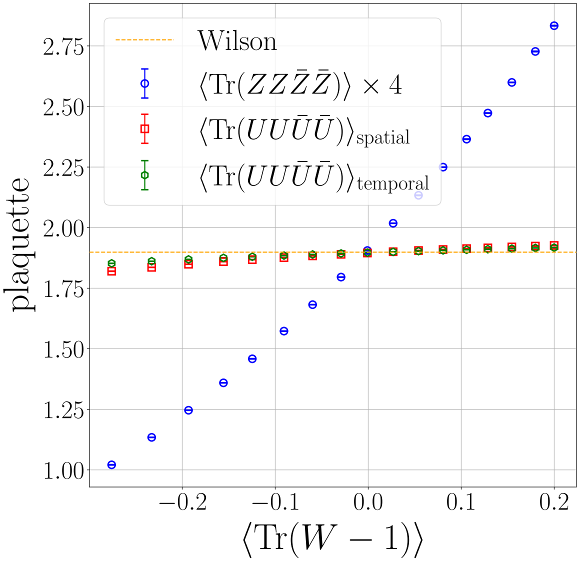
Eliminating the needs of large m^2 by a counter-term

$$-\gamma \cdot \sum_{j,\vec{n}} \mathrm{Tr}(\hat{Z}_{j,\vec{n}} \hat{\bar{Z}}_{j,\vec{n}})$$

Tuning the counter-radiative term parameter γ

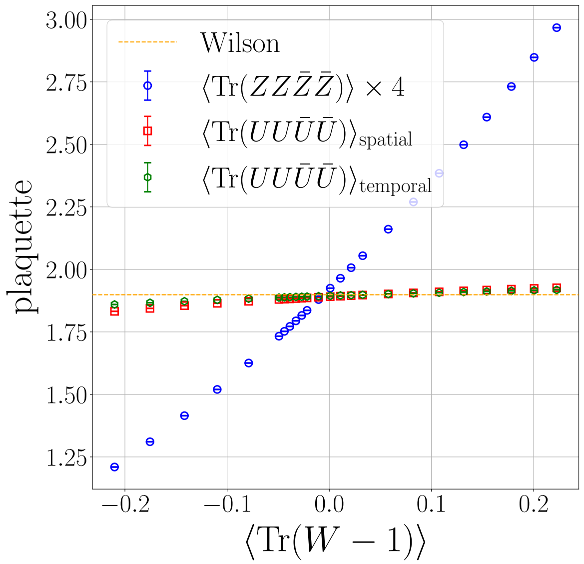
$H \quad m^2 = 50 \quad 8^2 \times 8 \quad a_s = a_t = 0.1$

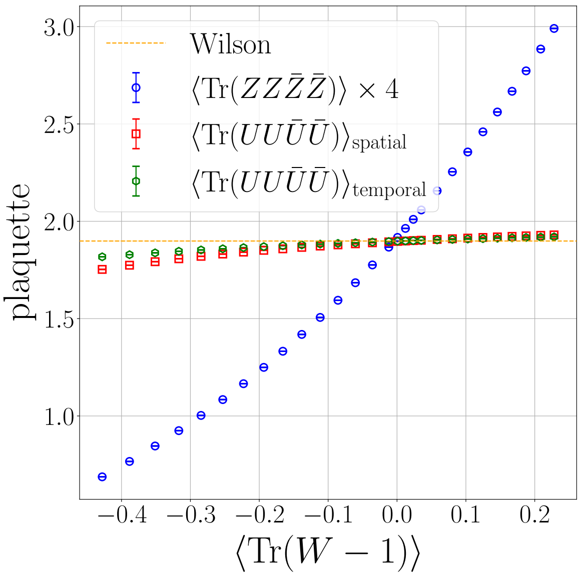




- Tuning γ we get agreement for much smaller values of m^2

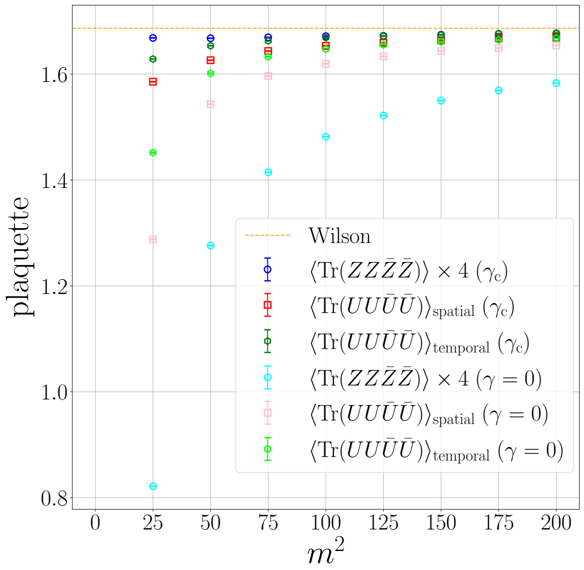
Tuning γ for H_1 and H_2

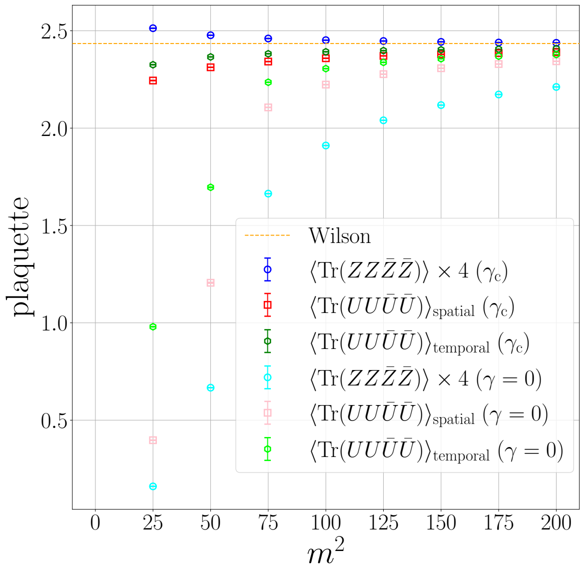




- H_1 $m^2 = 50$ $8^2 \times 8$ $a_s = a_t = 0.1$
- H_2 $m^2 = 500$ $8^2 \times 8$ $a_s = a_t = 0.1$

H with and without γ for SU(2) and SU(3)





- $SU(2)$ $\quad 8^2 \times 8$ $\quad a_s = a_t = 0.3$
- $SU(3)$ $\quad 8^2 \times 8$ $\quad a_s = a_t = 0.2$

Conclusions

Take-home message

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- Orbifold lattice reproduce Kogut-Susskind in a controlled limit ($m^2 \to \infty$)
- Lattice simulations confirm no issues in extracting the K-S limit for SU(2) and SU(3) in (2+1)d

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Ongoing work:

- Explore Orbifold lattice properties, like phase diagram etc.
- Quantify the bosons truncation effects for quantum simulations
- Quantum simulation of Orbifold lattice for SU(2) and SU(3) systems


4th QuantHEP @ Queen Mary
University of London
LONDON, JULY 13-16, 2026
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Southampton

Invited speakers:

Alessio Celi (Universitat Autònoma de Barcelona)
Bipasha Chakraborty (University of Southampton)
Luca Dellantonio (University of Exeter)
Elisa Ercolessi (University of Bologna)
Lena Funcke (University of Bonn)
Tomoya Hayata (Keio University)
Philipp Hauke (University of Trento)
Junichi Haruna (University of Osaka)
Joachim Kopp (University of Mainz)
Sarah Malik (University College London)
Enrique Rico Ortega (CERN)
Johann Ostmeyer (University of Bonn)
Masahito Yamazaki (University of Tokyo)
Enrico Rinaldi (Quantinuum)
German Rodrigo (Instituto de Física Corpuscular)
Andreas Schaefer (University of Regensburg)
Alessandro Roggero (University of Trento)
Sofia Vallecorsa (CERN)
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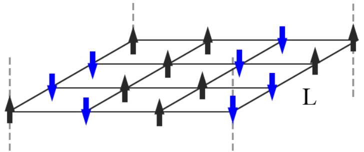
Thank you all for your time!

Time for Questions!

Backup Slides

Classical Vs Quantum resources

Let's consider a simple 3D spin model system with N spin-1/2



$\longrightarrow$ Hamiltonian $2^N \times 2^N$
(Hamiltonian approach is not feasible)

Let's estimate the computational resources needed to study a system with N spins:

$Z = \int dx\, dy\, dz\, e^{-S} \longrightarrow \approx 2^N$

Classical computer: 2^N memory slots $\Longrightarrow$ Quantum computer: N qubits

---

Let's double the system: N spins $\longrightarrow 2N$ spins

Classically

- $2^N \longrightarrow \left(2^{2N}\right) = \left(2^N\right)^2$

(escape importance sampling Monte Carlo)

Quantum

- N qubits $\longrightarrow 2N$ qubits

Hamiltonian formalism is feasible on quantum computer!

The gauge invariance of the SU(N) orbifold lattice

- The $\hat{H}$ is invariant under the local U(N) transformation:

$$\hat{Z}_{j,\vec{n}}\,,\ \ \hat{P}_{j,\vec{n}} \quad \rightarrow \quad \Omega^{-1}_{\vec{n}}\hat{Z}_{j,\vec{n}}\Omega_{\vec{n}+\hat{j}} \qquad \Omega^{-1}_{\vec{n}}\hat{P}_{j,\vec{n}}\Omega_{\vec{n}+\hat{j}}$$

- The generator of the transformation such that $[\hat{G}_{\vec{n}}, \hat{H}] = 0$ is:

$$\hat{G}_{\vec{n},pq} \equiv i\sum_{j=1}^{3}\left(-\hat{Z}_{j,\vec{n}}\hat{\bar{P}}_{j,\vec{n}} + \hat{P}_{j,\vec{n}}\hat{\bar{Z}}_{j,\vec{n}} - \hat{\bar{Z}}_{j,\vec{n}-\hat{j}}\hat{P}_{j,\vec{n}-\hat{j}} + \hat{\bar{P}}_{j,\vec{n}-\hat{j}}\hat{Z}_{j,\vec{n}-\hat{j}}\right)_{pq}$$

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The introduced counter-term, partially "compensate" the Jacobian contribution:

$$-\gamma \cdot \frac{g_d^2}{a^{d-1}} \sum_{j,\vec{n}} \mathrm{Tr}(\hat{Z}_{j,\vec{n}} \hat{\overline{Z}}_{j,\vec{n}}) \sim \mathrm{Tr}(Z\,\overline{Z}) = \mathrm{Tr}\left(e^{2ag\phi}\right) = N + 2ag\,\mathrm{Tr}(\phi) + 2(ag)^2\mathrm{Tr}(\phi^2) + \cdots$$

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